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AN INTEGRAL REPRESENTATION ON THE PATH SPACE FOR SCATTERING LENGTH

Dedicated to Professor N. Ikeda on the occasion of his sixtieth birthday

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0. The *scattering length* Γ is the limit of the scattering amplitude $f_k(e, e')$ as the wave number k tends to 0. It is independent of the choice of unit 3-vectors e and e'. The scattering amplitude is defined as the unique constant $f_k(e, e')$ such that there holds the asymptotics

$$\phi_k(x) \sim e^{ik\langle u,e\rangle} + f_k(e, e')\, e^{ik\langle u,e'\rangle}/|x| \quad \text{as} \quad |x| \to \infty \quad \left(e' = \frac{x}{|x|}\right)$$

for a solution ϕ_k, called the *scattering solution*, of the equation

$$\Delta\phi_k - v\phi_k = -k^2\,\phi_k\,,$$

where v is a given potential which is assumed to be nonnegative and integrable. As M. Kac proved,

$$\text{(1)} \qquad \Gamma = \frac{1}{2\pi}\int_{R^3} v(x)\,\phi_0(x)\,dx$$

where $\phi_0(x)$ is the solution of

$$\text{(2)} \qquad \phi_0(x) = 1 - \frac{1}{2\pi}\int_{R^3} \frac{v(y)\,\phi_0(y)}{|x-y|}\,dy\,.$$

In [4], M. Kac gave the formula

$$\text{(3)} \qquad \Gamma = \frac{1}{2\pi}\lim_{t\to\infty}\frac{1}{t}\int_{R^3} E_x[1-\exp\,(-\int_0^t v(w(s))\,ds)]\;dx$$

where E_x denotes the expectation with respect to the three dimensional Brownian motion starting at x. He conjectured that

(C1) the scattering length $\Gamma = \Gamma(\alpha v)$ for the potential αv has limit as α goes to infinity and

(C2) the limit, say γ_U, is independent of the choice of potential v and depends only on the support $U = \{x;\, v(x) > 0\}$.

The purpose of the present note is to prove the conjecture C1–2 by giving an integral representation of the scattering length $\Gamma(v)$ on the path space W, where $W=C((-\infty, +\infty), \boldsymbol{R}^3)$ for the above case.

1. Let us state the result in a little more general setup. Consider a transient Markov process with state space R which admits a reversible invariant measure λ. Assume that R is a Polish space, λ is a Radon measure on R and the path is continuous. Now we define the *scattering length* by the formula

$$\Gamma(v) = \lim_{t\to\infty} \frac{1}{t} \int_R E_x[1-\exp(-\int_0^t v(w(s))\,ds)]\,\lambda(dx) \tag{3)$'$ (}$$

for continuous functions v with compact support on R, where E_x denotes the expectation with respect to the Markov process starting at x. A proof of the existence is given in Lemma 2 below.

By the reversibility the path may be considered to be defined for both positive and negative time and then, given a starting point $x=w(0)$ at time 0, the process $w(-t)$, $t\geqq 0$, is an independent copy of $w(t)$, $t\geqq 0$. So we take

$$W = C((-\infty, +\infty), R)$$

and define a measure Λ on the path space W by

$$\int_W \Lambda(dw)\,\Phi(w) = \int_R \lambda(dx) \int_W P_x(dw)\,\Phi(w)$$

for bounded Borel function Φ on W, where P_x denotes the law of the Markov process starting at x at the initial time 0.

Theorem. *Let v be a nonnegative continuous function with compact support on R. Then,*

$$\Gamma(v) = \int_{S(v)} \Lambda(dw) \frac{1-\exp(-\int_{-\infty}^{+\infty} v(w(t))\,dt)}{\int_{-\infty}^{+\infty} v(w(t))\,dt}\, v(w(0)) \tag{4}$$

where

$$S = S(v) = \{w;\ \int_{-\infty}^{+\infty} v(w(t))\,dt > 0\}\ . \tag{5}$$

REMARK 1. It is known [4] that $\Gamma(v)\leqq C(K)$, where $C(K)$ is the electrostatic capacity of the closure K of the set

$$U = \{x;\ v(x)>0\}$$

for the 3-dimensional Brownian motion. Similar bounds hold in general cases. Hence, $\Gamma(v)$ is finite if so is the capacity $C(K)$.

Corollary. *Let u and v be two nonnegative continuous function with common support U. Then the limit $\lim_{\alpha\to\infty} \Gamma(\alpha u)$ exists and is equal to $\lim_{\alpha\to\infty} \Gamma(\alpha v)$.*

Consequently, the conjecture (C1–2) is true.

REMARK 2. The proof given below works for certain nonegative Borel functions v, such as the indicator of a compact set which is the closure of its interior. Thus Corollary is also valied for such functions.

Proof of Corollary. From the formula (4) it follows that the monotone limit $\lim_{\alpha\to\infty} \Gamma(\alpha v)$ exists and is equal to

$$\int_S \Lambda(dw) \frac{v(w(0))}{\int_{-\infty}^{+\infty} v(w(t))\,dt} . \tag{6}$$

Since λ is an invariant measure for the Markov process (W, P_x), the measure Λ is invariant under the time shift $w(t)\to w(t+s)$ for any s. Furthermore, it is invariant under the time reversion $w(t)\to w(-t)$ by the reversibility of λ.

Keeping in mind these properties and the facts that S is common for u and v and that the fucntions

$$\int_{-\infty}^{+\infty} v(w(t))\,dt \quad \text{and} \quad \int_{-\infty}^{+\infty} u(w(t))\,dt$$

are invariant under either of the shift and the reversion, we obtain

$$\begin{aligned}
&\lim \Gamma(\alpha v) \\
&= \int_S \Lambda(dw) \frac{v(w(0))}{\int_{-\infty}^{+\infty} v(w(t))\,dt} \\
&= \int_S \Lambda(dw) \frac{v(w(0))}{\int_{-\infty}^{+\infty} v(w(t))\,dt} \frac{\int_{-\infty}^{+\infty} u(w(s))\,ds}{\int_{-\infty}^{+\infty} u(w(t))\,dt} \\
&= \int_{-\infty}^{+\infty} ds \int_S \Lambda(dw) \frac{v(w(0))}{\int_{-\infty}^{+\infty} v(w(t))\,dt} \frac{u(w(s))}{\int_{-\infty}^{+\infty} u(w(t))\,dt} \\
&= \int_{-\infty}^{+\infty} ds \int_S \Lambda(dw) \frac{v(w(-s))}{\int_{-\infty}^{+\infty} v(w(t))\,dt} \frac{u(w(0))}{\int_{-\infty}^{+\infty} u(w(t))\,dt} \\
&\qquad\qquad\qquad\qquad \text{(by shift invariance)} \\
&= \int_{-\infty}^{+\infty} ds \int_S \Lambda(dw) \frac{v(w(s))}{\int_{-\infty}^{+\infty} v(w(t))\,dt} \frac{u(w(0))}{\int_{-\infty}^{+\infty} u(w(t))\,dt} \\
&\qquad\qquad\qquad\qquad \text{(by the reversibility)} \\
&= \int_S \Lambda(dw) \frac{u(w(0))}{\int_{-\infty}^{+\infty} u(w(t))\,dt} \frac{\int_{-\infty}^{+\infty} v(w(s))\,ds}{\int_{-\infty}^{+\infty} v(w(t))\,dt}
\end{aligned}$$

$$= \int_S \Lambda(dw) \frac{u(w(0))}{\int_{-\infty}^{+\infty} u(w(t))\,dt} = \lim \Gamma(\alpha u)\,.$$

Consequently, we obtain Corollary.

2. Now let us proceed to the proof of Theorem. At first we give another expression for the scattering length Γ. For this sake we prepare the following:

Lemma 1. *For a continuous function v with compact support on R,*

$$\int_0^T dt\, E_x[v(w(t)) \exp\{-\int_0^t v(w(s))\,ds\}] \tag{7}$$
$$= 1 - E_x[\exp\{-\int_0^T v(w(t))\,dt\}] \quad (0 \leqq T \leqq \infty)\,.$$

Proof. For $T < \infty$ the left hand side is equal to

$$\int_0^T dt\, E_x[-\frac{d}{dt} \exp\{-\int_0^t v(w(s))\,ds\}]\,,$$

which is equal to the right hand side. For $T = \infty$ the convergence is assured by the monotonicity, or, directly, by the transience:

$$\int_0^\infty E_x[v(w(t))]\,dt < \infty\,.$$

Lemma 2. *Let v be a nonnegative continuous function with compact support on R. Then,*

$$\Gamma(v) = \int_W \Lambda(dw)\, v(w(0)) \exp\{-\int_0^\infty v(w(t))\,dt\} \tag{8}$$
$$= \int_R \lambda(dx)\, v(x)\, E_x[\exp\{-\int_0^\infty v(w(t))\,dt\}]\,.$$

Proof. Note that

$$E_x[1 - \exp\{-\int_0^t v(w(s))\,ds\}]$$
$$= E_x[\int_0^t ds\, v(w(s)) \exp\{-\int_0^s v(w(r))\,dr\}]\,.$$

Integrating this against λ, we obtain

$$\frac{1}{t}\int_W \Lambda(dw)\,[1 - \exp\{-\int_0^t v(w(s))\,ds\}]$$
$$= \frac{1}{t}\int_0^t ds \int_W \Lambda(dw)\, v(w(s)) \exp\{-\int_0^s v(w(r))\,dr\}$$
$$= \frac{1}{t}\int_0^t ds \int_W \Lambda(dw)\, v(w(0)) \exp\{-\int_{-s}^0 v(w(r))\,dr\}$$

(by the shift invariance)

$$= \frac{1}{t}\int_0^t ds \int_W \Lambda(dw)\, v(w(0)) \exp\{-\int_0^s v(w(r))\, dr\}$$

(by the reversibility)

$$\to \int_W \Lambda(dw)\, v(w(0)) \exp\{-\int_0^\infty v(w(t))\, dt\}$$

as $t \to \infty$, as is desired.

The formula (8) together with (7) enables us to compute the derivative of the functional Γ at v, which will be denoted by $D\Gamma(v)$:

$$D\Gamma(v)f = \lim_{t \downarrow 0} \frac{1}{t}\{\Gamma(v+tf) - \Gamma(v)\} \tag{9}$$

for nonnegative continuous functions f with compact support on R. One can remove the restriction that f is nonnegative and may prove that $D\Gamma(v)$ is the Fréchet derivative. But here we only need the Gateaux derivative from the right, whose existence is obvious from the formula (7) by virtue of the transience.

Lemma 3. *The following formula holds for* $D\Gamma(v)$:

$$\begin{aligned} D\Gamma(v)f &= \int_R \lambda(dx) f(x) (E_x[\exp\{-\int_0^\infty v(w(t))\, dt\}])^2 \\ &= \int_W \Lambda(dw) f(w(0)) \exp\{-\int_{-\infty}^\infty v(w(t))\, dt\}\,. \end{aligned} \tag{10}$$

Proof. Let us differentiate the second expression for Γ in (7). Let us write

$$g(x) = E_x[\exp\{-\int_0^\infty v(w(s))\, ds\}]\,.$$

Then,

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0+} \Gamma(v+tf) \\ &= \int_R \lambda(dx) f(x) g(x) \\ &\quad + \int_R \lambda(dx)\, v(x)\, E_x[-\int_0^\infty f(w(s))\, ds \exp\{-\int_0^\infty v(w(t))\, dt\}] \\ &= \int_R \lambda(dx) f(x) g(x) \\ &\quad -\int_0^\infty ds \int_W \Lambda(dw)\, v(w(0)) f(w(s)) \exp\{-\int_0^\infty v(w(t))\, dt\}\,. \end{aligned}$$

Now the second term can be written as

$$-\int_0^\infty ds \int_W \Lambda(dw)\, v(w(-s)) \exp\{-\int_{-s}^0 v(w(t))\, dt\}\, f(w(0)) \exp\{-\int_0^\infty v(w(s))\, ds\}$$

(by the shift invariance)

$$= -\int_0^\infty ds \int_W \Lambda(dw)\, v(w(-s)) \exp\{-\int_{-s}^0 v(w(t))\, dt\}\, f(w(0))\, g(w(0))$$

(by the Markov property)

$$= -\int_0^\infty ds \int_R \lambda(dx) f(x)\, g(x)\, E_x[v(w(s)) \exp\{-\int_0^s v(w(t))\, dt\}]$$

(by the reversibility)

$$= -\int_R \lambda(dx) f(x)\, g(x)\, [1-g(x)]$$

by virtue of Lemma 2. Consequently,

$$D\Gamma(v) f = \int_R \lambda(dx) f(x)\, g(x)^2 .$$

Finally, by the reversibility we obtain the expression

$$g(x)^2 = (E_x[\exp\{-\int_0^\infty v(w(s))\, ds\}])^2 = E_x[\exp\{-\int_{-\infty}^\infty v(w(s))\, ds\}] .$$

The proof is completed.

Proof of Theorem. From Lemma 3 it follows that

$$\frac{d}{d\alpha} \Gamma(\alpha v) = \int_S \Lambda(dw)\, v(w(0)) \exp\{-\alpha \int_{-\infty}^\infty v(w(s))\, ds\} + \int_{S^c} \Lambda(dw)\, v(w(0)) .$$

Note that $\Gamma(\alpha v) \to 0$ as $\alpha \to 0$ and that the second term in the right hand side vanishes because of the definition of the set S and the continuity of the path. Consequently, we obtain

$$\Gamma(v) = \int_0^1 d\alpha \int_S \Lambda(dw)\, v(w(0)) \exp\{-\alpha \int_{-\infty}^\infty v(w(s))\, ds\}$$
$$= \int_S \Lambda(dw)\, v(w(0)) \frac{1-\exp\{-\int_{-\infty}^\infty v(w(s))\, ds\}}{\int_{-\infty}^\infty v(w(s))\, ds\}} .$$

Hence the proof is completed.

REMARK 3. In the case of three dimensional Brownian motion the constant γ_U with $U=\text{int } K$ coincides for "nice" compacts K called *semiclassical* by Kac [2] (cf. [3] for counter-example) with the electrostatic capacity $C(K)$, for which a similar result to (3) (and more) was obtained earlier by F. Spitzer [7]. A further historical remark can be found in [6].

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