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ON SOME ABSTRACT DEGENERATE PROBLEMS OF PARABOLIC TYPE-3: APPLICATIONS TO LINEAR AND NONLINEAR PROBLEMS

Dedicated to Professor B. Pini on his 70th birthday

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0. Introduction

In papers [11], [12] (which will be called henceforth part I and part II, respectively) we developed some abstract results about existence and uniqueness of solutions of Cauchy problems related to degenerate evolution equations which might be put under the general pattern

$$\frac{d}{dt}(M(t)u(t))+L(t)u(t)=f(t,u(t)),\ \text{for every}\ t\in[0,\tau] \tag{0.1}$$

where L, M are linear operators, possibly depending upon t (we shall call it the *time variable*, as opposed to the variable x of the functional space in which $u(t)$ lies, to be referred to as the *space variable*); more precise assumptions will be made in each particular case.

In parts I and II the technique involved deeper abstraction, leading to study of (0.1) as a particular case of more general, quite algebraically-looking equations:

$$BMu+Lu=F(u) \tag{0.2}$$

or even

$$BMu=f(u); \tag{0.3}$$

here, we recall or restate in the specific case those results: while doing so, we hope to make this part reasonably self-contained.

A remark about style: the assumptions which work are sometimes rather lengthy, the details cumbersome; we choose therefore not to seek maximal generality; in some applications, we do not emphasize degeneration, even though we might introduce it throughout, and stress nonlinearity. Sometimes assumptions could be weakened: we often label 'regular' (='of cass $C^{(1)}$') something which could have been differentiable, plus something better, or even less, as well. At first, problems and equations are formally stated, i.e. without precise assumptions about objects involved: these will be made clear after appropriate discussion, and the cooperative reader should understand better the aim of some postulations.

Here is the plan of this part: §1 contains a discussion about linear problems related to (0.1), that is, the case f independent of u. In 1.4 (Example 1) we study the nondegenerate linear case M=identity, L=strongly uniformly elliptic; in 1.5 (Example 2) a more specific framework is involved, as $L(t)$, $M(t)$ are related to quadratic forms in Hilbert spaces. These two examples are rather general models, recorded for future reference. A first application, to degenerate integrodifferential equations follows (1.9). It is nearly impossible to give a full bibliographical discussion of this topic, studied extensively by many authors: we refer mainly to the beautiful papers of Lunardi and Sinestrari [19], and Lunardi [18]: in the former the ambient space is h^{α} (little-hoelder continuous functions), but the calculations may be adapted to the C^{α} case (personal communication of Prof. A. Lunardi to the second author); in the latter a more general type of kernel than ours is considered, but in the framework of α-hoelder continuity. In §2 we show how to reduce into form (0.1) some other equations, and obtain new existence results (theorems 4 and 5), in case of a truly nonlinear right-hand side. In §3, 4 we give the main applications: to semilinear equations (3.1), degenerate semilinear equations in Hilbert space setting (3.2), fully nonlinear equations in spaces of continuous functions (3.3), degenerate parabolic equations (3.4), higher dimensional problems (3.5, 3.6), abstract Navier-Stokes equations (4.1 and 4.2); finally, in 4.3, we discuss shortly the equation $u'(t)=F(-A(t)u(t))$: references will be given when appropriate.

Notations are rather standard: see however part I; as a general remark, we use the ∂-notation for partial differentials even in the abstract case: this should cause no confusion.

Unless otherwise stated, constants $C, k, \cdots$ are meant to be positive numbers independent of the relevant parameters; they are subject to numerical change even from step to step.

1. Linear Problems

Recall that we are concerned with concrete instances of the abstract equation

$$BMu+Lu = F(u) \tag{1}$$

where B is a closed invertible linear operator in E_1, L, M are closed linear operators from E_2 into E_1 (two complex Banach spaces), F is a (possibly nonlinear) mapping from a subset of E_2 into E_1. The equation

$$BMu = f(u) \tag{2}$$

may be put into the form (1) conveniently if e.g. f is differentiable at a point $u_0 \in E_2$, since then (2) becomes

$$BMu = f'(u_0)\,u + G(u)$$

with $G(u) = f(u) - f'(u_0)\,u$.

Let us now restate the assumptions of parts I (§3: $(H1)-(H3)$) and II (§3, Theorem 1) concerning (1):

(A) $\mathcal{D}(B)$ is dense in E_1 and $\forall\, z \in \boldsymbol{C}$, $|\pi - \arg z| \leq \phi < \pi/2$, $B - zI$ is invertible, with $\|(B-zI)^{-1};\ \mathcal{L}(E_1)\| \leq C(1+|z|)^{-1}$;

(B) L, M are two closed linear operators from E_2 into E_1, with L invertible and $\mathcal{D}(L) \subseteq \mathcal{D}(M)$; moreover, if ϕ is as above, and $\varepsilon > 0$ is small, we suppose that $\forall\, z \in \boldsymbol{C}$, $|\arg z| < \pi - \phi + \varepsilon$ there exists $L(zM+L)^{-1}$ and $\|L(zM+L)^{-1};\ \mathcal{L}(E_1)\| \leq C$;

(C) let Γ be a path in the complex plane parametrized by $t \to t\exp(\pm i\Phi)$, $t \geq a_0 > 0$ (we put $\Phi = \pi - \phi + \varepsilon/2$ for short), and by $t \to a_0 \exp(it)$, $|t| \leq \Phi$; we set $V = V_\theta = (E_1; \mathcal{D}(B))_{\theta,\infty}$ with $0 < \theta < 1$, $T = ML^{-1}$, and assume that there exists θ such that (s.t.) for every $z \in \Gamma$ the commutator $[B; (zT+I)^{-1}]$ has bounded extensions as an operator from E_1 into itself and from V_θ into itself as well, with bounds $\max\ \{\|[B; (zT+I)^{-1}];\ \mathcal{L}(E_1)\|, \|[B; (zT+I)^{-1}];\ \mathcal{L}(V)\|\} < C(1+|z|)^\sigma$ with suitable $\sigma \in [0, 1[$;

(D) Let $r > 0$ and write S_1 for the closed V-ball at the origin with radius r. Assume the existence of κ, β s.t. $0 < \beta < \kappa$,
$\|F(L^{-1}h);\ V\| \leq r\kappa\ \forall\, h \in S_1$,
$\|F(L^{-1}h_1) - F(L^{-1}h_2);\ V\| \leq \beta\|h_1 - h_2;\ V\|$
$\forall\, h_1, h_2 \in S_1$, with θ held fixed as previously.

We obtained in Part II (§3, Theorem 1):

Theorem 1. *Let* (A) *through* (D) *hold true. Then* (1) *has exactly one solution u with Lu in $V = V_\theta$.*

We may restate easily the preceding theorem in the linear case, in which case assumption (D) may be consistently weakened.

Proposition 1. *Assume* (A), (B), (C), *and let $F(u) = Ku + f$ with $f \in V_\theta$ and $K \in \mathcal{B}(\mathcal{L}(L), E_1)$: If $KL^{-1} \in \mathcal{L}(V_\theta)$ has norm suitably small, then* (1) *is uniquely solvable in such a way that $Lu \in V_\theta$.*

Note that, if KL^{-1} commutes with B (that is, for every $u \in \mathcal{D}(B)$, $KL^{-1}u \in \mathcal{D}(B)$ and $KL^{-1}Bu = BKL^{-1}u$) then $KL^{-1} \in \mathcal{L}(V_\theta)$ by interpolation.

1.1. First of all, let us write down the linear problem we shall be concerned with:

$$(LP) \qquad \begin{cases} \dfrac{d}{dt}(M(t)u(t)) + L(t)u(t) = f(t),\ t \in [0, \tau] \\ (M(t)u(t))_{t=0} = w_0 \end{cases}$$

and record for future reference some assumptions about it:
$\{L(t), t\in[0, \tau]\}$, $\{[M(t), t\in[0, \tau]\}$ are two families of closed linear operators from Y into X (X, Y complex Banach spaces) such that:

(i) $L(t)$ is invertible for every $t\in[0, \tau]$;

(ii) $\mathcal{D}(L(t))\subseteq\mathcal{D}(M(t))$ for every $t\in[0, \tau]$;

(iii) $t\to M(t)\,(L(t))^{-1} = T(t)$ is a continuous mapping $[0, \tau]\to\mathcal{L}(X)$;

(iv) $t\to(L(t))^{-1}$ is continuous as a mapping $[0, \tau]\to\mathcal{L}(X, Y)$;

(v) $\|(zT(t)+I)^{-1}; \mathcal{L}(X)\| = \|L(t)\,(zM(t)+L(t))^{-1}; \mathcal{L}(X)\|\leq C$
$\forall z\in\boldsymbol{C}$ with $\operatorname{Re} z\geq 0$, $\forall t\in[0, \tau]$;

(vi) $t\to M(t)\,(L(t))^{-1} = T(t)$ is a $C^{(1)}$ mapping $[0, \tau]\to\mathcal{L}(X)$;
$\forall z\in\boldsymbol{C}$ with $\operatorname{Re} z\geq 0$
$$\|\frac{\partial}{\partial t}(zT(t)+I)^{-1}; \mathcal{L}(X)\|\leq C(1+|z|)^{1-\rho}, \quad \rho\in]0, 1];$$

(vii) $\|T'(t)-T'(s); \mathcal{L}(X)\|\leq C|t-s|^{\varepsilon}, \qquad \varepsilon\in]0, 1]$.

Presently, let us keep f, w_0 undefined.

1.2. Lemma. *Under* (i)—(vii), *the estimate*
$$\|\frac{\partial}{\partial t}(zT(t)+I)^{-1}-\frac{\partial}{\partial s}(zT(s)+I)^{-1}; \mathcal{L}(X)\|\leq C|t-s|^{\varepsilon}(1+|z|)^{2-\rho},$$
holds for every $t, s\in[0, \tau]$, $z\in\boldsymbol{C}$ *with* $\operatorname{Re} z\geq 0$ *with exponents* ε, ρ, *as in* (vi)—(vii).

Proof. We simply work out the calculations:
$$\frac{\partial}{\partial t}(zT(t)+I)^{-1}-\frac{\partial}{\partial s}(zT(s)+I)^{-1} =$$
$$= z\{(zT(s)+I)^{-1}\,T'(s)\,(zT(s)+I)^{-1}-(zT(t)+I)^{-1}\,T'(t)\,(zT(t)+I)^{-1}\}\,.$$
Since
$$(zT(s)+I)^{-1}-(zT(t)+I)^{-1} = \int_t^s \frac{\partial}{\partial x}(zT(x)+I)^{-1}\,dx\,,$$
we obtain
$$\|(zT(t)+I)^{-1}-(zT(s)+I)^{-1}; \mathcal{L}(X)\|\leq C\,|t-s|\,(1+|z|)^{1-\rho},$$
whence the lemma follows easily.

1.3. Let us now choose $E_1=C_0[0, \tau; X]$, $\mathcal{D}(B)=\{u\in C_0^{(1)}[0, \tau; X]; u'(0)=0\}$ with $Bu=u'$; it is well known that $(E_1, \mathcal{D}(B))_{\theta,\infty}$ is $C_0^{(\theta)}[0, \tau; X]$, the space of continuous X-valued functions u on $[0, \tau]$ s.t. $u(0)=0$ and
$$\|u; C_0^{(\theta)}[0, \tau; X]\| = \max_{0\leq t\leq\tau}\|u(t); X\|+\sup_{\substack{0\leq t\leq\tau\\ t\neq\tau}}\frac{\|u(t)-u(s); X\|}{|t-s|^{\theta}},$$
to be written henceforth $\|\,\|u\|\,\|_{\theta}$, is finite.

Fix $\omega\in]0, \varepsilon]$: for every $u\in C_0^{(\omega)}[0, \tau; X]$ and $z\in \boldsymbol{C}$ with Re $z\geq 0$

$$\begin{aligned}&\|[\frac{\partial}{\partial t}(zT(t)+I)^{-1}]u(t)-[\frac{\partial}{\partial s}(zT(s)+I)^{-1}]u(s); X\|\\&\leq C|t-s|^{\varepsilon}(1+|z|)^{2-\rho}\|u(t); X\|+C(1+|z|)^{1-\rho}\|u(t)-u(s); X\|\end{aligned}$$

by the lemma, so

$$\begin{aligned}&|t-s|^{-\omega}\|[\frac{\partial}{\partial t}(zT(t)+I)^{-1}]u(t)-[\frac{\partial}{\partial s}(zT(s)+I)^{-1}]u(s); X\|\\&\leq C(1+|z|)^{2-\rho}\|\,\|u\|\,\|_{\omega}.\end{aligned}$$

Hence, by (vi)

$$\|[B; (zT+I)^{-1}]; \mathcal{L}(E_1)\|<C(1+|z|)^{1-\rho},$$
$$\|[B; (zT+I)^{-1}]; \mathcal{L}(V_{\omega})\|<C(1+|z|)^{2-\rho},$$

since now the commutator is a multiplication by $\dfrac{\partial}{\partial t}(zT(t)+I)^{-1}$.

After repeated interpolations ($(E_1, V_{\omega})_{\sigma,\infty}=V_{\omega\sigma}\ \forall\sigma\in]0, 1[$), we get

$$\|[B; (zT+I)^{-1}]; \mathcal{L}(V_{\sigma\omega})\|<C(1+|z|)^{1-\rho+\sigma}.$$

We therefore claim that an estimate like (C) holds in every $V_{\nu}(0<\nu<\rho\varepsilon)$ if $\rho>\sigma>0$. Note that at best, when ρ, ε equal 1, we may allow $0<\nu<1$. So, from Theorem 1, we deduced

Theorem 2. *Let $0<\nu<\rho\varepsilon$,* (i)-(vii) *hold, and fix any $f\in C^{(\nu)}[0, \tau; X]$, $w_0\in X$ s.t. $w_0(=T(0)v_0)\in\mathcal{R}(T(0))$ and $(f(0)-v_0-T'(0)v_0)\in\mathcal{R}(T(0))$. Then* (LP) *has one and only one strict solution u s.t. $L(\cdot)u(\cdot)\in C^{(\nu)}[0, \tau; X]$.*

1.4. Theorem 2 has some interest even if (LP) is not truly degenerate:

EXAMPLE 1. Let Ω be a bounded $C^{(m)}$ domain (=open connected set) in $\boldsymbol{R}^n$. We assume that

$$A(t, x; D)=\sum_{|\alpha|\leq 2m}a_{\alpha}(t, x)D^{\alpha}, \quad x\in\bar{\Omega}$$

is strongly elliptic, uniformly in $t\in[0, \tau]$ and that for every t the a_{α}'s with $|\alpha|=2m$ are continuous in $\bar{\Omega}$, while the lower degree terms a_{α} with $|\alpha|<2m$ are only assumed to be $L^{\infty}(\Omega)$, all being $C^{(1)}$ in t, with

$$\max_{k=0,1}\max_{|\alpha|\leq 2m}\sup_{x\in\Omega}\left|\frac{\partial^k}{\partial t^k}a_{\alpha}(t, x)-\frac{\partial^k}{\partial t^k}a_{\alpha}(s, x)\right|\leq L|t-s|^{h}$$

for every $t, s\in[0, \tau]$, h being a suitable exponent in $]0, 1]$.

Introduce next a normal system of boundary operators

$$B_j=B_j(t, y; D)=\sum_{|\beta|\leq m(j)}b_{j,\beta}(t, y)D^{\beta}\quad(y\in\partial\Omega, j=1, \cdots, m),$$

and suppose that the assumptions in [28, p. 140] hold.

If we fix p, $1<p<+\infty$, and define

$$\begin{aligned}
&\mathcal{D}(L(t)) = \{u\in W^{2m,p}(\Omega),\ B_j(t,\cdot\,;D)\,u(\cdot) = 0\ \forall t\in[0,\tau],\ \forall j = 1,\cdots,m\}\,,\\
&L(t)\,u = A(t,\cdot\,;D)\,u(\cdot),\ \forall t\in[0,\tau],\ u\in\mathcal{D}(L(t))\,,\\
&M(t) = \text{the identity operator } I,\\
&X = Y = L^p(\Omega):
\end{aligned}$$

it is known that (i)—(vii) are satisfied [28, pp. 140–144], so Theorem 2 applies to the L^p realization of the problem

$$\begin{cases} \dfrac{d}{dt}\,u(t)+A(t,\cdot,D)\,u(t) = f(t),\ t\in[0,\tau] \\ (u(t))_{t=0} = w_0 \end{cases}$$

if w_0, f are 'good' data in the sense of Theorem 2.

P. Acquistapace and B. Terreni [2] have shown that under similar conditions, it is possible to deduce all estimates (i)—(vii) when $L^p(\Omega)$ is replaced by $C(\overline{\Omega})$, so our theorem applies within this framework as well, provided we choose suitable f, w_0.

1.5. Kato and Tanabe in their basic work [15] gave a quite general example of a family of operators $\{L(t);\ t\in[0,\tau]\}$ with time dependent domains for which all the preceding conditions hold with $\rho=1/2$. They developed it in terms of a sesquilinear form in a Hilbert space; we give now another example which is reminescent of Kato and Tanabe's one.

EXAMPLE 2. W, V, H are complex separable Hibert spaces with dense and continuous inclusions $V\subsetneq W\subsetneq H$; we identify H alone with its own antidual, thus obtaining $H\subsetneq W'\subsetneq V'$ densely and continuously.

For every $t\in[0,\tau]$, let sesquilinear forms $a_0(t;\cdot,\cdot)$, $a_1(t;\cdot,\cdot)$ in V, W respectively, be given; suppose that for every u, v, x, y, t

$$\begin{aligned}
&|a_0(t;u,v)|\le C\|u;V\|\,\|v;V\|\\
&|a_1(t;x,y)|\le C\|x;W\|\,\|y;W\|\,,\\
&\mathrm{Re}\ a_0(t;u,u)\ge C\|u;V\|^2\,,\\
&a_1(t;x,x)\ge 0
\end{aligned}$$

As for the t-dependence, assume that $t\to a_0(t;u,v)$, $t\to a_1(t;x,y)$ are $C^{(1)}$ functions for every x, y, u, v, and are s.t.

$$\begin{aligned}
&|a_0'(t;u,v)|\le C\|u;V\|\,\|v;V\|\\
&|a_1'(t;x,y)|\le C\,|a_1(t;x,y)|\\
&|a_0'(t;u,v)-a_0'(s;u,v)|\le C\|u;V\|\,\|v;V\|\,|t-s|^{\varepsilon}
\end{aligned}$$

$$|a_1'(t; x, y) - a_1'(s; x, y)| \leq C\|x; W\| \, \|y; W\| \, |t-s|^{\varepsilon}$$

with an exponent ε in $]0, 1]$ and constants independent of time, for every x, y, u, v, t, s. Here the primes denote differentiation with respect to time.

For any $t \in [0, \tau]$, we define

$$V = \mathcal{D}(L(t)),\ \langle (L(t))\, u, v \rangle_V = a_1(t; u, v) \quad \text{and}$$
$$W = \mathcal{D}(M(t)),\ \langle (M(t))\, x, y \rangle_W = a_1(t; x, y)\ (\forall u, v \in V, \forall x, y \in W)\,.$$

It is easy to check that (i)—(vii) hold with $\rho = 1$; note only that

$$(L(t)^{-1})' = -L(t)^{-1} L'(t) L(t)^{-1}\,, \quad \text{and}$$
$$\|[L(t)^{-1} - L(s)^{-1}] f; V\| \leq C\,|t-s|\,\|f; V'\|\,;$$

since

$$\|[L(t) - L(s)]\, u; V'\| \leq C\,|t-s|\,\|u; V\|\,,$$

it follows that

$$\|[L'(t) - L'(s)]; \mathcal{L}(V, V')\| \leq C\,|t-s|^{\varepsilon}\,.$$

Quite similarly,

$$\|[M(t) - M(s)]; \mathcal{L}(V, V')\| \leq C\,|t-s|$$

and

$$\|[M'(t) - M'(s)]; \mathcal{L}(V, V')\| \leq C\,|t-s|^{\varepsilon}\,.$$

We obtain information about variational solutions of (LP), when M, L are sesquilinear forms: in other terms, our solution to (LP) is also, in particular, a solution of

$$-(w_0, v(0)) - \int_0^{\tau} a_1(t, u(t), v'(t))\, dt + \int_0^{\tau} a_0(t, u(t), v(t))\, dt = \int_0^{\tau} f(t)\, v(t)\, dt$$

for every 'test function' $v \in C^{(1)}[0, \tau; V]$ s.t. $v(\tau) = 0$, $(\cdot, \cdot)$ being the duality between V and V'.

Now, look for a bounded domain Ω in $\boldsymbol{R}^n$ with regular boundary $\partial\Omega$, and put

$$a_0(t; u, v) = \sum_{i,j=1}^{n} \int_{\Omega} [a_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j} + c(t, x)\, u\bar{v}]\, dx$$

where $u, v \in H_0^1(\Omega) = V \subset L^2(\Omega) = H$. The coefficients are supposed to be continuous on $[0, \tau] \times \bar{\Omega}$ with Lipschitz continuous time derivatives, and s.t. $\forall (t, x) \in [0, \tau] \times \bar{\Omega}$

$$\sum_{i,j=1}^{n} a_{ij}(t, x)\, z_i \bar{z}_j \geq \gamma \sum_{i=1}^{n} |z_i|^2 \quad \forall z_1, \cdots, z_n \in \boldsymbol{C}\,,$$
$$c(t, x) \geq 0\,.$$

As for a_1, choose

$$a_1(t, u, v) = \int_\Omega m(t, x)\, u(x)\, \bar{v}(x)\, dx$$

with m continuous and nonnegative on $[0, \tau]\times\bar{\Omega}$. The key condition on m, which allows an application of Theorem 2 is:

$$(3) \qquad \begin{cases} \text{there exists } \dfrac{\partial m}{\partial t} \text{ as a continuous function, and} \\ \left| \displaystyle\int_\Omega \frac{\partial m}{\partial t}(t, x)\, u(x)\, \bar{v}(x)\, dx \right| \le \left| \displaystyle\int_\Omega m(t, x)\, u(x)\, \bar{v}(x)\, dx \right| \end{cases}$$

Clearly, (3) holds if we may separate variables, i.e. $m(t, x) = k(t)\, m_1(x)$: in this case we require m_1 continuous and nonnegative, k of class $C^{(1)}$ and nonnegative, and moreover $|k'(t)| \le Ck(t)$ for every $t \in [0, \tau]$. Another possible choice for a_1 is

$$a_1(t, u, v) = \int_\Omega \sum_{i,j=1}^{n} [b_{ij}(t, x) \frac{\partial u}{\partial x_i}(x) \frac{\partial \bar{v}}{\partial x_j}(x)]\, dx$$

if $\sum_{i,j=1}^{n} b_{ij}(t, x)\, z_i\, \bar{z}_j \ge 0\ \forall (t, x) \in [0, \tau]\times\bar{\Omega}$ and $\forall z_1, \cdots, z_n \in \boldsymbol{C}$: clearly, the b_{ij}'s and their derivatives $\partial b_{ij}/\partial t$ must satisfy natural conditions.

We end the section with some remarks.

REMARKS. 1.5.1. If the M's have uniformly bounded inverses in $\mathcal{L}(W, W')$ (i.e., $C\|u; W\| \le \|M(t)\, u; W'\| \le C'\|u; W\|$ for every u in $V = W$), we obtain, with our notations, $a_1(t; u, u) \ge C\|u; W\|^2$; now, an assumption $|a_1'(t; u, v)| \le C\|u; W\|\, \|v; W\|$ implies readily

$$\|(zT(t)+I)^{-1}; \mathcal{L}(W')\|,\ \|\frac{\partial}{\partial t}(zT(t)+I)^{-1}; \mathcal{L}(W')\| \le C\,.$$

Hence, if

$$\begin{cases} a_1(t; u, u) \ge C\|u; W\|^2 \\ |a_1'(t; u, v) - a_1'(s; u, v)| \le C\|u; W\|\, \|v; W\|\, |t-s|^{\varepsilon} \\ \text{for every } u, v \in W \text{ and for every } t, s \in [0, \tau] \end{cases}$$

Theorem 2 applies immediately: this fact allows us to handle Sobolev-type equations.

1.5.2. Let $M \in \mathcal{L}(H)$ be a nonnegative (bounded) operator. If we put $a_1(u, v) = \langle Mu, v\rangle_H$ and a_0 is as before at the beginning of Example 2, we get

$$\operatorname{Re} \lambda a_1(u, u) + a_0(t; u, u) \ge C\|u; V\|^2 \quad \text{for every } u \in V:$$

using again Theorem 2, we may then study degenerate equations of the type

$$\frac{d}{dt}(Mu(t))+L(t)\,u(t)=f(t),\ t\in[0,\tau]\,,$$

when f is in $C^{(\nu)}[0,\tau;V']$.

1.6. Now, we turn to a problem of the form

$$\text{(NLP)}\qquad \begin{cases} \dfrac{d}{dt}(M(t)\,u(t))+L(t)\,u(t)=f(t,u(t)),\ t\in[0,\tau] \\ (M(t)u(t))_{t=0}=w_0 \end{cases}$$

where the families $\{L(t);t\in[0,\tau]\}$, $\{M(t);t\in[0,\tau]\}$ are as before.

We assume that there exists a continuously embedded Banach space $Y_1\hookrightarrow Y$, $\eta>0$, exponents α, β in $]0,1]$ s.t.

(H) $\quad \|L(t)^{-1}-L(s)^{-1};\mathcal{L}(X,Y_1)\|\leq C\,|t-s|^{\alpha}$ for every $t,s\in[0,\tau]$;

(K) $\quad (t,y)\rightarrow f(t,y)$ is a $C^{(1)}$ mapping from $[0,\tau]\times\boldsymbol{U}$ into X, $\boldsymbol{U}$ being an Y_1-neighborhood of $u_0\in Y_1\cap\mathcal{D}(L(0))$; f satisfies
$\|\frac{\partial f}{\partial x}(t,x)-\frac{\partial f}{\partial y}(s,y);\mathcal{L}(Y_1,X)\|\leq C(|t-s|^{\beta}+\|x-y;Y_1\|)$
for every $t,s\in[0,\tau]$ and $\forall x,y\in\boldsymbol{U}$
$\|\frac{\partial f}{\partial x}(0,u_0);\mathcal{L}(Y_1,X)\|\leq\eta$;

(L) $\quad w_0(=M(0)\,u_0)\in\mathcal{R}(M(0))$ and $f(0,u_0)-(T'(0)+I)\,L(0)\,u_0\in\mathcal{R}(T(0))$.

Part II (§4, Theorem 2) then allows us to state

Theorem 3. *Let* (i)-(vii) *together with* (H), (K), (L) *hold, and assume* $0<\nu<\rho\mathcal{E}$, $\nu\leq\alpha,\beta\leq 1$. *If* τ,η *are sufficiently small, then* (NLP) *has a unique strict solution* u *s.t.* $L(\cdot)\,u$, $d/dt\,(M(\cdot)\,u(\cdot))$ *are in* $C^{(\nu)}[0,\tau;X]$.

1.7. Let now see what Theorem 3 implies in an interesting linear case. Let $\{L(t);t\in[0,\tau]\}$ be a family of operators with time-independent domain D; suppose moreover M is a closed time-independent operator from Y into X s.t.

$$(4)\qquad \begin{cases} L'(t)\in C[0,\tau;\mathcal{L}(D,X)] \\ \|L(0)\,(zM+L(0))^{-1};\mathcal{L}(X)\|\leq C\ \ \forall z\in\boldsymbol{C},\ \mathrm{Re}\,z\geq 0\,. \end{cases}$$

Note that no further condition is actually required on $L(t)$, $t>0$.

The linear problem

$$\text{(LP1)}\qquad \begin{cases} \dfrac{d}{dt}(Mu(t))+L(t)\,u(t)=h(t),\ t\in[0,\tau] \\ (Mu(t))_{t=0}=w_0 \end{cases}$$

may be seen as a particular case of (NLP) with $L(0)$ instead of $L(t)$; we put

$$f(t,u)=[L(0)-L(t)]\,u+h(t),\, t\in[0,\tau],\, u\in D\,.$$

Suppose $D=Y_1$ (see (H), (K)): then $(\partial f/\partial u)(0,u_0)=0$ and (K) holds if $h\in C^{(1)}[0,\tau;X]$ and

$$(5)\quad \left\|\frac{\partial f}{\partial x}(t,x)-\frac{\partial f}{\partial y}(s,y);\mathcal{L}(D,X)\right\|=\|L(t)-L(s);\mathcal{L}(D,X)\|\le C\,|t-s|^{\beta}$$

for every $t,s\in[0,\tau]$, $\forall\, x,y\in D$. Since

$$L(0)\,[L(t)^{-1}-L(s)^{-1}]=-L(0)\,L(t)^{-1}[L(t)-L(s)]\,L(0)^{-1}[L(0)\,L(s)^{-1}]\,,$$

(5) becomes the well-known Tanabe condition (see e.g. [28, p. 118])

$$\|L(0)\,[L(t)^{-1}-L(s)^{-1}];\mathcal{L}(X)\|\le C\,|t-s|^{\beta}$$

when there exist $L(t)^{-1}$ together with constants a, b such that

$$a\|L(t)x\|\le\|L(0)x\|\le b\|L(t)x\|\quad \forall\, x\in D,\ t\in[0,\tau]\,.$$

Statement (L) becomes the following compatibility condition at $t=0$:

$$(6)\qquad h(0)-L(0)\,u_0\in\mathcal{R}(ML(0)^{-1})\,.$$

So, we have shown

Theorem 3′. *Let* (4), (5), (6) *hold*: *then for every* $h\in C^{(1)}[0,\tau;X]$, (LP1) *has a unique strict solution in* $[0,\tau]$, *provided* τ *is sufficiently small. In this case, strict solution means*: $L(\cdot)\,u(\cdot)$ *is of class* C^{ν} *with* $0<\nu\le\beta$ *if* $\beta<1$, $0<\nu<1$ *if* $\beta=1$.

A crucial remark is that under (4), (5), (6) we can only hope to find a *local* result, as one will understand by means of the following simple counterexample.

Let $-A$ generate an analytic semigroup in X, and ε be >0; then consider

$$u'(t)+\varepsilon Au(t)=(t-2\varepsilon)\,Au(t)\,,\quad t\in]0,+\infty[,\ u(0)=u_0\,;$$

there is no strict solution on any $[0,\tau]$ if $\tau>2\varepsilon$; nevertheless, our Theorem 3 applies, since

$$\frac{\partial f}{\partial u}(0,u_0)=-2\varepsilon A\,.$$

1.8. If it is possible to apply Proposition 1, we may weaken the assumptions about regularity in Theorem 3′. If $h\in C^{(\nu)}[0,\tau;X]$ with $0<\nu\le\beta\le1$, $\nu<1$, $h(0)-v_0=h(0)-L(0)\,u_0=ML(0)^{-1}\,v_1$, and

$$F(w)\,(t)=[L(0)-L(t)]\,L(0)^{-1}\,w(t)+[L(0)-L(t)]\,L(0)^{-1}\,[v_0+tv_1]+h(t)-h(0)$$
$$-tv_1=K(w)\,(t)+\eta(t)\,,$$

it will suffice to observe that

$$\begin{aligned}
&\eta\in C_0^{(\nu)}[0,\tau;X] \quad \text{and} \quad (KL^{-1}w)(t)=[L(0)-L(t)]\,L(0)^{-1}w(t) \text{ imply}\\
&\|(KL^{-1}w)(t);X\|\le Ct^{\beta}\|w(t);X\|\le C'\tau^{\beta+\nu}\|w;C_0^{(\nu)}[0,\tau;X]\|, \text{ so}\\
&|t-s|^{-\nu}\,\|(KL^{-1})\,w(t)-(KL^{-1})\,w(s);X\|\\
&\quad=|t-s|^{-\nu}\,\|(I-L(t)\,L(0)^{-1})\,[w(t)-w(s)]-(L(t)-L(s))\,L(0)^{-1}w(s);X\|\\
&\quad\le C''\{\tau^{\beta}\|w;C_0^{(\nu)}[0,\tau;X]\|+|t-s|^{\beta-\nu}\,\tau^{\nu}\|w;C_0^{(\nu)}[0,\tau;X]\|\}\\
&\quad\le C'''\,\tau^{\beta}\|w;C_0^{(\nu)}[0,\tau;X]\|\,.
\end{aligned}$$

If we choose a suitably small τ, we obtain

Corollary 1. *Let* $0<\nu\le\beta$, $\nu<1$, *and suppose that* (4) *holds, without differentiability, together with* (5) *and* (6). *If* $h\in C^{(\nu)}[0,\tau;X]$, $h(0)-L(0)\,u_0\in\mathcal{R}(M(L(0))^{-1})$, *then, if* $\tau>0$ *is suitably small, there exists a unique strict solution* u *of* (LP1) *on* $[0,\tau]$, *such that* $L(\cdot)\,u(\cdot)\in C^{(\nu)}[0,\tau;X]$.

EXAMPLE 3. If we turn now to the pattern of Example 2, with the same notations V, W, H, a_0, a_1, suppose a_1 independent of t, and

$$(E)\qquad \begin{cases}
|a_0(t;u,v)|\le C\|u;V\|\,\|v;V\|\\
|a_1(x,y)|\le C\|x;W\|\,\|y;W\|\\
\operatorname{Re}\,a_0(t;u,u)\ge C\|u;V\|^2\\
a_1(x,x)\ge 0\\
|a_0(t;u,v)-a_0(s;u,v)|\le C\|u;V\|\,\|v;V\|\,|t-s|^{\beta}\\
|a_0'(t;u,v)-a_0'(s;u,v)|\le C\|u;V\|\,\|v;V\|\,|t-s|^{a}
\end{cases}$$

for every x,y,u,v,t,s ($x\in W$ and so on) with suitable exponents $a,\beta\in]0,1]$, we see that conditions (4) and (5) are satisfied: see [28, pp. 144–145]; if u_0, h do satisfy (6) we may apply theorem 3′ to (LP1). It is also easy to apply corollary 1, under suitable conditions.

1.9. APPLICATION 1. (Integrodifferential Equations). We turn now to the study of degenerate integrodifferential equations (IDEs for short).

Let us see presently how to work with Proposition 1 and Corollary 1 to Theorem 3′ in the case of a problem

$$(\mathrm{LP2})\qquad \begin{cases}
\dfrac{d}{dt}\,(M(t)\,u(t))+L(t)\,u(t)=\displaystyle\int_0^t K(t-s)\,u(s)\,ds+f(t),\ t\in[0,\tau]\\
(M(t)\,u(t))_{t=0}=w_0
\end{cases}$$

concerning a degenerate but linear IDE.

For the sake of simplicity, we suppose that L, M, satisfy (i)—(vii) with $\varepsilon=\rho=1$, and that the operators $K(t,s)\,L(s)^{-1}$ lie in $\mathcal{L}(X)$ $\forall\,(s,t)$ in the triangle

$\{(s,t)\in\boldsymbol{R}^2;\ 0\le s<t\le\tau\}=\Delta$: a particular case might be $K(t,s)=k(t-s)\,L(s)$ $\forall(s,t)\in\Delta$, with $k(u)\in\mathcal{L}(X)\ \forall u\in]0,\tau]$; moreover they must satisfy ($\circ$), the following set of conditions:

($\circ$): $\|K(t,s)\,L(s)^{-1};\mathcal{L}(X)\|\le C\,|t-s|^{-\gamma}, \forall(s,t)\in\Delta$, with $\gamma\in[0,1[$,
$\|K(t',t'-s)\,L(t'-s)^{-1}-K(t'',t''-s)\,L(t''-s)^{-1};\mathcal{L}(X)\|\le C\,|t'-t''|^{\alpha_1}$,
$\forall t',t''$ such that $0<t'\le\tau$, $0<t''\le\tau$, $0<s<t'$, $0<s<t''$
with suitable α_1, $0\le\alpha_1<1$ (such a kind of bounds are used e.g. in [20], [1]).

If

($\circ\circ$) $\quad f(0)-(I+T'(0))\,v_0\in\mathcal{R}(T(0))$, $v_0=L(0)\,u_0$ and $w_0=M(0)\,u_0$

then (LP2) may be translated into

$$\text{(LP3)}\qquad\begin{cases}\dfrac{d}{dt}(T(t)\,w(t))+w(t)=\displaystyle\int_0^t K(t,s)\,L(s)^{-1}\,w(s)\,ds+h(t),\ t\in[0,\tau]\\ (T(t)\,w(t))_{t=0}=0\end{cases}$$

where h is defined by

$$h(t)=f(t)+\int_0^t K(t,s)\,L(s)^{-1}\,[v_0+sv_1]\,ds-[I+T'(t)]\,v_0-[T(t)+tT'(t)+t]\,v_1,$$

and

$$T(0)\,v_1=f(0)-[I+T'(0)]\,v_0.$$

Let us choose f in $C^{(\theta)}[0,\tau;X]$ with $\theta\in]0,1[$, $\theta\le\min\{1-\gamma,\alpha_1\}$: if we look for an application of Proposition 1 we must only prove that $\forall\omega\in C_0^{(\theta)}[0,\tau;X]$ we get

$$\|F(\omega);C_0^{(\theta)}[0,\tau;X]\|\le C\|\omega;C_0^{(\theta)}[0,\tau;X]\|$$

for a suitably small conxtant C, where F is defined by

$$F(\omega)(t)=\int_0^t K(t,s)\,L(s)^{-1}\,\omega(s)\,ds\ \left(=\int_0^t K(t,t-s)\,L(t-s)^{-1}\,\omega(t-s)\,ds\right).$$

We estimate $F(\omega)(t)$ by

$$\begin{aligned}\|F(\omega)(t);X\|&\le C'\Big(\int_0^t s^{-\gamma}\,ds\Big)\sup_{0\le s\le t}\|\omega(s)\|\\&\le C''\,t^{1-\gamma}\,(\sup_{0\le s\le t}s^{-\theta}\|\omega(s)-\omega(0)\|)\,t^{\theta}\\&\le C'''\,\tau^{1-\gamma+\theta}\|\,\|\omega\|\,\|.\end{aligned}$$

Let us choose $0\le t''\le t'\le\tau$:

$$F(\omega)(t')-F(\omega)(t'') =$$
$$\int_0^{t''} [K(t', t'-s)\, L(t'-s)^{-1}-K(t'', t''-s)\, L(t''-s)^{-1}]\, \omega(t'-s)\, ds$$
$$+\int_0^{t''} [K(t'', t''-s)\, L(t''-s)^{-1}]\, [\omega(t'-s)-\omega(t''-s)]\, ds$$
$$+\int_{t''}^{t'} [K(t', t'-s)\, L(t'-s)^{-1}]\, [\omega(t'-s)]\, ds\,,$$

so

$$|t'-t''|^{-\theta}\|F(\omega)(t')-F(\omega)(t''); X\| \leq C\Big(\int_0^{t''} |t'-t''|^{\alpha_1-\theta}\, ds\Big)\, \tau^{\theta}\|\, \|\omega\|\, \|$$
$$+C\tau^{1-\gamma}\|\, \|\omega\|\, \|+C\Big(\int_{t''}^{t'} s^{-\gamma}\, ds\Big)\, \tau^{\theta}\|\, \|\omega\|\, \| \leq C''\, \tau^{1-\gamma}\|\, \|\omega\|\, \|\, |t'-t''|^{1-\gamma-\theta}\,.$$

We have obtained the following theorem.

Theorem 3′/A. *Let* (i) *through* (vii) *hold with* $\varepsilon=\rho=1$; *suppose* (°) *is true, and let* $0<\theta<1$, $\theta \leq \min\{1-\gamma, \alpha_1\}$. *Then* $\forall f \in C^{(\theta)}[0, \tau; X]$ *such that* (°°) *holds,* (LP2) *has a unique solution u defined on a suitably small interval* $[0, \tau]$, *such that* $d/dt\, M(.)\, u(.)$, $L(.)\, u(.)$ *are in* $C^{(\theta)}[0, \tau; X]$.

Now, we want to apply, while solving (LP2), our Corollary 1 to Theorem 3′; so we assume presently that (4), (5), (6) hold, apart possibly from differentiability of $L(t)$. Moreover we stipulate that $K(t, s) \in \mathcal{L}(D, X)$ for $0 \leq s < t \leq \tau$ and

(°°°) $\quad \|K(t_1, t_1-s); \mathcal{L}(D, X)\| \leq C\, |s|^{-\gamma}$,
$\forall s, t_1$ such that $0<s \leq t_1 \leq \tau$, with $\gamma \in [0, 1[$,
$\|K(t_1, t_1-s)-K(t_2, t_2-s); \mathcal{L}(D, X)\| \leq C\, |t_1-t_2|^{\alpha_1}$,
$\forall s, t_1, t_2$ such that $0<s \leq t_i \leq \tau$, i $=1, 2$, with $0 \leq \alpha_1 < 1$.

We rewrite the equation occurring in (LP2) as

$$\frac{d}{dt}\, Mu(t)+L(0)\, u(t) = -[L(t)-L(0)]\, u(t)+\int_0^t K(t, s)\, u(s)\, ds+f(t)\ ;$$

as before, we change the unknown u into $v=L(0)\, u$ and so, if we write $T_0= ML(0)^{-1}$, we are led to

$$\frac{d}{dt}\, T_0\, v(t)+v(t) = [I-L(t)\, L(0)^{-1}]\, v(t)+\int_0^t K(t, s)\, L(0)^{-1}\, v(s)\, ds+f(t)\,.$$

If $v(t)=v_0+tv_1+w(t)$, we obtain in turn

$$\frac{d}{dt}\, T_0\, w(t)+w(t) = [I-L(t)\, L(0)^{-1}]\, w(t)-T_0\, v_1-v_0+tv_1$$
$$+[I-L(t)\, L(0)^{-1}]\, [v_0+tv_1]$$
$$+\int_0^t K(t, s)\, L(0)^{-1}\, w(s)\, ds+f(t)+\int_0^t K(t, s)\, L(0)^{-1}[v_0+sv_1]\, ds\,.$$

We are looking for a solution $w\in C_0^{(\theta)}[0,\tau;X]$; so, assume $0<\theta<\beta$, $\theta\le\min\{\alpha_1, 1-\gamma\}$; then, we need only to show the following:
if F is defined by

$$F(w)(t)=\int_0^t K(t,s)\,L(0)^{-1}\,w(s)\,ds\,,$$

then $F\in\mathcal{L}(C_0^{(\theta)}[0,\tau;X])$ with small norm. Then we may apply the machinery developed in Corollary 1.

So, fix an arbitrary $w\in C_0^{(\theta)}[0,\tau;X]$ and estimate:

$$\|\int_0^t K(t,s)\,L(0)^{-1}\,w(s)\,ds\|\le C\tau^{\theta}(\int_0^t s^{-\gamma}\,ds)\|\,\|w\|\,\|\le C'\tau^{\theta+1-\gamma}\|\,\|w\|\,\|\,;$$

note that, for $t''<t'$

$$\begin{aligned}&\int_0^{t'} K(t',t'-s)\,L(0)^{-1}\,w(t'-s)\,ds-\int_0^{t''} K(t'',t''-s)\,L(0)^{-1}\,w(t''-s)\,ds\\
&\quad=\int_0^{t''}[K(t',t'-s)-K(t'',t''-s)]\,L(0)^{-1}\,w(t'-s)\,ds\\
&\quad+\int_0^{t''} K(t'',t''-s)\,L(0)^{-1}[w(t'-s)-w(t''-s)]\,ds\\
&\quad+\int_{t''}^{t'} K(t',t'-s)\,L(0)^{-1}\,w(t'-s)\,ds\,,\end{aligned}$$

so if we call A the X-norm of this vector, we get

$$\begin{aligned}A\le&(C\tau^{\theta+1}\|\,\|w\|\,\|\,|t'-t''|^{\alpha_1})+(C\int_0^t s^{-\gamma}\,ds\|\,\|w\|\,\|\,|t'-t''|^{\theta})\\
&+(C\int_{t''}^{t'} s^{-\gamma}\,ds)\,(\tau^{\theta}\|\,\|w\|\,\|)\,,\end{aligned}$$

and finally

$$\begin{aligned}&|t'-t''|^{-\theta}A\le C'(|t'-t''|^{\alpha_1-\theta}\,\tau^{\theta+1}+\tau^{\theta+1}+\tau^{1-\gamma}+\\
&|t'-t''|^{1-\gamma-\theta}\tau^{\theta})\|\,\|w\|\,\|\le C''\tau^{\beta}\|\,\|w\|\,\|\,.\end{aligned}$$

We have proved the following statement:

Theorem 3'/B. *Let* (4), (5), (6) *hold, except the differentiability of* $L(t)$: *suppose further* $K(t,s)\in\mathcal{L}(D,X)$ $\forall(s,t)\in\Delta$, *and* (°°°). *Let* $0<\theta<\beta$, $\theta\le\min\{\alpha_1, 1-\gamma\}$. *Then, if* τ *is small enough, for any* $f\in C_0^{(\theta)}[0,\tau;X]$ *there exists a unique strict solution u of* (LP2) *with* $d/dt\,(Mu(.))$, $L(.)\,u(.)$ *in* $C^{(\theta)}[0,\tau;X]$.

Of course, it is worth observing that, in fact,

$$t\to\int_0^t K(t,s)\,L(0)^{-1}(v_0+sv_1)\,ds$$

belongs to $C^{(\theta)}[0,\tau;X]$, as a consequence of the results already obtained, because $t\to(v_0+tv_1)$ lies in the same space.

REMARKS. 1.9.1. In the case L, M are t-independent, assume that M^{-1} exists and that $-LM^{-1}$ generates an analytic semigroup in X; some sufficient conditions are given in [10]. If $K(s)\in\mathcal{L}(\mathcal{D}(L), X)$, one may change the initial value problem

$$\text{(I)}\qquad \begin{cases} \dfrac{d}{dt}(Mu(t))+Lu(t)=\displaystyle\int_0^t K(t-s)\,u(s)\,ds+f(t),\ t\in[0,\tau] \\ (Mu(t))_{t=0}=w_0(=Mu_0) \end{cases}$$

into

$$\text{(II)}\qquad \begin{cases} \dfrac{d}{dt}v(t)+LM^{-1}v(t)=\displaystyle\int_0^t K(t-s)\,M^{-1}v(s)\,ds+f(t) \\ \qquad =\displaystyle\int_0^t K(t-s)\,L^{-1}(LM^{-1})\,v(s)\,ds+f(t),\ t\in[0,\tau] \\ (v(t))_{t=0}=w_0 \end{cases}$$

It is now possible to apply many classical results obtained by Da Prato and his school to (II), thus obtaining global solutions on $[0,\tau]$ and maximal regularity. For example, let us assume, following [6, p. 364], that

(i) $\mathcal{D}(K(s))=\mathcal{D}(L)\quad\forall s$

(ii) $s\to K(s)x$ (strongly) measurable $\forall x\in\mathcal{D}(L)$

(iii) $s\to\|K(s);\mathcal{L}(\mathcal{D}(L),X)\|$ bounded in $[0,\tau]$;

since $K(t-s)\,M^{-1}=(K(t-s)\,L^{-1})\,LM^{-1}$, it is an easy matter to infer that for any $f\in C^{(\theta)}[0,\tau;X]$ and $w_0\in\mathcal{D}(LM^{-1})$ such that

$$\text{(A)}\qquad f(0)-LM^{-1}w_0\in\mathcal{D}(-LM^{-1})_{\theta,\infty}$$

there is a (unique) v satisfying (II) and such that v', $LM^{-1}\in C^{(\theta)}[0,\tau;X]$. But now $w_0\in\mathcal{D}(LM^{-1})$ reads $w_0=Mu_0$, $u_0\in\mathcal{D}(L)$, so (A) becomes in turn $(f(0)-Lu_0)\in\mathcal{D}(-LM^{-1})_{\theta,\infty}$, that is, (see [2])

$$\text{(III)}\qquad f(0)-Lu_0=y,\ \sup_{t>0}(t^{\theta}\|L(tM+L)^{-1}y;X\|)<+\infty$$

It has been proved

Theorem 3′/C. *Assume that $-LM^{-1}$ generates an analytic semigroup-of course, $\mathcal{D}(L)\subseteq\mathcal{D}(M)$, and K satisfies* (i)–(ii)–(iii). *If $f\in C^{(\theta)}[0,\tau;X]$, $w_0=Mu_0$, $u_0\in\mathcal{D}(L)$, and finally*

$$\sup_{t>1}(t^{\theta}\|L(tM+L)^{-1}(f(0)-Lu_0);X\|)<+\infty\ ,$$

then (LP2) *has a unique global strict solution u such that $d/dt\,(Mu(\cdot))$, $Lu(\cdot)$ are in $C^{(\theta)}[0,\tau;X]$.*

Notice that, if $y\in\mathcal{R}(ML^{-1})$, $y=ML^{-1}x$, then (as $t>1$ is not restrictive)

$$t^{\theta} L(tM+L)^{-1} y = t^{\theta-1} L(tM+L)^{-1}(tM+L-L) L^{-1}x$$
$$= t^{\theta-1}[x-L(tM+L)^{-1}x] ,$$

and hence y satisfies (III).

1.10. Finally, we give a general condition entailing formula (4) in a complex Hilbert space X (with related inner product $\langle .,. \rangle_X$) in the case of time-independent operators $L(t)=L$, $M(t)=M$, and then an example of a concrete nonlinearity satisfying (K).

Assume therefore that M is a nonnegative self-adjoint operator in X, L is a positive self-adjoint operator in X s.t.

i) M is L-bounded with L-bound 0 (according to [13]);

ii) $\langle Lu, Mu \rangle_X \geq 0 \quad \forall u \in \mathcal{D}(L)$.

i) ensures that $zM+L$ is closed for every complex z; moreover, by ii)

$$\|(zM+L)\,u; X\|^2 \geq \|Lu; X\|^2 \quad \forall u \in \mathcal{D}(L), \forall z \in \boldsymbol{C}, \text{ s.t. } \operatorname{Re} z \geq 0 .$$

Hence, $zM+L$ has a closed range. On the other hand, in [31] it is shown that in this case

$$(zM+L)^* = \bar{z}M+L, \forall z \in \boldsymbol{C}, \textit{ s.t. } \operatorname{Re} z \geq 0 ;$$

thus, if $f \in (\mathcal{R}(zM+L))^{\perp}$-that is, $\langle (zM+L)\,u, f \rangle_X = 0 \ \forall u \in \mathcal{D}(L)$-then $f \in \mathcal{D}((zM+L)^*) = \mathcal{D}(L)$ and

$$\langle u, (\bar{z}M+L) f \rangle_X = 0 \quad \forall u \in \mathcal{D}(L) .$$

So we get $0 = \|(\bar{z}M+L)f; X\| \geq \|Lf; X\|$, that is, $f=0$; hence,

$$\|L(zM+L)^{-1}f; X\| \leq \|f; X\| \quad \forall f \in X ,$$
$$\|(zT+I)^{-1}; \mathcal{L}(X)\| \leq \text{constant } (T = ML^{-1})$$

in a sector of the complex plane containing the half-plane $\operatorname{Re} z \geq 0$.

Let us give now the example of nonlinearity we spoke about: let then F: $[0, 1] \times [0, 1] \times \boldsymbol{R} \to \boldsymbol{R}$, and assume the same as in [23, pp. 204, 205], namely, $F=F(s, t, u)$ is continuous together with its derivatives $\dfrac{\partial F}{\partial u}, \dfrac{\partial^2 F}{\partial u^2}$, with $\left|\dfrac{\partial^2 F}{\partial u^2}\right|$ bounded on $[0, 1] \times [0, 1] \times \boldsymbol{R}$.

Then define for $x \in X = L^2(]0, 1[)$

$$T(x)(s) = \int_0^1 F(s, t, x(t))\, dt ;$$

T turns out to be an everywhere differentiable operator $X \to X$, s.t.

$$T'(x)(h) = \int_0^1 \frac{\partial F}{\partial u}(\cdot, t, x(t))\, h(t)\, dt \quad \forall x, h \in X ,$$

([23, pp. 204–206]). An easy evaluation of $T'(x)(h)-T'(y)(h)$ yields the Lipschitz estimate

$$||T'(x)(h)-T'(y)(h); X|| \leq M||x-y; X||\, ||h; X|| \quad \forall x, y, h \in X,$$

since $L^\infty(]0,1[)\subsetneq X$. Finally, choose a closed operator K from X into X, s.t. $\mathcal{D}(L)\subseteq\mathcal{D}(K)$: then, if $f(u)=F(\varepsilon K u)$ (with a real ε, $|\varepsilon|$ small) we may see that assumption (K) is satisfied.

2. Nonlinear Problems: Preliminaries

2.1. Now, let us show how to reduce conveniently into the pattern (NLP) some problems which, at first glance, might seem a bit more general.

Let us write down the problem

$$\text{(NLP1)} \qquad \begin{cases} \dfrac{d}{dt}(M(t)\,u(t)) = g(t, u(t)),\ t\in[0,\tau] \\ (M(t)\,u(t))_{t=0} = w_0 = M(0)\,u_0\,. \end{cases}$$

Here, as usual, we introduce two Banach spaces X, Y_1, a family $\{M(t); t\in[0,\tau]\}$ of linear bounded operators from Y_1 into X, a map $g\colon [0,\tau]\times \boldsymbol{U}\to X$, where $\boldsymbol{U}$ is an Y_1-neighborhood of u_0. We assume that g is of class $C^{(1)}$ and that $\partial g/\partial u\,(t, u_0)=-\Lambda(t)\in\mathcal{L}(Y_1, X)$ for every $t\in[0,\tau]$.
We rewrite the equation in (NLP1) as

$$\frac{d}{dt}(M(t)\,u(t)) = -\Lambda(t)\,u(t)+\{g(t,u(t))+\Lambda(t)\,u(t)\},\ t\in[0,\tau]\,.$$

If $Y(t)$ is a subspace of Y_1 s.t. for every $t\in[0,\tau]$ the restriction of $\Lambda(t)$ to it, to be written henceforth $L(t)$, satisfies (i)-(vii), then Theorem 3 allows us to solve it in the form

$$\begin{cases} \dfrac{d}{dt}(M(t)\,u(t)) = -L(t)\,u(t)+\{g(t,u(t))+\Lambda(t)\,u(t)\},\ t\in[0,\tau]\,, \\ (M(t)\,u(t))_{t=0} = w_0\,, \end{cases}$$

at least if

$$\text{(N)} \qquad \begin{cases} w_0 = M(0)\,u_0,\ u_0\in\mathcal{D}(L(0)) = Y(0) \\ L(t) \text{ satisfies } (H) \\ ||\dfrac{\partial g}{\partial u}(t,u_1)-\dfrac{\partial g}{\partial u}(s,u_2);\ \mathcal{L}(Y_1,X)|| \leq C(|t-s|^\beta+||u_1-u_2;\ Y_1||) \\ \text{for every } t, s\in[0,\tau] \quad \text{and} \quad \forall u_1, u_2\in \boldsymbol{U} \\ g(0,u_0)-T'(0)\,L(0)\,u_0\in\mathcal{R}(T(0))\,. \end{cases}$$

Note that, if one introduces $F(t,u)=g(t,u)-(\partial g/\partial u)(t,u_0)\,u$, one gets $(\partial F/\partial u)(0,u_0)=0$; consequently we obtain

Theorem 4. *Let* (i)-(vii) *and* (N) *hold. If* $0<\nu<\rho\varepsilon$, $\nu\leq\alpha$, $\beta\leq 1$, *and* τ *is sufficiently small, then* (NLP1) *has a unique strict solution* u *s.t.* $L(\cdot)u$ *is in* $C^{(\nu)}[0, \tau; X]$.

2.2. If $M(t)=M$, $\mathcal{D}((\partial g/\partial u)(t, u))=D$ are independent of t and u, we may weaken (or, at least, simplify) assumptions in Theorem 4. If we set

$$L = -\frac{\partial g}{\partial u}(0, u_0)\colon D \to X,\ F(t, u) = g(t, u)+Lu\ ,$$

we get

$$\frac{\partial F}{\partial u}(t, u_1)-\frac{\partial F}{\partial u}(s, u_2) = \frac{\partial g}{\partial u}(t, u_1)-\frac{\partial g}{\partial u}(s, u_2)\ , \quad \text{for every} \quad t, s\in[0, \tau]$$

and $\forall u_1, u_2\in \boldsymbol{U}$

$$\frac{\partial F}{\partial u}(0, u_0) = 0\ ,$$

so, under the preceding assumptions, the following extension of Theorem 3′ holds.

Theorem 5. *Let the operators* M, L *satisfy*

$$\|L(zM+L)^{-1}; \mathcal{L}(X)\|\leq C \quad \forall z\in \boldsymbol{C}, \operatorname{Re} z\geq 0\ ;$$

as for g, *suppose it to be a* $C^{(1)}$ *map* $[0, \tau]\times \boldsymbol{U}\to X$, *where* $\boldsymbol{U}$ *is a neighborhood of* u_0 *in* $D=\mathcal{D}(L)$: *assume further that, for every* $t, s\in[0, \tau]$ *and* $\forall u_1, u_1\in \boldsymbol{U}$

$$\|\frac{\partial g}{\partial u}(t, u_1)-\frac{\partial g}{\partial u}(s, u_2); \mathcal{L}(D, X)\|\leq C(|t-s|^{\beta}+\|u_1-u_2; D\|)$$

$$w_0 = Mu_0, g(0, u_0)\,(= F(0, u_0)-(I+T'(0)\,L(u_0)))\in\mathcal{R}(ML^{-1}) = M(D)\ ;$$

then the conclusions of Theorem 4 *hold as well.*

2.3. A simple example will clarify the method. Let us look for regular u, v: $[0, \tau]\to \boldsymbol{R}$ (or $\boldsymbol{C}$) s.t.

$$\begin{cases} (u+v)'(t) &= -u(t)+(v(t))^2 \\ 0 &= -v(t)+1-(u(t))^2 \quad \text{for every} \quad t\in[0, \tau] \\ u(0)+v(0) = 0 & \end{cases}$$

Put $u_0=u(0)$, $v_0=v(0)$: then one gets easily $v_0=1-u_0^2$, $u_0^2-u_0-1=0$: here arise the compatibility conditions, as in Theorem 5. We write down the problem in matrix form and use the notations of that theorem; the jacobian matrix of g is

$$J_g(u_0, v_0) = \begin{bmatrix} -1, & 2v_0 \\ -2u_0, & -1 \end{bmatrix} = -\begin{bmatrix} 1, & 2u_0 \\ 2u_0, & 1 \end{bmatrix}$$

which is nonsingular. The condition $g(0, u_0)\in\mathcal{R}(ML^{-1})$ now becomes

$$\begin{bmatrix} -u_0+v_0^2 \\ -v_0+1-u_0^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 0 \end{bmatrix}$$

for suitable x, y in $\boldsymbol{R}$, that is, the very condition $v_0=1-u_0^2$. If these conditions do hold, we may apply Theorem 5; of course, the problem could be explicitly solved in u by trivial tricks and separation of variables.

2.4. As another example, consider

$$\text{(NLP2)} \qquad \begin{cases} \dfrac{d}{dt}(Mu(t)) = -L(t)\,u(t)+f(t, u(t)),\ t\in[0, \tau] \\ (Mu(t))_{t=0} = w_0 \end{cases}$$

when $D=\mathcal{D}(L(t))$ is time-independent: we may put it into the form

$$\begin{cases} \dfrac{d}{dt}(Mu(t))+L(0)\,u(t) = [L(0)-L(t)]\,u(t)+f(t, u(t)),\ t\in[0, \tau] \\ (Mu(t))_{t=0} = w_0 \end{cases}$$

Hence, if (4), (5) hold (the assumptions on $L'(t)$ may be dropped, if we invoke theorem 1 directly), f: $[0, \tau]\times \boldsymbol{U}\to X$ is $C^{(1)}$ (here, $\boldsymbol{U}$=a neighborhood of u_0 in D) and moreover

$$\text{(7)} \qquad \begin{cases} f(0, u_0)-L(0)\,u_0\in\mathcal{R}(ML(0)^{-1}) \\ \|\dfrac{\partial f}{\partial u}(t, u_1)-\dfrac{\partial f}{\partial u}(s, u_2);\ \mathcal{L}(D, X)\|\le C(|t-s|^{\beta}+\|u_1-u_2;\ D\|) \\ \text{for every } t, s\in[0, \tau] \ \text{ and } \ \forall\, u_1, u_2\in \boldsymbol{U} \\ \|\dfrac{\partial f}{\partial u}(0, u_0);\ \mathcal{L}(D, X)\| \text{ is small} \end{cases}$$

then (NLP2) has a unique strict local solution.

On the other hand, write

$$\begin{aligned} &-L(t)\,u(t)+f(t, u(t)) \\ &\quad= -L(0)\,u(t)+\frac{\partial f}{\partial u}(0, u_0)\,u(t)+[L(0)-L(t)]\,u(t)+[f(t, u(t)) \\ &\quad-\frac{\partial f}{\partial u}(0, u_0)\,u(t)]\,. \end{aligned}$$

We see that, if (4) holds with L replaced by $[L(0)-(\partial f/\partial u)(0, u_0)]$-perturbation results are in order, of course-(5) is true, $(\partial f/\partial u)$ satisfies the second assumption in (7) and

$$f(0, u_0)-L(0)\,u_0\in\mathcal{R}(M[L(0) -\frac{\partial f}{\partial u}(0, u_0)]^{-1})\,,$$

we may prove existence, uniqueness and regularity of solutions for (NLP2), without any assumption about the smallness of $\|(\partial f/\partial u)(0, u_0); \mathcal{L}(D, X)\|$.

3. Semilinear and Nonlinear Parabolic Problems

3.1. In the following, we will look for real-valued solutions under the assumption that the linearized problem does have real-valued solutions for real data.

APPLICATION 2. (Semilinear Equations). Let Ω be a bounded domain in $\boldsymbol{R}^n$ with regular boundary $\partial\Omega$; A, B_j are formal differential operators as in Example 1.

We want to study

$$\text{(SLP)}\quad \begin{cases} \dfrac{\partial}{\partial t}(t, x) = -A(t, x; D)\, u(t, x) + f(t, u(t, x), \cdots, D^{2m-1} u(t, x)), \\ \quad t \in [0, \tau],\ x \in \Omega \\ B_j(t, y; D)\, u(t, y) = 0,\ t \in [0, \tau],\ y \in \partial\Omega \\ u(0, x) = u_0(x), \quad x \in \Omega\,. \end{cases}$$

So fix $f \in C^{(2)}$, $f = f(t, X_1, \cdots, X_{2m})$ with arguments $t \in [0, \tau]$, X_j a real vector with n^{j-1} coordinates and call F the related substitution operator

$$F(t, u)(x) = f(t, u(x), Du(x) \cdots, D^{2m-1} u(x)), \quad t \in [0, \tau],\ x \in \Omega$$

to be defined on $W^{2m,p}(\Omega)$: $D^k u$ is the vector containing the n^k partial derivatives of u of order k, $1 \leq k \leq 2\text{m-}1$.

Formally, in matricial notations,

$$\begin{aligned} \Big(\frac{\partial F}{\partial u}(t, u) v\Big)(x) &= \frac{\partial f}{\partial X_1}(t, u(x), Du(x) \cdots, D^{2m-1} u(x))\, v(x) \\ &+ \frac{\partial f}{\partial X_2}(t, u(x), Du(x), \cdots, D^{m-1} u(x))\, Dv(x) + \cdots \\ &+ \frac{\partial f}{\partial X_{2m}}(t, u(x), Du(x) \cdots, D^{2m-1} u(x))\, D^{2m-1} v(x)\,, \end{aligned}$$

where $\partial f/\partial X_j$ ($j = 1, \cdots, 2\text{m}$) is the partial differential of f with respect to the $(j+1)$th variable.

We wish to apply Theorem 3 with $X = Y = L^p(\Omega)$, $1 < p < +\infty$, $Y_1 = W^{2m,p}(\Omega)$. Let $p > n$, $u_1, u_2 \in W^{2m,p}(\Omega)$ with norm $\leq R$; for small τ's we get, by Sobolev Embedding Theorem [22, p. 208]

$$\begin{aligned} &\Big|\frac{\partial f}{\partial X_j}(t, u_1(x), Du_1(x), \cdots, D^{2m-1} u_1(x)) - \frac{\partial f}{\partial X_j}(s, u_2(x), Du_2(x), \cdots, D^{2m-1} u_2(x))\Big| \\ &\leq C_R(|t-s| + \|u_1 - u_2;\ W^{2m,p}(\Omega)\|)\,. \end{aligned}$$

Now, the same theorem implies that F is differentiable in u; in fact,

$$
\begin{aligned}
&[F(t, u+h)-F(t, u)](x) \\
&= \int_0^1 \frac{d}{d\theta} f(t, u(x)+\theta h(x), \cdots, D^{2m-1} u(x)+\theta D^{2m-1} h(x))\, d\theta \\
&= \int_0^1 [\frac{\partial f}{\partial X_1}(t, u(x)+\theta h(x), \cdots, D^{2m-1} u(x)+\theta D^{2m-1} h(x))\, h(x)+\cdots \\
&+\frac{\partial f}{\partial X_{2m}}(t, u(x)+\theta h(x), \cdots, D^{2m-1} u(x)+\theta D^{2m-1} h(x))\, D^{2m-1} h(x)]\, d\theta ,
\end{aligned}
$$

so we get

$$
\begin{aligned}
&\left|\left[\frac{\partial F}{\partial u}(t, u_1)-\frac{\partial F}{\partial u}(s, u_2)\right] v(x)\right| \\
&\leq C_R(|t-s|+\|u_1-u_2;\ Y_1\|)\sum_{j=0}^{2m-1} |D^j v(x)|
\end{aligned}
$$

for every $t, s \in [0, \tau]$ if u_1, u_2 have Y_1-norms less than R.

Then (same t, s, u_1, u_2's)

$$
\begin{aligned}
&\left\|\frac{\partial F}{\partial u}(t, u_1)-\frac{\partial F}{\partial u}(s, u_2);\ \mathcal{L}(W^{2m,p}(\Omega), L^p(\Omega))\right\| \\
&\leq C_R(|t-s|+\|u_1-u_2;\ W^{2m,p}(\Omega)\|)
\end{aligned}
$$

if τ is small. Further, [28, Lemma 5.3.4, p. 142] implies that (H) holds with $\alpha=1$.

Hence, if

$$
\begin{aligned}
&u_0 \in \mathcal{D}(L(0)), \text{ the partial derivatives } \frac{\partial f}{\partial X_j}(0, u_0(x), \cdots) \text{ satisfy} \\
&\sup_{x\in\bar\Omega} \left|\frac{\partial f}{\partial X_j}(0, u_0(x), \cdots)\right| \leq k,\ k \text{ being a small constant,} \\
&\text{if } L(t) \text{ is the } L^p\text{-realization of } A(t), \qquad (8) \\
&x \to [f(0, u_0(x), \cdots)-(I+(\frac{d}{dt}(L(t))^{-1})_{t=0})\, v_0(x)] \text{ is in } \mathcal{D}(L(0)) \\
&\text{with} \quad v_0(x) = (L(0)\, u_0)(x) = A(0, x; D)\, u_0(x) ,
\end{aligned}
$$

which implies regularity for f, u_0, we are done.

As a Dirichlet problem, ($\mathcal{D}(L(t))=W^{2m,p}(\Omega) \cap W_0^{2m,p}(\Omega)$), (8) implies that

$$
x \to [f(0, u_0(x), \cdots, D^{2m-1} u_0(x))-L(0, x; D)\, u_0(x)]
$$

vanishes on $\partial\Omega$, together with its derivatives of order $\leq m-1$: we must look at $f(0, p_1, \cdots, p_{2m})$.

For the sake of clearness, take $m=1$; we want

$$
\begin{aligned}
&x \to [f(0, u_0(x), Du_0(x))-L(0, x; D)\, u_0(x)] \in W^{2,p}(\Omega) \\
&f(0, u_0(x), Du_0(x))-L(0, x; D)\, u_0(x) = 0,\ x \in \partial\Omega
\end{aligned}
$$

Since $u_0 \in W_0^{2,p}(\Omega)$, it vanishes on $\partial\Omega$; if, moreover, $L(0, x; D)\, u_0(x)=0$ on $\partial\Omega$, our assumption reads:

$$f,\, u_0 \text{ regular}, f(0, 0, \mathrm{p}) = 0 \text{ for every } p \in \boldsymbol{R}^n .$$

Such equations have been thoroughly studied by Pazy, Kielhöfer, Sinestrari-Vernole [22, 16, 24], but within t-independent domains: here we add time regularity results; moreover, the equations hold at $t=0$, too, which forces further compatibility and regularity conditions. The case of t-dependent domains has been deeply studied by Amann in many interesting papers with different techniques, also under more general assumptions: here we refer to [5], which deals with nonlinear *boundary conditions,* and [4], which studies the semilinear case.

A few final remarks are in order:

REMARKS. 3.1.1. As remarked after Theorem 5, the assumption about smallness of the derivatives $(\partial f/\partial X_j)\,(0, u_0(\cdot), \cdots)$ may be dropped if $\mathcal{D}(L(t))$ is time independent and $-[L(0)-(\partial F/\partial u)\,(0, u_0)]$ generates, for instance, an analytic semigroup in $L^p(\Omega)$: $L(t)$, $(\partial F/\partial u)\,(t, u)$ must however depend upon time as in (4), (5), (7):

3.1.2. We could even define a substitution operator F built up with highest-order derivatives:

$$\begin{aligned} F(t, u)\,(x) &= f(t, u(x), Du(x) \cdots, D^{2m-1}\, u(x)) \\ &\quad + \sum_{|\alpha|=2m} g_\alpha(t, \cdots, D^{2m-1}\, u(x))\, D^\alpha\, u(x) \end{aligned}$$

with suitably regular g_α's, since then $\partial F/\partial u$ may be evaluated using Sobolev's theory. We must require smallness of

$$\sup_{x\in\Omega} |\, g_\alpha(0, u_0(x), \cdots, D^{2m-1}\, u_0(x))\,| \ ,$$

$$\sup_{x\in\Omega} \left\| \frac{\partial g_\alpha}{\partial X_j}\,(0, u_0(x), \cdots, D^{2m-1}\, u_0(x)) \right\|$$

for every α and every j: assumptions about the smallness of the partial derivatives $\partial f/\partial X_j$ can be avoided by a perturbation argument.

3.2. (EXAMPLE 2 Again). Let us return into the framework of Example 2, as in 1.8. Two sesquilinear forms $a_0(t, u, v)$, $a_1(u, v)$ are given, $u, v \in V \hookrightarrow H$, $0 \leq t \leq \tau$.

Consider

$$\begin{cases} \dfrac{d}{dt}\,(Mu(t)) + L(t)\, u(t) = F(u(t)),\ t \in [0, \tau] \\ (Mu(t))_{t=0} = w_0 \end{cases}$$

where $L(t)$, M are linear operators connected to $a_0(t, \cdot, \cdot)$, a_1 respectively, as explained before.

We assume that $F \in C^{(1)}(V, H)$ and

$$\|F'(u_1)-F'(u_2); \mathcal{L}(V, H)\| \leq C_r \|u_1-u_2; V\| \quad \text{if} \quad \|u_i; V\| \leq r, i=1, 2 ;$$
$$\|F'(u_0); \mathcal{L}(V, H)\| \text{ is suitably small,}$$
$$w_0 = Mu_0, u_0 \in V, \quad \text{and} \quad [F(u_0)-(I+T'(0)) L(0) u_0] \in \mathcal{R}(ML(0)^{-1}) .$$

We want to clarify the kind of assumptions needed by means of a concrete particular case. Fix a bounded open domain Ω in $\boldsymbol{R}^n$ with regular boundary $\partial\Omega$ (or else put $\Omega=\boldsymbol{R}^n$). Let V be the space $H_0^m(\Omega)$ $(m \geq 1)$, and W an intermediate space, so that $V \subsetneq W \subsetneq L^2(\Omega)$; $a_0=a_0(t, u, v)$, $a_1=a_1(x, y)$ $(t \in [0, \tau], u, v \in H_0^m(\Omega), x, y \in W)$ will be the related sesquilinear forms, satisfying (E) in 1.8 with $\beta=1$: as before, call $L(t)$, M the operators (from V into V^*, from W into W^*, respectively) associated with $a_0(t, \cdot, \cdot)$, a_1.

Since we wish to apply Theorem 3, it is easy to see that (i)-(vii) and (H) are fulfilled in the case $X=V^*$, $Y=Y_1=V$, $\rho=1$, $\varepsilon=a$. Note that

$$\|L(t)-L(s); \mathcal{L}(V, V^*)\| \leq k|t-s| \qquad \text{for every} \quad t, s \in [0, \tau]$$

implies that (H) holds with $\alpha=1$ [28].

As for the nonlinear right-hand side, we put, with the same meaning as in 3.1 for D^j:

$$F(u)(x) = a(u(x), Du(x) \cdots, D^k u(x)) :$$

k is a nonnegative integer, $\leq m$, so F makes sense for every u in $H_0^m(\Omega)$, and a is a real valued $C^{(2)}$ function. We shall see in a moment that we need the condition

$$k+\frac{n}{2}<m . \tag{9}$$

If $u, v \in H_0^m(\Omega)=V$,

$$\begin{aligned}
&[F(u+v)-F(u)](x) \\
&= \int_0^1 \frac{d}{d\eta} a(u(x)+\eta v(x), \cdots, D^k u(x)+\eta D^k v(x)) \, d\eta \\
&= \int_0^1 \sum_{j=0}^k [\frac{\partial a}{\partial X_{j+1}} D^j v(x)] \, d\eta
\end{aligned}$$

where $\partial a/\partial X_{j+1}$ is evaluated at $(u(x)+\eta v(x), \cdots, D^k u(x)+\eta D^k v(x))$ for $j=0, \cdots, k$.

Suppose now that the norms in V of u, v are bounded by r: then

$$\left(\int_{\Omega}\{[F(u+v)-F(u)](x)-\sum_{j=0}^{k}[\frac{\partial a}{\partial X_{j+1}}D^j v(x)]\}^2\,dx\right)^{1/2}$$
$$\leq M(r)\left\{\int_{\Omega}[\sum_{j=0}^{k}\|D^j v(x)\|^2]\|v(x)\|^2\,dx)^{1/2}\right.$$
$$\left.+\cdots+(\int_{\Omega}[\sum_{j+0}^{k}\|D^j v(x)\|^2]\|D^k v(x)\|^2\,dx)^{1/2}\right\}$$
$$\leq M'(r)\{\sup_{x\in\Omega}|v(x)|+\cdots+\sup_{x\in\Omega}\|D^k v(x)\|\}\|v;V\|\leq M''(r)\|v;V\|^2\,.$$

(By (9), we may apply Sobolev Embedding Theorem [22, pp. 208, 222] and deduce the last estimate). So, F is differentiable as an application from V *into* H, and *a fortiori*, from V into V^*.

A new application of Sobolev Embedding Theorem yields also that F' is locally Lipschitz.

To summarize: we may apply Theorem 3 to abstract problems arising from

$$\begin{cases}\dfrac{\partial}{\partial t}(M(x,D)\,u(t,x))+L(t,x;D)\,u(t,x)=a(u(x),\cdots,D^k u(x))\,,\\ t\in[0,\tau],\,x\in\Omega\\ M(x,D)\,u(0,x)=M(x,D)\,u_0(x)\end{cases}$$

if we are looking for $u=u(t,\cdot)$ in $H_0^m(\Omega)$, (9) holds, and u_0 is regular. Besides trivial assumptions, we need

$$\sum_{j=0}^{k}\sup_{\Omega}|\frac{\partial a}{\partial X_{j+1}}(u_0,\cdots,D^k u_0)|\quad\text{to be small}$$

and, with the intended meaning for T,

$$[a(u_0,\cdots,D^k u_0)-(I+T'(0))\,L(0,\cdot;D)\,u_0]\in M(V)\,.$$

When $n=1$, k can reach the value $m-1$; if $L(t)\equiv L$ for simplicity, the last condition becomes

$$[a(u_0,\cdots,D^k u_0)-L(\cdot,D)\,u_0]=M(\cdot,D)\,w\,,$$

with $w\in H_0^m(\Omega)$.

3.3. Application 3. (Nonlinear Parabolic Equations). Let us apply Theorem 3 in the framework of spaces of continuous functions.

Put $C=C[0,1;\boldsymbol{C}]$ (or $C=C[0,1;\boldsymbol{R}]$ as well); as usual $\|f;C\|=\sup_{x\in[0,1]}|f(x)|$. Set now $C_{0,0}=\{\phi\in C;\,\phi(0)=\phi(1)=0\}$, and define A by

$$\mathcal{D}(A)=\{\phi\in C_{0,0};\,\phi'\in C,\,\phi''\in C_{0,0}\}\,,$$
$$A\phi=-\phi''\qquad\text{for every}\quad\phi\in\mathcal{D}(A)\,.$$

It is well-known [21, p. 312] that $-A$ is the infinitesimal gneerator of an analytic semigroup in $C_{0,0}$. Note [13, p. 172] that for every $n \in \boldsymbol{N}$

$$||u'; C|| \leq \frac{1}{n+2} ||u''; C|| + 2(n+1) ||u; C|| .$$

We are ready to develop an abstract version

$$\begin{cases} \dfrac{\partial}{\partial t} u(t, x) = \dfrac{\partial^2 u}{\partial x^2}(t, x) + \psi(t, u(t, x), \dfrac{\partial}{\partial x} u(t, x), \dfrac{\partial^2 u}{\partial x^2}(t, x)) \\ t \in [0, \tau],\ x \in]0, 1[\\ u(0, x) = u_0(x),\ x \in [0, 1] \\ u(t, 0) = u(t, 1) = \dfrac{\partial^2 u}{\partial x^2}(t, 0) = \dfrac{\partial^2 u}{\partial x^2}(t, 1) = 0,\ t \in [0, \tau] , \end{cases}$$

once we have fixed the nonlinearity ψ. We take it a $C^{(2)}$ function $[0, \tau] \times \boldsymbol{R}^3 \rightarrow \boldsymbol{R}$, and denote by F the related substitution operator, acting on $u \in C^{(2)}[0, 1; \boldsymbol{R}]$:

$$F(t, u)(x) = \psi(t, u(x), u'(x), u''(x)),\ t \in [0, \tau],\ x \in [0, 1] .$$

We perform the same tricks as before and we see easily that F is differentiable in u, with

$$[\frac{\partial F}{\partial u}(t, u) h](x) = \frac{\partial \psi}{\partial X_1} h(x) + \frac{\partial \psi}{\partial X_2} h'(x) + \frac{\partial \psi}{\partial X_3} h''(x) ,$$

the partial derivatives of ψ being evaluated at $(t, u(x), u'(x), u''(x))$.

So, if we wish to apply Theorem 3, we need assume that

$$\sup_{x \in [0,1]} | \frac{\partial \psi}{\partial X_3}(0, u_0(x), \cdots) | \text{ is small enough}, \quad \text{and}$$

(*) $$\psi(t, 0, p, 0) = \frac{\partial \psi}{\partial X_2}(t, 0, p, 0) = 0 \quad \text{for every} \quad p \in \boldsymbol{R},\ \forall t \in [0, \tau] ,$$

$$u_0 \in C^{(4)}[0, 1; \boldsymbol{C}],\ u_0^{(k)}(j) = 0,\ \forall j = 0, 1 \quad \text{and} \quad k = 0, 1, 2, 3, 4 .$$

REMARKS. 3.3.1. Thoerem 5 allows us to study more general problems. For instance, given the equation

(10) $$\frac{\partial}{\partial t} u(t, x) = g(t, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x)),\ t \in [0, \tau],\ x \in [0, 1] ,$$

with an initial value u_0 at $t=0$ and limit conditions, denote by G the substitution operator arising from g; if the latter is $C^{(2)}$, then the former is differentiable and, as before,

$$[\frac{\partial G}{\partial u}(t, u) v](x) = \frac{\partial g}{\partial X_1} v(x) + \frac{\partial g}{\partial X_2} v'(x) + \frac{\partial g}{\partial X_3} v''(x) ,$$

with partial derivatives evaluated at $(t, u(x), u'(x), u''(x))$; so the previous remark applies if e.g.

$$\inf_{x\in[0,1]} \frac{\partial g}{\partial X_3}(0, u_0(x), u_0'(x), u_0''(x))>0\,, \quad \text{and}$$

$$\frac{\partial g}{\partial X_2}(t, 0, p, 0) = g(t, 0, p, 0) = 0 \quad \text{identically on} \quad [0, \tau],\ \forall p\in \boldsymbol{R}\,,$$

$$u_0\in C^{(4)}[0, 1; \boldsymbol{C}],\ u_0^{(k)}(j) = 0,\ \forall j = 0, 1 \quad \text{and} \quad k = 0, 1, 2, 3, 4\,.$$

3.3.2. We can even study equation (10) in the space $C[0, 1]$ by means of semigroup generators with non dense domain [8]. In this case, the assumptions about g and u_0 can be relaxed; in fact, they turn out to be (compare Theorem 5)

$$\inf_{x\in[0,1]} \frac{\partial g}{\partial X_3}(0, u_0(x), u_0'(x), u_0''(x))>0\,, \quad \text{and}$$

$$g(0, 0, p, 0) = 0\ \forall p\in \boldsymbol{R}\,,$$

$$u_0\in C^{(4)}[0, 1],\ u_0(j) = u_0^{(2)}(j) = 0,\ \forall j = 0, 1\,.$$

3.3.3. We could improve a bit on the operator A too. We could introduce A_0, with (formally)

$$A_0 u(x) = -a(x)\,u''(x)+b(x)\,u'(x)+c(x)\,u(x)$$

(a, b, c regular enough), and then apply well-known perturbation results to $-A+A_0$.

As for the functions a, b, c, there are natural assumptions which work. We wish to discuss briefly some of them: we shall restrict to the case of continuous functions: so, we choose as ambient space either C (in which case A_0 is not densely defined), or $C_{0,0}$: moreover, we assume $a>0$ in [0,1]. There are many ways to prove generation properties in this case: see [13], [26], [27], [20], [7].

A very handy method is to note that $u\to -au''$ equipped with limit conditions does generate an analytic semigroup, and then apply Kato's perturbation theorem for analytic semigroups related to homogeneous limit conditions: this is possible in view of the estimate in [13, p. 172]: but if we choose $C_{0,0}$ as the ambient space, we have to assume $b(0)=b(1)=0$: we refer to [7, p. 378].

3.4. APPLICATION 4. (Degenerate Parabolic Equations). Consider now the problem ($k>0$ is a parameter)

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}+1\right)u(t, x) = \frac{\partial^2}{\partial x^2}u(t, x)-ku(t, x)+\psi(t, u(t, x), \frac{\partial}{\partial x}u(t, x), \frac{\partial^2}{\partial x^2}u(t, x))$$

$t\in[0, \tau]$, $x\in[0, \pi]$,

$$\left(\frac{\partial^2}{\partial x^2}+1\right)u(0, x) = w_0(x),\ x\in[0, \pi]\,,$$

$$u(t, 0) = u(t, \pi) = \frac{\partial^2}{\partial x^2}u(t, 0) = \frac{\partial^2}{\partial x^2}u(t, \pi) = 0,\ \forall t\in[0, \tau]\,,$$

where;:

a) the ambient space is $\tilde{C} = \{f \in C[0, \pi; \boldsymbol{C}]; f(0) = f(\pi) = 0\}$;

b) L is defined as the realization of $-\dfrac{\partial^2}{\partial x^2}$ with domain

$\mathcal{D}(L) = \{u \in C^{(2)}[0, \pi; \boldsymbol{C}]; u''(0) = u''(\pi) = u(0) = u(\pi) = 0\}$;

c) M is $\dfrac{\partial^2}{\partial x^2}+1$, plus the boundary conditions;

d) $w_0 = u_0''+u_0$, with $u_0 \in \mathcal{D}(L)$,

e) the nonlinearity ψ is a $C^{(2)}$ function $[0, \tau] \times \boldsymbol{R}^3 \to \boldsymbol{R}$ which satisfies

$\sup_{x \in [0,\pi]} \left|\dfrac{\partial \psi}{\partial X_3}(0\ u_0(x)\ \cdots)\right|$ is small enough,

$\psi(t, 0, p, 0) = \dfrac{\partial \psi}{\partial X_2}(t, 0, p, 0) = 0$ for every $p \in \boldsymbol{R}$, $\forall t \in [0, \tau]$,

f) the boundary value provlem

$\psi(0, u_0(x), u_0'(x), u_0''(x))+u_0''(x)-ku_0(x) = \left(\dfrac{d^2}{dx^2}+1\right) v(x),\ v \in \mathcal{D}(L)$

has a solution.

Note that, in particular, these conditions imply

$$\psi(0, 0, u_0'(0), 0) = \psi(0, 0, u_0'(\pi), 0) = 0\ ;$$

our third condition in e) is not necessary if we use the space $C[0, \pi; \boldsymbol{C}]$ instead of $\tilde{C}$.

Since we wish to apply Theorem 3, we need just observe that $z=0$ is a polar singularity for the resolvent $z \to (zI+M)^{-1}$, and that the preceeding problem is of the abstract type

$$\begin{cases} \dfrac{d}{dt}((K+I)\,u(t)) = (K-kI)\,u(t)+f(t, u(t)),\ t \in [0, \tau] \\ ((K+I)\,u(t))_{t=0} = w_0\,. \end{cases}$$

The point $z=0$ is in fact a polar singularity for $M=K+I$, that is, there exists $(M+zI)^{-1}$ for $0<|z|\leq \varepsilon$, and for such z's and a suitable $\alpha>0$,

$$\|(M+zI)^{-1}; \mathcal{L}(X)\| \leq \alpha |z|^{-1}\ ;$$

the change of variable

$$w(t) = \exp(-(\beta+1)\,t)\,u(t),\ \beta>0 \quad \text{a suitable positive number,}$$

changes the equation into the equivalent form

$$\frac{d}{dt}((K+I)\,w(t)) = -(K+I)\,((1+\beta)\,w(t))+g(t, w(t)),\ t \in [0, \tau]$$

and now the sought for estimate for $L(zM+L)^{-1}$ does hold in the half-plane $\operatorname{Re} z \geq 0$, if $L=k+1+\beta(K+I)$.

Theorem 3 then applies under assumptions on the nonlinearity ψ, which are quite similar to the ones discussed in 3.3.

3.5. Application 5. (Higher-Dimensional Parabolic Problems). Parabolic problems of the same kind as in 3.3 are clumsier and much more difficult to handle in more than one space dimension (this topic has been extensively studied by Da Prato and his school; see also [26], [27]).

However, let Ω be a bounded domain in $\boldsymbol{R}^n$ $(n>1)$ with regular boundary $\partial\Omega$: we replace the second derivative in 3.4 with a uniformly elliptic second-order operator on $\bar{\Omega}$, with coefficients as in [26]:

$$-(Au)(x)=\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x)+\sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i}+a(x)\,u(x)$$

Choose now as the ambient space $X=C(\bar{\Omega})$ equipped with the sup-norm, define A on $\mathcal{D}(A)=\{\phi\in W^{2,p}(\Omega);\ A\phi\in C(\bar{\Omega}),\ \phi=0 \text{ on } \partial\Omega\}$, and make the key assumption that $p>n$: from Stewart's results [26] it follows that $-A$ generates an analytic semigroup, which is not however strongly continuous at the origin, since $\overline{\mathcal{D}(A)}=\overline{C_0(\bar{\Omega})}\neq X$ (in this connection, see also [25]). Let f be of class $C^{(2)}(\boldsymbol{R},\boldsymbol{R})$. We state formally the problem as

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x)=-Au(t,x)+f(Au(t,x)),\ t\in[0,\tau],\ x\in\Omega \\ u(t,\cdot)\in\mathcal{D}(A) \quad \text{for every} \quad t\in[0,\tau] \\ u(0,x)=u_0(x),\ x\in\Omega\,. \end{cases}$$

Without loss of generality, suppose A has bounded inverse; the problem may be abstractly translated into

$$\begin{cases} \dfrac{d}{dt}(L^{-1}v(t))=-v(t)+f(v(t)),\ t\in[0,\tau] \\ (L^{-1}v(t))_{t=0}=u_0 \end{cases}$$

Call F the substitution operator relative to f: $F(v)(x)=f(v(x))$, $v\in C(\bar{\Omega})$. Then for every $h\in C(\bar{\Omega})$, as usual,

$$\begin{aligned} &F(v+h)(x)-F(v)(x)-f'(v(x))\,h(x) \\ &\quad=\int_0^1[f'(v(x)+\eta h(x))-f'(v(x))]\,h(x)\,d\eta\ ; \end{aligned}$$

so, if $v_0=Lu_0\in C(\bar{\Omega})$ and $\|v-v_0;\ C(\bar{\Omega})\|$, $\|h;\ C(\bar{\Omega})\|\leq r$, we may find a positive constant C, depending upon r, s.t.

$$\|F(v+h)-F(v)-[f'(v(\cdot))\,h(\cdot)];\ C(\bar{\Omega})\|\leq C\|h;\ C(\bar{\Omega})\|^2\,.$$

So F turns out to be differentiable, and

$$(F'(v)(h))(x) = f'(v(x))\, h(x), \quad \text{for every} \quad v, h \in C(\bar{\Omega}),\ x \in \bar{\Omega}\ ;$$

moreover, since

$$(F'(v_1)(h))(x) - (F'(v_2)(h))(x) = [f'(v_1(x)) - f'(v_2(x))]\, h(x)\,,$$

we may find also $C' = C'(r)$ s.t. for every $v_1, v_2 \in C(\bar{\Omega})$ with norm $\leq r$

$$\|F'(v_1) - F'(v_2);\ \mathcal{L}(C(\bar{\Omega}))\| \leq C' \|v_1 - v_2;\ C(\bar{\Omega})\|\,,$$

and Theorem 3 applies with $X = Y = Y_1 = C(\bar{\Omega})$; we must assume that

$$\sup_{x \in \Omega} |f'(v_0(x))| \text{ is suitably small,}$$
$$x \to [f(v_0(x)) - v_0(x))] \in \mathcal{D}(A)\ :$$

in particular, if v_0 vanishes on $\partial\Omega$ we must assume $f(0) = 0$.

A final remark: it is possible to generalize a bit, and introduce the equation

$$\frac{\partial}{\partial t} u(t, x) = -Au(t, x) + f(Bu(t, x)),\ t \in [0, \tau],\ x \in \Omega\ ;$$

here $B = B(x, D)$ is a differential operator of order ≤ 2: now we must assume that the abstract operator arising from BA^{-1} is in $\mathcal{L}(C(\bar{\Omega}))$, i.e. for every $u \in \mathcal{D}(A)$, $u \in \mathcal{D}(B)$ too, and

$$\|Bu;\ C(\bar{\Omega})\| \leq C \|Au;\ C(\bar{\Omega})\|\,,$$

a formally simple condition which is however difficult to work out explicitly.

3.6. Let us define

$$X = C_0^0(\boldsymbol{R}^n) = \{f \in C^{(0)}(\boldsymbol{R}^n);\ \lim_{|x| \to \infty} f(x) = 0\}$$

with the sup norm. It has been shown in [17, p. 309] that the operator Δ, the (distributional) laplacian with domain $\mathcal{D}(\Delta) = \{f \in X;\ \Delta f \in X\}$ generates a bounded, strongly continuous, analytic semigroup in X. Analogously, if $\Phi \in C^{(2\theta)}(\boldsymbol{R}^n)$, with $0 < \theta < 1/2$, satisfies

$$\Phi(x) \geq \varepsilon \geq 0 \quad \forall x \in \boldsymbol{R}^n\,,$$

then the operator K defined by

$$\mathcal{D}(K) = \mathcal{D}(\Delta),\ (Kf)(x) = \Phi(x)\, \Delta f(x)\ \forall x \in \boldsymbol{R}^n,\ \forall f \in \mathcal{D}(K)\,,$$

is the infinitesimal generator of an analytic semigroup in X ([17, p. 310]: see also [26]).

Let us now discuss how to apply our Theorem 3 to (an abstract form of) the problem ($t\in[0, \tau]$, $x\in\boldsymbol{R}^n$)

$$
(11) \qquad \begin{cases} \dfrac{\partial}{\partial t}\,((\Delta_x-k)\,u(t, x)) = (\Phi(x)\Delta_x-k')\,(\Delta-k'')\,u(t, x) \\ \qquad +\phi(t, u(t, x), \Delta_x\,u(t, x), \Delta_x^2\,u(t, x)) \\ x\to u(t, x) \quad x\to \Delta_x^2\,u(t, x) \quad \text{are in } X, \\ (\Delta_x\,u(t, x))_{t=0} = w_0(x) = \Delta u_0(x) \end{cases}
$$

where $k, k'>0$ and $k''\in\boldsymbol{R}$ are parameters, $u_0\in\mathcal{D}(\Delta^2)$: as for u, its properties should appear through the choice of functional spaces in (11).

First of all, with our usual notations, $||\lambda M(\lambda M+L)^{-1};\ \mathcal{L}(X)||$ is uniformly bounded on $\operatorname{Re}\lambda\geq 0$: thence, the assumptions on L, M in Theorem 3 hold true. Let us now turn to the regularity we need for ϕ: $[0, \tau]\times\boldsymbol{R}^3\to\boldsymbol{R}$; if such a mapping is $C^{(2)}$, then the substitution operator

$$
F(t, u)\,(x) = \phi\,(t, u(x), \Delta u(x), \Delta^2 u(x))\ (t\in[0, \tau], x\in\boldsymbol{R}^n, u\in\mathcal{D}(\Delta^2))
$$

satisfies

$$
(12) \qquad \begin{aligned} &F(t, u+h)\,(x)-F(t, u)\,(x) \\ &\quad = \int_0^1 \{\frac{\partial\phi}{\partial\xi_1}\,h(x)+\frac{\partial\phi}{\partial\xi_2}\,\Delta h(x)+\frac{\partial\phi}{\partial\xi_3}\,\Delta^2 h(x)\}\,d\sigma\,, \end{aligned}
$$

the space derivatives of ϕ being evaluated at

$$
(t, u(x)+\sigma h(x), \Delta u(x)+\sigma\Delta h(x), \Delta^2 u(x)+\sigma\Delta^2 h(x))\ ;
$$

moreover, we know that

$$
\begin{aligned} &|\frac{\partial\phi}{\partial\xi_i}\,(t', u)-\frac{\partial\phi}{\partial\xi_i}\,(t'', v)|\leq C(R)\,[\,|t'-t''|+|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\,], \\ &i = 1, 2, 3 \end{aligned}
$$

$\forall t', t''\in[0, \tau]$, if $u=(u_1, u_2, u_3)$ and $v=(v_1, v_2, v_3)$ are in a fixed closed ball of radius $R>0$.

Choose u_1, u_2 in the space $\mathcal{D}(\Delta^2)$, with $||(\Delta-I)^2\,(u_i-u_0);\ X||\leq R$, $i=1, 2$; then, for $i=1, 2, 3$

$$
\begin{aligned} &|\frac{\partial\phi}{\partial\xi_i}\,(t', u_1(x), \Delta u_1(x), \Delta^2\,u_1(x))-\frac{\partial\phi}{\partial\xi_i}\,(t'', u_2(x), \Delta u_2(x), \Delta^2\,u_2(x))\,| \\ &\leq C'(R)\,[\,|t'-t''|+|u_1(x)-u_2(x)|+|\Delta(u_1(x)-u_2(x))|+|\Delta^2(u_1(x)-u_2(x))|\,] \\ &\leq C_1(R)\,[\,|t'-t''|+|u_1(x)-u_2(x)|+|(\Delta-I)\,(u_1(x)-u_2(x))| \\ &+|(\Delta-I)^2\,(u_1(x)-u_2(x))|\,]\,, \end{aligned}
$$

since then $||(u_i-u_0);\ X||\leq\alpha R$, $||\Delta(u_i-u_0);\ X||\leq\alpha R$, $i=1, 2$.

We see now that, up to a change of $C_1(R)$ to another bound $C''(R)$, we can obtain, with the same notations and integral representation as in (12),

$$|F(t, u+h)(x)-F(t, u)(x)-\{\frac{\partial\phi}{\partial\xi_1}h(x)+\frac{\partial\phi}{\partial\xi_2}\Delta h(x)+\frac{\partial\phi}{\partial\xi_3}\Delta^2 h(x)\}| \\ \leq C''(R))\{|h(x)|^2+|\Delta h(x)|^2+|\Delta^2 h(x)|^2\}$$

for every $h\in\mathcal{D}(\Delta^2)$ with $\|(\Delta-I)^2 h; X\|\leq R$, the space derivatives of ϕ being evaluated at $(t, u(x), \Delta u(x), \Delta^2 u(x))$. So the partial differential $\partial F/\partial u$ has the required regularity; hence, a standard perturbation argument shows that if

$$\sup_{x\in[0,\tau]}|\frac{\partial\phi}{\partial\xi_3}(0, u_0(x), \Delta u_0(x), \Delta^2 u_0(x))| \text{ is small enough,}$$

and the compatibility condition

$$x\to\{\phi(0, u_0(x), \Delta u_0(x), \Delta^2 u_0(x))+(\Phi(x)\Delta_x-k')(\Delta_x-k'')u_0(x)\}\in\mathcal{R}(ML^{-1})$$

holds that is, that function is in $\mathcal{D}(\Delta)$, then problem (11) can be solved by means of Theorem 3. For small τ we obtain solutions u s.t. $t\to(\Delta_x-k)u(t,\cdot)$, $(\Phi(\cdot)\Delta_x-k')(\Delta-k'')u(t,\cdot)$ are of class $C_0^{(\nu)}[0, \tau; X]$, $0<\nu<1$.

4. Navier-Stokes and Other Equations

4.1. APPLICATION 5. (Abstract Navier-Stokes Equation). Let us now consider the Navier-Stokes equation in $\boldsymbol{R}^n$, $n\geq 2$,

$$\text{(NSP0)}\quad\begin{cases}\dfrac{\partial}{\partial t}u(t, x)+\langle u(t, x), \nabla_x\rangle u(t, x)-\Delta_x u(t, x) = k(t, x)-\nabla_x q(t, x),\\ \quad t\in[0, \tau],\ x\in\boldsymbol{R}^n\\ \langle\nabla_x, u(t, x)\rangle = 0 \quad\text{for every}\quad t\in[0, \tau],\ x\in\boldsymbol{R}^n\\ u(0, x) = u_0(x),\ x\in\boldsymbol{R}^n\,.\end{cases}$$

as considered e.g. in [14].

Here the unknown q is a scalar function, k and the unknown u are n-tuples of functious $[0, \tau]\times\boldsymbol{R}^n\to\boldsymbol{R}$, u_0 is a given n-tuple of real-valued functions defined on $\boldsymbol{R}^n$: the data are supposed to be continuous functions (something less could work too).

In the framework of $X=(L^p(\boldsymbol{R}^n))^n$, $n<p<+\infty$, the key fact used classically in the solution of (NSPO) is the following (see again [14], where further related literature is quoted). If we define

$$\begin{cases}X_p \text{ is the closure of } \{u\in(C_0^{(\infty)}(\boldsymbol{R}^n))^n; \langle\nabla, u\rangle = 0\} \text{ in } X\\ P \text{ is the projection of } X \text{ onto } X_p\\ A \text{ is } -\Delta \text{ with domain } \mathcal{D}(A)=(W^{2,p}(\boldsymbol{R}^n))^n\,,\end{cases}$$

then $P\nabla=0$ and P, Δ commute in some sense.

In this section, we want to traduce abstractly the special features of the operators entering in (NSP0) which make successful such an approach, thus discussing an abstract version of (NSP0), to which it is possible to apply our general results. While doing so, we shall obtain cretain t-regularity results that are new to our knowledge.

First of all, we note that in (NSP0), by a standard trick, we could seek u, $\nabla_x q$ instead of u, q. This remark allows us to set (NSP0) into the following abstract framework.

Let X, Y be Banach spaces, A, C linear closed operators (from X into itself, from X into Y, respectively); if Y_1 is an intermediate Banach space between $\mathcal{D}(A)$ and X (that is, $\mathcal{D}(A)\subsetneq Y_1\subsetneq X$), we introduce continuous functions F: $[0, \tau]\times Y_1\to X$, h: $[0, \tau]\to X$, and $u_0\in\mathcal{D}(A)$: these notations held fixed, we want to study the abstract problem of Navier-Stokes type

$$
\text{(NSP)}\qquad \begin{cases} \dfrac{du(t)}{dt}+Au(t)=p(t)+F(t, u(\cdot))+h(t),\ t\in[0, \tau]\,, \\ Ku(t)=0 \quad \text{for every} \quad t\in[0, \tau]\,, \\ u(0)=u_0\,. \end{cases}
$$

We call a pair (u, p) a solution of (NSP) if $u\in C[0, \tau; \mathcal{D}(A)]\cap C^{(1)}[0, \tau; X]$, $u(t)\in\mathcal{D}(K)$ for every $t\in[0, \tau]$, $p\in C[0, \tau; X]$ and they do satisfy (NSP).
We then assume:

(*) $-A$ generates an analytic semigroup in X, X being reflexive;

(**) there exists an operator $\boldsymbol{A}$, such that its opposite generates an analytic semigroup in $\mathcal{L}(Y)$, and moreover $KAu=\boldsymbol{A}Ku$ if $u\in\mathcal{D}(KA)$;

(***) if $u\in C^{(1)}[0, \tau; X]$ and $u'(t)\in\mathcal{D}(K)$ $\forall t\in[0, \tau]$, then $Ku'=(Ku)'$;

if P is the projection operator from X onto $\boldsymbol{N}(K)$ along the closure of $\mathcal{R}(A)$,

(****) $u_0\in\mathcal{D}(A)\cap\boldsymbol{N}(K)$, so that $0=KAu_0=\boldsymbol{A}Ku_0$,
$\|u_0; Y_1\|$ is suitably small,
$u_1=-Au_0+P\{F(0, u_0)+h(0)\}\in\mathcal{D}(A)$;

(*****) $(t, y)\to F(t, y)$ is a $C^{(1)}$ mapping $[0, \tau]\times Y_1\to X$,
and there exists $\eta>0$ such that

$$\left\|\frac{\partial F}{\partial u}(t, u_1)-\frac{\partial F}{\partial u}(s, u_2); \mathcal{L}(Y_1, X)\right\|\leq C(|t-s|+\|u_1-u_2; Y_1\|)$$

$\forall s, t\in[0, \tau]$ and $\forall u_1, u_2\in Y_1$ with norms $\leq\eta$, $\dfrac{\partial F}{\partial u}(0, 0)=0$;

$h\in C^{(1)}[0, \tau; X]$, and $P\{F(0, 0)+h(0)\}\in\mathcal{D}(A)$.

Now, we can apply the machinery developed so far to the Cauchy problem

$$
\text{(NSP1)}\qquad \begin{cases} \dfrac{d}{dt}(u(t))+Au(t)=P[F(t, u(t))+h(t)],\ t\in[0, \tau] \\ (u(t))_{t=0}=u_0\,. \end{cases}
$$

We claim that the pair (u, p) is a solution of (NSP) if u satisfies (NSP1) and

$$p=p(t)=(P-I)\,[F(t, u(t))+h(t)]\;;$$

in fact this follows from two remarks:
(I) If (NSP1) has a solution u, then $\forall t\in[0, \tau]$,

$$\frac{d}{dt}(u(t))+Au(t) = F(t, u(t))+h(t)+p(t) ;$$

this is the very definition of p.
(II) $\forall t\in[0, \tau]$, we get $Ku(t)=0$.

To see this, let us define $v(t)=u(t)-u_0-tu_1$, $\forall t\in[0, \tau]$: we have only to prove that $Kv=0$. Now, v satisfies

$$\frac{d}{dt}(v(t))+Av(t) = P[F(t, v(t)+u_0+tu_1)+h(t)]-Au_0-u_1-tAu_1 ,$$

$\forall t\in[0, \tau]$. If we call $g(v)(t)$ the right-hand side, we can write v as a contour integral. If B is the time derivative with domain $\mathcal{D}(B)=\{u\in C_0^{(1)}[0, \tau; X]; u'(0)=0\}$,

$$v = \frac{1}{2\pi i}\int_\Gamma z^{-1}(zI+A)^{-1} B(B-zI)^{-1} g(v)\, dz ,$$

with a trivial meaning for Γ (see (C) in §1).

Since

$$Kg(v)(t) = -K[Au_0+u_1]-tKAu_1 = 0 \quad \forall t\in[0, \tau] ,$$

we see that

$$Kv(t) = \frac{1}{2\pi i}\int_\Gamma z^{-1}(zI+\boldsymbol{A})^{-1} \boldsymbol{B}(\boldsymbol{B}-zI)^{-1} Kg(v)\, dz = 0 ,$$

$\boldsymbol{B}$ being the time derivative defined in a similar manner as B, but in $C_0[0, \tau; Y]$.

It follows $Ku(t)=Kv(t)+tKu_1=-t\boldsymbol{A}Ku_0=0$. As a summary of the preceding discussion, we can then state

Theorem 6. *Let* (*)—(*****) *hold. Then* (NSP) *has a solution* (u, p) *such that* $\frac{du}{dt}$, Au *are in* $C^{(\theta)}[0, \tau; X]$, $0<\theta<1$.

A final remark: if $(\partial F/\partial u)(0, u_0)\in\mathcal{L}(Y_1, X)$ is a good perturbation of $-A$, in the sense that $-(A-(\partial F/\partial u)(0, u_0))$ still generates an analytic semigroup in X, we need not suppose that $\|u_0; Y_1\|$ is small, which implies that $\|(\partial F/\partial u)(0, u_0); \mathcal{L}(Y_1, X)\|$ is small too.

4.2. Let us return now to our Navier-Stokes problem (NSP0). We consider it as an abstract one in the framework of $X=(L^p(\boldsymbol{R}^n))^n$, $n<p<+\infty$, and set

$$F(w) = = -\langle w, \nabla_x\rangle w,\ w\in(W^{1,p}(\boldsymbol{R}^n))^n .$$

We need now check the regularity of $G(w)=P\langle w, \nabla_x\rangle w$. In fact, we assumed

$n<p$ mainly to secure that $W^{1,p}(\boldsymbol{R}^n)$ is a Banach algebra [3, p. 115] so that

$$\begin{aligned}
&\|G(u+h)-G(u)-P\langle h, \nabla\rangle u-P\langle u, \nabla\rangle h; (L^p(\boldsymbol{R}^n))^n\| \\
&\quad = \|P\langle h, \nabla\rangle h; (L^p(\boldsymbol{R}^n))^n\| \\
&\quad \leq C\|h; W\| \, \|h; (W^{1,p}(\boldsymbol{R}^n))^n\| \leq C\| h; (W^{1,p}(\boldsymbol{R}^n))^n\|^2 = C\| h; Y_1\|^2 .
\end{aligned}$$

For notational ease, W is here the space of bounded continuous functions $g: \boldsymbol{R}^n \to \boldsymbol{R}^n$ with the sup norm; in this connection, recall that if $\Omega \subset \boldsymbol{R}^n$ has the cone property [3, p. 66], $W^{1,p}(\Omega)$ is continuously embedded in the space of bounded continuous functions $\Omega \to \boldsymbol{R}$.

If $u_1, u_2 \in Y_1$ it is easy to see that

$$G'(u_i)\, h = P\langle h, \nabla\rangle u_i - P\langle u_i, \nabla\rangle h , \quad \mathrm{i} = 1, 2 .$$

We note the bound

$$\| [G'(u_1)-G'(u_2)]\, h; X\| \leq C\| h; Y_1\| \, \|u_1-u_2; Y_1\| ,$$

whence

$$\| G'(u_1)-G'(u_2); \mathcal{L}(Y_1, X)\| \leq C\|u_1-u_2; Y_1\| .$$

Finally, if $u \in (W^{3,p}(\boldsymbol{R}^n))^n$ and Δ_1 is the Laplace operator in $L^p(\boldsymbol{R}^n)$ with domain $W^{2,p}(\boldsymbol{R}^n)$, then $\langle \nabla, (\Delta u_1, \cdots, \Delta u_n)\rangle = \Delta_1 \langle \nabla, u\rangle$. Moreover, $-(A-G'(u_0))$ generates an analytic semigroup if $u_0 \in Y_1$, and we are done. All crucial assumptions of Theorem 6 are easily verifiable, in view of the remark following it.

4.3. APPLICATION 6. We conclude with a brief sketch of discussion about equations in the form $u' = F(-Au)$. In [30] W. von Wahl studied the global solvability of

$$\begin{cases}
\dfrac{\partial u(t, x)}{\partial t} - f(\Delta_x u(t, x)) = 0 \text{ on } \Omega , \\
u(t, x) = 0 \text{ on } \partial\Omega , \\
u(0, x) = \phi(x) .
\end{cases}$$

The main assumption was the strict positivity of f': we shall show how such a condition arises rather naturally within the framework built so far.

Let Ω be -as above- a bounded domain in $\boldsymbol{R}^n (n>1)$ with regular boundary $\partial\Omega$. Define second-order time dependent differential operators and boundary ones as in Example 1 (1.4)

$$\begin{cases}
-A(t, x; D) = \sum_{i,j=1}^n a_{ij}(t, x) \dfrac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(t, x) \dfrac{\partial}{\partial x_i} + a(t, x) , \\
(t, x) \in [0, \tau] \times \bar{\Omega} \\
B(t, y; D) = \sum_{i=1}^n b_i(t, y) \dfrac{\partial}{\partial x_i} + b(t, y), \ (t, y) \in [0, \tau] \times \partial\Omega
\end{cases}$$

We assume that the coefficients fulfill all assumptions labeled in [2] as (A1, 2), (B1, 2), (AB2, 3); we define then $A=A(t)$ by

$\mathcal{D}(A(t))=\{u\in C(\bar{\Omega})\cap W^{2,q}(\Omega);\ A(t,\cdot;D)u\in C(\bar{\Omega}),\ B(t,\cdot;D)u=0 \text{ on } \partial\Omega\}$, $A(t)u=A(t,\cdot;D)u$; we fix henceforth $q>n$.

Let $\phi\in C^{(1)}(\boldsymbol{R},\boldsymbol{R})$: on $C(\bar{\Omega})$ we define also the substitution operator F arising from ϕ in the usual manner: $F(v)(x)=\phi(v(x))$ for every $x\in\bar{\Omega}$ and $v\in C(\bar{\Omega})$.

Hence we translate the problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t}=\phi(-A(t,x;D)\,u(t,x)) & (t,x)\in[0,\tau]\times\bar{\Omega} \\ B(t,x;D)\,u(t,x)=0 & (t,x)\in[0,\tau]\times\partial\Omega \\ u(0,x)=u_0(x) & x\in\bar{\Omega} \end{cases}$$

into the abstract form

$$\text{(NLP3)}\qquad \begin{cases} \dfrac{du(t)}{dt}=F(-A(t)\,u(t))\,, & t\in[0,\tau] \\ u(0)=u_0\in C(\bar{\Omega}) \end{cases}$$

which is equivalent, under the change $A(t)\,u(t)=v(t)$, to

$$\begin{cases} \dfrac{d}{dt}(A(t)^{-1}v(t))=F(-v(t)), & t\in[0,\tau] \\ (A(t)^{-1}v(t))_{t=0}=u_0\in C(\bar{\Omega})\,. \end{cases}$$

F is trivially differentiable, and

$$(F'(v)\,h)\,(x)=\phi'(v(x))\,h(x) \quad \text{for every} \quad x\in\bar{\Omega},\ v,h\in C(\bar{\Omega})\,.$$

The condition $\phi'(t)>0$ for every $t\in\boldsymbol{R}$ implies that $F'(-A(0)\,u_0)\,A(t)$ enjoys the very same relevant properties as $A(t)$, so the problem discussed in [30] may be attacked, inasmuch local solvability is concerned, as a particular case of (NLP3).

Besides other condition easy to guess, we obtain existence if $F(-v_0)-Sv_0\in\mathcal{D}(A(0))$: v_0 is $A(0)\,u_0$ and S equals $((d/dt)\,(A(t)^{-1}))_{t=0}$. If $\mathcal{D}(A(t))\equiv D$ does not depend upon time this condition becomes simpler: $F(-v_0)\in D$, since then

$$S=-A(0)^{-1}A'(0)\,A(0)^{-1}\,.$$

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