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TRANSFORMATION LAW FOR THE SZEGÖ PROJECTORS ON CR MANIFOLDS

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Introduction. Let M be a strictly pseudoconvex CR manifold of dimension $2n+1$. In case a volume element is specified on M, the Szegö projector $S: L^2(M) \rightarrow L^2(M)$ is defined as the orthogonal projector onto the subspace $\ker \bar{\partial}_b$; thus S is not CR-invariant. Assuming that M is the boundary of a strictly pseudoconvex domain $\Omega \subset \boldsymbol{C}^{n+1}$, Fefferman [5] constructed a volume element on M, by using the complex Monge-Ampère operator, in such a way that a natural transformation law for the Szegö projectors holds under CR isomorphisms, cf. (4.1) below. The purpose of this note is to generalize his construction to the case in which M is not necessarily the boundary of a domain.

What we have to do is to seek a right condition on the vloume element on M so as to get the transformation law. Keeping in mind that volume element on M is uniquely determined by contact form, we first specify a family of locally defined contact forms—a step, due to Farris [2], of making Fefferman's construction intrinsic (cf. also Fefferman [4]). As is observed in Farris [2], this is equivalent to giving a family of $(n+1, 0)$-forms on M, closed and nonvanishing. In order to achieve our construction of volume element, it is at first necessary to assume the local existence of a nonvanishing closed $(n+1, 0)$-form in a neighborhood of every point on M. The simplest situation is that there exists a globally defined contact form θ obtained by gluing the $(n+1, 0)$-forms above; if the volume element is given by $\theta \wedge d\theta^n$, then the transformation law for the Szegö projectors (Theorem 1) is derived just as in Fefferman's construction. However, there is a topological obstruction to the global existence of such a contact form. The vanishing of $c(K_M)$, the Chern class of the canonical bundle with real coefficients, is a necessary condition for the global existence. It is not known whether this condition is sufficient (cf. Lee [6] and Remark 1 below); to avoid this difficulty we generalize the notion of the Szegö projector. We construct a complex line bundle, by using the assumption $c(K_M)=0$, *via* transition of the locally defined contact forms in order to define the space of L^2 sections of the bundle, the space on which the Szegö projector is acting; then, the required transformation law (Theorem 2) follows naturally.

In deriving the transformation law, we assume the topological condition $H^1(M, \boldsymbol{R})=0$. This assumption ensures the univalence of a "Jacobian" in Theorem 1, and, in Theorem 2, the uniqueness of the hermitian line bundle on which sections the Szegö projector acts.

The plan of this note is as follows. We introduce some definitions and notation in section 1. In section 2 we present a condition on contact forms which characterizes Fefferman's volume element in the general case, a natural condition leading to the transformation law. Then, we can state and prove, in section 3, our main result (Theorem 1)—a transformation law for the Szegö projectors in case M admits a globally defined contact form which satisfies the condition in section 2. In section 4 we apply the result of section 3 to the case of domains in $\boldsymbol{C}^{n+1}$ (which was treated by Fefferman). In section 5 we prove Theorem 2, a generalization of Theorem 1 to the case of CR manifolds satisfying $c(K_M)=0$ (see also Remark 2).

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1. CR manifolds. In order to state our results, we shall briefly recall definitions and notation. Let M be a real, $(2n+1)$-dimensional, orientable, C^∞ manifold. A *CR structure* on M is defined by giving a complex n-dimensional complex subbundle $T^{1,0}$ of the complexified tangent bundle $\boldsymbol{C}TM$ satisfying:

(i) $T^{1,0}\cap T^{0,1}=\{0\}$, where $T^{0,1}=\overline{T^{1,0}}$;

(ii) if X and Y are sections of $T^{1,0}$, so is $[X, Y]$.

We will assume that the structure is *strictly pseudoconvex*, that is, for some choice of a real one-form θ annihilating $T^{1,0}$, the *Levi form* $L_\theta(V, W)=-id\theta(V\wedge\overline{W})$ is positive definite on $T^{1,0}$. Such a one-form θ is called a *contact form associated with* the CR structure.

To define the Szegö projector, we need to give a contact form θ on M—a choice of contact form is called a *pseudohermitian structure* on M. A pseudohermitian structure permits us to define the Hilbert space $L^2(M)$ of square integrable functions with respect to the volume element $\theta\wedge d\theta^n$, so that the operator $\bar\partial_b$: $C^\infty(M)\to C^\infty(M, T^{0,1*})$ defined by $\bar\partial_b f=df|_{T^{0,1}}$ extends naturally to a closed operator in $L^2(M)$. Then, the Szegö projector is defined as the orthogonal projector onto the closed subspace ker $\bar\partial_b$, the space of L^2 *CR holomorphic functions*.

Recall that the *canonical bundle* of M is a complex line bundle K_M of $(n+1, 0)$-forms given by

$$K_M=\{\zeta\in\boldsymbol{C}\wedge^{n+1}T^*M;\ V\rfloor\zeta=0\quad\text{for}\quad V\in T^{0,1}\}\ .$$

We shall later use the following fact on the canonical bundle K_M, or, rather, $K_M^\times=K_M\setminus\{0\}$. Given closed sections ζ, ζ' of $K_M^\times$, we have $\zeta=f\zeta'$, where f is CR holomorphic.

2. Condition on contact forms. In order to derive the transformation law, we start with the observation of Farris [2] that a nonvanishing $(n+1, 0)$-form determines a contact form.

Proposition 2.1 ([2, Proposition 3.2]). *Let U be an open set in M. If $\zeta\in C^{\infty}(U, K_M^{\times})$ there exists a unique contact form θ_ζ defined on U such that*

$$(2.1)\qquad \theta_\zeta\wedge d\theta_\zeta^n=i^{n^2}n!\,\theta_\zeta\wedge(T\lrcorner\zeta)\wedge(T\lrcorner\bar{\zeta})\qquad \textit{whenever}\quad \theta_\zeta(T)=1,\ T\in TM\,.$$

Furthermore, if f is a nowhere zero complex valued function on U, then

$$(2.2)\qquad \theta_{f\zeta}=|f|^{2/(n+2)}\theta_\zeta\,.$$

This contact form θ_ζ is said to be *normalized with respect to ζ*. This normalization is used in order to define our condition.

DEFINITION 2.2. A pseudohermitian structure θ is said to satisfy *Condition F* if, in a neighborhood of every point, there exists a closed section ζ of $K_M^{\times}$ which normalizes θ.

The most important example of a pseudohermitian structure satisfying Condition F is the one induced by an embedding $M\subset \boldsymbol{C}^{n+1}$. In this case, $\zeta=dz^1\wedge\cdots\wedge dz^{n+1}$ gives a closed section of $K_M^{\times}$, and hence θ_ζ satisfies Condition F. We will see in section 4 that the associated volume element $\theta_\zeta\wedge d\theta_\zeta^n$ coincides with the volume element constructed by Fefferman.

REMARK 1. If $\dim M\geq 5$, then Condition F arises from a geometric problem which was posed by Lee [6]: Find a pseudohermitian structure for which the pseudohermitian Ricci tensor (i.e. the Webster Ricci tensor) is a scalar multiple of the Levi form. A pseudohermitian structure satisfying this condition is said to be *pseudo-Einstein*. This condition is nontrivial only when $\dim M\geq 5$. In this case, Lee showed in [6] that the pseudo-Einstein condition is equivalent to Condition F. For global existence of a pseudo-Einstein structure, he gave a simple necessary condition—the vanishing of the first Chern class of $T^{1,0}$ with real coefficients, or, equivalently, $c(K_M)=0$. (In three-dimensional case, it is easy to see that this is also a necessary condition for the existence of a pseudohermitian structure satisfying Condition F.) He conjectured that the Chern class condition is also sufficient, and proved it positively under some geometric restrictions.

3. Transformation law on pseudohermitian manifolds. We are now in a position to derive a transformation law for the Szegö projectors on pseudohermitian manifolds satisfying Condition F. Our result is the following:

Theorem 1. *Let (M_1, θ_1), (M_2, θ_2) be pseudohermitian manifolds satisfying*

Condition F and let Φ: $M_1 \to M_2$ be a CR isomorphism. Assume that $H^1(M_1, \boldsymbol{R}) = 0$. Then, there exists a CR holomorphic function f on M_1 which satisfies $\Phi^*\theta_2 = |f|^{2/(n+1)}\theta_1$, and the Szegö projectors transform according to

$$\boldsymbol{S}_1(f\cdot(\varphi\circ\Phi)) = f\cdot((\boldsymbol{S}_2\varphi)\circ\Phi) \qquad for \quad \varphi \in L^2(M_2)\,, \tag{3.1}$$

where $\boldsymbol{S}_j$ is the Szegö projector on M_j for $j=1, 2$.

Before proving this, we rewrite (3.1) in terms of kernel functions. Boutet de Monvel and Sjöstrand [1] showed that if M bounds a relatively compact set in a Stein manifold, or if M is compact and dim $M \geq 5$, then the Szegö projector is written as a Fourier integral operator. In this case, we can write (3.1) as

$$s_1(x, y) = f(x)s_2(\Phi(x), \Phi(y))\overline{f(y)} \qquad \text{for} \quad (x, y) \in M_1 \times M_1 \tag{3.2}$$

where s_j is the Schwartz kernel of $\boldsymbol{S}_j$ for $j=1, 2$.

Proof of Theorem 1. Since $\Phi^*\theta_2$ is a contact form on M_1, it follows that $\Phi^*\theta_2 = e^{2u}\theta_1$ with a real a valued function u on M_1. We first show that u is *CR pluriharmonic*, that is, u is locally the real part of a CR holomorphic function. Clearly θ_1 and $\Phi^*\theta_2$ satisfy Condition F, so that there are locally defined, closed sections ζ, ζ' of $K^\times_{M_1}$ which normalize θ_1, $\Phi^*\theta_2$, respectively. If we write $\zeta' = e^g\zeta$ with a CR holomorphic function g, then (2.2) gives $\Phi^*\theta_2 = e^{2\mathrm{Re}(g)/(n+2)}\theta_1$. Therefore u is locally the real part of $g/(n+2)$. Since $H^1(M_1, \boldsymbol{R}) = 0$, it follows from Lemma 3.1 of [6] that there exists a globally defined conjugate function v of u—a real valued function which makes $u+iv$ CR holomorphic. Then, the CR holomorphic function $f = e^{(n+1)(u+iv)}$ satisfies the first assertion. With such an f, we have, for any functions φ, $\psi \in L^2(M_2)$,

$$\begin{aligned}\int_{M_2} \varphi\bar{\psi}\,\theta_2\wedge d\theta_2^n &= \int_{M_1} \varphi\circ\Phi\cdot\overline{\psi\circ\Phi}\cdot\Phi^*(\theta_2\wedge d\theta_2^n) \\ &= \int_{M_1} f\cdot(\varphi\circ\Phi)\cdot\overline{f\cdot(\psi\circ\Phi)}\cdot\theta_1\wedge d\theta_1^n\,.\end{aligned}$$

Hence we can define an isomorphism $L^2(M_2) \to L^2(M_1)$ by $\varphi \mapsto f\cdot(\varphi\circ\Phi)$. Since f is CR holomorphic, it follows that this isomorphism preserves the space of CR holomorphic functions and thus commutes with the Szegö projectors. We therefore obtain (3.1).

4. Strictly pseudoconvex domains. In this section we shall recall Fefferman's construction of volume elements and view it from the standpoint of Theorem 1.

Fefferman, in the epilogue of [5], defined a volume element σ on the boundary of a strictly pseudoconvex domain $\Omega \subset \boldsymbol{C}^{n+1}$ by the normalization

$$\sigma\wedge d\psi = i^{(n+1)^2} n!\, J(\psi)^{1/(n+2)} dz\wedge \overline{dz}\,,$$

where ψ is a defining function of Ω measured as positive in Ω, $dz=dz^1\wedge\cdots\wedge dz^{n+1}$, and J denotes the complex Monge-Ampère operator defined by

$$J(\psi) = (-1)^{n+1}\det\begin{pmatrix} \psi & \partial\psi/\partial z^\alpha \\ \partial\psi/\partial\bar z^\beta & \partial^2\psi/\partial z^\alpha\partial\bar z^\beta \end{pmatrix}.$$

Then the associated contact form is computed by using an approximate solution of $J(\psi)=1$ to first order along $\partial\Omega$. Taking a defining function u of Ω which satisfies $J(u)=1$ on $\partial\Omega$, we consider the contact form $\theta_\Omega=(i/2)(\partial-\bar\partial)u$. After some calculation, we see that the volume element $\theta_\Omega\wedge d\theta_\Omega^n$ satisfies the normalization above. It was further shown by Farris [2] that θ_Ω is normalized with respect to dz. Thus the pseudohermitian manifold $(\partial\Omega, \theta_\Omega)$ satisfies Condition F.

We now apply Theorem 1. Let $\Phi\colon\Omega_1\to\Omega_2$ be a biholomorphic map between bounded strictly pseudoconvex domains. Since Φ has a smooth extension to $\bar\Omega_1$ by Fefferman's theorem [3], it follows that the holomorphic Jacobian J_Φ is defined on $\bar\Omega_1$ by $\Phi^*dz=J_\Phi\cdot dz$. Thus (2.2) gives

$$\Phi^*\theta_{\Omega_2} = |J_\Phi|^{2/(n+2)}\theta_{\Omega_1}\,.$$

In this case, we can take a branch of $(J_\Phi)^{(n+1)/(n+2)}$ as f in Theorem 1. Then (3.2) is written as

$$s_1(z, w) = (J_\Phi)^{(n+1)/(n+2)}(z)\, s_2(\Phi(z), \Phi(w))\,\overline{(J_\Phi)^{(n+1)/(n+2)}(w)} \tag{4.1}$$

for $(z, w)\in\partial\Omega_1\times\partial\Omega_1$. This formula is also valid for $(z, w)\in\Omega_1\times\Omega_1$, if we regard s_1 and s_2 as the Szegö kernels.

5. Transformation law on CR manifolds satisfying $c(K_M)=0$. In this section we generalize Theorem 1 to CR manifolds such that the real Chern classes of the canonical bundles vanish. For the purpose, instead of specifying a volume element, we construct a CR holomorphic line bundle $\boldsymbol{L}$ with a $(2n+1)$-form valued hermitian inner product. In view of (2.2), we make $\boldsymbol{L}$ by modifying $K_M^{\otimes(n+1)/(n+2)}$—it is rather symbolic; we assume the Chern class condition $c(K_M)=0$ in order that the $(n+2)$-th root makes sense (cf. Remark 2).

To construct such a CR holomorphic line bundle, we need an additional assumption that K_M is a CR holomorphic line bundle in a natural manner. This amounts to assuming that the canonical bundle admits a nonvanishing closed section in a neighborhood of every point.

We begin with the definition of the Szegö projector acting on sections of a hermitian line bundle. Let $\boldsymbol{L}$ be a CR holomorphic hermitian line bundle with $\boldsymbol{C}\wedge^{2n+1}T^*M$-valued pointwise inner product $\langle\,,\rangle$. The L^2 inner product is then defined by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle \qquad \text{for the sections } \varphi, \psi \text{ of } \boldsymbol{L}.$$

Note that $\langle \varphi, \psi \rangle$ is a $(2n+1)$-form on M. Denoting by $L^2(M, \boldsymbol{L})$ the Hilbert space of square integrable sections of $\boldsymbol{L}$, we can define the Szegö projector as the orthogonal projector of $L^2(M, \boldsymbol{L})$ onto the subspace ker $\bar{\partial}_b$, where $\bar{\partial}_b$ is regarded as an operator acting on the sections of the CR holomorphic line bundle $\boldsymbol{L}$.

Our result is the following:

Theorem 2. *Let M_1, M_2 be strictly pseudoconvex CR manifolds with the CR holomorphic canonical bundles such that the real Chern classes vanish, and assume that $H^1(M_j, \boldsymbol{R})=0$ for $j=1, 2$. Then there exists a CR holomorphic hermitian line bundle $\boldsymbol{L}_j$ on each M_j with the following property: for any CR isomorphism Φ: $M_1 \to M_2$, there exists an isomorphism $\Phi^\sharp$: $L^2(M_2, \boldsymbol{L}_2) \to L^2(M_1, \boldsymbol{L}_1)$ which commutes with the Szegö projectors, i.e.,*

$$\boldsymbol{S}_1 \circ \Phi^\sharp = \Phi^\sharp \circ \boldsymbol{S}_2 ,$$

where $\boldsymbol{S}_j$ is the Szegö projector defined on $L^2(M_j, \boldsymbol{L}_j)$ for $j=1, 2$.

It turns out that if the CR manifolds above admit pseudohermitian structures satisfying Condition F, then the bundles $\boldsymbol{L}_1$, $\boldsymbol{L}_2$ are trivial CR holomorphic bundles, and Theorem 2 is reduced to Theorem 1.

In what follows we give a procedure of constructing a CR holomorphic hermitian line bundle, unique up to isomorphisms, intrinsically from the CR structure—then the proof of Theorem 2 will be clear.

Assume at first we are given a CR holomorphic line bundle $\boldsymbol{L}$ on M with a system of transition functions $\{\mu_{j,k}\}$, with respect to an open covering $\{U_j\}$, such that

$$(5.1) \qquad |\mu_{j,k}|^{n+2} = |\lambda_{j,k}|^{n+1} \qquad \text{on} \quad U_j \cap U_k ,$$

where $\{\lambda_{j,k}\}$ is a system of transition functions of K_M which comes from local frames $\{\zeta_j\}$. Then we can naturally define a hermitian inner product $\langle \, , \, \rangle$ on $\boldsymbol{L}$. For φ, ψ in a fiber $\boldsymbol{L}_p$ over a point $p \in U_j$, we define

$$\langle \varphi, \psi \rangle_p = \varphi_j \overline{\psi_j} (\theta_j \wedge d\theta_j^n)_p \in (\boldsymbol{C} \wedge^{2n+1} T^*M)_p ,$$

where φ_j, ψ_j are fiber coordinates of φ, ψ over U_j, and θ_j is the contact form normalized by ζ_j. Since this defintion depends on a choice of the open set U_j, we must show that this inner product is well-defined. This easily follows from (2.2) and the transformation rule on $U_j \cap U_k$:

$$\varphi_j = \varphi_k \mu_{k,j} , \quad \psi_j = \psi_k \mu_{k,j} \quad \text{and} \quad \zeta_j = \lambda_{j,k} \zeta_k .$$

Therefore we have only to find a line bundle $\boldsymbol{L}$ satisfying (5.1). Then the required hermitian line bundle is given by $\boldsymbol{L}$ with the inner product defined as above. For this purpose we consider the exact sequence:

$$0 \to \boldsymbol{Z}_{n+2} \overset{\alpha}{\to} \mathcal{O}^* \overset{\beta}{\to} \mathcal{O}^* \to 0\,,$$

where $\mathcal{O}^*$ is the sheaf of nowhere zero CR holomorphic functions and α, β are defined by $\alpha(k)=e^{2\pi ik/(n+2)}$, $\beta(f)=f^{n+2}$. The existence of $\boldsymbol{L}$ follows from its induced cohomology exact sequence:

$$H^1(M, \mathcal{O}^*) \overset{\beta^*}{\to} H^1(M, \mathcal{O}^*) \overset{\delta}{\to} H^2(M, \boldsymbol{Z}_{n+2})\,. \tag{5.2}$$

Since $c(K_M)=0$, we can choose $\boldsymbol{F}\in H^1(M, S^1)$ in such a way that the integral Chern class of $\boldsymbol{F}\otimes K_M^{\otimes n+1}$ vanishes. Then we have $\delta(\boldsymbol{F}\otimes K_M^{\otimes n+1})=0$. Thus, by (5.2), there exists a CR holomorphic line bundle $\boldsymbol{L}$ such that

$$\boldsymbol{L}^{\otimes n+2} \cong \boldsymbol{F}\otimes K_M^{\otimes n+1}\,. \tag{5.3}$$

This implies (5.1). Moreover, we can select $\boldsymbol{L}$ uniquely in such a way that the integral Chern class vanishes. Here we use the assumption $H^1(M, \boldsymbol{R})=0$. Since there is ambiguity in the choice of the isomorphisms in (5.3), the hermitian inner product on $\boldsymbol{L}$ is not uniquely determined. Nevertheless, we can show, by using $H^1(M, \boldsymbol{R})=0$, that all such hermitian line bundles are isomorphic—the proof is essentially the same as that of Theorem 1. We have thus constructed a CR holomorphic hermitian line bundle unique up to isomorphisms.

REMARK 2. The existence of $\boldsymbol{L}$ satisfying (5.1) follows from a weaker Chern class condition: $c(K_M)=c(\boldsymbol{E}^{\otimes n+2})$ for some complex line bundle $\boldsymbol{E}$. However, there is no canonical choice of $\boldsymbol{L}$, and the ambiguity of the bundles is described by $H^1(M, S^1)$.

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