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BOUNDED EXPONENTIAL SUMS

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1. Introduction

T. Kamae has asked (personal communication) whether it is possible to find a sequence (a_k) of ± 1's such that the sums

$$\sum_{k=m}^{m+n} a_k e^{-ik\theta}$$

stay bounded (for all integers m and n with $n \geq 0$) for all $\theta \in [-\pi, \pi)$ (with the bound possibly depending on θ). We show that there is no such sequence. In fact, the only such real-valued sequences must be "essentially zero" in a sense explained below.

This conclusion is reached by adopting a dynamical viewpoint, applying the Spectral Theorem, and showing that every nonzero element of L^2 must have nonzero mean power at some frequency. This latter observation is equivalent to the triviality of the intersection of all the spaces of "twisted coboundaries" for a unitary operator.

2. Results

Suppose that $a=(a_k) \in \boldsymbol{R}^{\boldsymbol{Z}}$ is a doubly infinite sequence with the property that

$$\left|\sum_{k=m}^{m+n} a_k e^{-ik\theta}\right| \leq c(\theta) < \infty \qquad \text{for all } m \in \boldsymbol{Z}, \text{ all } n \geq 0, \text{ and all } \theta \in [-\pi, \pi)\,.$$

Taking $n=\theta=0$, we see that a is bounded and so takes values in a compact interval I. Let X denote the closure of the orbit of a under the shift transformation σ in the compact metric space $I^{\boldsymbol{Z}}$. Let μ be a shift-invariant Borel probability measure on X.

Given $x \in X$ and a block $B=b_0 \cdots b_n$ which appears in x, we can find a block $D=d_0 \cdots d_n$ in a such that $|b_i - d_i| < 1/(n+1)$ for $i=0, \cdots, n$. Consequently

$$\left|\sum_{k=0}^{n} b_k e^{-ik\theta}\right| \leq c(\theta)+1 \qquad \text{for all } \theta\,.$$

If $Tg = g \circ \sigma$ for $g \in L^2(X, \mu)$ and $f(x) = \pi_0 x = x_0$ for $x \in X$, we have then that

$$\|\sum_{k=0}^{n} T^k f e^{-ik\theta}\|_2 \leq c(\theta)+1 \qquad \text{for all } \theta \text{ and all } n \geq 0\,.$$

We will see that this is impossible unless $f=0$ in L^2. Since this cannot happen for a sequence a which assumes only finitely many values, all nonzero, the original question will be settled. For general sequences, the conclusion is that boundedness against all θ is possible only if projection onto the central coordinate is 0 a.e. with respect to every invariant measure on the orbit closure X of the sequence; that is, the only invariant probability measure on X is concentrated on the fixed point 0^∞. In this case we say that the sequence (a_k) is *essentially zero.*

Theorem. *Let H be a Hilbert space and $T\colon H\to H$ a unitary operator. For each $n=1, 2, \cdots, \theta\in[-\pi, \pi)$, and $f\in H$ let*

$$S_n^\theta f=\sum_{k=0}^{n-1} e^{-ik\theta}T^k f\,.$$

If $\sup_n \|S_n^\theta f\|<\infty$ for all θ, then $f=0$.

Proof. Applying the Spectral Theorem with common notations and conventions, we may write

$$Tf=\int_{-\pi}^{\pi} e^{i\lambda}dE(\lambda)f\,,$$

$$S_n^\theta f=\int_{-\pi}^{\pi}\sum_{k=0}^{n-1} e^{ik(\lambda-\theta)}dE(\lambda)f=\int_{-\pi}^{\pi}\frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}dE(\lambda)f\,,$$

and

$$\|S_n^\theta f\|^2=\int_{-\pi}^{\pi}\left|\frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}\right|^2 d\|Ef\|^2(\lambda)\,.$$

The following Lemma will show that such expressions cannot stay bounded for any positive measure (such as $\nu=\|E(\ )f\|^2$ if $f\neq 0$ a.e.), thereby completing the proof.

Lemma. *There is a constant $C>0$ such that if ν is a positive measure on $[-\pi, \pi)$, n is a positive integer, $\varepsilon>0$, and*

$$A_n(\theta)=\frac{1}{n}\int_{-\pi}^{\pi}\left|\frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}\right|^2 d\nu(\lambda)\,,$$

then $\nu\{\theta\in[-\pi, \pi)\colon A_n(\theta)<\varepsilon\}<\dfrac{\varepsilon}{C}$.

Proof. Let C_1 and C_2 be positive constants such that $|\alpha|<\pi$ implies that $C_1|\alpha|\leq|1-e^{i\alpha}|\leq C_2|\alpha|$. Then

$$A_n(\theta)\geq\frac{1}{n}\int_{\theta-(\pi/n)}^{\theta+(\pi/n)}\left|\frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}\right|^2 d\nu(\lambda)\geq\left[\frac{C_1}{C_2}\right]^2 n\nu\left(\theta-\frac{\pi}{n}, \theta+\frac{\pi}{n}\right).$$

Let $\varepsilon>0$ and $n>0$, and let $\delta=\delta(\varepsilon)=\nu\{\theta: A_n(\theta)<\varepsilon\}$. Suppose that $\delta>0$, since otherwise we are finished. Choose a compact set $K\subset\{\theta: A_n(\theta)<\varepsilon\}$ with $\nu(K)>\delta/2$. There are $\theta_1, \cdots, \theta_p\in K$ such that the intervals $\left(\theta_i-\frac{\pi}{n}, \theta_i+\frac{\pi}{n}\right)$ cover K and no more than two of them intersect at any point. Since the union of these intervals is contained in $(-2\pi, 2\pi)$, it follows that $p\frac{2\pi}{n}\leq 8\pi$, and hence $p\leq 4n$. Therefore

$$\nu(K)\leq\sum_{i=1}^{p}\nu\left(\theta_i-\frac{\pi}{n}, \theta_i+\frac{\pi}{n}\right)\leq p\left[\frac{C_2}{C_1}\right]^2\frac{1}{n}\varepsilon\leq 4\left[\frac{C_2}{C_1}\right]^2\varepsilon,$$

and $\delta<8(C_2/C_1)^2\varepsilon$, proving the Lemma and hence also the Theorem.

Corollary 1. *If ν is a positive measure on $[-\pi, \pi)$ and (n_j) is an increasing sequence of positive integers, then*

$$\limsup_{j\to\infty}\frac{1}{n_j}\int_{-\pi}^{\pi}\left|\frac{1-e^{in_j(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}\right|^2 d\nu(\lambda)>0 \qquad \text{for } \nu\text{-almost all } \theta .$$

Proof. For each $\varepsilon>0$, $\{\theta: \limsup A_{n_j}(\theta)=0\}\subset\{\theta: A_{n_j}(\theta)<\varepsilon$ for all large enough $j\}$, a set of measure less than ε/C by the Lemma.

Corollary 2. *Let H be a Hilbert space, $T: H\to H$ a unitary operator, and $0\neq f\in H$. Then there exists a frequency θ at which the "mean power" of f, defined by*

$$\bar{P}(\theta) = \limsup_{n\to\infty}\frac{1}{n}\,\|\sum_{k=0}^{n-1} e^{-ik\theta}T^k f\|^2 ,$$

is positive.

Corollary 3. *Let H be a Hilbert space and $T: H\to H$ a unitary operator. For each $\theta\in[-\pi, \pi)$ let*

$$\mathcal{B}_\theta = \{e^{i\theta}g-Tg: g\in H\}$$

be the space of "θ-twisted coboundaries" for T. Then $\bigcap_{\theta\in[\pi,\pi)}\mathcal{B}_\theta=\{0\}$.

Proof. If $f\in\mathcal{B}_\theta$, then $\{\|S_n^\theta f\|: n=1, 2, \cdots\}$ is bounded.

REMARK. As in [2], by considering fixed points of the operator $V_f^\theta g= e^{-i\theta}(f+Tg)$, one can show that in fact $f\in\mathcal{B}_\theta$ if and only if $\{\|S_n^\theta f\|: n=1, 2,\cdots\}$ is bounded. For further developments in this direction, see [1].

Corollary 4. *As in* [3], *define the "spectral notch" subshift $\sum(r, \theta)$ corresponding to $r>0$ and $\theta\in[-\pi, \pi)$ to be the set of all those $x\in\{-1, 1\}^{\mathbf{Z}}$ for which*

$$|\sum_{k=m}^{m+n} x_k e^{-ik\theta}| < r \qquad \text{for all } m \in \boldsymbol{Z} \text{ and all } n \geq 0\,.$$

Then $\bigcap_{\theta \in [-\pi, \pi)} \bigcup_{r>0} \Sigma(r, \theta) = \phi$.

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