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STIEFEL MANIFOLDS AS FRAMED BOUNDARIES

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Let $V_{n,q}$ denote the Stiefel manifold of orthogonal q-frames in F^n where $F=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$. We regard this space as the homogeneous space G_n/G_{n-q} of right cosets modulo $1_q \times G_{n-q}$ where G_k denotes the relevant group $SO(k)$, $SU(k)$ or $Sp(k)$. Then this space obtains a framing in a canonical way as mentioned below [3]. We denote this framing ambiguously by $\mathcal{F}$ and we write $[V_{n,q}, \mathcal{F}]$ for an element in π_*^S defined by the pair $(V_{n,q}, \mathcal{F})$ via the Thom-Pontrjagin construction. In this note we prove the following

Theorem. *Let* $1 \leqq q \leqq n-1$, $n-1$ *or* n *according as* $F=\boldsymbol{R}$, $\boldsymbol{C}$ *or* $\boldsymbol{H}$. *Then*

$$[V_{n,q}, \mathcal{F}] = 0 .$$

We denote by $\mathcal{R}$ the right invariant framing of G_n and by $\mathcal{R}^\alpha$ the framing obtained by twisting $\mathcal{R}$ by a representation α [5]. Also we write $[G_n, \alpha]$ for $[G_n, \mathcal{R}^\alpha]$. Let

$$\rho_n\colon G_n \subset GL(dn, \boldsymbol{R})$$

be the standard real representation of G_n where $d=\dim_{\boldsymbol{R}} F$. Then by the theorem we have

Corollary ([1], [5]).

$$[SO(n), (n-1)\rho_n] = 0,\ [SU(n), (n-1)\rho_n] = 0 \quad \textit{and} \quad [Sp(n), n\rho_n] = 0 .$$

REMARK. By taking $G_k=U(k)$ instead of $SU(k)$ we get $[U(n), n\rho_n]=0$ analogously.

The proof of the theorem uses the arguments parallel to [4]. Actually we construct a bounding manifold for $V_{n,q}=G_n/G_{n-q}$.

Let V_k denote the representation space of ρ_k. There is then the real vector bundle

$$\xi_k\colon G_n \times_{G_k} V_k \to G_n/G_k$$

for $k<n$. If $S(V_k)$ denote the unit sphere of V_k, then we have a canonical G_k-equivariant diffeomorphism.

$$G_k/G_{k-1} \approx S(V_k)\,.$$

This and $G_n/G_{n-q} = G_n \times_{G_{n-q+1}} G_{n-q+1}/G_{n-q}$ imply that the homogeneous fibre bundle

$$(1) \qquad G_{n-q+1}/G_{n-q} \to G_n/G_{n-q} \xrightarrow{\pi} G_n/G_{n-q+1}$$

is isomorphic to the sphere bundle of ξ_{n-q+1}. Hence we have

$$G_n/G_{n-q} \approx S(\xi_{n-q+1})$$

where $S(\xi_k)$ denotes the total space of the sphere bundle of ξ_k. Denote also by $D(\xi_k)$ the total space of the disc bundle of ξ_k. Evidently we then have

$$\partial D(\xi_{n-q+1}) \approx V_{n,q}\,.$$

To prove the theorem it therefore suffices to show that the framing $\mathscr{F}$ of $V_{n,q}$ extends over $D(\xi_{n-k+1})$.

So we first recall the framing of [3]. Let G be a compact connected Lie group and H a closed subgroup of G. Let $\tau(G/H)$ denote the tangent bundle of G/H. Consider the principal H-bundle

$$H \to G \xrightarrow{\pi} G/H\,.$$

Then we have a decomposition of the tangent bundle of G

$$\tau(G) \cong \pi^*\tau(G/H) \oplus \tau_H(G)$$

where $\tau_H(G)$ is the bundle of tangents along the fibres. This isomorphism is compatible with the right action of H, so that we obtain an isomorphism of vector bundles over G/H

$$\tau(G)/H \cong \tau(G/H) \oplus \tau_H(G)/H\,.$$

Let $\tau_g(G)$ denote the tangent space at $g \in G$ and $R_{g^{-1}}\colon \tau_g(G) \to \tau_e(G)$ denote the differential of right multiplication by g^{-1} where e is the identity element of G. Then the right invariant framing of G

$$\mathscr{R}\colon \tau(G) \cong G \times \tau_e(G)$$

is given by $\mathscr{R}(v) = (g, R_{g^{-1}}(v))$ where $v \in \tau_g(G)$. This gives

$$\tau(G)/H \cong G/H \times \tau_e(G)\,.$$

as vector bundles over G/H.

By ad_H we denote the adjoint representation of H on $\tau_e(H)$. We consider the differential $L_{a^{-1}}\colon \tau_a(gH) \to \tau_e(H)$ induced by the left multipliaction by a^{-1} where $a \in gH$. Then similarly we have

$$\tau_H(G)/H \cong G \times_H \tau_e(H)$$

as vector bundles over G/H where H acts on $\tau_e(H)$ via ad_H.

Combining these three bundle equations we have

$$G/H \times \tau_e(G) \cong \tau(G/H) \oplus G \times_{\mathrm{ad}_H} \tau_e(H) \tag{2}$$

as vector bundles over G/H. So we find that if ad_H is contained in the image of the restriction map $RO(G) \to RO(H)$ of real representation rings, then formula (2) gives rise to a framing of G/H.

Here we return to the framing of $V_{n,q}$. We consider the restrictions of ad_{G_n} and ρ_n to G_k for $k<n$. Now we write $\mathrm{ad}_k = \mathrm{ad}_{G_k}$ briefly. Then we have

Lemma 1. (i) $\rho_n|_{G_{n-1}} \cong \rho_{n-1} \oplus d\cdot 1$,
(ii) $\mathrm{ad}_n|_{G_{n-1}} \cong \mathrm{ad}_{n-1} \oplus \rho_{n-1} \oplus (d-1)\cdot 1$
where 1 *denotes the trivial* 1-*dimensional real representation.*

Proof. (i) is obvious. By observing the maximal root of G_n [2] we see that $c(\mathrm{ad}_n) = \lambda^2 \tilde{\rho}_n$, $(\lambda^1 \tilde{\rho}_n)(\lambda^{n-1} \tilde{\rho}_n) - 1$ or $(\lambda^1 \tilde{\rho}_n)^2 - \lambda^2 \tilde{\rho}_n$ where $\tilde{\rho}_n$ denotes the canonical complex representation $G_n \subset GL(n, \boldsymbol{C})$, $GL(n, \boldsymbol{C})$ or $GL(2n, \boldsymbol{C})$ according as $F = \boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. Here c is the complexification and $\lambda^i \tilde{\rho}_n$ is the i-th exterior power of $\tilde{\rho}_n$. For this fact, however, we refer to [6]. So we can readily obtain (ii) using (i).

From Lemma 1 it follows that for $k<n$

$$\begin{aligned} \mathrm{ad}_n|_{G_k} &\cong \mathrm{ad}_k \oplus (n-k)\,\rho_k \oplus ((n-k)(dn-dk+d-2)/2)\cdot 1\,, \\ \rho_n|_{G_k} &\cong \rho_k \oplus d(n-k)\cdot 1\,. \end{aligned} \tag{3}$$

Hence we get

$$\mathrm{ad}_n|_{G_k} \oplus s(n,k)\cdot 1 \cong \mathrm{ad}_k \oplus (n-k)\,\rho_n|_{G_k} \tag{4}$$

for $k<n$ where $s(n,k) = (n-k)(dn-dk-d+2)/2$.

Denote by W_k the representation space of ad_k. Then using (2) when $G = G_n$, $H = G_{n-q}$ and (4) we obtain a framing of $V_{n,q}$

$$\mathcal{F}\colon \tau(V_{n,q}) \oplus (V_{n,q} \times W_n) \oplus s(n, n-q)\cdot 1 \cong V_{n,q} \times (\tau_e(G_n) \oplus qV_n) \tag{5}$$

where 1 denotes the trivial real line bundle.

We now give this framing $\mathcal{F}$ more directly. Let W_k' be a direct summand of $W_n|_{G_k}$ such that $W_n|_{G_k} \cong W_k' \oplus W_k$ for $k<n$. Then we have

Lemma 2 ([1]). $\tau(V_{n,n-k}) \cong G_n \times_{G_k} W_k'$ *for* $k<n$.

Proof. There is an obvious isomorphism of vector bundles over $V_{n,n-k}$

$$G_n \times_{G_k} (W_n|_{G_k}) \cong V_{n,n-k} \times \tau_e(G_n)\,.$$

So consider the composite with the isomorphism of (2) when $G=G_n$, $H=G_k$

$$G_n\times_{G_k}(W_n|_{G_k})\cong\tau(V_{n,n-k})\oplus G_n\times_{G_k}W_k\,.$$

Then we see that this map sends identically the direct summand $G_n\times_{G_k}W_k$ to that on the right-hand side and isomorphically another direct summand $G_n\times_{G_k}W'_k$ to $\tau(V_{n,n-k})$, and so the result follows.

Using Lemma 2 we can interpret $\mathcal{F}$ of (5) as follows. By (4) we have

$$W'_{k-q}\oplus(W_n|_{G_{n-q}})\oplus s(n, n-q)\cdot 1\cong(W_n\oplus qV_n)|_{G_{n-q}}\,. \tag{6}$$

This gives rise to an isomorphism of real vector bundles associated with the principal G_{n-q}-bundle $G_n\to V_{n,q}$ with modules on both sides as fibres. It is easily seen that this isomorphism induces $\mathcal{F}$ of (5) precisely.

Proof of Theorem. By the second formula of (3) it follows that

$$\xi_{n-q+1}\oplus d(q-1)\cdot 1\cong V_{n,q-1}\times V_n\,.$$

Taking the sum of this and the framing $\mathcal{F}$ of $V_{n,q-1}$ we have

$$\begin{aligned}\tau(V_{n,q-1})\oplus\xi_{n-q+1}\oplus(V_{n,q-1}\times W_n)\oplus(s(n, n-q)-1)\cdot 1\\ \cong V_{n,q-1}\times(\tau_e(G_n)\oplus qV_n)\end{aligned}\tag{7}$$

since $s(n, n-q)=s(n, n-q+1)+d(q-1)+1$. Now from the above arguments about the fibre bundle of (1) it is clear that

$$\tau(V_{n,q})\oplus 1\cong\pi^*(\tau(V_{n,q-1})\oplus\xi_{n-q+1})$$

where π is the projection map of (1). Therefore by pulling the isomorphism of (7) back along π we obtain another framing of $V_{n,q}$

$$\mathcal{F}'\colon\tau(V_{n,q})\oplus(V_{n,q}\times W_n)\oplus s(n, n-q)\cdot 1\cong V_{n,q}\times(\tau_e(G_n)\oplus qV_n)\,.$$

Denote by $\tilde{\pi}$ the canonical projection map $D(\xi_{n-q+1})\to V_{n,q-1}$. Moreover we then have

$$\tau(D(\xi_{n-q+1}))\cong\tilde{\pi}^*(\tau(V_{n,q-1})\oplus\xi_{n-q+1})\,,$$

so that by pulling the isomorphism of (7) back along $\tilde{\pi}$ again we obtain a framing of $D(\xi_{n-q+1})$. Identifying $V_{n,q}$ with $S(\xi_{n-q+1})$, this framing is obviously an extension of $\mathcal{F}'$ over $D(\xi_{n-q+1})$ since $\tilde{\pi}|_{V_{n,q}}=\pi$. Hence it follows $[V_{n,q}, \mathcal{F}']=0$ and so it suffices to show that $\mathcal{F}$ agrees up to sign with $\mathcal{F}'$.

By (3) we have

$$W'_{n-q}\oplus 1\cong(W'_{n-q+1}\oplus V_{n-q+1})|_{G_{n-q}}\,.$$

Using this and Lemma 2 we can verify that formula (6) also gives rise to either

$\mathcal{F}'$ or $-\mathcal{F}'$ in the same way as the case of $\mathcal{F}$. Therefore this proves the theorem.

Proof of Corollary. Since $V_{n,n-1}=SO(n)$, $SU(n)$ and $V_{n,n}=Sp(n)$, we set $V_{n,q}=G_n$ in (5) and so in defining $\mathcal{F}$ we consider $\mathrm{ad}_{n-q}=0$. Hence by definition it follows that $\mathcal{F}$ is just the framing twisting $\mathcal{R}$ of G_n by $\mathrm{ad}_n-q\rho_n$, so that by the theorem it follows $[G_n, \mathrm{ad}_n-q\rho_n]=0$. Therefore we have $[G_n, q\rho_n]=0$ by [5].

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