ON THE COMPLEX STRUCTURE OF A MANIFOLD OF SECTIONS

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0. Introduction

There are two different classical approaches to the theory of deformations, one is developed by Kodaira and Spencer [6], and the other by Douady [2]. Both approaches have their own merits. For example, the former uses tensor analysis, which allows relations with other geometric structures e.g. riemannian metrics. On the other hand, the latter approach gives complex structures on the moduli spaces more directly than the former. These approaches are unified in a sense in [7] or [11]. In these papers, Banach spaces consisting of maps are effectively used to construct a moduli space, and a complex structure and a riemannian metric on the moduli space are canonically defined.

The purpose of this paper is to extend the idea of the unified approach. Our approach is similar to Douady's, but we consider not only spaces of complex analytic objects but also those of C^∞-objects, and discuss on almost complex structures as far as possible.

Theorem A. *Let E be a fiber bundle over a compact C^∞-manifold M. Assume that each fiber of E is a complex manifold. Then the space $C^\infty(E)$ of all sections forms an (infinite dimensional) complex manifold.*

Theorem B. *Let M be a compact almost complex manifold and N a complex manifold. The space of all holomorphic maps (i.e., maps whose derivatives commute with the almost complex structures) from M to N forms a complex analytic set of $C^\infty(M, N)$.*

This research was motivated by a discussion with A. Fischer. In particular, Theorem A and Proposition 1.4 were got as answers to his question. The author is grateful to him.

1. Definition of the complex structure

In this paper, we will consider spaces of C^∞-sections. We can analogously

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define spaces of H^r-sections, and the space $C^\infty(E)$ becomes an ILH-manifold. To prove propositions in this paper, we have to use the following technique: we prove $H^r(E)$ version of a proposition and take the intersection $\cap_{r=n}^\infty H^r(E)$. This method was developed by Omori [10], to which we refer for details. For example, we shall prove Proposition 1.6 in Hilbert space category, and can apply it to a space of C^∞-sections via ILH-category. For standard properties of infinite dimensional complex manifold, we refer to [2] and [5, Chap. IV].

To state Theorem A more precisely, we need some basic definitions. Throughout this paper, M denotes a compact C^∞-manifold. We call a fiber bundle E over M an *almost complex fiber bundle* if each fiber E_x is an almost complex manifold (which depends C^∞-ly on M). We call E a *complex fiber bundle* if each almost complex structure is integrable, in other words, if E is a differentiable family of complex manifolds with parameter space M. For an almost complex fiber bundle E, we denote by J_x the almost complex structure on each E_x and by VE the vertical distribution on E.

DEFINITION 1.1. Let $\pi: E \to M$ be an almost complex fiber bundle. We denote by $C^\infty(E)$ the set of all C^∞-sections. The space $C^\infty(E)$ is an (infinite dimensional) manifold and the tangent space $T_s C^\infty(E)$ at the point $s \in C^\infty(E)$ is the space $C^\infty(s^{-1}VE)$, that is, $\xi \in T_s C^\infty(E)$ is regarded as a map from M to TE such that $\xi(x) \in T_{s(x)}E$ and $(d\pi)\xi(x) = 0$ for each $x \in M$. We define an *almost complex structure* $\mathscr{J}$ on the space $C^\infty(E)$ by

$$(\mathscr{J}_s\xi)(x) = J_x\xi(x).$$

Theorem 1.2. *If E is a complex fiber bundle, then the almost complex manifold $(C^\infty(E), \mathscr{J})$ is a complex manifold.*

Now we will give two different proofs of Theorem 1.2. The first one directly defines a holomorphic coordinate system and the second one uses an infinite dimensional version of Newlander-Nirenberg's theorem.

Proof (First). By [9], for any point $e \in E$, there are an open neighbourhood W of e in E and a C^∞-map $f: W \to C^r$ ($2r = \dim_R E_x$) such that the map $\pi \times f: W \to \pi(W) \times f(W)$ is a diffeomorphism and that the restriction $f|(E_x \cap W)$ is holomorphic for each $x \in \pi(W)$. We fix a section $s_0 \in C^\infty(E)$. Then we can choose a finite open covering $\{W_\alpha\}_{\alpha \in A}$ of the image of s_0 in E and maps $\{f_\alpha: W_\alpha \to C^r\}_{\alpha \in A}$ with the property above. We define an open neighbourhood $\mathscr{U}(s_0)$ of s_0 in $C^\infty(E)$ by $\{s \in C^\infty(E); s(U_\alpha) \subset W_\alpha\}$, where $U_\alpha = \pi(W_\alpha)$, and define a map φ from $\mathscr{U}(s_0)$ into the complex vector space $\mathscr{V} = \bigoplus_{\alpha \in A} C^\infty(\pi(W_\alpha); C^r)$ by

$$\varphi(s)_\alpha(x) = f_\alpha(s(x)) \quad \text{for} \quad \alpha \in A, x \in U_\alpha.$$

Since we have, for $\xi \in T_s C^\infty(E)$,

$$(d\varphi)_s(\xi)_\alpha(x) = (df_\alpha)_{s(x)}\xi(x),$$

the map φ is an imbedding.   Moreover since $f_\alpha | (E_x \cap W_\alpha)$ is holomorphic,

$$(d\varphi)_s (\mathcal{J}\xi)_\alpha (x) = (df_\alpha)_{s(x)} \mathcal{J}_x \xi(x) = J_0((df_\alpha)_{s(x)} \xi(x)),$$

where J_0 is the linear almost complex structure of $\boldsymbol{C}^r$, we see that φ is holomorphic, i.e., $(d\varphi) \mathcal{J}\xi = \mathcal{J}_0(d\varphi) \xi$ where $\mathcal{J}_0$ is the linear almost complex structure of $\mathcal{CV}$.   As finite dimensional case, by the implicit function theorem, the image of φ is a complex submanifold $\mathcal{CV}'$ of $\mathcal{CV}$ and the almost complex structure $\mathcal{J}$ of $\mathcal{V}(s_0)$ coincides with the pull-back of the complex structure of $\mathcal{CV}'$ by the map φ.   It means that $\varphi: (\mathcal{V}(s_0), \mathcal{J}) \to (\mathcal{CV}', \mathcal{J}_0)$ gives a holomorphic coordinate system around the point $s_0 \in C^\infty(E)$.   Q.E.D.

The second proof is based on the following

Proposition 1.3 ([4]).   *The Nijenhuis tensor $N(\mathcal{J})$ of $\mathcal{J}$ vanishes if and only if each J_x is integrable.*

This proposition can be proved by usual tensor calculas, which we omit. Remark that this proposition gives the converse of Theorem A.   Observe also that, in Theorem B, the complex structure of N is essential, while the almost complex structure of M is not important as we will see in Remark 4.3.

Now the second proof is reduced to the following two propositions.

Proposition 1.4 (Newlander-Nirenberg's theorem).   *A real analytic almost complex structure on a real analytic Hilbert manifold is a complex structure if and only if its Nijenhuis tensor vansihes.*

Lemma 1.5.   *For a complex fiber bundle E, $C^\infty(E)$ is a real analytic manifold and the almost complex structure $\mathcal{J}$ is real analytic on $C^\infty(E)$.*

Proof (of Lemma 1.5).   We fix a section $s_0 \in C^\infty(E)$ and choose a fiber metric defined on an open neighbourhood of $\text{Im}(s_0)$ in E which is real analytic along each fiber and the convergence radius of the Taylor expansion is uniformly bounded from below on $\text{Im}(s_0)$.   Such a fiber metric can be constructed, for example, as follows.   Let $\{(W_\alpha, f_\alpha, U_\alpha)\}_{\alpha \in A}$ be as in the first proof of Theorem 1.2 and choose a partition of unity $\{u_\alpha\}$ subordinate to the open covering $\{U_\alpha\}$ of M.   Denoting by g_0 the flat metric on $\boldsymbol{C}^r$, we can define

$$g = \sum_{\alpha \in A} u_\alpha \cdot f_\alpha^* g_0.$$

Now define $J_x^* = (\exp_x)^* J_x$ using the exponential map: $T_{s_0(x)} E_x \to E_x$ of the metric g_x for each $x \in M$.   The tensor field J^* is defined on an open neighbourhood W of the zero-section of the vector bundle $s_0^{-1} VE$ over M and each J_x^* is real analytic with bounded convergence radius.   Therefore the tensor field $\mathcal{J}^*$ defined by J^* as Definition 1.1 has holomorphic extension and so is real anlaytic.   When

we choose another section $s_1 \in C^\infty(E)$, all transformations we used are real analytic on each fiber, hence the manifold $C^\infty(E)$ becomes a real analytic manifold and the tensor field $\mathscr{J}$ is real analytic.   See [7, Appendix].   Q.E.D.

In the finite dimensional real analytic category, Newlander-Nirenberg's theorem is proved easily as follows ([3]).   Take a holomorphic extension of the pair $(\mathbf{R}^{2r}, J)$ to $(\mathbf{C}^{2r}, J_C)$.   The holomorphic distribution D_C^+ on $\mathbf{C}^{2r}$ defined by $\{X \in T^+\mathbf{C}^{2r}; J_C X = \sqrt{-1}X\}$ is involutive if and only if the Nijenhuis tensor vanishes. Therefore by Frobenius' theorem in holomorphic category, there is a holomorphic map $f: \mathbf{C}^{2r} \to \mathbf{C}^r$ such that $D_C^+ = \mathrm{Ker}(df)$, and the restriction $f \mid \mathbf{R}^{2r}: \mathbf{R}^{2r} \to \mathbf{C}^r$ defines a holomorphic coordinate system of $(\mathbf{R}^{2r}, J)$ around the origin.   Thus to prove Proposition 1.4, the only point to check is Frobenius' theorem in infinite dimensional holomorphic category, which we prove in below.

Proposition 1.6 (Frobenius' theorem).   *Let X and Y be Hilbert spaces and $g: X \oplus Y \to L(X, Y)$ a holomorphic map defined on an open neighbourhood of the origin.   Let D be a local distribution on $X \oplus Y$ defined by $D(x, y) = \{(v, g(x, y) v); v \in X\}$.   If D is involutive, then there exists a holomorphic map $f: X \oplus Y \to Y$ such that $f(0, y) = y$ and $\{(v, (df)_{(x,y)}(v, 0))\} = D(x, f(x, y))$.*

Proof.   By [1], such a map f exists provided that we require only that f belongs in C^r-category.   Therefore it is sufficient to prove that the derivative df is $\mathbf{C}$-linear.   Denote by d_X, d_Y the derivatives for the direction X, Y, respectively.   Then, since

$$(d_X f)_{(x,y)} = g(x, f(x, y)),$$

$d_X f$ is $\mathbf{C}$-linear.   On the other hand, since

$$(d_X d_Y f)(v, w) = (d_Y d_X f)(w, v) = (d_Y g)((d_Y f) w, v)$$

for $v \in X$ and $w \in Y$, and g is holomorphic, we see that

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\begin{aligned}
(d_X[d_Y f, J_Y])(v, w) &= (d_X d_Y f)(v, J_Y w) - J_Y(d_X d_Y f)(v, w) \\
&= (d_Y g)((d_Y f) J_Y w, v) - J_Y(d_Y g)((d_Y f) w, v) \\
&= (d_Y g)([d_Y f, J_Y] w, v),
\end{aligned}
$$

where J_Y is the linear almost complex structure of Y.   Here the tensor field $[d_Y f, J_Y]$ vanishes on $0 \oplus Y$.   Thus the uniqueness of the solution of O.D.E. implies that $[d_Y f, J_Y]$ vanishes on an open neighbourhood of the origin of $X \oplus Y$, that is, also d_Y is $\mathbf{C}$-linear.   Q.E.D.

Finally in this section we remark that Definition 1.1 of the complex structure on $C^\infty(E)$ is natural.   For example, the following is easy to see.

Proposition 1.7.   *Let E and F be complex fiber bundles over M and $f: E \to F$*

a complex fiber map, i.e., holomorphic on each fiber.   Then the canonically induced map

$$\tilde{f}\colon C^{\infty}(E) \to C^{\infty}(F);\ s \mapsto f \circ s$$

is holomorphic.

This proposition applies in particular to the cases that E is a subbundle of F and that F is a quotient bundle of E.

2. Compatibility with other structures

We consider a complex fiber bundle E over M with another geometric structure on each fiber.   At first, let each fiber be a complex Lie group.

DEFINITION 2.1.   A complex fiber bundle E over M is called a *complex Lie group bundle* if each fiber is a complex Lie group and the group operations depend C^{∞}-ly on M.

Theorem 2.2.   *For a complex Lie group bundle E over M, the space $C^{\infty}(E)$ becomes an (infinite dimensional) complex Lie group.*

Proof.   We know that $C^{\infty}(E)$ is a complex manifold and easily see that it is a group.   We check that the multiplying operator is holomorphic, which implies also that the inverse operator is holomorphic by the implicit function theorem.   The multiplying operator is given by $m(s_1, s_2)(x) = s_1(x) \cdot s_2(x)$.   Let ξ_i be a tangent vector at s_i.   Then, since

$$(dm)_{(s_1, s_2)}(\xi_1, \xi_2)(x) = dR_{s_2(x)}\,\xi_1(x) + dL_{s_1(x)}\,\xi_2(x)\,,$$

and $dR_{s_2(x)}$ and $dL_{s_1(x)}$ commute with the almost complex structure, we get

$$(dm)_{(s_1, s_2)}(\mathcal{J}\xi_1, \mathcal{J}\xi_2)(x) = \mathcal{J}(dR_{s_2(x)}\,\xi_1 + dL_{s_1(x)}\,\xi_2)(x)$$
$$= (\mathcal{J}(dm)_{(s_1, s_2)}(\xi_1, \xi_2))(x)\,.$$

Q.E.D.

Next, we give

DEFINITION 2.3.   A complex fiber bundle E over M is called a *Kähler fiber bundle* if each fiber is a Kähler manifold and the fiber metric tensor depends C^{∞}-ly on M.   If a volume element v_M on the base manifold M is given, we define a metric on the space $C^{\infty}(E)$ by

$$\langle \xi_1, \xi_2 \rangle = \int_M g_x(\xi_1(x), \xi_2(x))\, v_M\,,$$

where g is the fiber metric of E and ξ_1, ξ_2 tangent vectors at $s \in C^{\infty}(E)$.

Theorem 2.4.   *For a Kähler fiber bundle E over M with a volume element on M, the space $C^{\infty}(E)$ becomes an (infinite dimensional) Kähler manifold.*

Proof.   It is clear that the metric $\langle\ ,\ \rangle$ is a hermitian metric.   We denote by ω the Kähler form of the fiber metric and by Ω that of $\langle\ ,\ \rangle$, and show that Ω is closed.   Let ξ_1, ξ_2 and ξ_3 be tangent vectors at $s \in C^\infty(E)$.   As in the second proof of Theorem 1.2, we may assume that E is a vector bundle over M (but the Kähler structure may be non-linear on each fiber).   If we extend ξ_i's parallel, then $[\xi_1, \xi_2]$ etc. vanish, and so

$$(d\Omega)(\xi_1, \xi_2, \xi_3) = \xi_1[\langle \xi_2, \mathscr{J}\xi_3\rangle] + \text{alt.}$$
$$= \xi_1 \int_M g(\xi_2, J\xi_3)\, v_M + \text{alt.} = \int_M \xi_1(x)\, [\omega(\xi_2, \xi_3)]\, v_M + \text{alt.}$$
$$= \int_M (d\omega)(\xi_1, \xi_2, \xi_3)\, v_M = 0\ .$$

<div align="right">Q.E.D.</div>

3. Basic Examples

The most typical examples are complex fiber bundles associated with principal fiber bundles and trivial fiber bundles.   Let $P \to M$ be a principal K-bundle and the Lie group K act on a complex manifold N as holomorphic transformations.   Then the associated fiber bundle $P \times_K N$ is a complex fiber bundle over M and $C^\infty(P \times_K N)$ becomes a complex manifold.   Moreover, if N is a complex Lie group and the action of K preserves the structure, then the space $C^\infty(P \times_K N)$ becomes a complex Lie group.   If N is a Kähler manifold and K acts as isometries and M is endowed with a volume element, then the space $C^\infty(P \times_K N)$ becomes a Kähler manifold.

In particular, when the Lie group K is trivial, the space $C^\infty(M, N)$ of all maps has the corresponding structure.

Remark that the transformation group bundle $P \times_{Ad^{-1}} K$ acts on $P \times_K N$ as automorphisms of fiber bundle.   Therefore, Proposition 1.7 implies that the (real) Lie group $C^\infty(P \times_{Ad^{-1}} N)$ acts on $C^\infty(P \times_K N)$ as holomorphic transformations.

As a special case, we consider the set of all almost complex structures on a compact manifold M.   Let N be the set of all linear complex structures on $\boldsymbol{R}^n$, where $n = \dim_{\boldsymbol{R}} M$.   The manifold N is a homogeneous complex manifold and the group $GL(n, \boldsymbol{R})$ acts as holomorphic transformations.   Thus the space $E = F(M) \times_{GL(n,\boldsymbol{R})} N$ becomes a complex fiber bundle over M and the set of all almost complex structures is nothing but the space $C^\infty(E)$, which is a complex manifold.   Moreover, since this complex structure is invariant under the action of the group $\mathscr{D}(M)$ of diffeomorphisms of M, the coset space $C^\infty(E)/\mathscr{D}(M)$ admits a complex space structure.   These complex structures are nothing but those given by the classical holomorphic coordinate system [8].

4. Compatibility with the classical deformation theory

In the previous section, we saw that at least the complex structure $\mathcal{J}$ is the same as the classical one in the theory of deformations of complex structures. However, to apply Definition 1.1 to deformation theory, we need one more analysis. That is, we have to show that the space of holomorphic maps is a complex analytic set. Remark that Definition 1.1 of complex structure $\mathcal{J}$ does not require any complex structure on the base manifold M.

Now let E be a complex fiber bundle over M and assume that E is a complex manifold. Since notations in below are complicated for general cases, we treat the trivial case $E = M \times N$ at first, and will reduce general cases to it later. Let J_N be the almost complex structure of N. For a map $s \in C^\infty(M, N)$, the differential ds is a map from TM to TN, and $J_N(ds)$ is defined by $J_N(ds): X \mapsto J_N((ds)X)$. Since TM is not compact, we regard ds as a cross section Ds of the bundle $T^*M \otimes TN (\to M \times N)$ over M. For $x \in M$, the fiber of the bundle $T^*M \otimes TN \to M$ has the structure of the fiber bundle

$$T_x^* M \otimes_R TN \to N ,$$

and choosing (real) basis of $T_x^* M$, it is identified with the fiber bundle

$$\underbrace{TN \oplus \cdots \oplus TN}_{n \text{ times}} \to N .$$

If we pull-back the complex structure of $(TN \oplus \cdots \oplus TN)$ to the space $T_x^* M \otimes_R TN$, it is independent of the choice of basis of $T_x^* M$ and is compatible with the operation J_N above. In that sense the fiber $T_x^* M \otimes_R TN$ is a holomorphic vector bundle over N, and has a complex structure. Thus D is a map from a complex manifold $C^\infty(M, N)$ to a complex manifold $C^\infty(T^*M \otimes TN)$.

Lemma 4.1. *The map D is holomorphic.*

Proof. By the definition of the complex structure $\mathcal{J}$, we may consider locally, i.e., we may replace M, N by open neighbourhoods of the origins of $\boldsymbol{R}^n, \boldsymbol{C}^r$, respectively. Then the space $T^*M \otimes TN$ is replaced by $\boldsymbol{R}^n + \boldsymbol{C}^r + \boldsymbol{R}^n \otimes \boldsymbol{C}^r$, and for a map $s: \boldsymbol{R}^n \to \boldsymbol{C}^r$, the map

$$Ds: \boldsymbol{R}^n \to \boldsymbol{R}^n + \boldsymbol{C}^r + \boldsymbol{R}^n \otimes \boldsymbol{C}^r$$

is given by

$$Ds(x) = (x, s(x), \partial_M s) ,$$

where ∂_M denotes the partial derivatives $(\partial/\partial x^1, \cdots, \partial/\partial x^n)$. Therefore for a one parameter family $\{s_t\}$, we see

$$\left(\frac{d}{dt}\,s_t\right)(x) = (s(x),\,\frac{d}{dt}\,s_t(x)) \in \boldsymbol{C}^r \times \boldsymbol{C}^r\,,$$

$$\left(\frac{d}{dt}\,Ds_t\right)(x) = \frac{d}{dt}\,(x,\,s_t(x),\,\partial_M\,s_t)$$

$$= (x,\,s(x),\,\partial_M s,\,0,\,\frac{d}{dt}\,s,\,\partial_M\frac{d}{dt}\,s)\,.$$

Thus

$$\left(\mathscr{J}\frac{d}{dt}\,s_t\right)(x) = (s(x),\,J_N\frac{d}{dt}\,s_t(x))\,,$$

$$\left(dD\left(\mathscr{J}\frac{d}{dt}\,s_t\right)\right)(x) = (x,\,s(x),\,\partial_M s,\,0,\,J_N\frac{d}{dt}\,s,\,\partial_M J_N\frac{d}{dt}\,s)\,,$$

$$\left(\mathscr{J}\,dD\left(\frac{d}{dt}\,s_t\right)\right)(x) = (x,\,s(x),\,\partial_M s,\,0,\,J_N\frac{d}{dt}\,s,\,J_N\,\partial_M\frac{d}{dt}\,s)\,.$$

Since J_N is constant in the coordinate system, $dD(\mathscr{J}(ds_t/dt))$ coincides with $\mathscr{J}\,dD(ds_t/dt)$.   Q.E.D.

Now we use an almost complex structure J_M of the base manifold M. The space $T^*M \otimes TN$ is decomposed to

$$H = \{\xi \in T^*M \otimes TN;\ \xi J_M = J_N\,\xi\}$$

and

$$A = \{\xi \in T^*M \otimes TN;\ \xi J_M = -J_N\,\xi\}$$

Here, it is easy to check that the mappings J_M and J_N are holomorphic on each fiber $T^*_x M \otimes TN$, and so the projection maps from $T^*_x M \otimes TN$ to H and A are holomorphic, and H_x and A_x are complex submanifolds of $T^*_x M \otimes TN$. Thus if we define a map $\bar{D}\colon C^\infty(M, N) \to C^\infty(A)$ as the A-part of D, then the map $\bar{D}$ is holomorphic by Proposition 1.7. Remarking that a map $s\colon M \to N$ is holomorphic if and only if $\bar{D}s = 0$, we get

Proposition 4.2. *If N is a complex manifold and M an almost complex manifold, then the space of all holomorphic maps is the inverse image of a point by a holomorphic map.*

REMARK 4.3. In the above proposition, we do not use so essentially the almost complex structure of M. In fact, from the proof, it is clear that the same conclusion holds even for an almost contact manifold M.

When we treat general complex fiber bundle E over M, we may consider the complex manifold $C^\infty(E)$ as a complex submanifold of $C^\infty(M, E)$ by Proposition 1.7, and can apply the above result. However, to ensure that there is a holomorphic section, we set a natural assumption that E is a holomorphic fiber bundle over a complex manifold M, i.e., E and M are complex manifolds and the projection map is holomorphic. We can state as

Theorem 4.4. *Let E be a holomorphic fiber bundle over a compact complex manifold M. Then the space of all holomorphic sections forms a complex analytic set of the complex manifold $C^\infty(E)$.*

It is clear that the complex structure of the set of holomorphic sections coincides with the classical one.

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