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YANG-MILLS CONNECTIONS OF HOMOGENEOUS BUNDLES

Dedicated to Professor Shingo Murakami on his 60th birthday

NORIHITO KOISO[1]

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0. Introduction

Let (M, g) be a compact riemannian manifold and P a principal fiber bundle over M with compact structure group K. A functional $\mathcal{F}_{YM}$ which maps a connetion ∇ of P to the square integral $\int_M |R^\nabla|^2 v_g$ of the norm of the curvature tensor of ∇ is called the Yang-Mills functional. A Yang-Mills connection is by definition a critical point of the functional $\mathcal{F}_{YM}$. Therefore there is some possibility that the so called direct method and the heat equation method can be applied to construct a Yang-Mills connection of P.

When the manifold M is an algebraic manifold and the group K is a unitary group, there is a strong relationship between the notion of stable vector bundles and that of Yang-Mills connections ([4]), and Donaldson shows the exsitence of a Yang-Mills connection by the heat equation method ([2]).

In this paper we consider homogeneous bundles as simple examples in order to see in what situations the direct method and the heat equation method can be applied to the existence problem of Yang-Mills connections. Let the riemannian manifold M be expressed as a homogeneous space G/H and the principal fiber bundle P as $G\times_\rho K$ using a Lie group homomorphism $\rho: H\to K$. The space C_G of all G-invariant connections forms a finite dimensional vector space. Corresponding to the direct method, we will prove the following

Theorem 1. *Assume that the Lie group H is connected. Then the function $\mathcal{F}_{YM}|C_G$ is proper if and only if one of the following conditions holds. (1) The fundamental group $\pi_1(M)$ of M is finite. (2) The Lie algebra $\mathfrak{k}$ of the structure grogp K has no trivial factor as H-module.*

This means that if (1) or (2) holds, then any minimizing sequence for the function $\mathcal{F}_{YM}|C_G$ has a convergent subsequence to a Yang-Mills connection.

[1] This work was done while the author was staying in Max-Planck-Institut für Mathematik in Bonn, to which he is grateful for the hospitality.

But if neither (1) nor (2) holds, a minimizing sequence may diverge to "∞".

However, even if neither (1) nor (2) holds, we can find a Yang-Mills connection by the heat equation—an ordinary differential equation in our case—method.

Theorem 2. *The heat equation with a G-invariant connection ∇_0 as the initial data has a solution ∇_t which is a bounded curve in the space C_G. In particular, the bundle P admits a Yang-Mills connection.*

As a particular case of Theorem 2, we will see what happens in the case of homogeneous complex situations. Finally, we will prove the mountain-pass Lemma for the function $\mathcal{F}_{YM}|C_G$. Remark that, when we consider Einstein's equation the corresponding statement to Theorem 2 does not hold, i.e., the solution diverges in general ([7, Introduction]).

1. Properness

We will prove Theorem 1 in this section. Let M be a compact homogeneous riemannian manifold G/H, where G is a compact Lie group and H is a closed subgroup. Denote by $\mathfrak{g}$, $\mathfrak{h}$ the Lie algebras of the Lie groups G, H respectively. Fix a bi-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ and denote by $\mathfrak{m}$ the orghogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. The riemannian metric of the space M is represented by an H-invariant inner product g on $\mathfrak{m}$. Define a principal fiber bundle $P=G\times_\rho K$ using a compact Lie group K and a homomorphism $\rho: H\to K$. The Lie algebra of K is denoted by $\mathfrak{k}$ and is endowed with a bi-invariant inner product $\langle\ ,\ \rangle$. The differential: $\mathfrak{h}\to\mathfrak{k}$ of the Lie group homomorphism ρ is denoted by the same symbol ρ. The space $\mathfrak{k}$ becomes an H-module and an H^0-module via ρ, where H^0 is the identity component of H. For basic facts about Lie groups, refer to [3].

As usual, we denote by $\mathfrak{g}'$ the semi-simple part of the Lie algebra $\mathfrak{g}$ and by $\mathfrak{z}(\mathfrak{g})$ its center. Let $\mathfrak{m}'$ be the projection image from $\mathfrak{g}'$ to $\mathfrak{m}$. The vector space $\mathfrak{m}$ decomposes as H-module:

$$\mathfrak{m} = \mathfrak{m}' \oplus \mathfrak{m} \cap \mathfrak{z}(\mathfrak{g})\,, \tag{1.1}$$

which corresponds to the decomposition of the universal covering of M into a compact manifold and a vector space. Therefore the fundamental group $\pi_1(M)$ is finite if and only if $\mathfrak{m}\cap\mathfrak{z}(\mathfrak{g})$ vanishes. When the Lie algebra $\mathfrak{k}$ decomposes into the semi-simple part and the center, the functional $\mathcal{F}_{YM}$ correspondingly decomposes. These facts reduce the proof of Theoerm 1 to the following propositions.

Proposition 1.1. *Assume that the Lie group H is connected. If the space $\mathfrak{k}$ has a trivial factor as H-module and the space $\mathfrak{m}\cap\mathfrak{z}(\mathfrak{g})$ does not vanish, then the function $\mathcal{F}_{YM}|C_G$ is not proper.*

Proposition 1.2. *If one of the followong conditions holds, then the function* $\mathcal{F}_{YM}|C_G$ *is proper.*

1) *The space* $\mathfrak{k}$ *has no trivial factor as* H^0*-module.*
2) *The Lie algebra* $\mathfrak{k}$ *is commutative and the space* $\mathfrak{m}\cap\mathfrak{z}(\mathfrak{g})$ *vanishes.*
3) *The Lie algebra* $\mathfrak{k}$ *is semi-simple and the space* $\mathfrak{m}\cap\mathfrak{z}(\mathfrak{g})$ *vanishes.*

We will give proofs of these propositions by a series of lemmas. The following lemma is fundamental.

Lemma 1.3. [5, Chapter II Theorem 11.7]. *The space* C_G *is canonically identified with the space of all H-homomorphisms* $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k})$, *and the curvature tensor* $R^A\in\mathrm{Hom}_H(\wedge^2\mathfrak{m}, \mathfrak{k})$ *of an element* $A\in\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k})$ *is given by*

$$R^A(v, w) = [A(v), A(w)]-A([v, w]_{\mathfrak{m}})-\rho([v, w]_{\mathfrak{h}})\,, \tag{1.2}$$

where $(\ \ )_{\mathfrak{h}}$ *and* $(\ \ )_{\mathfrak{m}}$ *denote the components with respect to the decomposition* $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$.

From now on an element of the space $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k})$ is identified with a connection of P, and so the function $\mathcal{F}_{YM}|C_G$ is regarded as

$$\mathcal{F}_{YM}(A) = \mathrm{Vol}(M)\times|R^A|^2\,. \tag{1.3}$$

Since the properness of the function $\mathcal{F}_{YM}|C_G$ is independent of the choice of inner products of $\mathfrak{m}$, we may assume that the inner product g is the restriction of $\langle\ ,\ \rangle$ on $\mathfrak{g}$ in this section.

Proof (of Proposition 1.1). The assumption implies that there are non-zero elements X in a trivial factor of the H-module $\mathfrak{k}$ and v_0 in $\mathfrak{m}\cap\mathfrak{z}(\mathfrak{g})$. Then we can define an element A in $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k})$ by $A(v)=\langle v, v_0\rangle X$, which satisfies $R^{\lambda A}=R^0$ for any real number λ by formula (1.2). Q.E.D.

We decompose $\mathfrak{m}$ as H^0-module into the trivial factor $\mathfrak{m}_0$ and the sum $\mathfrak{m}_1$ of the irreducible factors. Then we have inclusions:

$$\mathfrak{m}_1\subset\mathfrak{m}' \quad\text{and}\quad \mathfrak{m}\cap\mathfrak{z}(\mathfrak{g})\subset\mathfrak{m}_0\,. \tag{1.4}$$

Lemma 1.4. *There exist positive constants* c_1, c_2 *and* c_3 *such that for any* $A\in\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k})$ *it holds that*

$$|R^A|\geq c_1|A|\mathfrak{m}_1|^2-c_2|A|\mathfrak{m}'|-c_3\,. \tag{1.5}$$

Proof. We set $[A\wedge A](v, w)=[A(v), A(w)]$ and observe that if $[A\wedge A]=0$, then $A(\mathfrak{m}_1)=0$. In fact

$$\begin{aligned}0 &= \langle[A(\mathfrak{m}), A(\mathfrak{m})], \rho(\mathfrak{h})\rangle = \langle A(\mathfrak{m}), [\rho(\mathfrak{h}), A(\mathfrak{m})]\rangle \\ &= \langle A(\mathfrak{m}), A([\mathfrak{h}, \mathfrak{m}])\rangle = \langle A(\mathfrak{m}), A(\mathfrak{m}_1)\rangle\,.\end{aligned} \tag{1.6}$$

Therefore if we set $c_1 = \inf\{|[A\wedge A]|\,;\, A\in \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k}),\ |A|=1\}$, then $c_1>0$. For the second term $A([v, w]_{\mathfrak{m}})$ of formula (1.2), it depends only on $A|\mathfrak{m}'$.

Q.E.D.

Proof (of Proposition 1.2. (1)). Since the space $\mathfrak{k}$ has no trivial factor as H^0-module, the space $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k})$ coincides with $\mathrm{Hom}_H(\mathfrak{m}_1, \mathfrak{k})$. Thus $A, A|\mathfrak{m}'$ and $A|\mathfrak{m}_1$ coincide in Lemma 1.4. Q.E.D.

Proof (of Proposition 1.2 (2)). Let A be any element of $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k})$. Since the Lie algebra $\mathfrak{k}$ is commutative, the first term $[A\wedge A]$ of formula (1.2) vanishes. Also, since $\mathfrak{k}$ is trivial as H^0-module, $A(\mathfrak{m}_1)=0$. On the other hand, since $\mathfrak{m}\cap\mathfrak{z}(\mathfrak{g})=0$, it holds that $\mathfrak{m}=\mathfrak{m}'=[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}+\mathfrak{m}_1$. Therfore if $A\neq 0$, then the second term $A([\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}})\neq 0$. Thus we can define a positive number c_1 by $\inf\{|A|[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}|\,;\, A\in\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k}),\ |A|=1\}$ and, setting c_2 to be the norm of the third term, we get $|R^A|\geq c_1|A|-c_2$. Q.E.D.

To prove the case of semi-simple Lie algebra $\mathfrak{k}$, we introduce the following usual notations. For a reductive Lie algebra $\mathfrak{i}$, $\mathfrak{t}(\mathfrak{i})$ denotes a Cartan subalgebra. When $\mathfrak{i}$ is semi-simple and endowed with a bi-invariant inner product $\langle\ ,\ \rangle$, we denote by $\Delta(\mathfrak{i})$ its root system as a subset of $\mathfrak{t}(\mathfrak{i})$ and characterize root vectors $X_\alpha\in\mathfrak{i}$ for $\alpha\in\Delta(\mathfrak{i})$ by (1) $[u, X_\alpha]=\langle u, \alpha\rangle X_{-\alpha}$ for all $u\in\mathfrak{t}(\mathfrak{i})$ and (2) $[X_\alpha, X_{-\alpha}]=\alpha$. The following lemma will be proved later.

Lemma 1.5. *Let $\mathfrak{k}$ be a compact semi-simple Lie algebra. For an element (w_0, w_1, w_2) of $\mathfrak{k}^3$ we define an element (u_0, u_1, u_2) of $\mathfrak{k}^3$ by*

$$u_0 = [w_1, w_2]-w_0\,,\quad u_1 = [w_2, w_0]-w_1\,,\quad u_2 = [w_0, w_1]-w_2\,, \tag{1.7}$$

then this map: $\mathfrak{k}^3\to\mathfrak{k}^3$ is proper.

Note that $\mathfrak{m}_0$ becomes a subalgebra of $\mathfrak{g}$, i.e., $[\mathfrak{m}_0, \mathfrak{m}_0]\subset\mathfrak{m}_0$, and $[\mathfrak{m}_0, \mathfrak{m}_1]$ is contained in $\mathfrak{m}_1$. Since Cartan subalgebras $\mathfrak{t}(\mathfrak{h})$ and $\mathfrak{t}(\mathfrak{m}_0)$ commute, there is a Cartan subalgebra $\mathfrak{t}(\mathfrak{g})$ which contains $\mathfrak{t}(\mathfrak{h})$ and $\mathfrak{t}(\mathfrak{m}_0)$. The space $\mathfrak{t}(\mathfrak{g})$ decomposes into the center $\mathfrak{z}(\mathfrak{g})$ and a Cartan subalgebra $\mathfrak{t}(\mathfrak{g}')$. It admits also an orthogonal decomposition:

$$\mathfrak{t}(\mathfrak{g}) = \mathfrak{t}(\mathfrak{h})\oplus\mathfrak{t}(\mathfrak{m}_0)\oplus\mathfrak{t}(\mathfrak{g})\cap\mathfrak{m}_1\,. \tag{1.8}$$

We denote by $\mathfrak{t}(\mathfrak{g}')_0$ the image of the orthogonal projection from $\mathfrak{t}(\mathfrak{g}')$ to $\mathfrak{t}(\mathfrak{m}_0)$.

Lemma 1.6. *Denoting by $(\mathfrak{m}_0)'$ the semi-smiple part of $\mathfrak{m}_0$, we get*

$$\mathfrak{t}(\mathfrak{g}')_0+(\mathfrak{m}_0)' = \mathfrak{m}'\cap\mathfrak{m}_0\,. \tag{1.9}$$

Proof. It is clear that the left hand side is contained in the right hand side. Let v be an element of the right hand side which is orthogonal to the left hand side. Then v is an element of the center $\mathfrak{z}(\mathfrak{m}_0)$, and is orthogonal to $\mathfrak{t}(\mathfrak{g}')$. Therefore we see that $v \in \mathfrak{z}(\mathfrak{g})$ and so by (1.1) we conclude that $v=0$. Q.E.D.

We restate Proposition 1.2 (3) as follows for later use.

Proposition 1.7. *If the Lie algebra $\mathfrak{k}$ is semi-simple, then $|A|\mathfrak{m}'|$ is estimated from above by using $|R^A|$.*

Proof. First remark that, by Lemma 1.5, if we take $v_0, v_1, v_2 \in \mathfrak{m}$ with $[v_0, v_1]_{\mathfrak{m}}=v_2$, $[v_1, v_2]_{\mathfrak{m}}=v_0$, $[v_2, v_0]_{\mathfrak{m}}=v_1$, then $|A(v_i)|$'s are estimated by using $|R^A|$. Therefore we can get an estimation of $A|(\mathfrak{m}_0)'$ because the space $(\mathfrak{m}_0)'$ is spanned by its roots and root vectors. Next we decompose a root $\alpha \in \Delta(\mathfrak{g}')$ by (1.8) and denote by α_0, α_1 the $\mathfrak{t}(\mathfrak{m}_0)$, $\mathfrak{t}(\mathfrak{g}) \cap \mathfrak{m}_1$-component, respectively, and set $\alpha'=\alpha_0+\alpha_1$. The vector α' is the $\mathfrak{m}$-component of α, and belongs to $\mathfrak{m}'$.

Now assume that $\alpha_0 \neq 0$. Then

$$
\begin{aligned}
(1.10) \qquad \pm|\alpha_0|^2 \cdot X_{\pm\alpha} = [\alpha_0, X_{\pm\alpha}] &= [\alpha_0, (X_{\pm\alpha})_{\mathfrak{h}}]+[\alpha_0, (X_{\pm\alpha})_{\mathfrak{m}}] \\
&= [\alpha_0, (X_{\pm\alpha})_{\mathfrak{m}}] \in \mathfrak{m}\,,
\end{aligned}
$$

and so $X_{\pm\alpha} \in \mathfrak{m}$. Setting $v_0=|\alpha'|^{-2}\alpha'$, $v_1=|\alpha'|^{-1}X_\alpha$, $v_2=|\alpha'|^{-1}X_{-\alpha}$, we can get an estimation of $A(\alpha')$ by the previous remark. Moreover, since $A(\mathrm{Ad}_h\alpha')=\mathrm{Ad}_{\rho(h)}A(\alpha')$ for $h \in H^0$, we get an estimation of $|A(\alpha_0)|$ by

$$
(1.11) \qquad |A(\alpha_0)| = \left|A\left(\int_{H^0}(\mathrm{Ad}_h\alpha')\,dh\right)\right| \leq \int_{H^0}|A(\mathrm{Ad}_h\alpha')|\,dh = |A(\alpha')|\,,
$$

where dh is Haar measure of H^0. Since the space $\mathfrak{t}(\mathfrak{g}')_0$ is spanned by such α_0's, we get an estimation of $A|\mathfrak{t}(\mathfrak{g}')_0$. Combining with the estimation of $A|(\mathfrak{m}_0)'$, we get an estimation of $A|(\mathfrak{m}' \cap \mathfrak{m}_0)$ by Lemma 1.6. Finally, using Lemma 1.4 and the inequality: $|A|\mathfrak{m}'| \leq |A|(\mathfrak{m}' \cap \mathfrak{m}_0)|+|A|\mathfrak{m}_1|$ from (1.4), we get an estimation of $A|\mathfrak{m}_1$, therefore of $A|\mathfrak{m}'$. Q.E.D.

Proof (of Lemma 1.5). We set $c=\max\{|u_0|, |u_1|, |u_2|\}$ and $l=|w_0|$, and show that l is bounded from above by using c. In the following, c_i's mean positive constants which depend only on c and do not depend on l. First we see that

$$
\begin{aligned}
(1.12) \qquad |w_1|^2-c|w_1| \leq \langle w_1, w_1+u_1\rangle &= \langle w_1, [w_2, w_0]\rangle \\
&= \langle [w_1, w_2], w_0\rangle = \langle w_0+u_0, w_0\rangle \leq l^2+cl\,,
\end{aligned}
$$

and so $|w_1| \leq l+c$. By the same way we see also that $|w_2| \leq l+c$.

We choose a Cartan subalgebra $\mathfrak{t}(\mathfrak{k})$ containing w_0 and a linear order $>$ of $\mathfrak{t}(\mathfrak{k})$ so that if $\langle w_0, \alpha\rangle>0$, then $\alpha>0$. Denote by $\Pi=\{\alpha_i\}$ the fundamental root system. Since Π is basis of $\mathfrak{t}(\mathfrak{k})$, whose pattern is independent of the choice of orders, it holds that

$$(1.13)\qquad |\sum_{\alpha_i\in\Pi} x_i\,\alpha_i| \geq c_1 \sum_{\alpha_i\in\Pi} |x_i| \quad \text{for any } (x_i)\,.$$

We set $w_1=z+\sum a_\alpha X_\alpha$, where z is an element of $\mathfrak{t}(\mathfrak{k})$ and the summation is taken for roots $\alpha\in\Delta(\mathfrak{k})$. Then we see that

$$(1.14)\qquad [w_0, w_1] = \sum a_\alpha\langle\alpha, w_0\rangle X_{-\alpha}\,,$$

$$(1.15)\qquad w_2 = [w_0, w_1]-u_2 = \sum a_\alpha\langle\alpha, w_0\rangle X_{-\alpha}-u_2\,,$$

$$(1.16)\qquad [w_0, w_2] = -\sum a_\alpha\langle\alpha, w_0\rangle^2 X_\alpha-[w_0, u_2]\,,$$

from which we get

$$(1.17)\qquad u_1 = [w_2, w_0]-w_1 = \sum a_\alpha\langle\alpha, w_0\rangle^2 X_\alpha+[w_0, u_2]-z-\sum a_\alpha X_\alpha\,,$$

$$(1.18)\qquad \sum a_\alpha(\langle\alpha, w_0\rangle^2-1)\,X_\alpha = z+u_1-[w_0, u_2]\,.$$

Since $\{X_\alpha;\ \alpha\in\Delta(\mathfrak{k})\}$ are orthogonal, it follows that

$$(1.19)\qquad |a_\alpha|\,|\langle\alpha, w_0\rangle^2-1| \leq |X_\alpha|^{-1}((l+c)+c+c_2\,cl)\leq c_3(l+1)\,.$$

Therefore, if $\langle\alpha, w_0\rangle^2\geq 2$, then

$$(1.20)\qquad |a_\alpha|\langle\alpha, w_0\rangle^2\leq 2c_3(l+1)\,.$$

Furthermore, since

$$(1.21)\qquad \begin{aligned}[w_1, w_2]_{\mathfrak{t}(\mathfrak{k})} &= [w_1, \textstyle\sum a_\alpha\langle\alpha, w_0\rangle X_{-\alpha}-u_2]_{\mathfrak{t}(\mathfrak{k})}\\ &= [z+\textstyle\sum a_\alpha X_\alpha, \sum a_\alpha\langle\alpha, w_0\rangle X_{-\alpha}]_{\mathfrak{t}(\mathfrak{k})}-[w_1, u_2]_{\mathfrak{t}(\mathfrak{k})}\\ &= \textstyle\sum (a_\alpha)^2\langle\alpha, w_0\rangle\cdot\alpha-[w_1, u_2]_{\mathfrak{t}(\mathfrak{k})}\,,\end{aligned}$$

we get

$$(1.22)\qquad \begin{aligned}w_0 &= [w_1, w_2]_{\mathfrak{t}(\mathfrak{k})}-(u_0)_{\mathfrak{t}(\mathfrak{k})}\\ &= \textstyle\sum (a_\alpha)^2\langle\alpha, w_0\rangle\cdot\alpha-[w_1, u_2]_{\mathfrak{t}(\mathfrak{k})}-(u_0)_{\mathfrak{t}(\mathfrak{k})}\,.\end{aligned}$$

Now for a positive number ε, we define a subset Π_ε of the fundamental root system Π by

$$(1.23)\qquad \Pi_\varepsilon = \{\alpha_i\in\Pi;\ \langle\alpha_i, w_0\rangle <\varepsilon|w_0|\,|\alpha_i|\}\,.$$

The number ε will be fixed later independently of l. In the following, the constants c_i are also independent of ε. Put

$$(1.24)\qquad S = \Delta(\mathfrak{k})\cap \sum_{\alpha_i\in\Pi_\varepsilon} \boldsymbol{Z}\alpha_i$$

An element β of $\Delta(\mathfrak{k})-S$ can be represented as $\sum m_i\,\alpha_i\ (\alpha_i\in\Pi)$, where all m_i are non-negative or all are non-positive. Hence

$$(1.25) \qquad |\langle\beta, w_0\rangle| = \sum |m_i\langle w_0, \alpha_i\rangle| \\ \geq \langle w_0, \alpha_i\rangle \quad \text{for some } \alpha_i \in \Pi - \Pi_\varepsilon \\ \geq c_4\, \varepsilon l\,.$$

Therefore if $l \geq 1$ and $\sqrt{2}(c_4\, \varepsilon)^{-1}$, then, by (1.20), we see that

$$(1.26) \qquad (a_\beta)^2 |\langle\beta, w_0\rangle| \leq (2c_3(l+1))^2 |\langle\beta, w_0\rangle|^{-3} \\ \leq (2c_3(l+1))^2 (c_4\, \varepsilon l)^{-3} \\ \leq c_6\, \varepsilon^{-3}\,.$$

We represent $\sum_{\alpha\in S}(a_\alpha)^2\langle\alpha, w_0\rangle\cdot\alpha$ as $\sum s_i\alpha_i$ $(\alpha_i \in \Pi_\varepsilon)$. Since

$$(1.27) \qquad \sum_{\alpha\in S}(a_\alpha)^2\langle\alpha, w_0\rangle\cdot\alpha = \sum_{\alpha\in S, \alpha>0}((a_\alpha)^2+(a_{-\alpha})^2)\langle\alpha, w_0\rangle\cdot\alpha\,,$$

all s_i are nonnegative, and so

$$(1.28) \qquad \langle w_0, \sum s_i\alpha_i\rangle = \sum s_i\langle w_0, \alpha_i\rangle < \sum s_i |w_0|\,|\alpha_i|\,\varepsilon \\ \leq c_6 \varepsilon l \sum s_i\,.$$

On the other hand, from (1.13), (1.22) and (1.26), if l is greater than 1 and $\sqrt{2}(c_4\varepsilon)^{-1}$, then

$$(1.29) \qquad c_1 \sum s_i \leq |\sum s_i\alpha_i| \\ = |w_0 - \sum_{\beta\notin S}(a_\beta)^2\langle\beta, w_0\rangle\cdot\beta + [w_1, u_2]_{\mathfrak{t}(\mathfrak{k})} + (u_0)_{\mathfrak{t}(\mathfrak{k})}| \\ \leq l + c_7\varepsilon^{-3} + c_8 l + c_9\,.$$

Combining it with (1.28),

$$(1.30) \qquad \langle w_0, \sum s_i\alpha_i\rangle \leq c_{10}\varepsilon l^2 + (\text{polynomial of } l \text{ of order } 1).$$

Therefore, again using (1.22) and (1.26), we see that if $l \geq 1$, $\sqrt{2}(c_4\varepsilon)^{-1}$, then

$$(1.31) \qquad l^2 = \langle w_0, w_0\rangle \\ = \langle w_0, \sum_{\beta\notin S}(a_\beta)^2\langle\beta, w_0\rangle\cdot\beta + \sum s_i\alpha_i - [w_1, u_2]_{\mathfrak{t}(\mathfrak{k})} - (u_0)_{\mathfrak{t}(\mathfrak{k})}\rangle \\ \leq c_{10}\varepsilon l^2 + c\,|[w_0, w_1]| + (\text{polynomial of } l \text{ of order } 1) \\ = c_{10}\varepsilon l^2 + c\,|w_2 + u_2| + (\text{polynomial of } l \text{ of order } 1) \\ \leq c_{10}\varepsilon l^2 + (\text{polynomial of } l \text{ of order } 1).$$

Thus choosing ε so that $c_{10}\varepsilon < 1/2$, we get the desired estimation of l. Q.E.D.

2. Gradient Flow

We consider the heat equation for the functional $\mathcal{F}_{YM}$ with respect to the

L_2 inner product, which becomes

$$\frac{d}{dt}\nabla_t = -(\mathrm{grad}\,\mathscr{F}_{YM})_{\nabla_t} = (\nabla_t)^k(R^{\nabla_t})_{ki}\,. \tag{2.1}$$

If we choose $\nabla_0 \in \mathcal{C}_G$ as the initial data of this equation, then the solution ∇_t is a curve in $\mathcal{C}_G$ and coincides with the solution of the ordinary differential equation defined by the vector field $-\mathrm{grad}(\mathscr{F}_{YM}|\mathcal{C}_G)$. As is easily computed from formula (1.2), the equation is given by (up to constant multiplication of time variable t),

$$\begin{aligned}\frac{d}{dt}A_i = {} & \sum [A_j, [A_j, A_i]] - \sum C_j{}^k{}_i [A_j, A_k] - \sum C_j{}^s{}_i [A_j, \rho_s] \\ & + \frac{1}{2}\sum C_j{}^i{}_k [A_j, A_k] - \frac{1}{2}\sum C_j{}^i{}_k C_j{}^l{}_k A_l - \frac{1}{2}\sum C_j{}^i{}_k C_j{}^s{}_k \rho_s,\end{aligned} \tag{2.2}$$

where we take orthonormal basis $\{v_i\}$ of $\mathfrak{m}$ with respect to g and baiss $\{v_s\}$ of $\mathfrak{h}$, and set

$$\begin{aligned} & A_i = A(v_i),\ \rho_s = \rho(v_s)\,, \\ & [v_i, v_j] = \sum C_i{}^k{}_j v_k + \sum C_i{}^s{}_j v_s\,. \end{aligned} \tag{2.3}$$

All the summations are taken for $j, k, l, s,$ which appear twice in the terms.

We will prove Theorem 2 for equation (2.2). Denote by $A(t)$ the solution. First of all, as we see from equation (2.1) or (2.2), when the Lie algebra $\mathfrak{k}$ decomposes as $\mathfrak{k}' \oplus \mathfrak{z}(\mathfrak{k})$, the solution also decomposes, and the $\mathfrak{z}(\mathfrak{k})$-component of $A_0(t)$ is constant. Therefore we may assume that *the Lie algebra* $\mathfrak{k}$ *is semi-simple*. Then, by Proposition 1.7, the norm of $A(t)|\mathfrak{m}'$ is estimated from above by using $|R^{A(0)}|$. Therefore, denoting by $(\mathfrak{m}')^\perp$ the orthogonal compliment of $\mathfrak{m}'$ in $\mathfrak{m}$ with respect to g, it is sufficient to prove that $A(t)|(\mathfrak{m}')^\perp$ is bounded. To show it, we choose an arbitrary unit vector v_0 in $(\mathfrak{m}')^\perp$, choose orthonormal basis $\{v_i;\ 0 \le i < \dim \mathfrak{m}\}$ of $\mathfrak{m}$ containing v_0, and prove that $A_0(t)$ is bounded. For $A_0(t)$, equation (2.2) is simplified as

$$\frac{d}{dt}A_0 = \sum [A_j, [A_j, A_0]] - \sum C_j{}^k{}_0 [A_j, A_k]\,. \tag{2.4}$$

In fact the structure constants $C_j{}^0{}_k$ vanish in equation (2.2) because $[\mathfrak{m}, \mathfrak{m}]_\mathfrak{m} \subset \mathfrak{m}'$. Moreover, since the inner product g is H-invariant and v_0 is orghogonal to $\mathfrak{m}'$, the vector v_0 is an element of $\mathfrak{m}_0$, and as the remark following Lemma 1.5, $[\mathfrak{m}_0, \mathfrak{m}] \subset \mathfrak{m}$, which implies that the structure constants $C_j{}^s{}_0$ also vanish.

Next, since the equations do not depend on the choice of inner products on $\mathfrak{k}$, we may assume that the root vetcors X_α of $\mathfrak{k}$ are unit.

Now we define a function L on the vector space $\mathfrak{k}$. Let 2δ be the sum of all positive roots of $\mathfrak{k}$. We represent 2δ as $2\delta = \sum n_i \alpha_i$. Let $\{\omega_i;\ 1 \le i \le r\}$ be the fundamental weight system of $\mathfrak{k}$, and set $\xi_i = (n_i)^{-1}\omega_i$. For $w \in \mathfrak{k}$, we define

$$L(w)=\max\{\langle \mathrm{Ad}_\gamma w,\ \xi_i\rangle;\ 1\leq i\leq r,\ \gamma\in K\}\ . \tag{2.5}$$

Lemma 2.1. *For $w\in\mathfrak{k}$, the value $L(w)$ is realized by $\gamma\in K$ such that $\mathrm{Ad}_\gamma w$ belongs to the positive Weyl chamber $\bar{W}$. In particular L is a norm of $\mathfrak{k}$.*

Proof. From the assumption, for any $X\in\mathfrak{k}$,

$$0=\langle[X,\mathrm{Ad}_\gamma w],\ \xi_i\rangle=\langle X,\ [\mathrm{Ad}_\gamma w,\ \xi_i]\rangle\ . \tag{2.6}$$

Therefore $\mathrm{Ad}_\gamma w$ and ξ_i belong to the same abelian subalgebra of $\mathfrak{k}$. Since all Cartan subalgebras are conjugate, we may assume that $\mathrm{Ad}_\gamma w\in\mathfrak{t}(\mathfrak{k})$. If $\langle\mathrm{Ad}_\gamma w,\ \alpha_j\rangle<0$ for some $\alpha_j\in\Pi$, then, taking $\eta\in K$ which gives the reflection with respect to α_j, for any ξ_k we see that

$$\begin{aligned}\langle\mathrm{Ad}_{\eta\gamma} w,\ \xi_k\rangle&=\langle\mathrm{Ad}_\eta(\mathrm{Ad}_\gamma w),\ \xi_k\rangle\\&=\langle\mathrm{Ad}_\gamma w-2|\alpha_j|^{-2}\langle\alpha_j,\ \mathrm{Ad}_\gamma w\rangle\cdot\alpha_j,\ \xi_k\rangle\\&=\langle\mathrm{Ad}_\gamma w,\ \xi_k\rangle-2|\alpha_j|^{-2}\langle\alpha_j,\ \mathrm{Ad}_\gamma w\rangle\,(n_j)^{-1}\,\delta_{jk}\\&\geq\langle\mathrm{Ad}_\gamma w,\ \xi_k\rangle\ .\end{aligned} \tag{2.7}$$

That is, when $\mathrm{Ad}_\gamma w$ is mapped into $\bar{W}$ by the Weyl group, the value $L(w)$ is still realized. Q.E.D.

We reduced the problem to the case that the Lie algebra $\mathfrak{k}$ is semi-simple in order to use the following

Lemma 2.2. *Let $\mathfrak{k}$ be semi-simple. There exists a positive number ε with the following property. Let w be a unit vector in the positive Weyl chamber $\bar{W}$. If $\langle w,\ \alpha_i\rangle<\varepsilon$, then there is ξ_j such that $\langle w,\ \xi_j\rangle>\langle w,\ \xi_i\rangle$.*

Proot. Set $w=\sum x_k\alpha_k$. Then all x_k are positive and $\langle w,\ \xi_k\rangle=(n_k)^{-1}x_k$. Assume that $\langle w,\ \xi_k\rangle\leq\langle w,\ \xi_i\rangle$ for all k. Then, since $\langle\alpha_j,\ \alpha_i\rangle\leq 0$ for $j\neq i$,

$$\begin{aligned}\langle w,\ \alpha_i\rangle&=\sum_j\langle\alpha_j,\ \alpha_i\rangle x_j\geq\langle\alpha_i,\ \alpha_i\rangle x_i+\sum_{j\neq i}\langle\alpha_j,\ \alpha_i\rangle n_j(n_i)^{-1}x_i\\&=(n_i)^{-1}\sum_j\langle n_j\alpha_j,\ \alpha_i\rangle x_i=(n_i)^{-1}\langle 2\delta,\ \alpha_i\rangle x_i>0\ ,\end{aligned} \tag{2.8}$$

because 2δ belongs to the open positive Weyl chamber W. Thus the conclusion follows from the continuity. Q.E.D.

Proof (of Theorem 2). It is sufficient to prove that $L(A_0(t))$ is bounded. Since $A_0(t)$ is real analytic, $L(A_0(t))$ is continuous and, by Lemma 2.1, piecewisely represented as

$$L(A_0(t))=\langle\mathrm{Ad}_{\gamma(t)}A_0(t),\ \xi_1\rangle\ ,\quad \mathrm{Ad}_{\gamma(t)}A_0(t)\in\bar{W}\ , \tag{2.9}$$

where $\gamma(t)$ is a real analytic curve of K and ξ_1 is taken by renumbering of suffix.

We may assume that $\gamma(t)=1$ at a time $t=t_0$ by changing the Cartan subalgebra $\mathfrak{t}(\mathfrak{k})$ if necessary. We set $A_j=u_j+\sum_\alpha x_j^\alpha X_\alpha$, where $u_j\in\mathfrak{t}(\mathfrak{k})$ and $\alpha\in\Delta(\mathfrak{k})$. At the time $t=t_0$, we see that

$$\frac{d}{dt}L(A_0)=\langle\frac{d}{dt}A_0,\xi_1\rangle+\langle[\frac{d}{dt}\gamma,A_0],\xi_1\rangle=\langle\frac{d}{dt}A_0,\xi_1\rangle. \tag{2.10}$$

Thus assigning (2.4), the last expression

$$\begin{aligned}
&=-\sum_{j\neq 0}\langle[A_0,A_j],[\xi_1,A_j]\rangle-\sum_{j,k\neq 0}C_j{}^k{}_0\langle[\xi_1,A_j],A_k\rangle\\
&=-\sum_{j\neq 0}\langle\sum_{\alpha\in\Delta(\mathfrak{k})}x_j^\alpha\langle\alpha,A_0\rangle X_{-\alpha},\sum_{\alpha\in\Delta(\mathfrak{k})}x_j^\alpha\langle\alpha,\xi_1\rangle X_{-\alpha}\rangle\\
&\quad-\sum_{j,k\neq 0}C_j{}^k{}_0\langle\sum_{\alpha\in\Delta(\mathfrak{k})}x_j^\alpha\langle\alpha,\xi_1\rangle X_{-\alpha},u_k+\sum_{\alpha\in\Delta(\mathfrak{k})}x_k^\alpha X_\alpha\rangle\\
&=-\sum_{\alpha\in\Delta(\mathfrak{k}),\alpha>0}\{\sum_{j\neq 0}\langle\alpha,\xi_1\rangle\langle\alpha,A_0\rangle(x_j^\alpha)^2+\sum_{k\neq 0}\langle\alpha,\xi_1\rangle\langle\alpha,A_0\rangle(x_k^{-\alpha})^2\\
&\qquad+\sum_{j,k\neq 0}(C_j{}^k{}_0-C_k{}^j{}_0)\langle\alpha,\xi_1\rangle x_j^\alpha x_k^{-\alpha}\}\,.
\end{aligned}$$

This summation is taken only for positive roots $\alpha\in\Delta(\mathfrak{k})$ such that $\langle\alpha,\xi_1\rangle\neq 0$. If we represent such α as $\sum m_i\alpha_i$, then $m_1\geq 1$ and all $m_i\geq 0$, and so $\langle\alpha,\xi_1\rangle\geq|\omega_1|^{-1}$. Therefore by Lemma 2.2, it holds that $\langle\alpha,A_0\rangle\geq\varepsilon|A_0|$. In fact, if $|\langle\alpha,A_0\rangle|<\varepsilon|A_0|$, then $|\langle\alpha_1,A_0\rangle|<\varepsilon|A_0|$ and so $\langle A_0,\xi_i\rangle>\langle A_0,\xi_1\rangle$ for some i, which contradicts the maximality of $\langle A_0,\xi_1\rangle$. We regard the last expression as a quadratic form of (x_j^α) and $(x_k^{-\alpha})$, and see that, if $L(A_0)$ is sufficiently large, and so is $|A_0|$, then the coefficients of $(x_j^\alpha)^2$ and $(x_k^{-\alpha})^2$ are sufficiently greater than that of $x_j^\alpha x_k^{-\alpha}$, which implies the non-positivity of the last expression.

Q.E.D.

Remark 2.3. From the boundedness of $A(t)$, we see that any subsequence of $A(t)$ has a subsequence which converges to a Yang-Mills connection. It seems to the author that $A(t)$ itself converges. At least it is clear that if the closure of the set $\{A(t);t\in\boldsymbol{R}^+\}$ contains an isolated Yand-Mills connection, then $A(t)$ converges. Here we mean by isolated to be isolated modulo the action of the normalizer group $N_K(\rho(H))$ of $\rho(H)$ in K.

3. Appendix

First, we consider the relation between equation (2.2) and holomorphic vector bundles. Let M be an algebraic manifold and P a principal $U(r)$-bundle. Take the complexification $GL(r,\boldsymbol{C})$ of the compact Lie group $U(r)$, and complexify P to a principal $GL(r,\boldsymbol{C})$-bundle $P^{\boldsymbol{C}}$. There is a one-to-one correspondence between holomorphic structures $\bar\partial$ of $P^{\boldsymbol{C}}$ and connections ∇ of P whose curvature tensor R^∇ are of type (1,1). Kobayashi shows that if the corresponding connection ∇ to a holomorphic structure $\bar\partial$ is a Yang-Mills connection, then $\bar\partial$ is semi-stable ([4]). Conversely, the following holds. Let $\bar\partial_0$ be a holomorphic structure and ∇_0 the corresponding connection. Heat equation (2.1) with initial data ∇_0 has a unique solution ∇_t, whose curvature tensors are of type (1,1). Let $\bar{\partial}_t$ be the corresponding holomorphic structure. Then all $\bar{\partial}_t$ are conjugate to $\bar{\partial}_0$ under automorphisms of P^C. Moreover if $\bar{\partial}_0$ is stable, then both ∇_t and $\bar{\partial}_t$ converge, and lim $\bar{\partial}_0$ is conjugate to $\bar{\partial}_0$ ([2]). In our homogeneous situation, we get

Corollary 3.1. *Let M and P be as above with homogeneous assumption. Let $\bar{\partial}_0$ be an invariant holomorphic structure. Then the solution $\bar{\partial}_t$ has a convergent subsequence. But if $\bar{\partial}_0$ is not semi-stable, then the limit of $\bar{\partial}_t$ is not conjugate to $\bar{\partial}_0$.*

Next we consider the so called mountain-pass lemma. Let S be a manifold and f a function of S. We say that *the mountain-pass lemma holds* on the pair (S, f) if it has the following property: If there are relative minima $x_1, x_2 \in S$ of f which are not contained in a connected component of the critical point set, then there exists an unstable critical point $x_3 \in S$.

Theorem 3.2. *The mountain-pass lemma holds on $(C_G, \mathcal{F}_{YM} | C_G)$.*

EXAMPLE 3.3. Assume that G is semi-simple and set $H = \{\mathrm{id}\}$, $K = G$ and $\rho = \mathrm{id}$. Then the space C_G is identified with $\mathrm{End}_{\boldsymbol{R}}(\mathfrak{g})$, and $\mathcal{F}_{YM}(A) = 0$ if and only if A is a Lie algebra homomorphism. Therefore $A = 0$ and $A = \mathrm{id}$ are critical points of $\mathcal{F}_{YM} | C_G$, and belong to different connected components. Thus we can conclude, by the mountain-pass lemma, that there exists another unstable Yang-Mills connection in C_G. When the riemannian metric g on G is bi-invariant, it is easy to get such an unstable Yang-Mills connection, say $A = (1/2)\,\mathrm{id}$. However it is not clear to see the existence of such a connection for a general left invariant metric on G without our Theorem.

As in the above example, if the space S is a vector space and if the function f is (by Theorem 1) proper, then the mountain-pass lemma holds by [1, (VI 6.1)]. For general case, i.e., when the fundamental group $\pi_1(M)$ may be infinite, we use the next lemma. Let V be a finite dimensional vetcor space, $\bar{S}$ a closed convex domain of V and f a smooth function on V. A point x in $\bar{S}$ is said to be *critical in* $\bar{S}$ if and only if one of the following conditions is satisfied. (1) x is an interior point of $\bar{S}$ and is critical for f. (2) x is a boundary point of $\bar{S}$ and it holds that $(df)_x(y-x) \geq 0$ for all $y \in \bar{S}$. The following is a finite dimensional version of Struwe's mountain-pass lemma, where the Palais-Smale condition is equivalent to the properness.

Lemma 3.4. ([6, Chapter II Theorem 1.13]). *If the function $f | \bar{S}$ is proper, then the mountain-pass lemma holds replacing critical by critical in $\bar{S}$.*

Proof (of Theorem 3.2). Let $\{v_i;\, 1 \leq i \leq k\}$ be orthonormal basis of $(\mathfrak{m}')^{\perp}$ and $\{v_i;\, k < i \leq n\}$ that of $\mathfrak{m}'$. We regard the vector space $V = \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{k})$ as a subspace of $\mathfrak{k}^n=\{(A_1,\cdots,A_k,A_{k+1},\cdots,A_n)\}$, where $A_i=A(v_i)$. Using the decomposition: $\mathfrak{k}=\mathfrak{k}'\oplus\mathfrak{z}(\mathfrak{k})$ and the function L on $\mathfrak{k}'$ defined by (2.5), we set

$$\bar{S}=\{(A_i)\in V;\ L((A_i)_{\mathfrak{k}'})\leq c \text{ and } |(A_i)_{\mathfrak{z}(\mathfrak{k})}|\leq c \text{ for } 1\leq i\leq k\}\,, \tag{3.1}$$

where c is a sufficiently large constant. Then $\bar{S}$ is a closed convex domain of V and by Propostioon 1.7 the function $\mathcal{F}_{YM}|C_G$ is proper on $\bar{S}$. If $A\in\partial\bar{S}$ is critical in $\bar{S}$, then, by definition, $(d\mathcal{F}_{YM}|C_G)_A(B-A)\geq 0$ for all $B\in\bar{S}$. But Proof of Theorem 2 implies the opposite inequality, provided that c is sufficiently large. Thus $A\in\bar{S}$ is critical in $\bar{S}$ if and only if A is critical in the usual sense, and so the proof reduces to Lemma 3.4. Q.E.D.

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College of General Education
Osaka University
Toyonaka, Osaka 560
Japan