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EXPANSIVE HOMEOMORPHISMS OF COMPACT SURFACES ARE PSEUDO-ANOSOV

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Expansiveness is a very important notion for the investigation of chaotic behaviors in dynamical systems. Let (X, d) be a compact metric space and $f: X \to X$ be a homeomorphism. We say thta f is *expansive* with *expansive constant* $c>0$ if for each pair (x, y) of distinct points of X there is an integer $n \in \boldsymbol{Z}$ such that $d(f^n(x), f^n(y))>c$.

The dynamics we are interested in dealing with are expansive homeomorphisms on compact surfaces. The following is one of the results related to our investigation. Any compact orientable surface with positive genus admits an expansive homeomorphism (T. O'Brien and W. Reddy [18]).

The notion of pseudo-Anosov was introduced by W. Thurston [21], in order to classify diffeomorphisms of compact surfaces up to isotopy. A pseudo-Anosov diffeomorphism is an expansive homeomorphism which is a diffeomorphism except at finitely many points (singular points), and it is an Anosov diffeomorphism if it is on the 2-torus. The notion of pseudo-Anosov can be well defined for homeomorphisms to admit differential structures so that the homeomorphisms become pseudo-Anosov diffeomorphisms (A. Casson and S. Bleiler [1]). Pseudo-Anosov diffeomorphisms have been studied by many people, for example, A. Fathi, F. Landenbach and V. Poénaru [3], M. Gerber and A. Katok [5], M. Gerber [4], J. Lewowicz [12], J. Lewowicz and E. Lima de Sá [14] and so on.

A question arises naturally as to whether compact surfaces admit expansive homeomorphisms which are not pseudo-Anosov homeomorphisms. For the question we shall give an answer as follows.[1)]

Theorem 1. *Every expansive homeomorphism of a compact surface must be pseudo-Anosov.*

This is a result announced in [10]. After this theorem is established, by using Euler-Poincaré's formula and Kneser's Theorem (cf. [3,7]), we can give an answer to a problem (raised by Hedlund) of whether expansive homeomorphisms exist on compact surfaces. The precise statement is as follows (announced in [9]).

1) J. Lewowicz [13] obtained the same result by a different method.

Theorem 2. *There exist no expansive homeomorphisms on the 2-sphere, the projective plane and the Klein bottle.*

As constructed in [18], every compact orientable surface of positive genus admits a pseudo-Anosov diffeomorphism. Recently R. Penner [19] gave examples of pseudo-Anosov diffeomorphisms on compact non-orientable surfaces (for example, the connected sum of two Klein bottles) by generalizing Thurston's construction.

1. Definitions and Preliminaries

Throughout this paper, "surface" will mean a connected, two dimensional, C^∞ Riemannian manifold without boundary and a compact surface will be denoted by M. The natural numbers, the real numbers and the complex numbers will be denoted by $\boldsymbol{N}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ respectively.

For $p \in \boldsymbol{N}$ let $\pi_p: \boldsymbol{C} \to \boldsymbol{C}$ be the map which sends z to z^p. We define domains $\mathcal{D}_p$ $(p=1, 2, \cdots)$ of $\boldsymbol{C}$ by

$$\mathcal{D}_2 = \{z \in \boldsymbol{C}\colon |\mathrm{Re}\, z| < 1,\ |\mathrm{Im}\, z| < 1\}\,,$$
$$\mathcal{D}_1 = \pi_2(\mathcal{D}_2) \quad \text{and} \quad \mathcal{D}_p = \pi_p^{-1}(\mathcal{D}_1)\,.$$

It is easily checked that $\pi_p\colon \mathcal{D}_p \to \mathcal{D}_1$ is a p-fold branched cover for every $p \in \boldsymbol{N}$. Denote by $\mathcal{H}_2$ and $\mathcal{V}_2$ the horizontal and vertical foliations on $\mathcal{D}_2$ respectively. We define a decomposition $\mathcal{H}_1$ (resp. $\mathcal{V}_1$) of $\mathcal{D}_1$ as the projection of $\mathcal{H}_2$ (resp. $\mathcal{V}_2$) by $\pi_2\colon \mathcal{D}_2 \to \mathcal{D}_1$, and define a decomposition $\mathcal{H}_p$ (resp. $\mathcal{V}_p$) of $\mathcal{D}_p$ as the lifting of $\mathcal{H}_1$ (resp. $\mathcal{V}_1$) by $\pi_p\colon \mathcal{D}_p \to \mathcal{D}_1$.

A decomposition $\mathcal{F}$ of M is called a C^0 *singular foliation* if every $L \in \mathcal{F}$ is path connected and if for every $x \in M$ there are $p(x) \in \boldsymbol{N}$ and a C^0 chart φ_x: $U_x \to \boldsymbol{C}$ around x such that

(1) $\varphi_x(x) = 0$,

(2) $\varphi_x(U_x) = \mathcal{D}_{p(x)}$,

(3) φ_x sends each connected component of $U_x \cap L$ onto some element of $\mathcal{H}_{p(x)}$ unless $U_x \cap L = \phi$ for $L \in \mathcal{F}$.

Let $\mathcal{F}$ be a C^0 singular foliation on M. Each element of $\mathcal{F}$ is called a *leaf* and equipped with the *leaf topology*. The number $p(x)$ is called the *number of separatrices* at x. We say that x is a *regular point* if $p(x)=2$, and x is a *singular point* with *$p(x)$-separatrices* if $p(x) \neq 2$. Since M is compact, obviously the set S of all singular points is finite. We denote by $\mathcal{RF}$ the C^0 foliation on $M \backslash S$ obtained by taking singular points away from each leaf of $\mathcal{F}$. For materials of C^0 foliations on surfaces, the reader may refer to G. Hector and U. Hirsch [7]. If every leaf of $\mathcal{RF}$ is dense in M, then $\mathcal{F}$ is called *minimal*. We say that $\mathcal{F}$ is *orientable* (resp. *transversally orientable*) if $\mathcal{RF}$ is orientable (resp. transversally orientable).

A subset A of M is an *arc* (resp. *open arc*) if there is a C^0 embedding h from a compact (resp. open) interval I of $\boldsymbol{R}$ into M such that $h(I)=A$. Let $\mathcal{F}$ and S be as above. An arc A is a called a *transversal* of $\mathcal{F}$ if the interior of A is contained in $M\backslash S$ and if for every $x\in A\backslash S$ there is a C^0 chart φ_x: $U_x\to\boldsymbol{C}$ around x as above such that $P_r\circ\varphi_x$ is injective on $U_x\cap A$ where P_r denotes the projection from $\boldsymbol{C}$ onto the imaginary axis.

Let A_0 and A_1 be transversals of $\mathcal{F}$. We say $A_0\simeq A_1$ if there is a continuous map H: $[0, 1]\times[0, 1]\to M$ such that $H_0=H|_{[0,1]\times\{0\}}$ and $H_1=H|_{[0,1]\times\{1\}}$ are homeomorphisms from $[0, 1]\times\{0\}$ onto A_0 and from $[0, 1]\times\{1\}$ onto A_1 respectively, and such that if $L\in\mathcal{F}$ then $H^{-1}(L)=B\times[0, 1]$ for some $B\subset[0, 1]$. Let h: $[0, 1]\times\{0\}\to[0, 1]\times\{1\}$ be the homeomorphism which sends $(t, 0)$ to $(t, 1)$. When $A_0\simeq A_1$, the homeomorphism $H_1\circ h\circ H_0^{-1}$: $A_0\to A_1$ is called a *projection along the leaves*.

A *transverse invariant measure* μ for $\mathcal{F}$ is a collection $\{\mu_A$: A is a transversal$\}$ of finite Borel measures on all transversals of $\mathcal{F}$ such that $\mu_A|_{A'}=\mu_{A'}$ if $A'\subset A$ and such that $\mu_{A_1}\circ h=\mu_{A_0}$ if h: $A_0\to A_1$ is a projection along the leaves. A *measured* C^0 *foliation* $(\mathcal{F}, \mu)$ is a C^0 singular foliation $\mathcal{F}$ equipped with a transverse invariant measure μ.

We denote by $\mathcal{M}(\mathcal{F})$ the set of all transverse invariant measures for $\mathcal{F}$. For $\{\mu_A\}$, $\{\nu_A\}\in\mathcal{M}(\mathcal{F})$ and $a\geq 0$, we write $\{\mu_A\}+\{\nu_A\}=\{\mu_A+\nu_A\}$ and $a\{\mu_A\}=\{a\mu_A\}$. Then $\mathcal{M}(\mathcal{F})$ is closed with respect to these operations. Let f: $M\to M$ be a homeomorphism. Then f sends $\mathcal{F}$ to a C^0 singular foliation $\mathcal{F}'$. If A' is a transversal of $\mathcal{F}'$ then $f^{-1}(A')$ is a transversal of $\mathcal{F}$. Hence we can define a map f_*: $\mathcal{M}(\mathcal{F})\to\mathcal{M}(\mathcal{F}')$ by $f_*(\{\mu_A\})=\{\mu_A\circ f^{-1}\}$. Clearly $f_*(a\mu+b\nu)=af_*(\mu)+bf_*(\nu)$ for μ, $\nu\in\mathcal{M}(\mathcal{F})$ and a, $b\geq 0$.

When f sends $\mathcal{F}$ to $\mathcal{F}'$ $(f(\mathcal{F})=\mathcal{F}')$ and $f_*(\mu)=\mu'$, we write $f(\mathcal{F}, \mu)=(\mathcal{F}', \mu')$.

Let $\mathcal{F}$ and $\mathcal{F}'$ be C^0 singular foliations on M. We say that $\mathcal{F}$ is *transverse* to $\mathcal{F}'$ if $\mathcal{F}$ and $\mathcal{F}'$ have the same number $p(x)$ of separatrices at all $x\in M$ and if every $x\in M$ has a C^0 chart φ_x: $U_x\to\boldsymbol{C}$ such that

(1) $\varphi_x(x)=0$,

(2) $\varphi_x(U_x)=\mathcal{D}_{p(x)}$,

(3) φ_x sends each connected component of $U_x\cap L$ onto some element of $\mathcal{H}_{p(x)}$ unless $U_x\cap L=\phi$ for $L\in\mathcal{F}$,

(4) φ_x sends each connected component of $U_x\cap L'$ onto some element of $\mathcal{V}_{p(x)}$ unless $U_x\cap L'=\phi$ for $L'\in\mathcal{F}'$.

Let $\mathcal{F}$ and $\mathcal{F}'$ be transverse C^0 singular foliations on M, and let S be the set of all singular points. If A is an arc in a leaf of $\mathcal{F}$ (resp. $\mathcal{F}'$) and the interior of A is contained in $M\backslash S$, then it is easily checked that A is a transversal of $\mathcal{F}'$ (resp. $\mathcal{F}$).

A homeomorphism f of M is called *pseudo-Anosov* if there are a constant

$\lambda>1$ and a pair $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$ of transverse measured C^0 foliations with the number of separatrices at each singular point greater than 2 and with every finite Borel measure of μ^s and of μ^u non-atomic and positive on all non-empty open sets such that

$$f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s), \quad f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u).$$

(This means that f preserves the transverse C^0 singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$; it contracts all arcs in the leaves of $\mathcal{F}^s$ by λ^{-1} and it expands all arcs in the leaves of $\mathcal{F}^u$ by λ).

It is no difficult to check that every pseudo-Anosov homeomorphism is expanvive.

Let f be a homeomorphism of a compact metric space (X, d). For $x \in X$ we define the *stable set* $W^s(x)$ and the *unstable set* $W^u(x)$ by

$$W^s(x) = \{y \in X: d(f^n(x), f^n(y)) \to 0 \quad \text{as} \quad n \to \infty\},$$
$$W^u(x) = \{y \in X: d(f^n(x), f^n(y)) \to 0 \quad \text{as} \quad n \to -\infty\}$$

and put

$$\mathcal{F}_f^\sigma = \{W^\sigma(x): x \in X\} \quad (\sigma = s, u).$$

Then $\mathcal{F}_f^\sigma$ is a decomposition of X and $f(\mathcal{F}_f^\sigma) = \mathcal{F}_f^\sigma$. If X is a compact surface and f is pseudo-Anosov, then it is easily checked that every leaf L of the associate C^0 singular foliation $\mathcal{F}^\sigma$ coincides with $W^\sigma(x)$ for all $x \in L$, that is, $\mathcal{F}^\sigma = \mathcal{F}_f^\sigma$.

For the proof of Theorem 1 we prepare the following

Proposition A. *Let $f\colon M \to M$ be an expansive homeomorphism. Then $\mathcal{F}_f^\sigma$ $(\sigma = s, u)$ have the following properties;*

(1) *$\mathcal{F}_f^\sigma$ is a C^0 singular foliation,*

(2) *every leaf $W^\sigma(x) \in \mathcal{F}_f^\sigma$ is homeomorphic to $L_p = \{z \in \boldsymbol{C}\colon \operatorname{Im}(z^{p/2}) = 0\}$ for some $p \geq 2$,*

(3) *$\mathcal{F}_f^s$ is transverse to $\mathcal{F}_f^u$,*

(4) *$\mathcal{F}_f^\sigma$ is minimal.*

If Proposition A is established, then the transverse invariant measures μ^σ for $\mathcal{F}_f^\sigma$ $(\sigma = s, u)$ and the stretching factor $\lambda > 1$ of f are obtained from the following proposition. These facts prove Theorem 1.

Proposition B. *Let $f\colon M \to M$ be a homeomorphism and let $\mathcal{F}^s$ and $\mathcal{F}^u$ be transverse C^0 singular foliations on M. If $f(\mathcal{F}^\sigma) = \mathcal{F}^\sigma$ and $\mathcal{F}^\sigma$ is minimal for $\sigma = s, u$, then there are a constant $\lambda > 0$ and transverse invariant measures μ^σ for $\mathcal{F}^\sigma$ $(\sigma = s, u)$ with every finite Borel measure of μ^σ non-atomic and positive on all non-empty open sets such that $f_*(\mu^s) = \lambda^{-1}\mu^s$ and $f_*(\mu^u) = \lambda\mu^u$.*

As above let X be a compact metric space. For $x\in X$ and $\varepsilon>0$ we put

$$B_\varepsilon(x)=\{y\in X: d(x,y)\leq\varepsilon\}\ ,$$
$$U_\varepsilon(x)=\{y\in X: d(x,y)<\varepsilon\}\ ,$$
$$S_\varepsilon(x)=\{y\in X: d(x,y)=\varepsilon\}\ .$$

If in particular X is a compact surface, for $\varepsilon>0$ small enough $B_\varepsilon(x)$, $U_\varepsilon(x)$ and $S_\varepsilon(x)$ are a disk, an open disk and a circle respectively. In the case when X is generally connected and locally connected, by using Theorem 2.4 of [6, p. 95] we may assume that $B_\varepsilon(x)$ is connected for all $x\in X$ and $\varepsilon>0$.

Let $f\colon X\to X$ be a homeomorphism. For $x\in X$ and $\varepsilon>0$ we define the *local stable set* $W^s_\varepsilon(x)$ and the *local unstable set* $W^u_\varepsilon(x)$ by

$$W^s_\varepsilon(x)=\{y\in X: d(f^n(x),f^n(y))\leq\varepsilon,\ n\geq 0\}\ ,$$
$$W^u_\varepsilon(x)=\{y\in X: d(f^n(x),f^n(y))\leq\varepsilon,\ n\leq 0\}\ .$$

Obviously $W^\sigma_\varepsilon(x)$ is a closed subset of X for $\sigma=s,u$.

Let $f\colon X\to X$ be expansive with expansive constant $c>0$. Then it is checked that for every $\varepsilon>0$ there is $N>0$ such that

$$f^n\,W^s_c(x)\subset W^s_\varepsilon(f^n(x))\,,\quad f^{-n}\,W^u_c(x)\subset W^u_\varepsilon(f^{-n}(x)) \tag{1.1}$$

for all $n\geq N$ and all $x\in X$ (see R. Mañé [15]). Hence

$$W^s(x)=\bigcup_{n\geq 0}f^{-n}\,W^s_\varepsilon(f^n(x))\,,\quad W^u(x)=\bigcup_{n\geq 0}f^n\,W^u_\varepsilon(f^{-n}(x)) \tag{1.2}$$

for all $x\in X$ and all $0<\varepsilon\leq c$.

For the proof of Proposition A, we will need to investigate the topological structures of $W^\sigma_\varepsilon(x)$ ($\sigma=s,u$). To do this, we require that $W^\sigma_\varepsilon(x)$ is connected. It is difficult to directly verify, however, whether $W^\sigma_\varepsilon(x)$ is connected even if X is a compact surface, and so we restrict our attention to the connected component of x in $W^\sigma_\varepsilon(x)$, which is denoted by $C^\sigma(x)$.

The following proposition will play an important role in the proof of Proposition A.

Proposition C. *Let $f\colon X\to X$ be an expansive homeomorphism. If X is non-trivial, connected and locally connected, then for every $\varepsilon>0$ there is $\delta>0$ such that for all $x\in X$*

$$S_\delta(x)\cap C^\sigma_\varepsilon(x)\neq\phi\quad(\sigma=s,u)\,.$$

2. Proof of Proposition C

Before we start the proof of Proposition C, we prepare several lemmas.

Let (X,d) be a compact metric space as before and denote by $\mathcal{C}(X)$ the set of all non-empty closed subsets of X.

Lemma 2.1 ([2, p. 439]). *If X is connected and $A \in \mathcal{C}(X)$ with $A \neq X$, then every connected component of A intersects the boundary of A in at least one point.*

The Hausdorff metric for $\mathcal{C}(X)$ is defined by

$$H(A, B) = \inf\{\varepsilon>0: N_\varepsilon(A) \supset B,\ N_\varepsilon(B) \supset A\} \quad (A, B \in \mathcal{C}(X))$$

where $N_\varepsilon(A)$ denotes the ε-neighborhood of A in X. The following result is well known.

Lemma 2.2 ([11, p. 45]). *$\mathcal{C}(X)$ is a compact space under H.*

As before let $f: X \to X$ be a homeomorphism and $W^\sigma(x)$ $(\sigma=s, u)$ be defined for f.

Lemma 2.3. *Let $\varepsilon>0$ be arbitrary. Suppose that a sequence $\{x_i\}_{i\in N}$ of X converges to $x_\infty \in X$, and that a sequence $\{B_i\}_{i\in N}$ of $\mathcal{C}(X)$ converges to $B_\infty \in \mathcal{C}(X)$. If $B_i \subset W^\sigma_\varepsilon(x_i)$ for all $i \in \boldsymbol{N}$ $(\sigma=s, u)$, then $B_\infty \subset W^\sigma_\varepsilon(x_\infty)$.*

Proof. We give the proof for $\sigma=s$. Let $z \in B_\infty$. Since $B_i \to B_\infty$, there is a sequence $\{y_i\}_{i\in N}$ with $y_i \in B_i$ for all $i \in \boldsymbol{N}$ such that $y_i \to z$ as $i \to \infty$. Since $B_i \subset W^s_\varepsilon(x_i)$, we have that $d(f^n(x_i), f^n(y_i)) \le \varepsilon$ for all $n \ge 0$. Since $x_i \to x$ and $y_i \to z$, it follows that $d(f^n(x_\infty), f^n(z)) \le \varepsilon$ for all $n \ge 0$. This means that $z \in W^s_\varepsilon(x_\infty)$, and therefore $B_\infty \subset W^s_\varepsilon(x_\infty)$. The conclusion for $\sigma=u$ is also obtained.

The above lemma is generalized as follows.

Lemma 2.4. *Let $\{x_i\}_{i\in N}$, x_∞, $\{B_i\}_{i\in N}$ and B_∞ be as in Lemma 2.3. Then the following hold;*

(1) *if $f^n(B_i) \subset B_\varepsilon(f^n(x_i))$ for all $0 \le n \le i$ and all $i \in \boldsymbol{N}$, then $B_\infty \subset W^s_\varepsilon(x_\infty)$,*

(2) *if $f^{-n}(B_i) \subset B_\varepsilon(f^{-n}(x_i))$ for all $0 \le n \le i$ and all $i \in \boldsymbol{N}$, then $B_\infty \subset W^u_\varepsilon(x_\infty)$.*

Proof. This is very similar to the proof of Lemma 2.3 and so we omit the proof.

Hereafter we assume that $f: X \to X$ be expansive with expansive constant $c>0$.

Lemma 2.5 ([14]). *Suppose that $0<\varepsilon\le c/2$. Then there exists $0<\delta\le\varepsilon$ such that*

(1) *if $d(x, y) \le \delta$ and $\varepsilon \le \max\{d(f^i(x), f^i(y)): 0 \le i \le n\} \le 2\varepsilon$, then $d(f^n(x), f^n(y)) \ge \delta$,*

(2) *if $d(x, y) \le \delta$ and $\varepsilon \le \max\{d(f^i(x), f^i(y)): -n \le i \le 0\} \le 2\varepsilon$, then $d(f^{-n}(x), f^{-n}(y)) \ge \delta$.*

The following is easily obtained from Lemma 2.5.

Lemma 2.6. *For $0<\varepsilon\le c/2$, let $0<\delta\le\varepsilon$ be as in Lemma 2.5. Suppose* *that A is a connected subset of X and that $x\in A$. Then the following hold;*

(1) *if $A\subset B_\delta(x)$, $f^i(A)\cap S_\varepsilon(f^i(x))\neq\phi$ for some $0\leq i\leq n$ and $f^i(A)\subset B_{2\varepsilon}(f^i(x))$ for all $0\leq i\leq n$, then $f^n(A)\cap S_\delta(f^n(x))\neq\phi$,*

(2) *if $A\subset B_\delta(x)$, $f^i(A)\cap S_\varepsilon(f^i(x))\neq\phi$ for some $-n\leq i\leq 0$ and $f^i(A)\subset B_{2\varepsilon}(f^i(x))$ for all $-n\leq i\leq 0$, then $f^{-n}(A)\cap S_\delta(f^n(x))\neq\phi$.*

Lemma 2.7. *For $0<\varepsilon\leq c/2$, let $0<\delta\leq\varepsilon$ be as in Lemma 2.5. Let $\{x_i\}_{i\in\boldsymbol{Z}}$ be a sequence of X and let $\Delta(x_i)$ denote the connected component of x_i in $B_\delta(x_i)\cap f^{-i}B_{\delta/2}(f^i(x_i))$ for all $i\in\boldsymbol{Z}$. Then the following hold;*

(1) *if for a sequence $\{j\}$ of $\boldsymbol{Z}$ with $j\to\infty$*

$$\lim_{j\to\infty} x_j = x_\infty \quad \text{and} \quad \lim_{j\to\infty}\Delta(x_j) = \Delta_\infty\,,$$

then $\Delta_\infty\subset W^s_\varepsilon(x_\infty)$,

(2) *if for a sequence $\{j\}$ of $\boldsymbol{Z}$ with $j\to-\infty$*

$$\lim_{j\to-\infty} x_j = x_{-\infty} \quad \text{and} \quad \lim_{j\to-\infty}\Delta(x_j) = \Delta_{-\infty}\,,$$

then $\Delta_{-\infty}\subset W^u_\varepsilon(x_{-\infty})$.

Proof. First we prove (1). Since $\Delta(x_j)\subset B_\delta(x_j)$, we have $\Delta_\infty\subset B_\delta(x_\infty)$, and hence $\Delta_\infty\subset B_\varepsilon(x_\infty)$. To obtain (1), assume that $\Delta_\infty\not\subset W^s_\varepsilon(x_\infty)$. Then by the definition of $W^s_\varepsilon(x_\infty)$ there is $k_0>0$ such that $f^{k_0}(\Delta_\infty)\not\subset B_\varepsilon(f^{k_0}(x_\infty))$. Take $\varepsilon<\lambda\leq 2\varepsilon$ such that $f^{k_0}(\Delta_\infty)\not\subset B_\lambda(f^{k_0}(x_\infty))$. Since $\Delta_\infty\subset B_\varepsilon(x_\infty)$, there is $0<k_1\leq k_0$ such that $f^i(\Delta_\infty)\subset U_\lambda(f^i(x_\infty))$ for all $0\leq i\leq k_1-1$ and $f^{k_1}(\Delta_\infty)\not\subset U_\lambda(f^{k_1}(x_\infty))$. Since $x_j\to x_\infty$ and $\Delta(x_j)\to\Delta_\infty$, we can find $l>k_1$ such that $f^i(\Delta(x_l))\subset B_\lambda(f^i(x_l))$ for all $0\leq i\leq k_1-1$ and $f^{k_1}(\Delta(x_l))\not\subset B_\varepsilon(f^{k_1}(x_l))$.

Let A_{k_1} denote the connected component of x_l in

$$f^{-k_1}[f^{k_1}(\Delta(x_l))\cap B_\varepsilon(f^{k_1}(x_l))]\,.$$

Then we have

$$f^i(A_{k_1})\subset B_\lambda(f^i(x_l)) \quad (0\leq i\leq k_1)\,. \tag{2.1}$$

Since $f^{k_1}(\Delta(x_l))$ is connected and $f^{k_1}(\Delta(x_l))\not\subset B_\varepsilon(f^{k_1}(x_l))$, it follows from Lemma 2.1 that

$$f^{k_1}(A_{k_1})\cap S_\varepsilon(f^{k_1}(x_l))\neq\emptyset\,. \tag{2.2}$$

For $k>k_1$ define A_k as the connected component of x_l in $f^{-k}[f^k(A_{k-1})\cap B_\varepsilon(f^k(x_l))]$. Then

$$\Delta(x_l)\supset A_{k_1}\supset A_{k_1+1}\supset\cdots\supset A_k\supset\cdots$$

and by (2.1) it is easily checked that

$$f^i(A_k)\subset B_\lambda(f^i(x_l)) \quad (0\leq i\leq k)\,. \tag{2.3}$$

Now we claim that $f^k(A_k)\cap S_\delta(f^k(x_l))\neq\emptyset$ for $k>k_1$. Indeed, if $A_k\neq A_{k-1}$, then $f^k(A_{k-1})\not\subset B_\varepsilon(f^k(x_l))$, and hence $f^k(A_k)\cap S_\varepsilon(f^k(x_l))\neq\emptyset$ (see Lemma 2.1). Since $0<\delta\leq\varepsilon$, we have $f^k(A_k)\cap S_\delta(f^k(x_l))\neq\emptyset$. For the case when $A_k=A_{k-1}$, put $i_0=\min\{i\colon A_i=A_k\}$. Clearly $k_1\leq i_0<k$. If $i_0=k_1$, then $f^{i_0}(A_k)\cap S_\varepsilon(f^{i_0}(x_l))\neq\emptyset$ by (2.2). If $i_0>k_1$, then $A_{i_0}\neq A_{i_0-1}$, and hence $f^{i_0}(A_k)\cap S_\varepsilon(f^{i_0}(x_k))\neq\emptyset$. In any case, $f^{i_0}(A_k)\cap S_\varepsilon(f^{i_0}(x_l))\neq\emptyset$. Since $\Delta(x_l)\supset A_k$, it is clear that $A_k\subset B_\delta(x_l)$. Combining these facts and (2.3), by Lemma 2.6 (1) we obtain $f^k(A_k)\cap S_\delta(f^k(x_l))\neq\emptyset$. Therefore the above claim holds.

Since $l>k_1$, consequently $f^l(A_l)\cap S_\delta(f^l(x_l))\neq\emptyset$, which contradicts $A_l\subset\Delta(x_l)$. Therefore (1) holds. In the same way, we obtain (2).

Lemma 2.8. *If X is non-trivial, connected and locally connected, then for all $0<\varepsilon\leq c/2$ and all $x\in X$*

$$\operatorname{int} W^\sigma_\varepsilon(x)=\emptyset\quad(\sigma=s,u)$$

where $\operatorname{int} W^\sigma_\varepsilon(x)$ *denotes the interior of* $W^\sigma_\varepsilon(x)$ *in* X.

Proof. If the proof is given for $\sigma=u$, then the conclusion for $\sigma=s$ is obtained in the same way. Thus we give the proof only for $\sigma=u$. Fix $0<\varepsilon\leq c/2$ and $x\in X$. Let $0<\delta\leq\varepsilon$ be as in Lemma 2.5. To show the case of $\sigma=u$, assuming that $y\in\operatorname{int} W^u_\varepsilon(x)\neq\emptyset$, we can take $0<\gamma\leq\delta$ such that $B_{2\gamma}(y)\subset\operatorname{int} W^u_\varepsilon(x)$. Then we claim that for every $0<\eta\leq\gamma$ there is $n>0$ such that $f^n B_\eta(z)\supset B_{\delta/2}(f^n(z))$ for all $z\subset B_\gamma(y)$. If this is established, then we can derive a contradiction as follows. Since X is non-trivial and connected, we see easily that for $k>0$ there are $0<\eta\leq\gamma$ and $p_i\subset B_\gamma(y)$ $(i=1,2,\cdots,k)$ such that $B_\eta(p_i)\cap B_\eta(p_j)=\emptyset$ for $i\neq j$. The claim ensures the existence of $n>0$ such that $f^n B_\eta(p_i)\supset B_{\delta/2}(f^n(p_i))$ for $i=1,2,\cdots,k$. Hence $B_{\delta/2}(f^n(p_i))\cap B_{\delta/2}(f^n(p_j))=\emptyset$ for $i\neq j$, which means that X contains mutually disjoint k balls with radius $\delta/2$. Since k is arbitrary, this contradicts that X is compact.

To conclude the lemma, it only remains to prove the above claim. Assume that the claim does not hold. Then we can take $0<\eta\leq\gamma$ such that for every $n>0$ there is $z_n\in B_\gamma(y)$ such that $f^n B_\eta(z_n)\not\supset B_{\delta/2}(f^n(z_n))$. Let $\Delta(z_n)$ denote the connected component of z_n in $B_\delta(z_n)\cap f^{-n}B_{\delta/2}(f^n(z_n))$. Since $\eta\geq\delta$ and $B_{\delta/2}(f^n(z_n))$ is connected, by using Lemma 2.1 we can check easily that $\Delta(z_n)\cap S_\eta(z_n)\neq\emptyset$. By Lemma 2.2 there is a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j}\to z_\infty\in B_\gamma(y)$ and $\Delta(z_{n_j})\to\Delta_\infty\in\mathcal{C}(X)$ as $n_j\to\infty$. Then $\Delta_\infty\cap S_\eta(z_\infty)\neq\emptyset$.

On the other hand, $\Delta_\infty\subset W^s_\varepsilon(z_\infty)$ by Lemma 2.7 (1). Since $0<\eta\leq\gamma$ and $z_\infty\in B_\gamma(y)$ and since $B_{2\gamma}(y)\subset W^u_\varepsilon(x)$, we have that $B_\eta(z_\infty)\subset W^u_\varepsilon(x)$ and hence $W^s_\varepsilon(z_\infty)\cap W^u_\varepsilon(x)\supset\Delta_\infty\cap B_\eta(z_\infty)$. Since $0<\varepsilon\leq c/2$, by expansiveness $W^s_\varepsilon(z_\infty)\cap W^u_\varepsilon(x)=\{z_\infty\}$, and hence $\Delta_\infty\cap B_\eta(z_\infty)=\{z_\infty\}$. This contradicts that $\Delta_\infty\cap S_\eta(z_\infty)\neq\emptyset$. Therefore our claim holds.

Proof of Proposition C. Since $C^\sigma_\varepsilon(x)\subset C^\sigma_{\varepsilon'}(x)$ for $0<\varepsilon<\varepsilon'$, it is sufficient to give the proof for $0<\varepsilon\leq c/4$. Let $0<\delta\leq\varepsilon$ be as in Lemma 2.5. We prove the case of $\sigma=s$. To do this, fix $x\in X$ and put $x(i)=f^i(x)$ for $i\geq 0$. Then there is a subsequence $\{j\}$ of $\{i\}$ such that $x(j)$ converges to some $x_\infty\in X$ as $j\to\infty$. Since int $W^u_{2\varepsilon}(x_\infty)=\emptyset$ by Lemma 2.8, for $0<\eta\leq\delta$ we can take $m_\eta>0$ such that $f^{-m_\eta}B_{\eta/2}(x_\infty)\not\subset B_{2\varepsilon}(f^{-m_\eta}(x_\infty))$. Then $m_\eta\to\infty$ as $\eta\to 0$. Choose $j_\eta\geq m_\eta$ with $d(x(j_\eta), x_\infty)\leq\eta/2$. Then the diameter of $f^{-m_\eta}B_\eta(x(j_\eta))$ is greater than 2ε. Hence there is $0<n_\eta\leq j_\eta$ such that $f^{-i}B_\eta(x(j_\eta))\subset B_\varepsilon(x(j_\eta-i))$ for all $0\leq i\leq n_\eta-1$ and $f^{-n_\eta}B_\eta(x(j_\eta))\not\subset B_\varepsilon(x(j_\eta-n_\eta))$.

For $0\leq k\leq i$, let $\Delta_k(x(i-k))$ denote the connected component of $x(i-k)$ in

$$B_\varepsilon(x(i-k))\cap f^{-1}B_\varepsilon(x(i-k+1))\cap\cdots\cap f^{-k+1}B_\varepsilon(x(i-1))\cap f^{-k}B_\delta(x(i))\,.$$

By the choice of n_η we ee easily that $\Delta_{n_\eta}(x(j_\eta-n_\eta))$ contains the connected component $C(x(j_\eta-n_\eta))$ of $x(j_\eta-n_\eta)$ in $B_\varepsilon(x(j_\eta-n_\eta))\cap f^{-n_\eta}B_\eta(x(j_\eta))$. Since $B_\eta(x(j_\eta))$ is connected and $f^{-n_\eta}B_\eta(x(j_\eta))\not\subset B_\varepsilon(x(j_\eta-n_\eta))$, we have by Lemma 2.1 that $C(x(j_\eta-n_\eta))\cap S_\varepsilon(x(j_\eta-n_\eta))\neq\emptyset$, and therefore $\Delta(0)\cap S_\varepsilon(x(j_\eta-n_\eta))\neq\emptyset$ where $\Delta(0)=\Delta_{n_\eta}(x(j_\eta-n_\eta))$.

For $k>0$ define $\Delta(k)$ as the connected component of $x(j_\eta-n_\eta-k)$ in $f^{-1}(\Delta(k-1))\cap B_\varepsilon(x(j_\eta-n_\eta-k))$. Then it is easily checked that

$$f^i(\Delta(j_\eta-n_\eta))\subset B_\varepsilon(x(i))\quad(0\leq i\leq j_\eta-1)\,,\tag{2.4}$$

$$f^{j_\eta}(\Delta(j_\eta-n_\eta))\subset B_\delta(x(j_\eta))\,.\tag{2.5}$$

We claim that $f^i(\Delta(j_\eta-n_\eta))\cap S_\varepsilon(x(i))\neq\emptyset$ for some $0\leq i\leq j_\eta-n_\eta$. Indeed, let $f^i(\Delta(j_\eta-n_\eta))\cap S_\varepsilon(x(i))=\emptyset$ for all $0\leq i\leq j_\eta-n_\eta$. Since $\Delta(j_\eta-n_\eta)$ is the connected component of $x(0)$ in $f^{-1}(\Delta(j_\eta-n_\eta-1))\cap B_\varepsilon(x(0))$, by using Lemma 2.1 we have that $\Delta(j_\eta-n_\eta)=f^{-1}(\Delta(j_\eta-n_\eta-1))$. Hence $f(\Delta(j_\eta-n_\eta))=\Delta(j_\eta-n_\eta-1)$ and by induction $f^i(\Delta(j_\eta-n_\eta))=\Delta(j_\eta-n_\eta-i)$ for all $0\leq i\leq j_\eta-n_\eta$. Hence $f^{j_\eta-n_\eta}(\Delta(j_\eta-n_\eta))=\Delta(0)$, contradicting $\Delta(0)\cap S_\varepsilon(x(j_\eta-n_\eta))\neq\emptyset$. Therefore the claim holds.

Combining this claim, (2.4) and (2.5), it follows from Lemma 2.6 (2) that $\Delta(j_\eta-n_\eta)\cap S_\delta(x)\neq\emptyset$. Since $\Delta(j_\eta-n_\eta)\subset\Delta_{j_\eta}(x(0))=\Delta_{j_\eta}(x)$ by (2.4) and (2.5), consequently $\Delta_{j_\eta}(x)\cap S_\delta(x)\neq\emptyset$.

Since $j_\eta\to\infty$ as $\eta\to 0$, by Lemma 2.2 we can take a subsequence $\{j'_\eta\}$ of $\{j_\eta\}$ such that $\Delta_{j'_\eta}(x)$ converges to some $\Delta_\infty\in\mathcal{C}(X)$ as $j'_\eta\to\infty$. Then $\Delta_\infty\cap S_\delta(x)\neq\emptyset$ by the above result and Δ_∞ is connected because so is $\Delta_{j'_\eta}(x)$. By the definition of $\Delta_{j'_\eta}(x)$, $f^i(\Delta_{j'_\eta}(x))\subset B_\varepsilon(f^i(x))$ for all $0\leq i\leq j'_\eta$, and hence $\Delta_\infty\subset W^s_\varepsilon(x)$ by Lemma 2.4 (1). Hence $\Delta_\infty\subset C^s(x)$, and therefore $C^s_\varepsilon(x)\cap S_\delta(x)\neq\emptyset$.

3. Local connectedness of $C^\sigma_\varepsilon(x)$.

The aim of this section is to prove the following

Proposition 3.1. *Let $f\colon X\to X$ be an expansive homeomorphism with expan-*

sive constant $c>0$. *If* X *is a compact surface, then* $C_\varepsilon^\sigma(x)$ $(\sigma=s, u)$ *are locally connected for all* $x\in X$ *and all* $0<\varepsilon\le c/2$.

Proof. Fix $x\in X$ and $0<\varepsilon\le c/2$. Let $\delta>0$ be as in Proposition C. To obtain the conclusion for $\sigma=s$, assume that $C_\varepsilon^s(x)$ is not locally connected. Then we can take $y\in C_\varepsilon^s(x)$ and $\gamma>0$ small enough with $\gamma\le\delta/2$ such that the connected component of y in $C_\varepsilon^s(x)\cap B_\gamma(y)$ does not contain $C_\varepsilon^s(x)\cap B_\lambda(y)$ for all $\lambda>0$. Denote by $\mathcal{K}$ the set of all connected components of $C_\varepsilon^s(x)\cap B_\gamma(y)$. Since $C_\varepsilon^s(x)$ is connected, it follows from Lemma 2.1 that $K\cap S_\gamma(y)\neq\emptyset$ for all $K\in\mathcal{K}$.

Fix $0<t<\gamma$ and put $\mathcal{S}=\{K\in\mathcal{K}: K\cap B_t(y)\neq\phi\}$. Then by the choice of y and γ it is easily checked that $\mathcal{S}$ is an infinite set. Hence there is a sequence $\{K_i\}_{i\in N}$ of $\mathcal{S}$ with $K_i\cap K_j=\emptyset$ for $i\neq j$ such that K_i converges to some $K_\infty\in\mathcal{C}(C_\varepsilon^s(x)\cap B_\gamma(y))$ as $i\to\infty$ (Lemma 2.2). Since each K_i is connected, so is K_∞. Hence K_∞ is contained in a connected component of $C_\varepsilon^s(x)\cap B_\gamma(y)$. Therefore we may assume that $K_i\cap K_\infty=\emptyset$ for all $i\in \boldsymbol{N}$.

Since X is a compact surface and γ is small enough, $T=B_\gamma(y)\setminus U_t(y)$ is an annulus bounded by circles $S_\gamma(y)$ and $S_t(y)$. Since $K_i\cap S_\gamma(y)\neq\emptyset$, we take $a_i\in K_i\cap S_\gamma(y)$. Denote by L_i the connected component of a_i in $T\cap K_i$. Since K_i is connected and $K_i\cap B_t(y)\neq\emptyset$, there is $b_i\in L_i\cap S_t(y)\neq\emptyset$ (Lemma 2.1). Since $K_i\cap K_j=\emptyset$ for $i\neq j$, it is clear that $L_i\cap L_j=\emptyset$, $a_i\neq a_j$ and $b_i\neq b_j$. By Lemma 2.2 we have that $a_i\to a_\infty\in S_\gamma(y)$, $b_i\to b_\infty\in S_t(y)$ and $L_i\to L_\infty\in\mathcal{C}(T)$ as $i\to\infty$ (take subsequences if necessary). Then $a_\infty, b_\infty\in L_\infty$. Since $L_i\subset K_i$, clearly $L_\infty\subset K_\infty$. Since $K_i\cap K_\infty=\emptyset$, we have that $L_i\cap L_\infty=\emptyset$, $a_i\neq a_\infty$ and $b_i\neq b_\infty$.

Without loss of generality, we can choose the arcs $a_i\, a_\infty$ in $S_\gamma(y)$ jointing a_i and a_∞ such that

$$(3.1)\qquad a_1\, a_\infty \supsetneqq a_2\, a_\infty \supsetneqq \cdots \supsetneqq a_i\, a_\infty \supsetneqq \cdots$$

(take a subsequence of $\{a_i\}_{i\in N}$ if necessary). In the same way, choose the arcs $b_i\, b_\infty$ in $S_t(y)$ jointing b_i and b_∞ such that

$$(3.2)\qquad b_1\, b_\infty \supsetneqq b_2\, b_\infty \supseteqq \cdots \supseteqq b_i\, b_\infty \supseteqq \cdots .$$

Since $a_i\to a_\infty$ and $b_i\to b_\infty$, we have that $\operatorname{diam}(a_i\, a_\infty)\to 0$ and $\operatorname{diam}(b_i\, b_\infty)\to 0$ as $i\to\infty$.

Since L_i, L_{i+1} and L_∞ are connected and mutually disjoint, it is checked that the orientation of $a_i\, a_\infty$ from a_i to a_∞ must coincide with that of $b_i\, b_\infty$ from b_i to b_∞. Indeed, we can take mutually disjoint connected neighborhoods N_i, N_{i+1} and N_∞ of L_i, L_{i+1} and L_∞ in T respectively. Then there are an arc A_i in N_i jointing a_i and b_i such that A_i intersects $S_\gamma(y)$ (resp. $S_t(y)$) only at a_i (resp. b_i), and an arc A_∞ in N_∞ jointing a_∞ and b_∞ such that A_∞ intersects $S_\gamma(y)$ (resp. $S_t(y)$) only at a_∞ (resp. b_∞). Since $N_i\cap N_\infty=\emptyset$, obviously $A_i\cap A_\infty=\emptyset$. Hence $T\setminus$

$\{A_i \cup A_\infty\}$ is decomposed into two connected components U_1 and U_2. Since $a_{i+1} \in U_1 \cup U_2$., we may assume $a_{i+1} \in U_1$. If the orientation of $a_i\, a_\infty$ differs from that of $b_i\, b_\infty$, then $b_{i+1} \in U_2$ by (3.1) and (3.2). In this sase, every arc in N_{i+1} jointing a_{i+1} and b_{i+1} must intersect A_i or A_∞, which contradicts that N_i, N_{i+1} and N_∞ are mutually disjoint. Therefore the orientation of $a_i\, a_\infty$ must coincide with that of $b_i\, b_\infty$.

Since L_i is connected, we can take $z_i \in L_i$ for $i \geq 2$ such that $d(y, z_i) = t + (\gamma - t)/2$. Since $L_i \subset K_i \subset C^s_\varepsilon(x)$, obviously $z_i \in C^s_\varepsilon(x) \cap C^u_\varepsilon(z_i)$, and hence $C^s_\varepsilon(x) \cap C^u_\varepsilon(z_i) = \{z_i\}$ by expansiveness. Since $z_i \notin L_{i-1} \cup L_{i+1}$ and $L_{i-1} \cup L_{i+1} \subset C^s_\varepsilon(x)$, we have that $(L_{i-1} \cup L_{i+1}) \cap C^u_\varepsilon(z_i) = \emptyset$, and so $(L_{i-1} \cup L_{i+1}) \cap (C^u_\varepsilon(z_i) \cup L_i) = \emptyset$. Hence there are connected neighborhoods N_{i-1} and N_{i+1} of L_{i-1} and L_{i+1} in T respectively such that N_{i-1}, N_{i+1} and $C^u_\varepsilon(z_i) \cup L_i$ are mutually disjoint. We can take an arc A_{i-1} in N_{i-1} jointing a_{i-1} and b_{i-1} such that A_{i-1} intersects $S_\gamma(y)$ (resp. $S_t(y)$) only at a_{i-1} (resp. b_{i-1}), and an arc A_{i+1} in N_{i+1} jointing a_{i+1} and b_{i+1} such that A_{i+1} intersects $S_\gamma(y)$ (resp. $S_t(y)$) only at a_{i+1} (resp. b_{i+1}). Then A_{i-1}, A_{i+1} and $C^u_\varepsilon(z_i) \cup L_i$ are mutually disjoint. Denote by $a_{i-1}\, a_{i+1}$ the subarc of $a_{i-1}\, a_\infty$ jointing a_{i-1} and a_{i+1}, and by $b_{i-1}\, b_{i+1}$ the subarc of $b_{i-1}\, b_\infty$ jointing b_{i-1} and b_{i+1}. Then

$$\Gamma = A_{i-1} \cup A_{i+1} \cup a_{i-1}\, a_{i+1} \cup b_{i-1}\, b_{i+1}$$

is a simple closed curve. From the relation between the orientations of $a_{i-1}\, a_\infty$ and $b_{i-1}\, b_\infty$, it follows that Γ bounds a disk D in T. Then $L_i \subset D$ by (3.1) and (3.2). Since $z_i \in L_i$ and $z_i \notin \Gamma$, we see that z_i is an interior point of D.

Since $\gamma \leq \delta/2$ and $C^u_\varepsilon(z_i)$ is connected, we have by Proposition C that $S_\gamma(y) \cap C^u_\varepsilon(z_i) \neq \emptyset$, and hence $\Gamma \cap C^u_\varepsilon(z_i) \neq \emptyset$. Since $(A_{i-1} \cup A_{i+1}) \cap C^u_\varepsilon(z_i) = \emptyset$, it is clear that

$$C^u_\varepsilon(z_i) \cap a_{i-1}\, a_{i+1} \neq \emptyset \quad \text{or} \quad C^u_\varepsilon(z_i) \cap b_{i-1}\, b_{i+1} \neq \emptyset\,.$$

Without loss of generality, we may assume that

$$w_i \in C^u_\varepsilon(z_i) \cap a_{i-1}\, a_{i+1} \neq \emptyset \quad (i \geq 2)\,.$$

Since $\operatorname{diam}(a_i\, a_\infty) \to 0$, we see easily that $w_i \to a_\infty$ as $i \to \infty$. Since $z_i \in L_i$ and $L_i \to L_\infty$, we have that z_i converges to some $z_\infty \in L_\infty$ as $i \to \infty$ (take a subsequence if necessary). Then $d(y, z_\infty) = t + (\gamma - t)/2$. Since $w_i \in C^u_\varepsilon(z_i)$, it follows from Lemma 2.3 that $a_\infty \in W^u_\varepsilon(z_\infty)$. Since $a_\infty, z_\infty \in L_\infty \subset K_\infty \subset C^s_\varepsilon(x)$, we obtain by expansiveness that $a_\infty = z_\infty$, which contradicts that $a_\infty \in S_\gamma(y)$. Therefore $C^s_\varepsilon(x)$ is locally connected. In the same way, the conclusion for $\sigma = u$ is obtained.

4. Preliminary discussions

In this section we shall investigate the topological structure of $C^\sigma_\varepsilon(x)$ (which denotes the connected component of x in $W^\sigma(x)$).

As before let (X, d) be a compact metric space and $f: X \to X$ be an expansive homeomorphism with expansive constant $c>0$.

Lemma 4.1. *For every $0<\varepsilon\leq c$ there exists $\delta>0$ such that*

$$W_\varepsilon^\sigma(x)\cap B_\delta(x) = W^\sigma(x)\cap B_\delta(x) \quad (\sigma = s, u)$$

for all $x\in X$.

Proof. This is similar to that of Lemma V of [15].

Lemma 4.2. *Let $0<\varepsilon\leq c/2$ and let A and B be non-empty subsets of X. If $W_\varepsilon^s(x)\cap W_\varepsilon^u(y)\neq\emptyset$ for all $x\in A$ and $y\in B$, then $W_\varepsilon^s(x)\cap W_\varepsilon^u(y)$ consists of exactly one point $\alpha(x, y)$ and $\alpha: A\times B\to X$ is a continuous map.*

Proof. Since $0<\varepsilon\leq c/2$ and c is an expansive constant, $W_\varepsilon^s(x)\cap W_\varepsilon^u(y)$ must consist of exactly one point. To show that $\alpha: A\times B\to X$ is continuous, assume that a sequence $\{(x_i, y_i)\}_{i\in N}$ of $A\times B$ converges to $(x, y)\in A\times B$, and put $z_i=\alpha(x_i, y_i)$. Then there is a subsequence $\{z_j\}$ of $\{z_i\}$ such that z_j converges to some $z_\infty\in X$ as $j\to\infty$. Since $z_j\in W_\varepsilon^s(x_j)$, it follows from Lemma 2.3 that $z_\infty\in W_\varepsilon^s(x)$. In the same way, we have that $z_\infty\in W_\varepsilon^u(y)$, and therefore $z_\infty=\alpha(x, y)$. This shows that α is continuous.

Hereafter, let M be a compact surface and f be an expansive homeomorphism of M with expansive constant $c>0$.

Fix $x\in M$ and $0<\varepsilon\leq c/2$. The s-(u-)direction is written by σ for simplicity.

Lemma 4.3. *$C_\varepsilon^\sigma(x)$ is arcwise connected and locally arcwise connected.*

Proof. From Proposition 3.1 and Theorem 5.9 of [6], it follows that $C_\varepsilon^\sigma(x)$ is a Peano space. Hence the conclusion is obtained (Theorem 6.29 of [6]).

Lemma 4.4. *For each pair (y, z) of distinct points of $C_\varepsilon^\sigma(x)$ there exists a unique arc jointing y and z in $C_\varepsilon^\sigma(x)$.*

Proof. The existence of arcs follows from Lemma 4.3. We prove the uniqueness of the existence for $\sigma=s$. To do this, assume that there are two arcs jointing y and z in $C_\varepsilon^s(x)$. Then we can find a simple closed curve Γ in $C_\varepsilon^s(x)$. Let $0<\varepsilon'\leq c/2$ be a small number such that $B_{\varepsilon'}(w)$ is a disk for all $w\in M$, and choose $0<r\leq\varepsilon'$ such that $fB_r(w)\subset B_{\varepsilon'}(f(w))$ for all $w\in M$. By (1.1) there is $N>0$ such that $f^n(W_c^s(x))\subset W_r^s(f^n(x))$ for all $n\geq N$. Since $\Gamma\subset C_\varepsilon^s(x)\subset W_c^s(x)$ and $W_r^s(f^n(x))\subset B_r(f^n(x))$, we have that $f^n(\Gamma)\subset B_r(f^n(x))$ for all $n\geq N$. Since $B_r(f^N(x))$ is a disk and $f^N(\Gamma)$ is a simple closed curve in $B_r(f^N(x))$, we see that $f^N(\Gamma)$ bounds a disk D in $B_r(f^N(x))$. Now we claim that $f^i(D)\subset B_r(f^{N+i}(x))$ for all $i\geq 0$. Indeed, by the choice of r, we have $f(D)\subset B_{\varepsilon'}(f^{N+1}(x))$. Since $f^{N+1}(\Gamma)\subset B_r(f^{N+1}(x))$ and $f^{N+1}(\Gamma)$ is the boundary of $f(D)$, it follows that $f(D)\subset B_r(f^{N+1}(x))$

and by induction $f^i(D)\subset B_r(f^{N+i}(x))$ for all $i\geq 2$. The claim was obtained. But this implies that $D\subset W_r^s(f^N(x))$, thus contradicting Lemma 2.8 since $0<r\leq\varepsilon'\leq c/2$. Therefore an arc jointing y and z in $C_\varepsilon^s(x)$ is unique. The conclusion for $\sigma=u$ is also obtained.

Let y and z be distinct points of $C_\varepsilon^\sigma(x)$. We denote by $\sigma(y, z; x, \varepsilon)$ the arc from y to z in $C_\varepsilon^\sigma(x)$ (Lemma 4.4). Since $C_\varepsilon^\sigma(x)\subset C_{c/2}^\sigma(x)$, we have $\sigma(y, z; x, \varepsilon)=\sigma(y, z; x, c/2)$. For simplicity we omit ε in $\sigma(y, z; x, \varepsilon)$ and write

$$\sigma(y, z; x) = \sigma(y, z; x, \varepsilon)$$

We denote by $IC^\sigma(x)$ the union of all open arcs in $C_\varepsilon^\sigma(x)$ and define

$$BC_\varepsilon^\sigma(x) = C_\varepsilon^\sigma(x)\backslash(IC_\varepsilon^\sigma(x)\cup\{x\})\,.$$

That x belongs to $IC_\varepsilon^\sigma(x)$ will be proved later on (Lemma 4.13).

Lemma 4.5. $BC_\varepsilon^\sigma(x)\neq\emptyset$ *and*

$$C_\varepsilon^\sigma(x) = \bigcup_{b\in BC^\sigma(x)} \sigma(x, b; x)\,. \tag{4.1}$$

Proof. Since $C_\varepsilon^\sigma(x)\supsetneqq\{x\}$ by Proposition C, we take $y\in C^\sigma(x)\backslash\{x\}$ and define

$$\mathcal{S} = \{\sigma(x, z; x)\colon \sigma(x, y; x)\subset\sigma(x, z; x)\}\,.$$

Obviously $\mathcal{S}$ is an ordered set with respect to inclusion. By Zorn's lemma there is a totally ordered subset $\mathcal{S}_0$ such that each element of $\mathcal{S}\backslash\mathcal{S}_0$ is not upper bound of $\mathcal{S}_0$. Denote by L the union of all elements of $\mathcal{S}_0$. Then $y\in L$.

It is enough to prove that $L=\sigma(x, b; x)$ for some $b\in C_\varepsilon^\sigma(x)$. Indeed, by the choice of $\mathcal{S}_0$, $b\in BC_\varepsilon^\sigma(x)$ and so $BC_\varepsilon^\sigma(x)\neq\emptyset$. Since $y\in L=\sigma(x, b; x)$ and y is taken arbitrarily, (4.1) holds.

Let $\mathcal{U}$ be the set of all injective continuous maps from $[0, 1)$ to $C_\varepsilon^\sigma(x)$ and define

$$\mathcal{U}_L = \{\alpha\in\mathcal{U}\colon \alpha(0) = x,\ \alpha([0, 1))\subset L\}\,.$$

Then we claim that for every $\alpha\in\mathcal{U}_L$ there is $\sigma(x, z; x)\in\mathcal{S}_0$ such that $\alpha([0, 1))\subset\sigma(x, z; x)$. Indeed, if this is false, we can take $\alpha_\infty\in\mathcal{U}_L$ such that for every $\sigma(x, z; x)\in\mathcal{S}_0$ there is $t\in[0, 1)$ satisfying $\alpha_\infty(t)\notin\sigma(x, z; x)$. Since $\alpha_\infty([0, 1))\subset L$, we have $\alpha_\infty(t)\in L$ and so there is $\sigma(x, w; x)\in\mathcal{S}_0$ such that $\alpha_\infty(t)\in\sigma(x, w; x)$. Since $\mathcal{S}_0$ is totally ordered, it follows that $\sigma(x, z; x)\subset\sigma(x, w; x)$. Since $\alpha_\infty(0)=x$, we have by Lemma 4.4 that $\alpha_\infty([0, t])\supset\sigma(x, z; x)$, and hence $\alpha_\infty([0, 1))\supset\sigma(x, z; x)$. Since $\sigma(x, z; x)$ is arbitrary in $\mathcal{S}_0$, $\alpha_\infty([0, 1))=L$. Hence there is a sequence $\{z_i\}_{i\in\boldsymbol{N}}$ of L such that $\sigma(x, z_i; x)\subsetneqq\sigma(x, z_{i+1}; x)$ for all $i\in\boldsymbol{N}$ and $L=$

$\bigcup_{i\in N}\sigma(x, z_i; x)$. Obviously there is a subsequence $\{z_j\}$ of $\{z_i\}$ such that z_j converges to some $z_\infty\in C^\sigma_\varepsilon(x)$. We write $J=\sigma(x, z_\infty; x)\cap L$ when $z_\infty\neq x$, and $J=\{x\}$ when $z_\infty=x$. Then it is checked that $J\subsetneqq L$. Indeed, if not, then $z_\infty\neq x$ and $J=L$. Hence $L\subset\sigma(x, z_\infty; x)$. Since $L=\alpha_\infty([0, 1))$, obviously $L\subsetneqq\sigma(x, z_\infty; x)$, contradicting that L is the union of all elements of $\mathcal{S}_0$. Therefore $J\subsetneqq L$.

Combining this fact and Lemma 4.4, we see that J is either an arc or one point set. Hence $J\subsetneqq\sigma(x, z_l; x)$ for some $l\in N$. Since $\sigma(x, z_j; x)\supsetneqq\sigma(x, z_l; x)$ for $j>l$, by using Lemma 4.4 we can check that $\sigma(z_\infty, z_j; x)\supsetneqq\sigma(z_\infty, z_l; x)$ for $j>l$, and so

$$\operatorname{diam}(\sigma(z_\infty, z_j; x))\geq\operatorname{diam}(\sigma(z_\infty, z_l; x))>0 .$$

Since $z_j\to z_\infty$ as $j\to\infty$, this contradicts the fact that $C^\sigma_\varepsilon(x)$ is locally arcwise connected (Lemma 4.3). Therefore the above claim holds.

Since $L\subset C^\sigma_\varepsilon(x)$, there is a countable subset G of L such that the closure $\bar{G}$ of G in $C^\sigma_\varepsilon(x)$ contains L. Then we can construct $\alpha\in\mathcal{U}_L$ such that $\alpha([0, 1))\supset G$, because L is the union of elements of the totally ordered set $\mathcal{S}_0$. By the above result there is $\sigma(x, b; x)\in\mathcal{S}_0$ such that $\alpha([0, 1))\subset\sigma(x, b; x)$. Then $G\subset\sigma(x, b; x)$. Since $L\subset\bar{G}$ and $\sigma(x, b; x)\subset L$, we have easily that $L=\sigma(x, b; x)$. The proof is completed.

Lemma 4.6. *Let A be an arc in $C^\sigma_\varepsilon(x)$. If x is an end point of A, then there exists $b\in BC^\sigma_\varepsilon(x)$ such that $A\subset\sigma(x, b; x)$.*

Proof. Let y be another end point of A. Since $y\in C^\sigma_\varepsilon(x)$, by Lemma 4.5 there is $b\in BC^\sigma_\varepsilon(x)$ such that $y\in\sigma(x, b; x)$. Then the conclusion is obtained by Lemma 4.4.

Let a, b and c be points of $C^\sigma_\varepsilon(x)$ such that $a\neq b$ and $a\neq c$. We write $\sigma(a, b; x)\sim\sigma(a, c; x)$ if $\sigma(a, b; x)\cap\sigma(a, c; x)\supsetneqq\{a\}$. In this case, $\sigma(a, b; x)\cap\sigma(a, c; x)$ is a subarc of both $\sigma(a, b; x)$ and $\sigma(a, c; x)$ (Lemma 4.4). Hence "$\sim$" is an equivalence relation on $\{\sigma(x, b; x); b\in BC^\sigma_\varepsilon(x)\}$. We define

$$P_\varepsilon(x) = \#[\{\sigma(x, b; x)\colon b\in BC^\sigma_\varepsilon(x)\}/_\sim]$$

where $\#[\cdot]$ denotes the cardinal number of $\cdot$.

Lemma 4.7. *$P^\sigma_\varepsilon(x)=P^\sigma_{c/2}(x)$ (remark that ε is chosen such that $0<\varepsilon\leq c/2$ as promised before).*

Proof. Using Lemma 4.1, we can find $\delta>0$ such that $W^\sigma_\varepsilon(x)\cap B_\delta(x)=W^\sigma_{c/2}(x)\cap B_\delta(x)$. Let C be the connected component of x in $W^\sigma_\varepsilon(x)\cap B_\delta(x)$. Then $C\subset C^\sigma_\varepsilon(x)\cap B_\delta(x)$ and hence C is the connected component of x in $C^\sigma_\varepsilon(x)\cap B_\delta(x)$. Since $W^\sigma_\varepsilon(x)\cap B_\delta(x)=W^\sigma_{c/2}(x)\cap B_\delta(x)$, it is easily checked that C is the connected component of x in $C^\sigma_{c/2}(x)\cap B_\delta(x)$. Therefore the connected component of x in $C^\sigma_\varepsilon(x)\cap B_\delta(x)$ coincides with that of x in $C^\sigma_{c/2}(x)\cap B_\delta(x)$. Combining this fact and Lemma 4.6, we see that $P^\sigma_\varepsilon(x)=P^\sigma_{c/2}(x)$.

As above let $x\in M$ and $\sigma=s, u$. Since $P^\sigma_\varepsilon(x)$ is independent of $\varepsilon(0<\varepsilon\leq c/2)$ by Lemma 4.7, we omit ε and write

$$P^\sigma(x)=P^\sigma_\varepsilon(x)\,.$$

Now we define

$$\mathrm{Sing}^\sigma(f)=\{x\in M\colon P^\sigma(x)\geq 3\}\,.$$

Lemma 4.8. $\mathrm{Sing}^\sigma(f)$ *is a finite set* ($\sigma=s, u$).

Proof. We give the proof for $\sigma=s$. If this is done, then the conclusion for $\sigma=u$ is obtained in the same way. Let c be an expansive constant for f as before and fix $0<\varepsilon\leq c/6$ small enough. Let $0<\delta\leq\varepsilon$ be as in Proposition C and Lemma 2.5. To show that $\mathrm{Sing}^s(f)$ is finite, let Λ be the set of points $x\in M$ with the property that $C^s_\varepsilon(x)$ contains distinct three points a_1, a_2, a_3 such that

$$s(x, a_k; x)\not\sim s(x, a_l; x)\quad(k\neq l)\,,$$
$$s(x, a_k; x)\cap S_\delta(x)\neq\emptyset\quad(k=1, 2, 3)\,.$$

Then it follows from Lemma 4.6 that $\Lambda\subset\mathrm{Sing}^s(f)$.

First we show that $\#[\Lambda]\geq\#[\mathrm{Sing}^s(f)]$. Let $x\in\mathrm{Sing}^s(f)$. Then $P^s(x)\geq 3$. By the definition of $P^s(x)$ there are $a_k\in C^s_\varepsilon(x)$ $(k=1, 2, 3)$ such that $s(x, a_k; x)\not\sim s(x, a_l; x)$ for $k\neq l$ and $s(x, a_k; x)\subset B_\delta(x)$ for $k=1, 2, 3$. Since $0<\delta\leq\varepsilon\leq c/6$, we can find $m_k>0$ such that $f^{-i}[s(x, a_k; x)]\subset B_\varepsilon(f^{-i}(x))$ for $0\leq i<m_k$ and $f^{-m_k}[s(x, a_k; x)]\not\subset B_\varepsilon(f^{-m_k}(x))$. Let $A^k(m_k)$ denote the connected component of $f^{-m_k}(x)$ in $f^{-m_k}[s(x, a_k; x)]\cap B_\varepsilon(f^{-m_k}(x))$. Then we can see easilyt tha $A^k(m_k)$ is an arc in $C^s_\varepsilon(f^{-m_k}(x))$ such that $f^{-m_k}(x)$ is an end point, and that $A^k(m_k)\cap S_\varepsilon(f^{-m_k}(x))\neq\emptyset$.

For $i>m_k$ define $A^k(i)$ as the connected component of $f^{-1}(x)$ in $f^{-1}(A^k(i-1))\cap B_\varepsilon(f^{-1}(x))$. As above the result obtained for $A^k(m_k)$ is established for $A^k(i)$ $(i>m_k)$, that is, $A^k(i)$ is an arc in $C^s(f^{-1}(x))$ such that $f^{-1}(x)$ is an end point.

Since $A^k(m_k)\cap S_\varepsilon(f^{-m_k}(x))\neq\emptyset$, it is easily checked that $f^{i-j}(A^k(i))\cap S_\varepsilon(f^{-j}(x))\neq\emptyset$ for some j with $m_k\leq j\leq i$. Note that $f^i(A^k(i))\subset s(x, a_k; x)\subset B_\delta(x)$. Combine these facts and Lemma 2.6 (2). Then we see that $A^k(i)\cap S_\delta(f^{-i}(x))\neq\emptyset$ for $i\geq m_k$. Since $s(x, a_k; x)\not\sim s(x, a_l; x)$, obviously $A^k(i)\not\sim A^l(i)$ for $k\neq l$. We write $m_0=\max\{m_1, m_2, m_3\}$ for simplicity. Then we have that $f^i(x)\in\Lambda$ for $i\geq m_0$.

Hence an injection from $\mathrm{Sing}^s(f)$ to Λ is defined as follows. For $x\in\mathrm{Sing}^s(f)$ consider the orbit $O_f(x)$ of x by f and put $S=\bigcup_{x\in\mathrm{Sing}^s(f)}O_f(x)$. Obviously $\mathrm{Sing}^s(f)\subset S$. For $x\in\mathrm{Sing}^s(f)$ we define

$$\xi(f^i(x)) = f^i(x) \quad (i \geq 0) \qquad\qquad \text{if} \quad x \in \mathrm{Per}(f),$$

$$\xi(f^i(x)) = \begin{cases} f^{-m_0+2i}(x) & (i<0) \\ f^{-m_0}(x) & (i=0) \\ f^{-m_0-2i+1}(x) & (i>0) \end{cases} \qquad \text{if} \quad x \notin \mathrm{Per}(f)$$

where $\mathrm{Per}(f)$ denotes the set of all periodic points of f. Then the right hand sides of the above relations belong to Λ and $\xi\colon S \to \Lambda$ is an injection, from which an injection from $\mathrm{Sing}^s(f)$ to Λ is obtained. Hence we have $\#[\Lambda] \geq \#[\mathrm{Sing}^s(f)]$.

To obtain that $\mathrm{Sing}^s(f)$ is finite, we assume that this is false. Then Λ is an infinite set by the above result. Hence we can take $p \in M$ such that $\Lambda \cap U_{\delta/4}(p)$ is infinite. Applying Zorn's lemma, we can choose a subset Λ_0 of $\Lambda \cap U_{\delta/4}(p)$ with the properties that if $x, y \in \Lambda_0$ and $x \neq y$ then $C^s_\varepsilon(x) \cap C^s_\varepsilon(y) = \emptyset$, and that if $x \in [\Lambda \cap U_{\delta/4}(p)] \setminus \Lambda_0$ then there is $y \in \Lambda_0$ such that $C^s_\varepsilon(x) \cap C^s_\varepsilon(y) \neq \emptyset$. Then one of the following must hold;

(I) Λ_0 is infinite,

(II) Λ_0 is finite.

In any case we can derive a contradiction as follows.

Case (I). Since Λ_0 is infinite and $\Lambda_0 \subset U_{\delta/4}(p)$, there is a sequence $\{x_i\}_{i \in N}$ of Λ_0 with $x_i \neq x_j$ for $i \neq j$ such that x_i converges to some $x_\infty \in B_{\delta/4}(p)$ as $i \to \infty$. Since $\Lambda_0 \subset \Lambda \cap U_{\delta/4}(p)$, obviously $x_i \in \Lambda \cap U_{\delta/4}(p)$. By the choice of Λ we can take $a^i_k \in C^s_\varepsilon(x_i)$ $(k=1, 2, 3)$ such that $s(x_i, a^i_k; x_i) \not\sim s(x_i, a^i_l; x_i)$ for $k \neq l$ and such that $a^i_k \in S_{\delta/2}(p)$ and $s(x_i, a^i_k; x_i) \subset B_{\delta/2}(p)$ for $k=1, 2, 3$. Since δ is small enough, $S_{\delta/2}(p)$ is a circle, and hence $\{a^i_k\}^3_{k=1}$ cut $S_{\delta/2}(p)$ in three open arcs.

We claim that if $i \neq j$ then $\{a^j_k\}^3_{k=1}$ is contained in an open arc of $S_{\delta/2}(p) \setminus \{a^i_k\}^3_{k=1}$. Indeed, write $\Sigma_i = \bigcup_{k=1}^{3} s(x_i, a^i_k; x_i)$. Since $s(x_i, a^i_k; x_i) \not\sim s(x_i, a^i_l; x_i)$, we see easily that Σ_i is a trident curve with end points a^i_k $(k=1, 2, 3)$. Since $x_i \neq x_j$, by the choice of Λ_0, $C^s_\varepsilon(x_i) \cap C^s_\varepsilon(x_j) = \emptyset$ and hence $\Sigma_i \cap \Sigma_j = \emptyset$. Since Σ_i and Σ_j are in a disk $B_{\delta/2}(p)$, we have that Σ_i is contained in a connected component of $B_{\delta/2}(p) \setminus \Sigma_j$, from which the claim is obtained.

Let $I_k (k=1, 2, 3)$ be the open arcs in which $\{a^1_k\}^3_{k=1}$ cut $S_{\delta/2}(p)$. By the above result $\{a^i_k\}^3_{k=1} \subset I_{k(i)}$ for all $i \neq 1$ where $k(i) = 1, 2$ or 3. We take the minimal arc A_i in $I_{k(i)}$ such that $A_i \supset \{a^i_k\}^3_{k=1}$. Let $A_i \cap A_j \neq \emptyset$ for $i \neq j$. Then $A_j \subset I_{k(i)}$. Since $\{a^j_k\}$ is contained in an open arc of $S_{\delta/2}(p) \setminus \{a^i_k\}^3_{k=1}$ by the above result, it is eas,ly checked that there is an implication between A_i and A_j. Note that $\{a^i_k\}^3_{k=1}$ cut A_i in two open arcs J^1_i and J^2_i. If $A_j \subset A_i$ then either $A_j \subset J^1_i$ or $A_j \subset J^2_i$ must hold. Consequently we have proved that there is a family $\{A_{i_l}\}^\infty_{l=1}$ such that one of the following cases holds.

(a) $\qquad A_{i_l} \cap A_{i_m} = \emptyset \qquad \text{for} \quad l \neq m$

(b) $\quad A_{i_{l+1}} \subset J^1_{i_l}$ or $A_{i_{l+1}} \subset J^2_{i_l}$ for $l \in \boldsymbol{N}$.

Since A_i is a miniaml arc such that $A_i \supset \{a^i_k\}^3_{k=1}$, we may assume that end points of A_i are a^i_1 and a^i_3, and write $a^i_1 a^i_3 = A_i$. Since $a^i_2 \in a^i_1 a^i_3$, we denote by $a^i_1 a^i_2$ the subarc in $a^i_1 a^i_3$ jointing a^i_1 and a^i_2. The notation $a^i_2 a^i_3$ is also defined. Then the interiors of $a^i_1 a^i_2$ and $a^i_2 a^i_3$ are equal to J^1_i or J^2_i respectively. Under these notations, without loss of generality we can rewrite the cases (a) and (b) as follows:

$$(\mathrm{I_a}) \qquad a^i_1 a^i_3 \supset a^j_1 a^j_3 = \emptyset \qquad \text{for} \quad i \neq j,$$

$$(\mathrm{I_b}) \qquad a^i_1 a^i_2 \supset a^{i+1}_1 a^{i+1}_3 \qquad \text{for all} \quad i \in \boldsymbol{N}.$$

We can assume that the orientation of $a^i_1 a^i_3$ from a^i_1 to a^i_3 coincides with that of $a^1_1 a^1_3$ from a^1_1 to a^1_3 for all i (by taking a subsequence of $\boldsymbol{N}$ if necessary).

Case $(\mathrm{I_a})$. Since $x_1 \in U_{\delta/4}(p)$ and $a^i_2 \in S_{\delta/2}(p)$, we can take $z_i \in s(x_i, a^i_2; x_i) \cap S_{3\delta/8}(p) \neq \emptyset$ for $i \in \boldsymbol{N}$. By Lemma 2.2 there are $a_\infty \in S_{\delta/2}(p)$, $z_\infty \in S_{3\delta/8}(p)$ and $\Delta_\infty \in \mathcal{C}(B_{\delta/2}(p))$ such that a^i_2, z_i and $s(x_i, a^i_2; x_i)$ converge to a_∞, z_∞ and Δ_∞ as $i \to \infty$ respectively (take a subsequence if necessary). Since $a^i_2, z_i \in s(x_i, a^i_2; x_i)$, we have $a_\infty, z_\infty \in \Delta_\infty$. Since $s(x_i, a^i_2; x_i) \subset W^s_\varepsilon(x_i)$ and $x_i \to x_\infty$, $\Delta_\infty \subset W^s_\varepsilon(x_\infty)$ by Lemma 2.3 and therefore $a_\infty, z_\infty \in W^s_\varepsilon(x_\infty)$.

On the other hand, let Σ_i be as above. Then Σ_i is a trident curve in the disk $B_{\delta/2}(p)$ with end points $a^i_k (k=1, 2, 3)$. Since $z_i \in s(x_i, a^i_2; x_i) \subset \Sigma_i \subset W^s_\varepsilon(x_i)$, by expansiveness $C^u_\varepsilon(z_i) \cap \Sigma_i = \{z_i\}$. Since δ is as in Proposition C and $z_i \in S_{3\delta/8}(p)$, we have that $C^u_\varepsilon(z_i) \cap S_{\delta/2}(p) \neq \emptyset$. Note that $a^i_2 \in a^i_1 a^i_3$. Then we can find $w_i \in C^u_\varepsilon(z_i) \cap a^i_1 a^i_3 \neq \emptyset$. By $(\mathrm{I_a})$ it is easily checked that $\mathrm{diam}(a^i_1 a^i_3) \to 0$ as $i \to \infty$. Since $w_i, a^i_2 \in a^i_1 a^i_3$ and $a^i_2 \to a_\infty$, clearly w_i converges to a_∞ is $i \to \infty$. Since $w_i \in C^u_\varepsilon(z_i)$ and $z_i \to z_\infty$, by Lemma 2.3 we conclude that $a_\infty \in W^u_\varepsilon(z_\infty)$. Since $a_\infty, z_\infty \in W^s_\varepsilon(x_\infty)$, by expansiveness $a_\infty = z_\infty$, thus contradicting that $a_\infty \in S_{\delta/2}(p)$ and $z_\infty \in S_{3\delta/8}(p)$.

Case $(\mathrm{I_b})$. Write $T = B_{\delta/2}(p) \setminus U_{\delta/4}(p)$. Then T is an annulus bounded by circles $S_{\delta/4}(p)$ and $S_{\delta/2}(p)$. Since $x_i \in U_{\delta/4}(p)$ and $a^i_k \in S_{\delta/2}(p)$, there are $b^i_k \in s(x_i, a^i_k; x_i) \cap S_{\delta/4}(p)$ ($i \in \boldsymbol{N}$ and $k=1, 2, 3$) such that $s(b^i_k, a^i_k; x_i) \subset T$. By Lemma 2.2 we can find $a_\infty \in S_{\delta/2}(p)$, $b_\infty \in S_{\delta/4}(p)$ and $\Delta_\infty \in \mathcal{C}(T)$ such that a^i_2, b^i_2 and $s(b^i_2, a^i_2; x_i)$ converge to a_∞, b_∞ and Δ_∞ as $i \to \infty$ respectively. Clearly $a_\infty, b_\infty \in \Delta_\infty$. Since $s(b^i_k, a^i_k; x_i) \subset W^s_\varepsilon(x_i)$ and $x_i \to x_\infty$, we have by Lemma 2.3 that $\Delta_\infty \subset W^s_\varepsilon(x_\infty)$.

By $(\mathrm{I_b})$ we have $a_\infty \in a^1_1 a^1_3$ and $a_\infty \neq a^i_2, a^i_3$. As above let aa' denote the subarc of $a^1_1 a^1_3$ jointing a and a' for $a, a' \in a^1_1 a^1_3$. By using the relation between the orientations of $a^1_1 a^1_3$ and $a^i_1 a^i_3$, we have that $a^i_2 \in a_\infty a^i_3 (i \in \boldsymbol{N})$, and then

$$a_\infty a^i_3 \supset a_\infty a^i_2 \supset a_\infty a^{i+1}_3 \supset a_\infty a^{i+1}_2 \quad (\forall i \in \boldsymbol{N}). \tag{4.2}$$

In the same fashion, we can choose the arcs in $S_{\delta/4}(p)$ such that

$$(4.3) \qquad b_\infty b_3^i \supset b_\infty b_2^i \supset b_\infty b_3^{i+1} \supset b_\infty b_2^{i+1} \quad (\forall i \in \boldsymbol{N}).$$

Since $a_2^i \in S_{\delta/2}(p)$ and $b_2^i \in S_{\delta/4}(p)$, there is $z_i \in s(b_2^i, a_2^i; x_i) \cap S_{3\delta/8}(p) \neq \emptyset$ for all $i \in \boldsymbol{N}$. Since δ is as in Proposition C, it follows that $C_\varepsilon^u(z_i) \cap S_{\delta/2}(p) \neq \emptyset$. Combining this fact, (4.2) and (4.3), by expansiveness we have that

$$C_\varepsilon^u(z_i) \cap a_2^{i+1} a_3^i \neq \emptyset \quad \text{or} \quad C_\varepsilon^u(z_i) \cap b_2^{i+1} b_3^i \neq \emptyset .$$

Without loss of generality, we assume that $w_i \in C_\varepsilon^u(z_i) \cap a_2^{i+1} a_3^i \neq \emptyset$ for all $i \in \boldsymbol{N}$. Since $a_2^i \to a_\infty$, $\operatorname{diam}(a_2^{i+1} a_3^i) \to 0$ as $i \to \infty$ by (4.2) and hence w_i converges to a_∞ as $i \to \infty$. Since $z_i \in S_{3\delta/8}(p)$, z_i converges to some $z_\infty \in S_{3\delta/8}(p)$ as $i \to \infty$ and then $z_\infty \in \Delta_\infty$ since $z_i \in s(a_2^i, a_2^i; x_i)$ and $s(b_2^i, a_2^i; x_i) \to \Delta_\infty$. Since $w_i \in C_\varepsilon^u(z_i)$, we have $a_\infty \in W_\varepsilon^u(z_\infty)$ (Lemma 2.3). Since $a_\infty, z_\infty \in \Delta_\infty$ and $\Delta_\infty \subset W_\varepsilon^s(x_\infty)$, by expansiveness $a_\infty = z_\infty$, thus contradicting that $a_\infty \in S_{\delta/2}(p)$ and $z_\infty \in S_{3\delta/8}(p)$.

Case (II). Since Λ_0 is finite and Λ is infinite, by the choice of Λ_0 we can take $y \in \Lambda_0$ and a sequence $\{x_i\}_{i \in N}$ of Λ with $x_i \neq x_j$ for $i \neq j$ such that $C_\varepsilon^s(x_i) \cap C_\varepsilon^s(y) \neq \emptyset$ for all $i \in \boldsymbol{N}$. Then $C_\varepsilon^s(x_i) \subset C_{3\varepsilon}^s(y)$ for all $i \in \boldsymbol{N}$. Since $x_i \in \Lambda$, by the choice of Λ we can take $a_k^i \in C_\varepsilon^s(x_i)$ $(k=1, 2, 3)$ such that $s(x_i, a_k^i; x_i) \not\sim s(x_i, a_l^i; x_i)$ for $k \neq l$ and $a_k^i \in S_{\delta/2}(p)$ for $k=1, 2, 3$. Let $K = \{a_k^i \colon i \in \boldsymbol{N}, k=1, 2, 3\}$. If K is finite, then $\{a_k^i\}_{k=1}^3 = \{a_k^j\}_{k=1}^3$ for some $i \neq j$. In this case, there are two arcs in $C_{3\varepsilon}^s(y)$ jointing x_i and x_j, which contradicts Lemma 4.4 since $0 < 3\varepsilon < c/2$. Hence K must be infinite, and so there are a subsequence $\{x_l\}$ of $\{x_i\}$ and a sequence $\{a^l\}$ of K with $a^l \neq a^{l'}$ for $l \neq l'$ such that x_l and a^l converge to some $x_\infty \in B_{\delta/4}(p)$ and some $a_\infty \in S_{\delta/2}(p)$ as $i \to \infty$ respectively. Then $x_\infty, a_\infty \in C_{3\varepsilon}^s(y)$ because $x_l, a^l \in C_{3\varepsilon}^s(y)$. Since $C_{3\varepsilon}^s(y)$ is locally arcwise connected by Lemma 4.3, there are arcwise connected neighborhoods U and V of x_∞ and a_∞ in $C_{3\varepsilon}^s(y)$ such that $U \cap V = \emptyset$, respectively. Then $x_l, x_{l'} \in U$ and $a^l, a^{l'} \in V$ for sufficiently large l and l' with $l \neq l'$. This implies the existence of two arcs in $C_{3\varepsilon}^s(y)$ jointing x_l and $x_{l'}$. But this contradicts Lemma 4.4.

Lemma 4.9. *Let $x \in M$ and $\sigma = s, u$. If $P^\sigma(x) \geq 3$, then $x \in \operatorname{Per}(f)$.*

Proof. Assume that $P^s(x) \geq 3$ and take $0 < \varepsilon \leq c/2$. Since $fW_\varepsilon^s(x) \subset W_\varepsilon^s(f(x))$, clearly $fC_\varepsilon^s(x) \subset C_\varepsilon^s(f(x))$ and so by Lemma 4.6, $P^s(f(x)) \geq 3$. Inductively $P^s(f^i(x)) \geq 3$ for all $i \geq 2$, and therefore by lemma 4.8, $x \in \operatorname{Per}(f)$. We obtain also that $P^u(x) \geq 3$ implies $x \in \operatorname{Per}(f)$.

Lemma 4.10. *For every $x \in M$, $P^\sigma(x)$ is finite $(\sigma = s, u)$.*

Proof. Fix $0 < \varepsilon \leq c/2$ and let $0 < \delta \leq \varepsilon$ be as in Lemma 2.5. Assume that $P^\sigma(x)$ is infinite for some $x \in M$. Then $x \in \operatorname{Per}(f)$ by Lemma 4.9. Now we write

$$B = \{b \in BC_\varepsilon^\sigma(x) \colon \sigma(x, b; x) \cap S_\delta(x) \neq \emptyset\} .$$

Since $P^{\sigma}(x)$ is infinite and $x\in\mathrm{Per}(f)$, as the first part of the proof of Lemma 4.8 we can prove that there is an infinite subset B' of B such that $\sigma(x, b_1; x)\not\sim\sigma(x, b_2; x)$ for $b_1, b_2\in B'$ with $b_1\neq b_2$ (use Lemma 4.6). Since $C^{\sigma}_{\varepsilon}(x)$ is locally arcwise connected by Lemma 4.3 and B' is infinite, there is an arcwise connected subset U of $C^{\sigma}_{\varepsilon}(x)$ such that diam $(U)<\delta$ and U contains distince points b_1, b_2 of B'. Hence $\sigma(x, b_1; x)\cup\sigma(x, b_2; x)\subset U$ by Lemma 4.4. Since $\sigma(x, b_1; x)\cap S_{\delta}(x)\neq\emptyset$, we have that $\mathrm{diam}(U)\geq\delta$, thus contradicting $\mathrm{diam}(U)<\delta$.

Let $x\in M$ and $0<\varepsilon\leq c/2$ and let $y\in C^{\sigma}_{\varepsilon}(x)\setminus\{x\}$. We say that y is a *branch point* of $C^{\sigma}_{\varepsilon}(x)$ if there are distinct points a_1, a_2 of $BC^{\sigma}_{\varepsilon}(x)$ such that $\sigma(x, a_1; x)\cap\sigma(x, a_2; x)=\sigma(x, y; x)$. Noet that $\sigma(x, y; x)\subsetneqq\sigma(x, a_i; x)$ $(i=1, 2)$. We obtain in proving the following lemma that if y is a branch point of $C^{\sigma}_{\varepsilon}(x)$ then $y\in\mathrm{Sing}^{\sigma}(f)$.

Lemma 4.11. *For $x\in M$ and $0<\varepsilon\leq c/4$, $C^{\sigma}_{\varepsilon}(x)$ has at most one branch point $(\sigma=s, u)$. If $P^{\sigma}(x)\geq 3$, then $C^{\sigma}_{\varepsilon}(x)$ has no branch points.*

Proof. Assume that y is an branch point of $C^{\sigma}_{\varepsilon}(x)$. Since $C^{\sigma}_{2\varepsilon}(y)\supset C^{\sigma}_{\varepsilon}(x)$, by Lemma 4.6 we see that $P^{\sigma}(y)\geq 3$. Therefore $y\in\mathrm{Per}(f)$ by Lemma 4.9 and so every branch point of $C^{\sigma}_{\varepsilon}(x)$ is a periodic point. By this fact and (1.1), we obtain that $C^{\sigma}_{\varepsilon}(x)$ has at most one branch point. The conclusion of the second statement is easily obtained in the same way.

Lemma 4.12. *For $x\in M$ and $0<\varepsilon\leq c/4$, $BC^{\sigma}_{\varepsilon}(x)$ is a finite set $(\sigma=s, u)$.*

Proof. The conclusion is easily obtained from Lemmas 4.10 and 4.11.

Lemma 4.13. *For every $x\in M$, $P^{\sigma}(x)\geq 2$ $(\sigma=s, u)$.*

Proof. If the proof is given for $\sigma=s$, then the conclusion for $\sigma=u$ is obtained in the same way and so we prove the case of $\sigma=s$. Since $BC^{s}_{c/2}(x)\neq\emptyset$ by Lemma 4.5, obviously $P^{s}(x)\geq 1$ for all $x\in M$. Hence it is enough to show that $P^{s}(x)\neq 1$ for all $x\in M$.

Assume that there is $x\in M$ such that $P^{s}(x)=1$. Then by using Lemma 4.11 we can find $0<3\varepsilon\leq c/4$ such that $C^{s}_{3\varepsilon}(x)$ is an arc, and then $C^{s}_{3\varepsilon}(x)=s(x, z; x)$ where $\{z\}=BC^{s}_{3\varepsilon}(x)$. Since $C^{s}_{\varepsilon}(x)\subset C^{s}_{3\varepsilon}(x)$, $C^{s}_{\varepsilon}(x)=s(x, y; x)$ for some $y\in s(x, z; x)$.

Let $0<2\delta\leq\varepsilon$ be as in Proposition C. Then we can take $a\in s(x, y; x)\cap S_{\delta}(x)\neq\emptyset$ and $b\in C^{u}_{\varepsilon}(x)\cap S_{\delta}(x)\neq\emptyset$ such that $s(x, a; x)\setminus\{a\}\subset U_{\delta}(x)$ and $u(x, b; x)\setminus\{b\}\subset U_{\delta}(x)$, *i.e.*, $L=s(x, a; x)\cup u(x, b; x)$ intersects $S_{\delta}(x)$ only at a and b. Since $s(x, a; x)\cap u(x, b; x)=\{x\}$ by expansiveness, L is an arc in $B_{\delta}(x)$, and so $B_{\delta}(x)$ is cut in two components U_1 and U_2 by L.

Now we claim that there are $q\in s(x, a; x)\setminus\{x, a\}$ and $q_i\in U_i (i=1, 2)$ such that $q_1, q_2\in C^{u}_{\varepsilon}(q)$ and $u(q_1, q_2; q)\subset U_{\delta}(x)$. Indeed, take $p\in s(x, a; x)$ with $d(x, p)$

$=\delta/2$. Then $s(x,a;x)\cap C^u_{\delta/4}(p)=\{p\}$ by expansiveness. If $w\in u(x,b;x)\cap C^u_{\delta/4}(p)\neq\emptyset$, then $w\in W^u_\varepsilon(x)\cap W^u_\varepsilon(p)$ and so $x,p\in W^u_\varepsilon(w)$. Since $p\in s(x,a;x)\subset C^s_\varepsilon(x)$, by expansiveness $x=p$, which contradicts $d(x,p)=\delta/2$. Hence $u(x,b;x)\cap C^u_{\delta/4}(p)=\emptyset$, and therefore $L\cap C^u_{\delta/4}(p)=\{p\}$. Combining this fact and Proposition C, we have that $U_1\cap C^u_{\delta/4}(p)\neq\emptyset$ or $U_2\cap C^u_{\delta/4}(p)\neq\emptyset$, *i,e.*, one of the following three cases holds:

(I) $q_i\in U_i\cap C^u_{\delta/4}(p)\neq\emptyset \quad (i=1,2)$,

(II) $U_1\cap C^u_{\delta/4}(p)=\emptyset$ and $U_2\cap C^u_{\delta/4}(p)\neq\emptyset$,

(III) $U_1\cap C^u_{\delta/4}(p)\neq\emptyset$ and $U_2\cap C^u_{\delta/4}(p)=\emptyset$.

For the case (I), the above claim holds since $q_1,q_2\in C^u_{\delta/4}\subset C^u_\varepsilon(p)$ and $u(q_1,q_2;q)\subset C^u_{\delta/4}(p)\subset U_\delta(x)$. For the case (II), we take a sequence $\{p_i\}_{i\in N}$ of U_1 such that p_i converges to p as $i\to\infty$. By Lemma 2.2, $C^u_{\delta/4}(p_i)$ converges to some $\Delta_\infty\in\mathcal{C}(M)$ (take a subsequence if necessary) and then $p\in\Delta_\infty\subset C^u_{\delta/4}(p)$ by Lemma 2.3. By using Proposition C, we have $\{p\}\subsetneqq\Delta_\infty$ and hence $\Delta_\infty\cap U_2\neq\emptyset$. So $q_2\in U_2\cap C^u_{\delta/4}(p_l)\neq\emptyset$ for sufficiently large $l\in N$ with $d(p,p_l)\leq\delta/10$ and then $u(p_l,q_2;p_l)\subset B_{7\delta/20}(p)\subset U_\delta(x)$. Combining this and the fact that $p_l\in U_1$ and $q_2\in U_2$, we can find $q\in[s(x,a;x)\setminus\{x,a\}]\cap u(p_l,q_2;p_l)\neq\emptyset$. Since $C^u_{\delta/4}(p_l)\subset C^u_\varepsilon(q)$, obviously $p_l,q_2\in C^u_\varepsilon(q)$ and $u(p_l,q_2;p_l)=u(p_l,q_2;q)$. Therefore the above claim holds for (II). In the same way, we obtain that the above claim holds also for (III).

Take $q\in s(x,a;x)\setminus\{x,a\}$ and $q_i\in U_i (i=1,2)$ as in the above claim. We note that $q\in u(q_1,q_2;q)$. Since 2δ is chosen as in Proposition C, there are $t_i\in S_\delta(x)\cap C^s_\varepsilon(q_i)$ $(i=1,2)$ such that $s(q_i,t_i;q_i)\setminus\{t_i\}\subset U_\delta(x)$. By expansiveness it is easily checked that

$$s(q_1,t_1;q_1)\cap s(q_2,t_2;q_2)=\emptyset\,,$$
$$s(q_1,t_1;q_1)\cap u(q_1,q_2;q)=\{q_1\}\,,$$
$$s(q_2,t_2;q_2)\cap u(q_1,q_2;q)=\{q_2\}\,.$$

If t_1t_2 is an arc in $S_\delta(x)$ jointing t_1 and t_2, then we have that

$$\Gamma=t_1t_2\cup s(q_1,t_1;q_1)\cup u(q_1,q_2;q)\cup s(q_2,t_2;q_2)$$

is a simple closed curve in $B_\delta(x)$. Since $s(x,q;x)\subset s(x,a;x)\setminus\{a\}\subset U_\delta(x)$, obviously $S_\delta(x)\cap s(x,q;x)=\emptyset$. By expansiveness $s(q_i,t_i;q_i)\cap s(x,q;x)=\emptyset$ $(i=1,2)$ and $u(q_1,q_2;q)\cap s(x,q;x)=\{q\}$, and therefore $\Gamma\cap s(x,q;x)=\{q\}$.

Ler D be the disk in $B_\delta(x)$ bounded by Γ. Then we can assume that $s(x,q;x)\subset D$ (retake the arc t_1t_2 in $S_\delta(x)$ if necessary). Since $\Gamma\cap s(x,q;x)=\{q\}$, there is a neighborhood U of $s(x,q:x)$ in D such that $U\cap\Gamma\subset u(q_1,q_2;q)$. Since $s(x,q;x)\subset C^s_{3\varepsilon}(x)=s(x,z;x)$, by expansiveness $s(x,z;x)\cap u(q_1,q_2;q)=\{q\}$. Hence there is a neighborhood U' of $s(x,z;x)$ in M such that U contains the connected component V of $s(x, q; x)$ in $D\cap U'$. Then $s(x, q; x)=s(x, z; x)\cap V$. We claim that there is a conneated neighborhood W of $s(x, q; x)$ in D such that $W\subset V$ and $C_\varepsilon^s(w)\subset U'$ for all $w\in W$. Indeed, if this is false, then we can find a sequence $\{w_i\}_{i\in N}$ of D such that $C_\varepsilon^s(w_i)\not\subset U'$ for all $i\in \boldsymbol{N}$ and w_i converges to some $w_\infty\in s(x, q; x)$ as $i\to\infty$. By Lemma 2.2, $C_\varepsilon^s(w_i)$ converges to some $\Delta_\infty\in \mathcal{C}(M)$ as $i\to\infty$ and then $\Delta_\infty\subset W_\varepsilon^s(w_\infty)$ (by Lemma 2.4). Since

$$w_\infty\in s(x, q; x)\subset s(x, a; x)\subset s(x, y; x) = C_\varepsilon^s(x)\,,$$

obviously $W_\varepsilon^s(w_\infty)\subset W_{3\varepsilon}^s(x)$. Therefore $\Delta_\infty\subset W_{3\varepsilon}^s(x)$ and so $\Delta_\infty\subset C_{3\varepsilon}^s(x)=s(x, z; x)$, contradicting that $C_\varepsilon^s(w_i)\not\subset U'$ for all $i\in \boldsymbol{N}$ and U' is a neighborhood of $s(x, z; x)$ in M. Therefore the conclusion is obtained.

Since $W\subset D\subset B_\delta(x)$, by Proposition C there is $e\in S_\delta(x)\cap C_\varepsilon^s(w)\neq\emptyset$ for every $w\in W$. Since Γ is the boundary of D, it follows that $s(w, e; w)\cap\Gamma\neq\emptyset$. Hence there is $t\in s(w, e; w)\cap\Gamma$ such that $s(w, t; w)\subset D$. Since $w\in W$, we have that $s(w, t; w)\subset U'$. By the choice of V, $s(w, t; w)\subset V\subset U$. Since $U\cap\Gamma\subset u(q_1, q_2; q)$, we have that $t\in u(q_1, q_2; q)$ and therefore $C_\varepsilon^s(w)\cap u(q_1, q_2; q)\neq\emptyset$.

Now we write

$$W_i = \{w\in W\backslash s(x, q; x)\colon C_\varepsilon^s(w)\cap u(q, q_i; q)\neq\emptyset\}\quad (i=1, 2)\,.$$

Then $W_1\cup W_2=W\backslash s(x, q; x)$ as we saw above. It is checked that $W_1\cap W_2=\emptyset$. Indeed, let $w\in W_1\cap W_2\neq\emptyset$. Then $C_\varepsilon^s(w)\cap u(q_1, q_2; q)=\{q\}$ by expansiveness. Hence $s(w, q; w)\subset V$. Since $q\in s(x, y; x)=C_\varepsilon^s(x)$ and $q\in C_\varepsilon^s(w)$, we have that $C_\varepsilon^s(w)\subset C_{3\varepsilon}^s(x)=s(x, z; x)$, and so $s(w, q; w)\subset s(x, z; x)$. Since $s(x, z; x)\cap V= s(x, q; x)$ we have that $s(w, q; w)\subset s(x, q; x)$, thus contradicting $w\in W_i$, Therefore $W_1\cap W_2=\emptyset$.

We claim that W_i is closed in $W\backslash s(x, q; x)$ for $i=1, 2$. Indeed, take a sequence $\{w_i\}_{i\in N}$ of W_1 such that w_i converges to some $w_\infty\in W\backslash s(x, q; x)$ as $i\to\infty$. Then there are $e_i\in C_\varepsilon^s(w_i)\cap u(q, q_i; q)$ $(i\in \boldsymbol{N})$ and $e_\infty\in C_\varepsilon^s(w_\infty)\cap u(q_1, q_2; q)$. By Lemma 4.2, e_i converges to e_∞ as $i\to\infty$. Hence $e_\infty\in u(q, q_1; q)$ and so $w_\infty\in W_1$, which means that W_1 is closed in $W\backslash s(x, q; x)$. We obtain also that W_2 is closed in $W\backslash s(x, d; x)$.

Since W is conneated, so is $W\backslash s(x, q; x)$ and hence $W_i=W\backslash s(x, q; x)$ for $i=1$ or 2 by the above results. Without loss of generality, we may assume that $W_1=W\backslash s(x, q; x)$. Then for $w\in u(q, q_2; q)\backslash\{q\}$ there is a sequence $\{w_i\}_{i\in N}$ of W_1 such that w_i converges to w as $i\to\infty$. Since $w_i\in W_1$, there is $e_i\in C_\varepsilon^s(w_i)\cap u(q, q_1; q)\neq\emptyset$ for every $i\in \boldsymbol{N}$. By Lemma 4.2, e_i converges to some $e_\infty\in u(q, q_1; q)$ as $i\to\infty$. Since $e_i\in W_\varepsilon^s(w_i)$, we have $e_\infty\in W_\varepsilon^s(w)$ by Lemma 2.3. Since e_∞, $w\in u(q_1, q_2; q)$, by expansiveness $e_\infty=w$, thus contradicting that $e_\infty\in u(q, q_1; q)$ and $w\in u(q, q_2; q)\backslash\{q\}$. The conclusion for $\sigma=s$ was obtained.

Lemma 4.14. *For every* $0<\delta\leq c/4$ *there exists* $0<\delta\leq\varepsilon$ *such that*

$$S_\delta(x) \cap \sigma(x, b; x) \neq \emptyset \quad (\sigma = s, u)$$

for all $x \in M$ and all $b \in BC_\varepsilon^\sigma(x)$.

Proof. By Lemma 4.1 there is $0<\delta\leq\varepsilon$ such that $W_{2\varepsilon}^\sigma(x) \cap B_\delta(x) = W_\varepsilon^\sigma(x) \cap B_\delta(x)$. To obtain the conclusion, assume that $\sigma(x, b; x) \subset U_\delta(x)$ for some $x \in M$ and $b \in BC_\varepsilon^\sigma(x)$. Then there is $0<\gamma<\delta$ such that $\sigma(x, b; x) \subset U_\gamma(x)$. Since $b \in C_\varepsilon^\sigma(x)$, we have $C_\varepsilon^\sigma(b) \subset C_{2\varepsilon}^\sigma(x)$ and hence $C_\varepsilon^\sigma(b) \cap B_{\delta-\gamma}(b) \subset C_{2\varepsilon}^\sigma(x) \cap B_{\delta-\gamma}(b)$. Since $B_{\delta-\gamma}(b) \subset B_\delta(x)$ and x and b are jointed by the arc $\sigma(x, b; x)$ in $U_\delta(x)$, the connected component of b in $C_{2\varepsilon}^\sigma(x) \cap B_{\delta-\gamma}(b)$ is contained in that of x in $C_{2\varepsilon}^\sigma(x) \cap B_\delta(x)$. Since $W_{2\varepsilon}^\sigma(x) \cap B_\delta(x) = W_\varepsilon^\sigma(x) \cap B_\delta(x)$, we see easily that the connected component of x in $C_{2\varepsilon}^\sigma(x) \cap B_\delta(x)$ coincides with that of x in $C_\varepsilon^\sigma(x) \cap B_\delta(x)$. Therefore the connected component of b in $C_\varepsilon^\sigma(b) \cap B_{\delta-\gamma}(b)$ is contained in that of x in $C_\varepsilon^\sigma(x) \cap B_\delta(x)$. Therefore $P^\sigma(b)=1$, which contradicts Lemma 4.13. The proof is completed.

For $0<\varepsilon\leq c/4$, let $0<\delta\leq\varepsilon$ be as in Lemma 4.14. By Lemma 4.11 for $x \in M$ we can take $0<\varepsilon(x)<\delta/2$ small enough such that $C_\varepsilon^\sigma(x) \cap B_{\varepsilon(x)}(x)$ has no branch points ($\sigma=s, u$), and define then

$$S_{\varepsilon(x)}^\sigma(x) = \{a \in S_{\varepsilon(x)}(x) \cap C_\varepsilon^\sigma(x) \colon \sigma(x, a; x) \setminus \{a\} \subset U_{\varepsilon(x)}(x)\} . \tag{4.4}$$

We note that $S_{\varepsilon(x)}(x)$ is a circle for every $x \in M$.

Lemma 4.15. *For every $x \in M$, $\#[S_{\varepsilon(x)}^\sigma(x)] = P^\sigma(x)$ ($\sigma=s, u$).*

Proof. The conclusion is easily obtained from Lemmas 4.6 and 4.14.

Lemma 4.16. *For every $x \in M$, $S_{\delta(x)}^\sigma(x)$ is a finite set with at least two points ($\sigma=s, u$). Let I_i^s $(1\leq i\leq l)$ be the open arcs in which $S(x)_{\varepsilon(x)}^s(x)$ cut $S_{\varepsilon(x)}(x)$. Then every $y \in S_{\varepsilon(x)}^u(x)$ is contained in some $I_i^s \in \{I_i^s \colon 1\leq i\leq l\}$. Choose from $S_{\varepsilon(x)}^u(x)$ another point different from y. Then the point is not contained in the same I_i^s. Exchanging s and u, one has the same result.*

Proof. The first staement is obtained from Lemmas 4.8, 4.13 and 4.15. Since $S_{\varepsilon(x)}^\sigma(x) \subset C_\varepsilon^\sigma(x)$, by expansiveness $S_{\varepsilon(x)}^s(x) \cap S_{\varepsilon(x)}^u(x) = \emptyset$ and hence each point of $S_{\varepsilon(x)}^u(x)$ is in some I_i^s. To obtain that distinct two points of $S_{\varepsilon(x)}^u(x)$ are not in the same I_i^s, assume that there are distinct points $a, b \in S_{\varepsilon(x)}^u(x)$ such that $a, b \in I_i^s$ for some i. We denote by ab the subarc in I_i^s jointing a and b. Then it is easily checked that

$$\Gamma = ab \cup u(x, a; x) \cup u(x, b; x)$$

is a simple closed curve in $B_{\varepsilon(x)}(x)$, and so it bounds a disk D in $B_{\varepsilon(x)}(x)$. Put $\Sigma = \bigcup_{z \in S_{\varepsilon(x)}^s(x)} s(x, z; x)$. By the definition of $S_{\varepsilon(x)}^s(x)$, we have that $\Sigma \subset B_{\varepsilon(x)}(x)$ and Σ intersects $S_{\varepsilon(x)}(x)$ only at $S_{\varepsilon(x)}^s(x)$. Since $ab \subset I_i^s$, by expansiveness $\Sigma \cap \Gamma = \{x\}$

and $\Sigma\cap(D\backslash\Gamma)=\emptyset$. For $\varepsilon(x)/2$ choose $\gamma>0$ as in Lemma 4.14. Since $\Sigma\cap D=\{x\}$, there is a neighborhood U of Σ in $B_{\varepsilon(x)}(x)$ such that $U\cap D\subset B_{\gamma/4}(x)$. Since $\Sigma\cap(D\backslash\Gamma)=\emptyset$, we can take a sequence $\{x_i\}_{i=N}$ of $D\backslash\Gamma$ with $d(x, x_i)<\gamma/4$ such that x_i converges to x as $i\to\infty$. Then by Lemma 2.2, $C^s_{\varepsilon(x)/2}(x_i)$ converges to some $\Delta_\infty\in\mathcal{C}(M)$ as $i\to\infty$ (take a subsequence if necessary). Since $\Delta_\infty\subset C^s_{\varepsilon(x)/2}(x)\subset U_{\varepsilon(x)}(x)$, we have that $\Delta_\infty\subset\Sigma$ and hence $C^s_{\varepsilon(x)/2}(x_l)\subset U$ for sufficiently large l. Since γ is as in Lemma 4.14, by Lemma 4.13 there are a_1, $a_2\in C^s_{\varepsilon(x)/2}(x_l)\cap S_{\gamma/2}(x)$ such that $s(x_l, a_1; x_l)\not\sim s(x_l, a_2; x_l)$. Since $C^s_{\varepsilon(x)/2}(x_l)\subset U$ and $U\cap D\subset B_{\gamma/4}(x)$, it follows that a_1, $a_2\notin D$. Hence $s(x_l, a_k; x_l)\cap\Gamma\neq\emptyset\,(k=1, 2)$ and since $s(x_l, a_k; x_l)\subset C^s_{\varepsilon(x)/2}(x_l)\subset U_{\varepsilon(x)}(x)$, we have

$$s(x_l, a_k; x_l)\cap[u(x, a; x)\cup u(x, b; x)]\neq\emptyset \quad (k=1, 2)$$

which contradicts expansiveness.

Lemma 4.17. *$P^s(x)=P^u(x)$ for all $x\in M$.*

Proof. The conclusion is easily obtained from Lemmas 4.15 and 4.16.

Lemma 4.18. *Let $0<\varepsilon\leq c/8$. For every $x\in M$ there exists $0<\eta<\varepsilon(x)$ such that if*

$$y\in B_\eta(x)\backslash\bigcup_{a\in S^\sigma_{\varepsilon(x)}(x)}\sigma(x, a; x)\quad(\sigma=s, u)$$

then $C^\sigma_\varepsilon(y)$ is an arc.

Proof. Using Lemmas 4.8 and 4.13, we can find $\eta_0>0$ such that $P^\sigma(y)=2$ for all $y\in B_{\eta_0}(x)\backslash\{x\}$. If the lemma is false, for $n\in\boldsymbol{N}$ there is

$$y_n\in B_{\eta_0/n}(x)\backslash\bigcap_{a\in S^\sigma_{\varepsilon(x)}(x)}\sigma(x, a; x)$$

such that $C^\sigma_\varepsilon(y_n)$ is not an arc. Since $P^\sigma(y_n)=2$, $C^\sigma_\varepsilon(y_n)$ has a branch point z_n and then $z_n\in\mathrm{Sing}^\sigma(f)$. By Lemma 4.8, we can assume that $z_n=z$ for all $n\in\boldsymbol{N}$. By Lemma 2.2 there is a subsequence $\{n\}$ of $\boldsymbol{N}$ such that $C^\sigma_\varepsilon(y_n)$ converges to some $\Delta_\infty\in\mathcal{C}(M)$ as $n\to\infty$. Since $y_n\to x$, $\Delta_\infty\subset C^\sigma_\varepsilon(x)$ by Lemma 2.4. Since $z=z_n\in C^\sigma_\varepsilon(y_n)$, obviously $z\in\Delta_\infty$ and so $z\in C^\sigma_\varepsilon(x)$. Hence $x\in C^\sigma_{2\varepsilon}(z)$. Since $z\in C^\sigma_\varepsilon(y_n)$, $y_n\in C^\sigma_{2\varepsilon}(z)$. Since $0<2\varepsilon\leq c/4$, we note that $C^\sigma_{2\varepsilon}(z)$ is a finite uniion of arcs (lemmas 4.5 and 4.12). Since $y_n\to x$, there is an arc A_n in $C^\sigma_{2\varepsilon}(z)$ with sufficiently small diameter such that y_n, $x\in A_n$. Since $z\in C^\sigma_\varepsilon(x)$, $C^\sigma_{2\varepsilon}(z)\subset C^\sigma_{3\varepsilon}(x)$ and so $A_n\subset C^\sigma_{3\varepsilon}(x)$. By Lemma 4.1 it is easily checked that $A_n\subset C^\sigma_\varepsilon(x)$. Hence $A_n\subset\bigcup_{a\in S^\sigma_{\varepsilon(x)}(x)}\sigma(x, a; x)$, thus contradicting the choice of y_n.

5. Proof of (1), (2) and (3) in Proposition A

In this section we shall give the proof of (1), (2) and (3) of Proposition A.

As before let $f: M \to M$ be an expansive homeomorphism with expansive constant $c>0$. Fix $0<\varepsilon\leq c/8$.

For $x\in M$ let $P^{\sigma}(x)$ $(\sigma=s, u)$ be as in §4. By Lemma 4.17, $P^{s}(x)=P^{u}(x)$ and so we define

$$p(x)=P^{\sigma}(x) \quad (\sigma=s, u).$$

By Lemmas 4.10 and 4.13 we have that $2\leq p(x)<\infty$ for all $x\in M$.

Next, let $x\in M$. Then we can construct a C^0 chart $\varphi_x: U_x \to \boldsymbol{C}$ as follows:

Construction of U_x. Let $\varepsilon(x)>0$ be as in §4 and define $S^{\sigma}_{\varepsilon(x)}(x)$ $(\sigma=s, u)$ as in (4.4). Then $S^{\sigma}_{\varepsilon(x)}(x)$ is a subset of a cirlce $S_{\varepsilon(x)}(x)$. Since $\#[S^{\sigma}_{\varepsilon(x)}(x)]=p(x)$ by Lemma 4.15 and $2\leq p(x)<\infty$, we have that $S^{\sigma}_{\varepsilon(x)}(x)$ cut $S_{\varepsilon(x)}(x)$ in $p(x)$ open arcs $I^{\sigma}_{i}(1\leq i\leq P(x))$. From Lemma 4.16 it follows that

$$S^{s}_{\varepsilon(x)}(x)\subset\bigcup_{i=1}^{p(x)} I^{u}_{i}, \quad S^{u}_{\varepsilon(x)}(x)\subset\bigcup_{i=1}^{p(x)} I^{s}_{i}.$$

Since $\#[S^{\sigma}_{\varepsilon(x)}(x)]=p(x)$ $(\sigma=s, u)$, we see by Lemma 4.16 that $S^{s}_{\varepsilon(x)}(x)\cap I^{u}_{i}$ is exactly one point a^{s}_{i} and $S^{u}_{\varepsilon(x)}(x)\cap I^{s}_{i}$ is exactly one point a^{u}_{i} for every $1\leq i\leq p(x)$. Since each I^{σ}_{i} is an open arc of $S_{\varepsilon(x)}(x)\backslash S^{\sigma}_{\varepsilon(x)}(x)$, we may assume that the boundary points of I^{s}_{i} are a^{s}_{i} and a^{s}_{i+1}, and that the boundary points of I^{u}_{i} are a^{u}_{i-1} and a^{u}_{i}, where $a^{\sigma}_{p(x)+1}=a^{\sigma}_{1}$ and $a^{\sigma}_{0}=a^{\sigma}_{p(x)}$. Then $\{a^{s}_{i}\}\cup I^{s}_{i}\cup\{a^{s}_{i+1}\}$ and $\{a^{u}_{i-1}\}\cup I^{u}_{i}\cup\{a^{u}_{i}\}$ are arcs of $S_{\varepsilon(x)}(x)$, and so we denote them by $a^{s}_{i}\, a^{s}_{i+1}$ and $a^{u}_{i-1}\, a^{u}_{i}$ respeatively. Obviously $a^{u}_{i}\in a^{s}_{i}a^{s}_{i+1}$ and $a^{s}_{i}\in a^{u}_{i-1}\ a^{u}_{i}$ for $1\leq i\leq p(x)$. We denote by $a^{s}_{i}\ a^{u}_{i}$ the subarc of $a^{s}_{i}\ a^{s}_{i+1}$ jointing a^{s}_{i} and a^{s}_{i}. The notation $a^{u}_{i}\ a^{s}_{i+1}$ is also defined.

By the definition of $S^{\sigma}_{\varepsilon(x)}(x)$ $(\sigma=s, u)$ we have that the arc $\sigma(x, a^{\sigma}_{i}; x)$ is contained in a disk $B_{\varepsilon(x)}(x)$ and it intersects $S_{\varepsilon(x)}(x)$ only at a^{σ}_{i} for $1\leq i\leq p(x)$. Since $s(x, a^{s}_{i}; x)\cap u(x, a^{u}_{i}; x)=\{x\}$ by expansiveness, it follows that

$$\Gamma^{s}_{i}=a^{s}_{i}\ a^{u}_{i}\cup s(x, a^{s}_{i}; x)\cup u(x, a^{u}_{i}; x)$$

is a simple closed curve in $B_{\varepsilon(x)}(x)$, and so Γ^{s}_{i} bounds a disk D^{s}_{i} in $B_{\varepsilon(x)}(x)$. Also we have that

$$\Gamma^{u}_{i}=a^{u}_{i}\ a^{s}_{i+1}\cup u(x, a^{u}_{i}; x)\cup s(x, a^{s}_{i+1}; x)$$

bounds a disk D^{u}_{i} in $B_{\varepsilon(x)}(x)$. Since $p(x)\geq 2$, obviously $D^{s}_{i}\cap D^{u}_{i}=u(x, a^{u}_{i}; x)$ and $D^{u}_{i}\cap D^{s}_{i+1}=s(x, a^{s}_{i+1}; x)$ for $1\leq i\leq p(x)$.

Let $0<\eta<\varepsilon(x)$ be as in Lemma 4.18. For $1\leq i\leq p(x)$, take and fix $y_i\in s(x, a^{s}_{i}; x)$ such that $0<d(x, y_i)\leq\eta$. Then $C^{u}_{\varepsilon}(y_i)$ is an arc, and so we denote its end points by $b_i(1)$ and $b_i(2)$. Lemma 4.13 ensures that $y_i\neq b_i(k)$ $(k=1,2)$. Since $0<\varepsilon(x)<\delta/2$ and δ is as in Lemma 4.14 (see §4), it follows that $u(y_i, b_i(k); y_i)\cap S_{\varepsilon(x)}(x)\neq\emptyset$ for $k=1, 2$, and hence we can find $c_i(k)\in u(y_i, b_i(k); y_i)$ $(k=1, 2)$ such that $u(y_i, c_i(k); y_i)\subset B_{\varepsilon(x)}(x)$ and $u(y_i, c_i(k); y_i)\cap S_{\varepsilon(x)}(x)=\{c_i(k)\}$. Since $y_i\in s(x, a^{s}_{i}; x)$ and $y_i\neq x$, by expansiveness it is easily checked that

$$(5.1) \qquad \{u(x, a^u_{i-1}; x) \cup u(x, a^u_i; x)\} \cap u(y_i, c_i(k); y_i) = \emptyset \quad (k = 1, 2).$$

Combining (5.1) and the fact that $D^u_{i-1} \cup D^s_i$ is a disk in $B_{\varepsilon(x)}(x)$ bounded by

$$a^u_{i-1}\, a^s_i \supset a^s_i\, a^u_i \cup u(x, a^u_{i-1}; x) \cup u(x, a^u_i; x),$$

we see that $u(y_i, c_i(k); y_i) \subset D^u_{i-1} \cup D^s_i$ for $k=1,2$. By expansiveness, $u(y_i, c_i(k); y_i) \cap s(x, a^s_i; x) = \{y_i\}$, and therefore $u(y_i, c_i(k); y_i)\,(k=1,2)$ are contained in D^u_{i-1} or D^s_i respectively.

We deal with the case $u(y_i, c_i(1); y_i) \subset D^u_{i-1}$. In this case, by using Lemma 4.16 it is easily checked that $u(y_i, c_i(2); y_i) \subset D^s_i$. Note that $c_i(k) \in S_{\varepsilon(x)}(x)$ $(k=1,2)$. Then we have that $c_i(1) \in a^u_{i-1}\, a^s_i$ and $c_i(2) \in a^s_i\, a^u_i$.

Choose $z_i \in u(x, a^u_i; x)\,(1 \le i \le p(x))$ such that $0 < d(z_i, x) \le \eta$. Then $C^s_\varepsilon(z_i)$ is an arc. In the same way as above, we can find $d_i(k) \in C^s_\varepsilon(z_i)$ $(k=1,2)$ such that $d_i(1) \in a^s_i\, a^u_i$ and $s(z_i, d_i(1); z_i) \subset D^s_i$, and such that $d_i(2) \in a^u_i\, a^s_{i+1}\, u(z_i, d_i(2); z_i) \subset D^u_i$.

We claim that if $d(z_i, x)$ is sufficiently small, then

$$(5.2) \qquad s(z_i, d_i(1); z_i) \cap u(y_i, c_i(2); y_i) \neq \emptyset,$$

$$(5.3) \qquad s(z_i, d_i(2); z_i) \cap u(y_{i+1}, c_{i+1}(1); y_{i+1}) \neq \emptyset.$$

Indeed, $u(y_i, c_i(2); y_i)$ cuts D^s_i in two components $D^s_i(-)$ and $D^s_i(+)$ because $u(y_i, c_i(2); y_i)$ is contained in D^s_i and it intersects Γ^s_i only at two poonts $y_i, c_i(2)$. Since $c_i(2) \in a^s_i\, a^u_i$, it is clear that $a^s_i\, c_i(2) \setminus \{c_i(2)\}$ and $c_i(2)\, a^u_i \setminus \{c_i(2)\}$ is contained in $D^s_i(-)$ or $D^s_i(+)$ respectively, where $a^s_i\, c_i\,(2)$ and $c_i\,(2)\, a^u_i$ denote the subarcs of $a^s_i\, a^u_i$. Hence $c_i(2)\, a^u_i \setminus \{c_i(2)\} \subset D^s_i(+)$ whenever $a^s_i\, c_i(2) \setminus \{c_i(2)\} \subset D^s_i(-)$.

To show the above claim, assume that $d_i(1) \in c_i(2)\, a^u_i$ even if $d(x, z_i)$ is small enough. By Lemma 2.2 there is a sequence $\{z_i\}$ such that $d_i(1)$ and $s(z_i, d_i(1); z_i)$ converge to some $d_\infty \in c_i(2)\, a^u_i$ and some $\Delta_\infty \in \mathcal{C}(D^s_i)$ as $z_i \to x$, respectively. Then $\Delta_\infty \subset W^s_\varepsilon(x)$ by Lemma 2.3, and so $\Delta_\infty \subset C^s_\varepsilon(x)$. Since $x, d_\infty \in \Delta_\infty$, it follows that $s(x, d_\infty; x) \subset \Delta_\infty$. Since $\Delta_\infty \subset D^s_i$, obviously $s(x, d_\infty; x) \subset D^s_i$. Combining this and the fact that $d_\infty \notin s(x, a^s_i; x)$, we see that $s(x, a^s_i; x) \subset s(x, d_\infty; x)$. Since $d_\infty \in c_i(2)\, a^u_i$, this implies that $s(x, d_\infty; x)$ intersects $u(y_i, c_i(2); y_i)$ in at least two points, thus contradicting expansiveness. Therefore $d_i(1) \in a^s_i\, c_i(2) \setminus \{c_i(2)\}$ whenever $d(z_i, x)$ is small enough, i.e., $d_i(1) \in D^s_i(-)$. Since $z_i \in D^s_i(+)$, (5.2) holds. (5.3) is also obtained.

For $1 \le i \le p(x)$, take and fix $z_i \in u(x, a^u_i; x)$ such that (5.2) and (5.3) hold. Expansiveness ensures that the left sides of (5.2) and (5.3) are exactly one point $w_i(-)$ and $w_i(+)$ respectively. It is clear that

$$J^s_i = s(x, y_i; x) \cup u(y_i, w_i(-); y_i) \\ \cup s(z_i, w_i(-); z_i) \cup u(x, z_i; x)$$

is a simple closed curve in D^s_i. Hence J^s_i bounds a disk R^s_i in D^s_i. In the same way, we obtain that

$$J_i^u = u(x, z_i; x) \cup s(z_i, w_i(+); z_i) \\ \cup u(y_{i+1}, w_i(+); y_{i+1}) \cup s(x, y_{i+1}; x)$$

bounds a disk R_i^u in D_i^u. For $1 \leq i \leq p(x)$, we write

$$U_i^s = R_i^s \setminus \{u(y_i, w_i(-); y_i) \cup s(z_i, w_i(-); z_i)\} ,$$
$$U_i^u = R_i^u \setminus \{s(z_i, w_i(+); z_i) \cup u(y_{i+1}, w_i(+); y_{i+1})\} .$$

And define $U_x = \bigcup_{i=1}^{p(x)} (U_i^s \cup U_i^u)$. Since $\bigcup_{i=1}^{p(x)} (R_i^s \cup R_i^u)$ is a disk and its boundary is

$$\bigcup_{i=1}^{p(x)} [u(y_i, w_i(-); y_i) \cup s(z_i, w_i(-); z_i) \\ \cup s(z_i, w_i(+); z_i) \cup u(y_{i+1}, w_i(+); y_{i+1})] ,$$

it follows that U_x is an open disk which contains the point x. This U_x is our desire.

For $p \geq 2$ let $\mathcal{D}_p$, $\mathcal{H}_p$ and $\mathcal{V}_p$ be as in §1. To construct φ_x: $U_x \to \boldsymbol{C}$, we define the coordingates of $\mathcal{D}_p$ with respect to $\mathcal{H}_p$ and $\mathcal{V}_p$ as follows. Let R_θ: $\boldsymbol{C} \to \boldsymbol{C}$ denote the rotation which sends z to $e^{i\theta}z$, and write

$$H_p^i = R_{2\pi(i-1)/p}([0, 1]) , \quad V_p^v = R_{\pi/p}(H_p^i)$$

for $1 \leq i \leq p$. Then

$$L_p^h = \bigcup_{i=1}^{p} H_p^i \quad \text{and} \quad L_p^v = \bigcup_{i=1}^{p} V_p^i$$

are the elements of $\mathcal{H}_p$ and $\mathcal{V}_p$ through $0 \in \boldsymbol{C}$ respectively. We denote by $\mathcal{D}_{p,i}^h$ the closed subset of $\mathcal{D}_p$ which is enclosed with H_p^i and V_p^i, and by $\mathcal{D}_{p,i}^v$ the closed subset of $\mathcal{D}_p$ which is enclosed with V_p^i and H_p^{i+1}. Clearly

$$\mathcal{D}_p = \bigcup_{i=1}^{p} (\mathcal{D}_{p,i}^h \cup \mathcal{D}_{p,i}^v) ,$$
$$\mathcal{D}_{p,i}^h \cap \mathcal{D}_{p,i}^v = V_p^i , \quad \mathcal{D}_{p,i}^v \cap \mathcal{D}_{p,i+1}^h = H_p^{i+1} \quad (1 \leq i \leq p) .$$

Let $(z_1, z_2) \in H_p^i \times V_p^i$. Then the element of $\mathcal{V}_p$ through z_1 and the element of $\mathcal{H}_p$ through z_2 intersect in exactly one point $\alpha_i^h(z_1, z_2) \in \mathcal{D}_{p,i}^h$. It is easily checked that

$$\alpha_i^h : H_p^i \times V_p^i \to \mathcal{D}_{p,i}^h \quad (1 \leq i \leq p)$$

are homeomorphisms. By the same fashion we can define homeomorphisms

$$\alpha_i^v : V_p^i \times H_p^{i+1} \to \mathcal{D}_{p,i}^v \quad (1 \leq i \leq p) .$$

Construction of φ_x: $U_x \to C$. Let $1 \leq i \leq p(x)$. In the same way as in Construction of U_x, for $y \in s(x, y_i; x)$ we can find $c_y(k) \in C_\varepsilon^u(y)$ $(k=1, 2)$ such that

$$u(y, c_y(1); y) \subset R_{i-1}^u, \quad u(y, c_y(2); y) \subset R_i^s,$$
$$u(y, c_y(1); y) \cap s(z_{i-1}, w_{i-1}(+); z_{i-1}) = \{c_y(1)\},$$
$$u(y, c_y(2); y) \cap s(z_i, w_i(-); z_i) = \{c_y(2)\}.$$

And also for $z \in u(x, z_i; x)$ we can find $d_z(k) \in C_\varepsilon^s(z)$ $(k=1, 2)$ such that

$$s(z, d_z(1); z) \subset R_i^s, \quad s(z, d_z(2); z) \subset R_i^u,$$
$$s(z, d_z(1); z) \cap u(y_i, w_i(-); y_i) = \{d_z(1)\},$$
$$s(z, d_z(2); z) \cap u(y_{i+1}, w_i(+); y_{i+1}) = \{d_z(2)\}.$$

Let $(y, z) \in s(x, y_i; x) \times u(x, z_i; x)$. Then it is easily checked that

$$\begin{aligned} W_\varepsilon^u(y) \cap W_\varepsilon^s(z) &\supset C_\varepsilon^u(y) \cap C_\varepsilon^s(z) \\ &\supset u(y, c_y(2); y) \cap s(z, d_z(1); z) \neq \emptyset. \end{aligned}$$

Hence $W_\varepsilon^u(y) \cap W_\varepsilon^s(z)$ is exactly one point by expansiveness and the point is penoted by $\alpha_i^s(y, z)$. Since $u(y, c_y(2); y)$ and $s(z, d_z(1); z)$ are contained in R_i^s, we have $\alpha_i^s(y, z) \in R_i^s$, and therefore

$$\alpha_i^s \colon s(x, y_i; x) \times u(x, z_i; x) \to R_i^s \quad (1 \leq i \leq p(x))$$

are defined. By Lemma 4.2 and expansiveness α_i^s is continuous and injective. It is clear that

$$\begin{aligned} \alpha_i^s(y, x) &= y \qquad \text{if} \quad y \in s(x, y_i; x), \\ \alpha_i^s(x, z) &= z \qquad \text{if} \quad z \in u(x, z_i; x). \end{aligned}$$

Since $\alpha_i^s(y_i, z_i) = w_i(-)$, we have that

$$\alpha_i^s(s(x, y_i; x) \times \{z_i\}) = s(z_i, w_i(-); z_i),$$
$$\alpha_i^s(\{y_i\} \times u(x, z_i; x)) = u(y_i, w_i(-); y_i),$$

and hence α_i^s sends the boundary of $s(x, y_i; x) \times u(x, z_i; x)$ onto the boundary of R_i^s. Since R_i^s is a disk and α_i^s is continuous, the image of α_i^s coincides with R_i^s. Since α_i^s is injective, consequently α_i^s is a homeomorphism.

For $1 \leq i \leq p(x)$, we write

$$E_i^s = [s(x, y_i; x) \setminus \{y_i\}] \times [u(x, z_i; x) \setminus \{z_i\}]$$

and define

$$\beta_i^s = \alpha_i^s|_{E_i^s} \colon E_i^s \to U_i^s.$$

In the same way as above, we have that if $(z, y) \in u(x, z_i; x) \times s(x, y_{i+1}; x)$

then $W^s_\varepsilon(z)\cap W^u_\varepsilon(z)=\{\alpha^u_i(z,y)\}\subset R^u_i$. Hence homeomorphisms

$$\alpha^u_i\colon u(x, z_i; x)\times s(x, y_{i+1}; x)\to R^u_i \quad (1\leq i\leq p(x))$$

are obtained such that

$$\begin{aligned}
&\alpha^u_i(z, x) = z \qquad \text{if} \quad z\in u(x, z_i; x)\,,\\
&\alpha^u_i(x, y) = y \qquad \text{if} \quad y\in s(x, y_{i+1}; x)\,,\\
&\alpha^u_i(u(x, z_i; x)\times\{y_{i+1}\}) = u(y_{i+1}, w_i(+); y_{i+1})\,,\\
&\alpha^u_i(\{z_i\}\times s(x, y_{i+1}; x)) = s(z_i, w_i(+); z_i)\,.
\end{aligned}$$

So we write

$$E^u_i = [u(x, z_i; x)\setminus\{z_i\}]\times[s(x, y_{i+1}; x)\setminus\{y_{i+1}\}]\,,$$

and define

$$\beta^u_i = \alpha^u_i|_{E^u_i}\colon E^u_i\to U^u_i \quad (1\leq i\leq p(x))\,.$$

For $1\leq i\leq p(x)$, let $g^s_i\colon s(x, y_i; x)\setminus\{y_i\}\to H^i_{p(x)}$ and $g^u_i\colon u(x, z_i; x)\setminus\{z_i\}\to V^i_{p(x)}$ be homeomorphisms, and define

$$r^s_i\colon U^s_i\to \mathcal{D}^h_{h(x),i}\,,\quad r^u_i\colon U^u_i\to\mathcal{D}^v_{p(x),i}$$

by

$$\begin{aligned}
r^s_i &= \alpha^h_i\circ(g^s_i\times g^u_i)\circ(\beta^s_i)^{-1}\,,\\
r^u_i &= \alpha^v_i\circ(g^u_i\times g^s_{i+1})\circ(\beta^u_i)^{-1}
\end{aligned}$$

respectively. Then it is easily obtained that $r^\sigma_i (1\leq i\leq p(x),\ \sigma=s, u)$ are homeomorphisms with the following properties:

$$\gamma^s_i|_{U^s_i\cap U^u_i} = \gamma^u_i|_{U^s_i\cap U^u_i}\,,\quad \gamma^u_i|_{U^u_i\cap U^s_{i+1}} = \gamma^s_{i+1}|_{U^u_i\cap U^s_{i+1}}\,.$$

Therefore we can define a map $\varphi_x\colon U_x\to\mathcal{D}_{p(x)}$ by

$$\varphi_x|_{U^\sigma_i} = \gamma^\sigma_i \quad (1\leq i\leq p(x),\ \sigma = s, u)\,.$$

Obviously φ_x is a homeomorphism which sends x to 0. This φ_x is our desire.

Now we define $S=\{x\in M; p(x)\geq 3\}$. Since $p(x)=P^\sigma(x)$ ($\sigma=s, u$), we remark that $S=\mathrm{Sing}^\sigma(f)$ where $\mathrm{Sing}^\sigma(f)$ is as in §4.

Proof of (1), (2) and (3) in Proposition A. For $x\in M$, write

$$L^s(x, x) = \bigcup_{i=1}^{p(x)}[s(x, y_i; x)\setminus\{y_i\}]\,,$$

$$L^u(x, x) = \bigcup_{i=1}^{p(x)}[u(x, z_i; x)\setminus\{z_i\}]\,.$$

Obviously $L^\sigma(x, x)\subset C^\sigma_\varepsilon(x)\subset W^\sigma_\varepsilon(x)$ $(\sigma=s, u)$. By the construction of φ_x it is easily checked that

$$(5.4)\qquad \varphi_x(L^s(x, x)) = L^h_{p(x)}\,, \quad \varphi_x(L^u(x, x)) = L^v_{p(x)}$$

for all $x\in M$.

Let $x\in M$ and $1\leq i\leq p(x)$. For $z\in u(x, z_i; x)\setminus \{x, z_i\}$, write

$$L^s(x, z) = \beta^s_i([s(x, y_i; x)\setminus\{y_i\}]\times \{z\})\cup\beta^u_i(\{z\}\times[s(x, y_{i+1}; x)\setminus\{y_{i+1}\})\,.$$

Then by the definition of $\beta^\sigma_i(\sigma=s, u)$ we have that $L^s(x, z)\subset C^s_\varepsilon(z)\subset W^s_\varepsilon(z)$. By the construction of φ_x it is obtained easily that φ_x sends $L^s(x, z)$ onto an element of $\mathcal{H}_{p(x)}$. Combining this fact and (5.4), we see that $\varphi_x(\{L^s(x, z);\, z\in L^u(x, x)\})=\mathcal{H}_{p(x)}$ and hence

$$(5.5)\qquad U_x=\bigcup_{z\in L^u(x,x)} L^s(x, z)\quad \text{(disjoint union)}.$$

Since $L^s(x, z)\subset W^s_\varepsilon(z)$, by (5.5) and expansiveness it follows that $L^s(x, z)= U_x\cap W^s_\varepsilon(z)$ for all $z\in L^u(x, x)$.

Let $y\in s(x, y_i; x)\setminus\{z, y_i\}$ $(1\leq i\leq p(x))$ and write

$$L^u(x, y) = \beta^u_{i-1}([u(x, z_{i-1}; x)\setminus\{z_{i-1}\}]\times \{y\})\cup\beta^s_i(\{y\}\times[u(x, z_i; x)\setminus\{z_i\}])\,.$$

Then $L^u(x, y)\subset W^u_\varepsilon(y)$. In the same way as above, we have that $\varphi_x(\{L^u(x, y);\, y\in L^s(x, x)\}=\mathcal{V}_{p(x)}$, and hence

$$(5.6)\qquad U_x=\bigcup_{y\in L^s(x,x)} L^u(x, y)\quad \text{(disjoint union)}$$

and $L^u(x, y) = U_x\cap W^u_\varepsilon(y)$ for all $y\in L^s(x, x)$.

As in Proposition A, let $L_p=\{z\in \boldsymbol{C};\ \mathrm{Im}\ z^{p/2}=0\}$ $(p\geq 2)$. We show that for $x\in M$ and $\sigma=s, u$ there are $p\geq 2$ and an injective continuous map $j^\sigma_x\colon L_p\to M$ such that $j^\sigma_x(L_p)=W^\sigma(x)$. To do this, take a finite subset A of M such that $\{U_a;\, a\in A\}$ is a covering of M. Obviously $S\subset A$. Let $0<\rho<2\varepsilon$ be a Lebesgue number of $\{U_a;\, a\in A\}$. For $x\in M$ choose $a(x)\in A$ such that $B_\rho(x)\subset U_{a(x)}$. Then $a(x)=x$ if $x\in S$. Let $x\in M$ and put

$$M^\sigma(a(x), x) = U_{a(x)}\cap W^\sigma_{2\varepsilon}(x)\quad (\sigma = s, u)\,.$$

Then we have

$$(5.7)\qquad W^\sigma_\rho(x) = U_{a(x)}\cap W^\sigma_\rho(x)\subset M^\sigma(a(x), x)\quad (\sigma = s, u)\,.$$

By (5.5) and (5.6) there is $w\in L^\sigma(a(x), a(x))$ such that $x\in L^{\sigma'}(a(x), w)$ where $\sigma'=s$ (reps. $\sigma'=u$) if $\sigma=u$ (resp. $\sigma=s$). Since $L^{\sigma'}(a(x), w)=U_{a(x)}\cap W^{\sigma'}_\varepsilon(w)$, it is easily checked that $L^{\sigma'}(a(x), w)\subset M^{\sigma'}(a(x), x)$. Combining this fact, (5.5) and (5.6), by expansiveness we obtain that $L^{\sigma'}(a(x), w)=M^{\sigma'}(a(x), x)$.

By (1.2) we have that

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}\, W^s_\rho(f^n(x))\,, \quad W^u(x) = \bigcup_{n\geq 0} f^n\, W^u_\rho(f^{-n}(x))\,, \tag{5.8}$$

and by (1.1) there is $n_0>0$ such that

$$\begin{aligned} &f^{n_0}(M^s(a(x), x))\subset W^s_\rho(f^{n_0}(x))\,, \\ &f^{-n_0}(M^u(a(x), x))\subset W^u_\rho(f^{-n_0}(x))\,. \end{aligned} \tag{5.9}$$

So we put $g=f^{n_0}$ and write

$$\begin{aligned} s_n(x) &= g^{-n}[M^s(a(g^n(x)), g^n(x))]\,, \\ u_n(x) &= g^n[M^u(a(g^{-n}(x)), g^{-n}(x))]\,. \end{aligned}$$

Then from (5.7) and (5.9) it follows that

$$s_n(x)\subset g^{-n-1}\, W^s_\rho(g^{n+1}(x))\subset s_{n+1}(x)\,, \tag{5.10}$$

$$u_n(x)\subset g^{n+1}\, W^u_\rho(g^{-n-1}(x))\subset u_{n+1}(x)\,, \tag{5.11}$$

and therefore by (5.8)

$$W^\sigma(x) = \bigcup_{n\geq 0} \sigma_n(x) \quad (\sigma = s, u)\,. \tag{5.12}$$

Let $x\in S$. Since S is finite, we can assume that x is a fixed point of g. Since $a(x)=x$, it follows that $M^\sigma(a(x), x)=L^\sigma(x, x)$, and hence $\sigma_n(x)$ is homeomorphic to $L_{p(x)}$ for all $n\geq 0$. By (5.10), (5.11) and (5.12) we can construct an injective continuous map j^σ_x: $L_{p(x)}\to M$ such that $j^\sigma_x(L_{p(x)})=W^\sigma(x)$. Let $y\in S$ and let $x\in W^\sigma(y)$. Then $W^\sigma(x)=W^\sigma(y)$. Hence a bijective continuous map j^σ_x: $L_{p(y)}\to W^\sigma(x)\subset M$ is obtained. Let $x\in M\setminus \bigcup_{y\in S} W^s(y)$. Then it is easily checked that $M^s(a(g^n(x)), g^n(x))$ is an open arc for all $n\geq 0$. Hence by (5.10) and (5.12) we can construct an injective continuous map j^s_x: $L_2\to M$ such that $j^s_x(L_2)=W^s(x)$. In the same way, for $x\in M\setminus \bigcup_{y\in S} W^u(y)$ the map j^u_x: $L_2\to W^u(x)$ is obtained.

Therefore $W^\sigma(x)$ $(\sigma=s, u)$ are path connected.

To obtain that $\mathcal{F}^\sigma_f(\sigma=s, u)$ are C^0 singular foliations, it is enough to show that for $x, y\in M$ every connected component of $W^\sigma(x)\cap U_y$ is of form $L^\sigma(y, z)$. We give the proof for $\sigma=s$.

Let $w\in W^s(x)\cap U_y$. By (5.7) there is $z\in L^u(y, y)$ such that $w\in L^s(y, z)$. Since $L^s(y, z)\subset W^s_\varepsilon(z)$, there is $n_1>0$ such that $g^{n_1}(L^s(y, z))\subset W^s_{\rho/2}(g^{n_1}(w))$. Since $w\in W^s(x)$, we can assume $g^{n_1}(w)\in W^s_{\rho/2}(g^{n_1}(x))$. Then we have

$$g^{n_1}(L^s(y, z))\subset W^s_\rho(g^{n_1}(x))\,.$$

Since

$$W_p^s(g^{n_1}(x)) \subset M^s(a(g^{n_1}(x)), g^{n_1}(x)) \quad (\text{by } (5.6)),$$

by the definition of $s_{n_1}(x)$ we have $L^s(y, z) \subset s_{n_1}(x)$. Since $s_{n_1}(x) \subset W^s(x)$, $L^s(y, z) \subset W^s(x)$. Hence there is a subset $\{z_\lambda\}_{\lambda \in \Lambda}$ of $L^u(y, y)$ such that

$$W^s(x) \cap U_y = \bigcup_{\lambda \in \Lambda} L^s(y, z_\lambda).$$

Since $L^s(y, z_\lambda)$ is either an open arc or homeomorphic to $L^h_{p(y)}$, $(j_x^s)^{-1}(L^s(y, z_\lambda))$ is open in L_p where $j_x^s(L_p) = W^s(x)$. Note that $L^s(p, z_\lambda)$ $(\lambda \in \Lambda)$ are mutually disjoint (by (5.5)). Hence $\{z_\lambda\}_{\lambda \in \Lambda}$ is at most countable, and therefore each $L^s(y, z_\lambda)$ is a connected component of $W^s(x) \cap U_y$. It is obtained also that each connected component of $W^u(x) \cap U_y$ is of form $L^u(y, z)$. Therefore $\mathcal{F}_f^\sigma(\sigma = s, u)$ are C^0 singular foliations and S is the set of all singular points of $\mathcal{F}_f^\sigma$.

By the definition of j_x^σ we see that the toploogy of $W^\sigma(x)$ induced by j_x^σ coincides with the leaf topology. Hence each $W^\sigma(x)$ is homoemorphic to L_p $(p \geq 2)$. As we saw above, φ_x sends $L^s(x, z)$ onto an element of $\mathcal{H}_{p(x)}$ and φ_x sends $L^u(x, z)$ onto an element of $\mathcal{V}_{p(x)}$. Therefore $\mathcal{F}_f^s$ is transverse to $\mathcal{F}_f^u$.

6. Proof of (4) in Proposition A

Let $\mathcal{F}$ be a C^0 singular foliation on M and let S be the set of all singular points of $\mathcal{F}$. We recall that $\mathcal{RF}$ denotes the C^0 foliation on $M \backslash S$ obtained by taking singular points away from each leaf of $\mathcal{F}$. A simple closed curve Γ of $M \backslash S$ is called a *closed transversal* of $\mathcal{RF}$ if all subarcs of Γ are transversals of $\mathcal{F}$. Let A be a connected subset of a leaf of $\mathcal{RF}$. Clearly there is $L \in \mathcal{F}$ such that $A \subset L$. If $s \in L \cap S \neq \emptyset$ and if s is a boundary point of A in L, then we say that *A leads to s*.

As before let $f: M \to M$ be an expansive homeomorphism and let $\mathcal{F}_f^\sigma = \{W^\sigma(x): x \in M\}$ $(\sigma = s, u)$. From the results of §5 it follows that $\mathcal{F}_f^\sigma$ satisfies all of (1), (2) and (3) in Proposition A. Hereafter let S be the set of all sigular points of $\mathcal{F}_f^\sigma$. Define $\mathcal{RF}_f^\sigma$ as above. For the proof of (4) in Proposition A we prepare the following

Lemma 6.1. *Suppose that $\mathcal{F}_f^\sigma(\sigma = s, u)$ are orientable. If Γ is a closed transversal of $\mathcal{RF}_f^s$ (resp. $\mathcal{RF}_f^u$), then Γ intersects each leaf of $\mathcal{RF}_f^s$ (resp. $\mathcal{RF}_f^u$) in at least one point.*

For $x \in M \backslash S$ let $L^\sigma(x)$ denote the leaf of $\mathcal{RF}_f^\sigma$ through $x (\sigma = s, u)$. By Proposition A(2) we have that each $L^\sigma(x)$ is homeomorphic to $\boldsymbol{R}$. Suppose that $\mathcal{F}_f^\sigma$ $(\sigma = s, u)$ are orientable. Then an order relation for $L^\sigma(x)$ is defined as follows. Let $y, z \in L^\sigma(x)$. We say $y \leq_\sigma z$ if either $y = z$ or the arc in $L^\sigma(x)$ from y to z has the same orientation as that of $L^\sigma(x)$. When $y \leq_\sigma z$ and $y \neq z$, we write $y <_\sigma z$.

For $x \in M \backslash S$ we define

$$L_+^\sigma(x) = \{y \in L^\sigma(x) \colon x <_\sigma y\}\ ,$$
$$L_-^\sigma(x) = \{y \in L^\sigma(x) \colon y <_\sigma x\}\ .$$

For $y, z \in L^\sigma(x)$ with $y <_\sigma z$ we define

$$[y, z]_\sigma = \{w \in L^\sigma(x) \colon y \leq_\sigma w \leq_\sigma z\}$$

and write

$$[y, z)_\sigma = [y, z]_\sigma \backslash \{z\}\ , \quad (y, z]_\sigma = [y, z]_\sigma \backslash \{y\}\ ,$$
$$(y, z)_\sigma = [y, z]_\sigma \backslash \{y, z\}\ .$$

We call here *intervals* in leaves such subsets.

Lemma 6.2. *Let I and I' be intervals in leaves of $\mathcal{RF}_f^s$ and let $cl(I)$ and $cl(I')$ denote the closures of I and I' in the leaves of $\mathcal{F}_f^s$ respectively. Suppose that $cl(I)$ is compact. If $h\colon I \to I'$ is a map which sends $x \in I$ to $h(x) \in L_+^u(x)$ such that $(x, h(x)]_u \cap I' = \{h(x)\}$ and if in particular h is a homeomorphism, then $cl(I')$ is compact and there is a continuou map $H\colon [0, 1] \times [0, 1] \to M$ satisfying $H([0, 1] \times \{0\}) = cl(I)$ and $H([0, 1] \times \{1\}) = cl(I')$ such that for every $x \in M$*

$$\text{(1)}\quad H^{-1}(W^s(x)) = [0, 1] \times A \quad \textit{for some} \quad A \subset [0, 1]\ ,$$
$$\text{(2)}\quad H^{-1}(W^u(x)) = B \times [0, 1] \quad \textit{for some} \quad B \subset [0, 1]\ .$$

Exchange s and u. Then the same statement holds.

Proof. Fix $a \in I$. We first consider a subinterval J of I satisfying the following:

(a) $a \in J$,

(b) there is a continuous map $\varphi_J\colon J \times [a, h(a)]_u \to M \backslash S$ such that

(1) $\varphi_J(x, a) = x \quad (x \in J)$,

(2) $\varphi_J(a, y) = y \quad (y \in [a, h(a)]_u)$,

(3) $\varphi_J(x, \cdot)$ is a homeomorphism from $[a, h(a)]_u$ onto $[x, h(x)]_u$ for all $x \in J$,

(4) for every $L \in \mathcal{RF}_f^s$ there is $A \subset [a, h(a)]_u$ such that $\varphi_J^{-1}(L) = J \times A$.

Let $\mathcal{S}$ be the set of subintervals of I which obey the above properties. Since $\mathcal{RF}_f^s$ is transverse to $\mathcal{RF}_f^u$, we have $\mathcal{S} \neq \emptyset$ (cf. [7, p. 35]).

For $J \in \mathcal{S}$ let φ_J and φ'_J be as in (b). Then it is checked that $\varphi_J = \varphi'_J$. Indeed, let $\pi\colon \boldsymbol{R}^2 \to M \backslash S$ be the universal cover. Denote by $\overline{\mathcal{RF}_f^\sigma}$ $(\sigma = s, u)$ the lifts of $\mathcal{RF}_f^\sigma$ by π and let $\bar{L}^\sigma(x)$ be the leaf of $\overline{\mathcal{RF}_f^\sigma}$ through $x \in \boldsymbol{R}^2$. Since each leaf of $\mathcal{RF}_f^\sigma$ is homeomorphic to $\boldsymbol{R}$, we have that $\pi\colon \bar{L}^\sigma(x) \to L^\sigma(\pi(x))$ is a homeomorphism for all $x \in \boldsymbol{R}^2$. Fix $\bar{a} \in \pi^{-1}(a)$. Since $J \subset L^s(a)$ and $[a, h(a)]_u \subset L^u(a)$, we let

$$\bar{J}=(\pi\,|\,_{\bar{L}^s(\bar{a})})^{-1}(J)\,,\quad \bar{A}=(\pi\,|\,_{\bar{L}^u(\bar{a})})^{-1}([a,h(a)]_u)\,.$$

Then $\pi|_{\bar{J}}: \bar{J}\to J$ and $\pi|_{\bar{A}}: \bar{A}\to[a, h(a)]_u$ are homeomorphisms. Let $\bar{\varphi}_J$ and $\bar{\varphi}'_J$ $(: J\times[a, h(a)]_u\to \boldsymbol{R}^2)$ be the lifts of φ_J and φ'_J by π such that $\bar{\varphi}_J(a, a)=\bar{\varphi}'_J(a, a)=\bar{a}$, respectively. Then $\pi\circ\bar{\varphi}_J(J\times\{a\})=\varphi_J(J\times\{a\})=J\subset L^s(a)$. Since $\bar{a}\in\bar{\varphi}_J(J\times\{a\})$, it follows that $\bar{\varphi}_J(J\times\{a\})\subset\bar{L}^s(\bar{a})$, and hence $\bar{\varphi}_J(J\times\{a\})=\bar{J}$. In the same way, we have $\bar{\varphi}'_J(J\times\{a\})=\bar{J}$. For $x\in J$ it is easily checked that

$$\pi\,|\,_{\bar{J}}\circ\bar{\varphi}_J(x, a)=\bar{\varphi}_J(x, a)=x=\pi\,|\,_{\bar{J}}\circ\bar{\varphi}'_J(x, a)\,.$$

Since $\pi|_{\bar{J}}$ is a homeomorphism, we have that $\bar{\varphi}_J(x, a)=\bar{\varphi}'_J(x, a)$. In the same fashion, we have that $\bar{\varphi}_J(a, y)=\varphi'_J(a, y)$ for all $y\in[a, h(a)]_u$.

Since

$$\begin{aligned}\pi\circ\bar{\varphi}_J(\{x\}\times[a, h(a)]_u)&=\varphi_J(\{x\}\times[a, h(a)]_u)\\&=[x, h(x)]_u\subset L^s(x)\,,\end{aligned}$$

clearly $\bar{\varphi}_J(\{x\}\times[a, h(a)]_u)\subset\bar{L}^u(\bar{\varphi}_J(x, a))$. Also $\bar{\varphi}(J\times\{y\})\subset\bar{L}^s(\bar{\varphi}_J(a, y))$, and hence

$$\bar{\varphi}_J(x, y)\in\bar{L}^l(\bar{\varphi}_J(x, a))\cap\bar{L}^s(\bar{\varphi}_J(a, y))$$

for all $(x, y)\in J\times[a, h(a)]_u$. In the same way, we have

$$\bar{\varphi}'_J(x, y)\in\bar{L}^u(\bar{\varphi}'_J(x, a))\cap\bar{L}^s(\bar{\varphi}'_J(a, y))$$

for all $(x, y)\in J\times[a, h(a)]_u$. Note that the left hand sides of above relations are one point sets respectively (cf. [7, p. 66]). Since $\bar{\varphi}_J(x, a)=\bar{\varphi}'_J(x, a)$ and $\bar{\varphi}_J(a, y)=\bar{\varphi}'_J(a, y)$, we conclude that $\bar{\varphi}_J(x, y)=\bar{\varphi}'_J(x, y)$, and therefore $\varphi_J=\varphi'_J$.

By the above result we see that $\mathcal{S}$ is inductive, and hence there is a maximum J_∞ of $\mathcal{S}$. We can check that $J_\infty=I$. Indeed, let $b\in J_\infty$. Since $\mathcal{RF}^s_f$ is transverse to $\mathcal{RF}^u_f$, there are a connected neighborhood K of b in I and a continuous map ψ_K: $K\times[b, h(b)]_u\to M\backslash S$ such that

(1) $\psi_K(x, b)=x \qquad (x\in K)\,,$

(2) $\psi_K(b, y)=y \qquad (y\in[b, h(b)]_u)\,,$

(3) $\psi_K(x,\cdot)$ is a homeomorphism from $[b, h(b)]_u$ onto $[x, h(x)]_u$ for all $x\in K$,

(4) for every $L\in\mathcal{RF}^s_f$ there is $A\subset[b, h(b)]_u$ such that $\psi_K^{-1}(K)=K\times A$.

We define ψ: $K\times[a, h(a)]_u\to M\backslash S$ by

$$\psi(x, y)=\psi_K(x, \varphi_{J_\infty}(b, y))$$

Then it follows that $\psi=\varphi_{J_\infty}$ on $(K\cap J_\infty)\times[a, h(a)]_u$. Since J_∞ is a maximum of $\mathcal{S}$, we have $K\subset J_\infty$. Hence J_∞ is open in I. In the same way, we obtain that J_∞ is closed in I, and therefore $J_\infty=I$. By this result we can take a continiuous map φ: $I\times[a, h(a)]_u\to M\backslash S$ such that

(1) $\varphi(x, a) = x \quad (x\in I)$,

(2) $\varphi(a, y) = y \quad (y\in[a, h(a)])$,

(3) $\varphi(x, \cdot)$ is a homeomorphism from $[a, h(a)]_u$ onto $[x, h(x)]_u$ for all $x\in I$,

(4) for every $L\in\mathcal{RF}_f^s$ there is $A\subset[a, h(a)]_u$ such that $\varphi^{-1}(L) = I\times A$.

By (1) and (3) we have that $\varphi(x, h(a))=h(x)$ for all $x\in I$. Hence $\varphi(I\times\{h(a)\})=I'$.

Hereafter, let I be homeomorphic to $[0, 1)$ for simplicity.

It is easily proved that $cl(\varphi(I\times\{y\}))$ is compactf or all $y\in[a, h(a)]_u$. Indeed, assume that there is $b\in[a, h(a)]_u$ such that $cl(\varphi(I\times\{b\})$ is not compact. Then $\varphi(I\times\{b\})=L_+^s(b)\cup\{b\}$ and $L_+^s(b)$ leads to no singular points. Hence $\varphi(I\times\{b\})$ has the recurrent property, and so we can find $x, x'\in I$ with $x\neq x'$ such that $(x, h(x)]_u\cap(x', h(x')]_u\neq\emptyset$, thus contradicting.

By the above result, for all $y\in[a, h(a)]_u$ we can take a boundary point c_y of $\varphi(I\times\{y\})$ in the leaf of $\mathcal{F}_f^s$ such that $c_y\notin\varphi(I\times\{y\})$. Define φ': $cl(I)\times[a, h(a)]^n\to M$ by

$$\varphi'|_{I\times[a,h(a)]} = \varphi\,, \quad \varphi'(c, y) = c_y \quad (y\in[a, h(a)])$$

where $cl(I)=I\cup\{c\}$. Since $\mathcal{F}_f^s$ is transverse to $\mathcal{F}_f^u$, it is easily checked that φ' is continuous and for all $x\in M$ there are $A\subset cl(I)$ and $B\subset[a, h(a)]$ such that $\varphi'^{-1}(W^s(x))=cl(I)\times A$ and $\varphi'^{-1}(W^u(x))=B\times[a, h(a)]_u$. Since $\varphi(I\times\{h(a)\})=I'$, clearly $\varphi'(cl(I)\times\{h(a)\})=cl(I')$, and hence $cl(I')$ is compact. Let g^s: $[0, 1]\to cl(I)$ and g^u: $[0, 1]\to[a, h(a)]_u$ be homeomorphisms and define H: $[0, 1]\times[0, 1]\to M$ by $H(x, y)=\varphi'(g^s(x), g^u(y))$. Then H satisfies all the properties in Lemma 6.2.

Proof of Lemma 6.1. Let Γ be a closed transversal of $\mathcal{RF}_f^s$, and define

$$\mathcal{S} = \{x\in M\backslash S\colon L^s(x)\cap\Gamma\neq\emptyset\}\,.$$

Then $\mathcal{S}$ is open in $M\backslash S$. Clearly $L^s(x)\subset\mathcal{S}$ whenever $x\in\mathcal{S}$. To obtain the conclusion, it is enough to prove $\mathcal{S}=M\backslash S$. To do this, assume that $\mathcal{S}\subsetneqq M\backslash S$. Then there is a transversal T of $\mathcal{F}_f^s$ in a leaf of $\mathcal{RF}_f^u$ such that $T\supsetneqq T\cap\mathcal{S}\neq\emptyset$ and $T\cap\Gamma=\emptyset$. Let I be a connected component of $T\cap\mathcal{S}$ and a be a boundary point of I in T. Since $\mathcal{S}$ is open in $M\backslash S$, obviously $a\notin\mathcal{S}$ and so $L^s(a)\cap\Gamma=\emptyset$.

Claim I. *$L^s(a)\neq W^s(a)$, that is, $L^s(a)$ leads to a singular point.*

Proof. By retaking the orientation of $\mathcal{RF}_u^f$ if necessary, we can assume that a is the least upper bound of I. Take and fix $x_1\in I$. Then $[x_1, a)_u\subset I\subset\mathcal{S}$. Since $L^s(x_1)\cap\Gamma\neq\emptyset$, clearly either $L_+^s(x_1)\cap\Gamma\neq\emptyset$ or $L_-^s(x_1)\cap\Gamma\neq\emptyset$.

From now on we deal with the case $L_-^s(x_1)\cap\neq\emptyset$. Since $L^s(a)\cap\Gamma=\emptyset$ and

Γ is a closed transversal of $\mathcal{RF}_f^s$, there is $x_2 \in (x_1, a]_u$ such that $L_+^s(x_2) \cap \Gamma = \emptyset$ and $L_+^s(x) \cap \Gamma \neq \emptyset$ for all $x \in [x_1, x_2)_u$. Since $T \cap \Gamma = \emptyset$, we can define $\gamma: [x_1, x_2)_u \to \Gamma$ by $\gamma(x) \in L_+^s(x)$ and $(x, \gamma(x)]_s \cap \Gamma = \{\gamma(x)\}$. Then it follows that γ is continuous and locally injective.

We first prove that γ can be extended to a continuous map from $[x_1, x_2]_u$ to Γ. If this is false, then $[x_1, x_2)_u$ covers infinitely Γ through γ, and so there is a decomposition

$$[x_1, x_2)_u = [y_1, y_2)_u \cup [y_2, y_3)_u \cup \cdots \cup [y_i, y_{i+1})_u \cup \cdots \quad (y_1 = x_1)$$

such that $\gamma: [y_i, y_{i+1})_u \to \Gamma$ is a bijection for all $i \in \boldsymbol{N}$. Clearly $\gamma(y_i) = \gamma(x_1)$. From the definition of γ it follows that $y_i \in L_-^s(\gamma(x_1))$. Hence we can take the maximum y_{i_0} of $\{y_i\}_{i=1}^{\infty}$ in $L_-^s(\gamma(x_1))$. Then $(y_{i_0}, \gamma(x_1)]_s \cap [x_1, x_2)_u = \phi$. Hence it is checked that $(x, \gamma(x)]_s \cap [x_1, x_2)_u = \emptyset$ for all $x \in [y_{i_0}, y_{i_0+1})_u$. Indeed, let

$$\mathcal{A} = \{x \in [y_{i_0}, y_{i_0+1})_u : (x, \gamma(x)]_s \cap [x_1, x_2)_u \neq \emptyset\} .$$

and suppose that $\mathcal{A} \neq \emptyset$. Then there is the greatest lower bound w of $\mathcal{A}$. If $(w, \gamma(w)]_s \cap [x_1, x_2)_u \neq \emptyset$, then $w \neq y_{i_0}$. In this case, we have that $x_1 \notin (w, \gamma(w)]_s$, and hence $(w, \gamma(w)]_s \cap (x_1, x_2)_u \neq \emptyset$. Since $\mathcal{RF}_f^s$ is transverse to $\mathcal{RF}_f^u$, there is a neighborhood K of w in $(y_{i_0}, y_{i_0+1})_u$ such that $(x, \gamma(x)]_s \cap [x_1, x_2)_u \neq \emptyset$ for all $x \in K$. This contradicts that w is the greatest lower bound of $\mathcal{A}$, and therefore $(w, \gamma(w)]_s \cap [x_1, x_2)_u = \emptyset$. Since $L_+^s(x_2) \cap \Gamma = \emptyset$, $(w, \gamma(w)]_s \cap [x_1, x_2]_u = \emptyset$. Hence we can find a neighborhood K of x in $[y_{i_0}, y_{i_0+1})_u$ such that $(x, \gamma(x)]_s \cap [x_1, x_2)_u = \emptyset$ for all $x \in K$, thus contradicting. Therefore $\mathcal{A} = \emptyset$.

Combining the above result and the fact that $\gamma: [y_{i_0}, y_{i_0+1})_u \to \Gamma$ is bijective, we see that $(x, \gamma(x)]_s \cap [y_{i_0}, y_{i_0+1})_u \neq \emptyset$ for all $x \in [y_{i_0+1}, x_2)_u$. Hence there is a map $\alpha: [y_{i_0+1}, x_2)_u \to [y_{i_0}, x_2)_u$ such that $\alpha(x) \in L_+^s(x)$ and $(x, \alpha(x)]_s \cap [y_{i_0}, x_2)_u = \{\alpha(x)\}$. Let $\alpha(y_j) \neq y_{i_0}$ for all $j \geq i_0 + 1$, then $\alpha([y_{i_0+1}, x_2)_u) \subset (y_{i_0}, x_2)_u$. Since $\mathcal{RF}_f^s$ is transverse to $\mathcal{RF}_f^u$, it follows that $\alpha: [y_{i_0+1}, x_2)_u \to [y_{i_0}, x_2)_u$ is continuous. If $\alpha(y_j) = y_{i_0}$ for some $j \geq i_0 + 1$, then $\alpha(y_i) \neq y_{i_0}$ for all $i > j$. In this case, we have that $\alpha([y_{j+1}, x_2)_u) \subset (y_{i_0}, x_2)^u$, and hence $\alpha: [y_{j+1}, x_2)_u \to [y_{i_0}, x_2)_u$ is continuous. In any case, we can find $i_1 \geq i_0 + 1$ such that $\alpha: [y_{i_1}, x_2)_u \to [y_{i_0}, x_2)_u$ is continuous.

Note that α is locally injective. Then we have that $\alpha: [y_{i_1}, x_2)_u \to [y_{i_0}, x_2)_u$ is a C^0 embedding, and therefore it is extended to $\bar{\alpha}: [y_{i_1}, x_2]_u \to [y_{i_0}, x_2]_u$. Since the diagram

$$\begin{tikzcd}[column sep=small] {[y_{i_1}, x_2)_u} \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & & {[y_{i_0}, x_2)_u} \arrow[dl, "\gamma"] \\ & \Gamma & \end{tikzcd}$$

commutes and Γ is covered infinitely by $[x_{i_1}, x_2)_u$, we conclude that $\bar{\alpha}(x_2) = x_2$.

Combining this result and Lemma 6.2, we can find a simple closed curve in $W^s(x_2)$, which contradicts Proposition A(2). Therefore γ is extended to a map $\bar{\gamma}\colon [x_1, x_2]_u \to \Gamma$.

By using Lemma 6.2, we see that $\bar{\gamma}(x_2) \in W^s(x_2)$. Let l_2 be the arc in $W^s(x_2)$ jointing x_2 and $\bar{\gamma}(x_2)$. Since $L^s_+(x_2) \cap \Gamma = \emptyset$, we have $L^s_+(x_2) \subsetneqq l_2$, and so there is a singular point s_2 in l_2. To obtain the conclusion of Claim I, assume $L^s(a) = W^s(a)$. Then we have $x_2 \neq a$, and so $[x_1, x_2]_u \subsetneqq [x_1, a)_u$. Hence $L^s(x_2) \cap \Gamma \neq \emptyset$, and therefore $L^s_-(x_2) \cap \Gamma \neq \emptyset$. Let z_2 be the maximum of $L^s_-(x_2) \cap \Gamma$ in $L^s_-(x_2)$, and take the arc A_2 in $W^s(x_2)$ jointing s_2 and z_2. Then $x_2 \in A_2$ and $A_2 \cap \Gamma = \{z_2\}$.

By repeating the above argument, we can find a sequence $\{x_i\}_{i=2}^{\infty}$ in $[x_1, a)_u$ with $x_i <_u x_j$ for $i<j$ and a family $\{A_i\}$ of arcs in leaves of $\mathcal{F}^s_f$ such that $x_i \in A_i$, $A_i \cap \Gamma$ consists of one point z_i and end points of A_i are z_i and a singular point. Since S is finite, $\{A_i\}$ must be a finite set, which contradicts that $\{x_i\}$ is infinite. Therefore the conclusion of Claim I was obtained.

Since S is finite, we can find $k>0$ such that $g=f^k$ fixes all singular points and it preserves every leaf of $\mathcal{RF}^s_f$ and of $\mathcal{RF}^u_f$ which leads to a singular point. Let $L^\sigma(x)$ lead to a singular point $p(\sigma=s, u)$. Then $L^\sigma(x) \subset W^\sigma(p)$. By the definition of $W^\sigma(p)$ we haev that $g^n(x) \to p$ as $n \to \infty$ if $\sigma = s$, and that $g^n(x) \to p$ as $n \to -\infty$ if $\sigma = u$.

Now we take a transversal T' of $\mathcal{F}^s_f$ in a leaf of $\mathcal{RF}^u$ such that $T' \supsetneqq T' \cap \mathcal{S} \neq \emptyset$ and $T' \cap \Gamma = \emptyset$. Let I' be a connected component of $T' \cap \mathcal{S}$ and a be a boundary point of I' in T'. By retaking the orientation of $\mathcal{RF}^u$ if necessary, we may assume that a is the greatest lower bound of I'. By Claim I, $L^s(a)$ leads to a singular point (say, $s(a)$). Hence one of $L^s_+(a)$ or $L^s_-(a)$ leads to $s(a)$. Without loss of generality, we may assume that $L^s_-(a)$ leads to $s(a)$. Then $L^s_+(a)$ has the recurrent property. Hence we can find a transversal T of $\mathcal{F}^s_f$ in a leaf of $\mathcal{RF}^u_f$ with $T \cap \Gamma = \emptyset$ such that $L^s_+(a) \cap T$ has an accumulation point b in T. Then there is a sequence $\{x_i\}_{i \in N}$ of $L^s_+(a) \cap T$ such that

$$x_1 <_s x_2 <_s x_3 <_s \cdots$$

and x_i converges to b in T as $i \to \infty$. By taking subsequence if necessary, we have one of the following two csaes:

(A) $x_1 <_u x_2 <_u x_3 <_u \cdots <_u b$,

(B) $b <_u \cdots <_u x_3 <_u x_2 <_u x_1$.

We consider the case of (A). Since a is the greatest lower bound of I' in T', for $i \in \boldsymbol{N}$ we can take the connected component I_i of $\mathcal{S} \cap T$ such that x_i is the greatest lower bound of I_i. If y_i denotes the least upper bound of I_i, then I_i is expressed as $I_i = (x_i, y_i)_u$.

Claim II. $y_i \in L^s_+(y_1)$ *for all* $i \in \boldsymbol{N}$

Proof. Since $x_1 \notin \mathcal{S}$, obviously $L^s_+(x_1) \cap [x_2, y_2)_u = \{x_2\}$. Since $\mathcal{R}\mathcal{F}^s_f$ is transverse to $\mathcal{R}\mathcal{F}^u_f$, we can find $z \in (x_1, y_1]_u$ satisfying the following: for all $x \in [x_1, z)_u$ there is $\alpha(x) \in L^s_+(x)$ such that $(x, \alpha(x)]_s \cap [x_2, y_2)_u = \{\alpha(x)\}$. Let $z_\infty \in (x_1, y_1]_u$ be the least upper bound of such points z. Then we have the map α_∞ from $[x_1, z_\infty)_u$ to $[x_2, y_2)_u$ such that $(x, \alpha_\infty(x)]_s \cap [x_2, y_2)_u = \{\alpha_\infty(x)\}$. It is easily checked that α_∞ is a C^0 embedding. Hence α_∞ is extended to $\bar{\alpha}_\infty$: $[x_1, z_\infty]_u \to [x_2, y_2]_u$. Using Lemma 6 2, we see that $\bar{\alpha}_\infty(z_\infty) \in W^s(z_\infty)$, and hence there is an arc l in $W^s(z_\infty)$ which joints z_∞ and $\bar{\alpha}_\infty(z_\infty)$. If l contains a singular point p, by Lemma 6.2 there is $q \in [x_1, x_2]_s$ such that $L^u_+(q)$ leads to p. Then $L^u_+(q) \setminus \{q\} \subset \mathcal{S}$. Let g: $M \to M$ be as above. Clearly $g^n(q) \in L^u_+(q) \setminus \{q\} \subset \mathcal{S}$ for some $n<0$. Since $L^s(x_1)$ $(=L^s(a))$ leads to $s(a)$, we have that $g^n(q) \in g^n(L^s(x_1)) = L^s(x_1)$, which contradicts $L^s(x_1) \cap \mathcal{S} = \emptyset$. Therefore l contains no singular points, and so $\bar{\alpha}_\infty(z_\infty) \in L^s_+(z_\infty)$. Note that $z_\infty = y_1$ or $z_\infty \in I_1$. In the case when $z_\infty \in I_1$, we have that $\bar{\alpha}_\infty(z_\infty) \in I_2$, which contradicts the choice of z_∞, and hence $z_\infty = y_1$. Obviously $z_\infty \notin \mathcal{S}$ and so $\bar{\alpha}_\infty(z_\infty) = y_2$. Therefore $y_2 \in L^s_+(y_1)$. Inductively we obtain $y_i \in L^s_+(y_1)$ for $i \in \boldsymbol{N}$.

Since $L^s_-(a)$ leads to $s(a)$ and $x_1 \in L^s_+(a)$, $L^s_-(x_1)$ leads to $s(a)$. Hence we can take the arc A in $W^s(x_1)$ jointing $s(a)$ and x_1.

Since y_1 is a boundary point of I_1, $L^s(y_1)$ leads to a singular point (say, $s(y_1)$) by Claim I. Claim II ensures that $L^s_+(y_1)$ has the reccurent property, and hence $L^s_-(y_1)$ leads to $s(y_1)$. Let B denote the arc in $W^s(y_1)$ jointing $s(y_1)$ and y_1.

Note that $y_1 \notin \mathcal{S}$. Then $(x_1, y_1]_u \cap L^s_-(y_1) = \emptyset$ and so $(x_1, y_1]_u \cap B = \{y_1\}$. Since $\mathcal{F}^s_f$ is transverse to $\mathcal{F}^u_f$, it follows that there is $(z, x_1]_s \subset A$ such that if $x \in (z, x_1]_s$ then $(x, \beta(x)]_u \cap B = \{\beta(x)\}$ for some $\beta(x) \in L^u_+(x)$. Let $U_\infty \subset A$ be the maximum of such intervals $(z, x_1]_s$. Then we have the map β_∞: $U_\infty \to B$ such that $(x, \beta_\infty(x)]_u \cap B = \{\beta_\infty(x)\}$ for all $x \in U_\infty$. Since $\beta_\infty(U_\infty) \subset B \setminus \{s(y_1)\}$, it is easily checked that β_∞ is a C^0 embedding. Suppose that $L^s_-(x_1) \supsetneqq U_\infty$. Then $U_\infty = (z_\infty, x_1]_s$ for some $z_\infty \in L^s_-(x_1)$, and hence β_∞ is extended to $\bar{\beta}_\infty$: $[z_\infty, x_1]_s \to B$. If $\bar{\beta}_\infty([z_\infty, x_1]_s) \subsetneqq B$, then the arc l in $W^u(z_\infty)$ jointing z_∞ and $\bar{\beta}_\infty(z_\infty)$ must contain a singular point p. In this case $g^n(z_\infty)$ converges to p as $n \to -\infty$. By Lemma 6.2 we have that $g^n(z_\infty) \in \mathcal{S}$ for $n<0$ small enough, which contradicts that $f^n(z_\infty) \in L^s(x_1)$ and $L^s(x_1) \cap \mathcal{S} = \emptyset$. Therefore $\bar{\beta}_\infty([z_\infty, x_1]_s) = B$, and so $z_\infty \in W^u(\bar{\beta}_\infty(z_\infty)) = W^u(s(y_1))$. Hence $g^n(z_\infty)$ converges to $s(y_1)$ as $n \to -\infty$, and we see by Lemma 6.2 that $g^n(z_\infty) \in \mathcal{S}$ for $n<0$ small enough, which is a contradiction. Therefore $U_\infty = L^s_-(x_1)$.

By this result β_∞ is extended to $\bar{\beta}^\infty$: $A \to B$. By Lemma 6.2 it follows that $s(a)$ and $\bar{\beta}_\infty(s(a))$ are in $W^u(s(a))$. Hence $g^n(\bar{\beta}_\infty(s(a)))$ converges to $s(a)$ as $n \to -\infty$, and therefore $g^n(\bar{\beta}_\infty(s(a))) \in \mathcal{S}$ for $n<0$ small enough. But $g^n(L^s(y_1)) = L^s(y_1)$, which contradicts $L^s(y_1) \cap \mathcal{S} = \emptyset$. Therefore the conclusion of Lemma 6.1 was obtained.

Proof of (4) in Proposition A. Let π': $N \to M \setminus S$ be a finite cover such that the lifts $\hat{\mathcal{F}}_f^\sigma$ ($\sigma=s, u$) of $\mathcal{R}\mathcal{F}_f^\sigma$ by π' are orientable and a lift of f by π' exists (cf. [7, p. 17]). And let π: $\bar{M}\to M$ be the branched cover induced from π'. Then the lifts $\bar{\mathcal{F}}_f^\sigma$ ($\sigma=s, u$) of $\mathcal{F}_f^\sigma$ by π are orientable because $\mathcal{R}\bar{\mathcal{F}}_f^\sigma=\hat{\mathcal{F}}_f^\sigma$, and we can take a lift $\bar{f}$: $\bar{M}\to\bar{M}$ of f by π.

Let $\bar{W}_\varepsilon^\sigma(x)$ ($\sigma=s, u$) denote the local stable and unstable sets for $\bar{f}$. If $\varepsilon>0$ is small enougn, then for all $x\in\bar{M}$

$$\pi(\bar{W}_\varepsilon^\sigma(x)) = W_\varepsilon^\sigma(\pi(x)) \quad (\sigma=s, u),$$

which implies that $\bar{W}_\varepsilon^s(x)\cap\bar{W}_\varepsilon^u(x)=\{x\}$. Hence $\bar{f}$ is expansive. By using this fact it is easily checked that $\mathcal{F}_{\bar{f}}^\sigma=\overline{\mathcal{F}_f^\sigma}$ for $\sigma=s, u$.

To show that $\bar{\mathcal{F}}_f^s$ is minimal, let l be an arc in a leaf of $\mathcal{R}\bar{\mathcal{F}}_f^u$. Since $\bar{\mathcal{F}}_f^u=\mathcal{F}_{\bar{f}}^u$, $\bar{f}^n(l)$ has the recurrent property if $n<0$ enough small. Since $\bar{\mathcal{F}}_f^u$ is orientable, we can construct a closed transversal Γ of $\mathcal{R}\bar{\mathcal{F}}_f^s$ by deforming $\bar{f}^n(l)$ along the leaves of $\mathcal{R}\bar{\mathcal{F}}_f^s$ (cf. [7, p. 52]). By Lemma 6.1, Γ intersects every leaf of $\mathcal{R}\bar{\mathcal{F}}_f^s$ in at least one point, and hence so does $\bar{f}^n(l)$. Therefore l intersects every leaf of $\mathcal{R}\bar{\mathcal{F}}_f^s$. Since l is arbitrary, we see that $\bar{\mathcal{F}}_f^s$ is minimal, and therefore so is $\mathcal{F}_f^s$. The conclusion for $\sigma=u$ is also obtained.

7. Proof of Proposition B

As before let $\mathcal{M}(\mathcal{F})$ denote the set of all transverse invariant measures for a C^0 singular foliation $\mathcal{F}$. For the proof of Proposition B we establish the following

Lemma 7.1. *Let $\mathcal{F}$ be a C^0 singular foliation on M. If $\mathcal{F}$ is orientable and transversally orientable and if $\mathcal{F}$ is minimal, then the following hold*;

(1) *$\mathcal{M}(\mathcal{F})$ is non-trivial,*

(2) *if $\mu\in\mathcal{M}(\mathcal{F})$ is non-zero, then every finite Borel measure of μ is non-atomic and positive on all non-empty open sets,*

(3) *there is an injective map k from $\mathcal{M}(\mathcal{F})$ into a finite dimensional Euclidean space such that*

$$k(s\mu+t\nu) = sk(\mu)+tk(\nu)$$

for $\mu, \nu\in\mathcal{M}(\mathcal{F})$ and $s, t\geq 0$.

Proof. Let S be the set of all singular points of $\mathcal{F}$ and define $\mathcal{R}\mathcal{F}$ as before. For $x\in M\setminus S$ let $L(x)$ be the leaf of $\mathcal{R}\mathcal{F}$ through x. Since $\mathcal{F}$ is minimal, it follows that each $L(x)$ are homeomorphic to $\boldsymbol{R}$. Since $\mathcal{F}$ is orientable, we can give an order $\leq$ for $L(x)$ in the same way as in §6. Then the intervals $L_+(x)$, $L_-(x)$, $[y, z)$ and $(y, z]$ of $L(x)$ are defined (see §6).

Take and fix a transversal T of $\mathcal{F}$ with $T\cap S=\emptyset$ such that the end points a, b of T are not in same leaf of $\mathcal{R}\mathcal{F}$ and they are not in leaves of $\mathcal{R}\mathcal{F}$ which lead to singular points. Hereafter, we identify T with $[0, 1]$ for simplicity.

Let us define

$$D = \{x \in T \colon L_+(x) \cap T = \emptyset\} .$$

Since $\mathcal{F}$ is minimal, it is clear that if $x \in D$ then $L_+(x)$ leads to a singular point. Hence $L_+(x) \cap L_+(y) = \emptyset$ for $x, y \in D$ with $x \neq y$. Combining these and the fact that S is finite, we see that D is finite. By the choice of a and b, it follows that $D \cap \{a, b\} = \emptyset$.

Define $\gamma \colon T \setminus D \to T$ by $\gamma(x) \in L^s_+(x)$ and $(x, \gamma(x)] \cap T = \{\gamma(x)\}$. Then γ is injective. Since $L_-(a)$ and $L_-(b)$ intersect T, we have that $a, b \in \gamma(T \setminus D)$. Hence $\gamma^{-1}(\{a, b\}) \cap D = \emptyset$. Since D is finite and $\gamma^{-1}(\{a, b\})$ consist of two points, $F = D \cup \gamma^{-1}(\{a, b\})$ is finite, and hence F cuts T in finitely many subintervals $I_1, I_2, \cdots, I_m$. Then $\gamma|_{I_i}$ is continuous. Since γ is injective, we have that $\gamma|_{I_i}$ is a C^0 embedding for all $1 \leq i \leq m$.

Let $c \in \gamma^{-1}(\{a, b\})$. Then $c \notin D$. Hence we can take $i(c) \in \{1, 2, \cdots, m\}$ such that c is a boundary point of $I_{i(c)}$ and $\gamma|_{I_{i(c)} \cup \{c\}}$ is a C^0 embedding. For simplicity, denote $I_{i(c)} \cup \{c\}$ by $I_{i(c)}$. Then we have

$$(7.1) \qquad T \setminus D = I_1 \cup I_2 \cup \cdots \cup I_m \quad \text{(disjoint union)}.$$

Since γ is injective, clearly $\gamma(I_i) \cap \gamma(I_j) = \emptyset$ for $i \neq j$.

Let $b(i)$ be the least upper bound of $I_i (1 \leq i \leq m)$. If $b(i) \in D$, then we write $\bar{I}_i = I_i \cup \{b(i)\}$. If not, then we write $\bar{I}_i = I_i$. Combining (7.1) and the fact that $\{a, b\} \cap D = \emptyset$, we see that

$$T = \bar{I}_1 \cup \bar{I}_2 \cup \cdots \cup \bar{I}_m \quad \text{(disjoint union)}.$$

Since $\gamma|_{I_i}$ is a C^0 embedding, it is extended to a C^0 embedding $\gamma_i \colon \bar{I}_i \to T$. Let $b(i) \in D$. As we saw above, $L_+(b(i))$ leads to a singular point. This implies that $L_-(\gamma_i(b(i)))$ leads to the same singular point. Since $\mathcal{F}$ is transversally orientable, we have that $\gamma_i(b(i))$ is the least upper bound of $\gamma_i(\bar{I}_i)$. Since $\gamma_i(I_i) \cap \gamma_j(I_j) = \emptyset$ for $i \neq j$, $\gamma_i(\bar{I}_i) \cap \gamma_j(\bar{I}_j) = \emptyset$. Consider the set $D' = \{x \in T \colon L_-(x) \cap T = \emptyset\}$ and the map $\gamma' \colon T \setminus D' \to T$ defined by $\gamma'(x) \in L_-(x)$ and $[\gamma'(x), x) \cap T = \{\gamma'(x)\}$. Then we see that $\gamma'(T \setminus D') = T \setminus D$ and $\gamma' = \gamma^{-1}$, and therefore

$$T = \gamma_1(\bar{I}_1) \cup \gamma_2(\bar{I}_2) \cup \cdots \cup \gamma_m(\bar{I}_m) \quad \text{(disjoint union)}.$$

Define $\bar{\gamma} \colon T \to T$ by $\bar{\gamma}|_{\bar{I}_i} = \gamma_i$ for all i. By the above results $\bar{\gamma}$ is a bijection and $\bar{\gamma}|_{\bar{I}_i} (i = 1, 2, \cdots, m)$ are C^0 embeddings. We note that $F = D \cup \gamma^{-1}(\{a, b\})$ coincides with the set of all discontinuous points of $\bar{\gamma}$ and that $\bar{\gamma}^n(x) \notin F$ for all $x \in F$ and all $n \in \boldsymbol{Z}$ with $n \neq 0$. Since $\mathcal{F}$ is minimal, it is easily checked that $\bar{\gamma}$ is minimal.

Let $\mathcal{M}(T)$ be the set of all finite Borel measures on T and define

$$\mathcal{M}_{\bar{\gamma}}(T) = \{\mu \in \mathcal{M}(T) \colon \mu \text{ is } \bar{\gamma}\text{-invariant}\}.$$

Claim I. *$\mathcal{M}_{\bar{\gamma}}(T)$ is non-trivial.*

Proof. Let $C(T)$ be the set of all real valued continuous functions on T. Then $C(T)$ is a Banach algebra with norm

$$\|\xi\| = \sup_{x \in T} |\xi(x)|.$$

Take and fix $x_0 \in T$, and define for $n \geq 1$ and $\xi \in C(T)$

$$K_n(\xi) = \frac{1}{n} \sum_{i=1}^{n-1} \xi(\bar{\gamma}^i(x_0)).$$

Then $K_n \colon C(T) \to \boldsymbol{R}$ is a continuous linear map such that $K_n(1)=1$ and $K_n(\xi) \geq 0$ if $\xi(x) \geq 0$ for all $x \in T$. By Riez representation theorem, there is a Borel probability measure μ_n on T such that

$$K_n(\xi) = \int \xi d\mu_n \quad (\xi \in C(T)).$$

There are a subsequence $\{\mu_{n_j}\}$ and a Borel probability measure μ on T such that

$$\int \xi d\mu_{n_j} \to \int \xi d\mu \quad (\xi \in C(T)).$$

If ξ, $\xi \circ \bar{\gamma}^{-1} \in C(T)$, then

$$\int \xi d\mu = \int \xi \circ \bar{\gamma}^{-1} d\mu \tag{7.2}$$

since $|K_{n_j}(\xi \circ \bar{\gamma}^{-1}) - K_{n_j}(\xi)| = \dfrac{1}{n_j} |\xi \circ \bar{\gamma}^{n_j}(x_0) - \xi(x)| \leq \dfrac{2}{n_j} \|\xi\|$.

To obtain that μ is $\bar{\gamma}$-invariant, we first check that μ is non-atomic. To do this, assume that $\mu(\{y\}) > 0$ for some $y \in T \setminus F$. We can take $l \in \boldsymbol{N}$ such that $l\mu(\{y\}) > 1$. Since $\bar{\gamma}^n(x) \notin F$ for $x \in F$ and $n \neq 0$, we can assume that $\bar{\gamma}^i(y) \notin F$ for all $i \geq 0$. Take $\delta_n \in C(T)$ ($n=1, 2, \cdots$) such that $\delta_n(y)=1$ and $\delta_n \to 1_{\{y\}}$ ($n \to \infty$) where $1_{\{y\}}$ denotes the characteristic function. Then there is $N>0$ such that $\delta_n \circ \bar{\gamma}^{-i} \in C(T)$ for all $n \geq N$ and $0 \leq i \leq l-1$. By (7.2) we have

$$\int \delta_n d\mu = \int \delta_n \circ \bar{\gamma}^{-i} d\mu \quad (0 \leq i \leq l-1)$$

and hence by Lebesgue convergence theorem

$$\int 1_{\{y\}} d\mu = \int 1_{\{y\}} \circ \bar{\gamma}^{-i} d\mu \quad (0 \leq i \leq l-1).$$

which implies that $\mu(\{y\}) = \mu(\{\bar{\gamma}^i(y)\})$. Hence $\mu(\{\bar{\gamma}^i(y) \colon 0 \leq i \leq l-1\}) = l\mu(\{y\}) > 1$, a contradiction. Therefore $\mu(\{y\})=0$ for all $y \in T \setminus F$. Next, assume that

$\mu(F)>0$. Take $\xi_n \in C(T)$ $(n=1, 2, \cdots)$ such that $\xi_n|_F=0$ and $\xi_n \to 1_{T\setminus F}$ as $n\to\infty$. Then $\xi_n \circ \bar{\gamma}^{-1} \in C(T)$ and $\xi_n\circ\bar{\gamma}^{-1} \to 1_{T\setminus\bar{\gamma}(F)}$. By (7.2) we have

$$\int \xi_n \, d\mu = \int \xi_n \circ \bar{\gamma}^{-1} \, d\mu$$

and hence

$$\int 1_{T\setminus F} \, d\mu = \int 1_{T\setminus\bar{\gamma}(F)} \, d\mu$$

which implies that $0<\mu(F)=\mu(\bar{\gamma}(F))$. Since $\bar{\gamma}(F)\subset T\setminus F$, we have $\mu(\bar{\gamma}(F))=0$, a contradiction. Therefoıe $\mu(F)=0$ and so μ is non-atomic.

Let $\xi_n \in C(T)$ $(n=1, 2, \cdots)$ be as above. Then $\xi_n \xi, (\xi_n\xi)\circ\bar{\gamma}^{-1} \in C(T)$ for $\xi \in C(T)$. By (7.2) we have

$$\int \xi_n \, \xi d\mu = \int (\xi_n \xi)\circ\bar{\gamma}^{-1} \, d\mu$$

and hence

$$\int 1_{T\setminus F} \, \xi d\mu = \int (1_{T\setminus F} \, \xi)\circ\bar{\gamma}^{-1} \, d\mu \, .$$

Since μ is non-atomic, we have

$$\int \xi d\mu = \int \xi\circ\bar{\gamma}^{-1} \, d\mu \quad (\xi \in C(T)),$$

which implies that μ is $\bar{\gamma}$-invariant. The proof of Claim I is completed.

Recall that T is expressed as the disjoint union of subintervals $\bar{I}_i (1\le i\le m)$. We define $\iota\colon \mathcal{M}_{\bar{\gamma}} \to \boldsymbol{R}^m$ by

$$\iota(\mu) = (\mu(\bar{I}_1), \mu(\bar{I}_2), \cdots, \mu(\bar{I}_m)) \, .$$

Then it follows that

$$\iota(s\mu+t\nu) = s\iota(\mu)+t\iota(\nu)$$

for $\mu, \nu \in \mathcal{M}(\mathcal{F})$ and $s, t\ge 0$.

Claim II. *ι is injective.*

Proof. It is enough to show that if $\iota(\mu)=\iota(\nu)$, then $\mu=\nu$. To do this, let

$$\mathcal{P}^2 = \{\bar{I}_{i_1} \cap \bar{\gamma}(\bar{I}_{j_2})\colon 1\le i_1, i_2\le m\} \, .$$

Then each element of $\mathcal{P}^2$ is a subinterval subinterval of T and $\mathcal{P}^2$ is a decomposition of T. So we write $\mathcal{P}^2=\{J_1, J_2, \cdots, J_{2m}\}$ where each index of J_i obeys the order of T. Then it is easily checked that for $1\le j\le 2m$, $K_j=J_1\cup J_2\cup\cdots\cup J_j$ is the union of elements of $\{\bar{I}_i\}_{i=1}^m$ or of elements of $\{\bar{\gamma}(\bar{I}_i)\}_{i=1}^m$. Since μ is $\bar{\gamma}$-invariant, $\mu(\bar{I}_i)=\mu(\bar{\gamma}(\bar{I}_i))$ for $1\le i\le m$, and hence

$$\mu(K_j) = \mu(\bar{I}_{l_1}) + \mu(\bar{I}_{l_2}) + \cdots + \mu(\bar{I}_{l_j})$$

for some $1 \leq l_1 < l_2 < \cdots < l_j \leq m$. Since $\iota(\mu) = \iota(\nu)$, it follows that $\mu(K_j) = \nu(K_j)$ for $1 \leq j \leq 2m$, and therefore

$$\mu(J_j) = \nu(J_j) \quad (1 \leq j \leq 2m).$$

Next we write

$$\begin{aligned}\mathcal{P}^3 &= \{J_{j_1} \cap \bar{\gamma}(J_{j_2}) \colon 1 \leq j_1, j_2 \leq 2m\} \\ &= \{I_{i_1} \cap \bar{\gamma}(I_{i_2}) \cap \bar{\gamma}^2(I_{i_3}) \colon 1 \leq i_1, i_2, i_3 \leq m\}.\end{aligned}$$

Then we can easily prove that $\mu(I) = \nu(I)$ for all $I \in \mathcal{P}^3$. Inductively, letting

$$\mathcal{P}^n = \{I_{i_1} \cap \bar{\gamma}(I_{i_2}) \cap \cdots \cap \bar{\gamma}^{n-1}(I_{i_n}) \colon 1 \leq i_l \leq m \quad (l = 1, 2, \cdots, n)\},$$

we have that

$$\mu(I) = \nu(I) \quad (I \in \mathcal{P}^n, n \geq 1). \tag{7.3}$$

Note that the set of all boundary points of elements of $\mathcal{P}^n$ coincides with

$$F_n = F \cup \bar{\gamma}(F) \cup \cdots \cup \bar{\gamma}^{n-1}(F).$$

Since $\bar{\gamma}$ is minimal, we have that $F_\infty = \bigcup_{n=1}^{\infty} F_n$ is dense in T. Therefore every open set of T is expressed as a disjoint union of at most countable elements of $\bigcup_{n=1}^{\infty} \mathcal{P}^n$. Combining this result and (7.3), we obtain $\mu = \nu$.

Claim III. *There is a bijection* $\tau \colon \mathcal{M}_{\bar{\gamma}} \to \mathcal{M}(\mathcal{F})$ *such that*

$$\tau(s\mu + t\nu) = s\tau(\mu) + t\tau(\nu)$$

for $\mu, \nu \in \mathcal{M}_{\bar{\gamma}}$ *and* $s, t \geq 0$.

Proof. Let A be a transversal of $\mathcal{F}$. We can choose a finite decomposition $\{A_i\}_{i=1}^n$ of A and a family $\{T_i\}_{i=1}^n$ of subintervals of T such that there is a projection $h_i \colon A_i \to I_i$ along the leaves for $1 \leq i \leq n$. Indeed, let $x \in A$ be a regular point. Since $L_+(x)$ or $L_-(x)$ lead to no singular points, we may assume that $L_+(x)$ leads to no singular points. Then there is $t(x) \in L_+(x) \cap T \setminus \{a, b\}$ since $\mathcal{F}$ is minimal. Since A is a transversal of $\mathcal{F}$, it follows that there is a projection along the leaves which maps a neighborhood of x in A onto a neighborhood of $t(x)$ in T. For the case when $x \in A$ is a singular point, take a transversal A'_x of $\mathcal{F}$ with $A'_x \subset M \setminus S$ such that there is a projection along the leaves which maps A'_x onto a neighborhood of x in A. Then we can find a projection along leaves which maps a neighborhood of x in A onto a subinterval of T. Therefore we can choose a finite decomposition $\{A_i\}_{i=1}^n$ of A and a family $\{T_i\}_{i=1}^n$ of subintervals of T which satisfy our desire.

Since each h_i: $A_i \to T_i$ is a homeomorphism, for $\mu \in \mathcal{M}_{\bar{\gamma}}(T)$ we can define a finite Borel measure μ_A on A by

$$\mu_A = \sum_{i=1}^{n} \mu|_{T_i} \circ h_i \, .$$

Since μ is $\bar{\gamma}$-invariant, it is checked that μ_A is independent of the choice of $(\{A_i\}, \{T_i\}, \{h_i\})$. Indeed, let h be a projection along the leaves from a subarc A' of A onto a subinterval T' of T. Then $\alpha = h_i|_{A_i \cap A'} \circ (h|_{A_i \cap A'})^{-1}$ is a projection along the leaves which maps a subinterval $h(A_i \cap A')$ onto a subinterval $h_i(A_i \cap A')$. By the definition of $\bar{\gamma}$ we can find a finite set E of $h(A_i \cap A')$ such that if J is a component of $h(A_i \cap A') \setminus E$ then $\alpha|_J$ is equal to $\bar{\gamma}^n|_J$ for some $n \in \boldsymbol{Z}$. Since μ is $\bar{\gamma}$-invariant, we have

$$\mu|_J \circ h|_{h^{-1}(J)} = \mu_A|_{h^{-1}(J)}$$

and so

$$\mu|_{h(A_i \cap A') \setminus E} \circ h|_{(A_i \cap A') \setminus h^{-1}(E)} = \mu_A|_{(A_i \subset A') \setminus h^{-1}(E)} \, .$$

Since μ and μ_A are non-atomic, we have

$$\mu|_{h(A_i \cap A')} \circ h|_{A_i \cap A'} = \mu_A|_{A_i \cap A'}$$

which means that the definition of μ_A is independent of the choice of $(\{A_i\}, \{T_i\}, \{h_i\})$.

We next show that $\{\mu_A$: A is a transversal$\}$ is a transverse invariant measure for $\mathcal{F}$. To do this, let A and B be transversals of $\mathcal{F}$ and let h: $A \to B$ be a projection along leaves. Then we can take decompositions $\{A_i\}_{i=1}^n$ of A and $\{B_i\}_{i=1}^n$ of B such that $h(A_i) = B_i (1 \leq i \leq n)$ and such that for $1 \leq i \leq n$ there is a projection f_i (resp. g_i) along the leaves which maps A_i (resp. B_i) onto a subinterval of T. Clearly $g_i \circ h|_{A_i} \circ f_i^{-1}$ is a projection along leaves for $1 \leq i \leq n$. By the definitions of μ_A and μ_B, we see that $\mu_A|_{A_i} = (\mu_B|_{B_i}) \circ h|_{A_i}$, and therefore $\mu_A = \mu_B \circ h$.

Define τ: $\mathcal{M}_{\bar{\gamma}} \to \mathcal{M}(\mathcal{F})$ by

$$\tau(\mu) = \{\mu_A\colon A \text{ is a transversal}\} \, .$$

Then τ satisfies all the properties in Claim III.

By Claims II and III there is an injection k from $\mathcal{H}(\mathcal{F})$ to $\boldsymbol{R}^m$ such that

$$k(s\mu + t\nu) = sk(\mu) + tk(\nu)$$

for $\mu, \nu \in \mathcal{M}(\mathcal{F})$ and $s, t \geq 0$. Hence Lemma 7.1(3) holds. Lemma 7.1(1) is obtained from Claim I. Note that if $\mu \in \mathcal{M}_{\bar{\gamma}}$ is non-zero then μ is non-atomic and positive on all non-empty open sets. Then Lemma 7.1(2) is easily checked. The proof of Lemma 7.1 is completed.

Proof of Proposition B. Let us take a p-fold branched cover $\pi: \bar{M} \to M$ ($p=1, 2$, or 4) such that the lifts $\bar{\mathcal{F}}^\sigma (\sigma=s, u)$ of $\mathcal{F}^\sigma$ are orientable and there is a lift $\bar{f}: \bar{M} \to \bar{M}$ of f. Clearyl $\bar{f}$ preserves $\bar{\mathcal{F}}^\sigma$. Since $\mathcal{F}^\sigma$ is minimal, it follows that $\bar{\mathcal{F}}^\sigma$ are minimal.

By Lemma 7.1(3) there is an injective map $k: \mathcal{M}(\mathcal{F}^s) \to \boldsymbol{R}^m$ for some $m \geq 1$ such that $k(s\mu+tv)=sk(\mu)+tk(v)$ for $\mu, v \in \mathcal{M}(\mathcal{F}^\sigma)$ and $s, t \geq 0$. Clearly the image V of k is a convex cone of $\boldsymbol{R}^m$. Define $\bar{f}'_*: V \to V$ by $\bar{f}'_* = k \circ \bar{f}_* \circ k^{-1}$. Then $\bar{f}'_*$ is continuous. Note that $V \cap S^{m-1}$ is a disk where S^{m-1} denotes the unit sphere of $\boldsymbol{R}^m$. By Brouwer's fixed point theorem, the map $V \cap S^{m-1} \to V \cap S^{m-1}$ which sends x to $\bar{f}'_*(x)/||\bar{f}'_*(x)||$ ($||\cdot||$ denotes the Euclidean norm) has a fixed point. This ensures the existence of $\mu^s \in \mathcal{M}(\bar{\mathcal{F}}^s)$ such that $\bar{f}_*(\bar{\mu}^s) = \lambda^s \bar{\mu}^s$ for some $\lambda^s > 0$. We can find also $\bar{\mu}^u \in \mathcal{M}(\bar{\mathcal{F}}^u)$ such that $\bar{f}_*(\bar{\mu}^u) = \lambda^u \bar{\mu}^u$ for some $\lambda^u > 0$. By Lemma 7.1(2) every finite Borel measure of $\bar{\mu}^s$ and of $\bar{\mu}^u$ is non-atomic and positive on all non-empty open sets.

Let A^s be a transversal of $\mathcal{F}^s$ and take all lifts $\bar{A}^s_1, \cdots, \bar{A}^s_p$ of A^s by $\pi: \bar{M} \to M$. Then we define a finite Borel measure μ_{A^s} on A^s by

$$\mu_{A^s} = \bar{\mu}_{\bar{A}^s_1} \circ (\pi \mid_{\bar{A}^s_1})^{-1} + \cdots + \bar{\mu}_{\bar{A}^s_p} \circ (\pi \mid_{\bar{A}^s_p})^{-1} .$$

By homotopy lifting property we have that

$$\{\mu_{A_s}: A^s \text{ is a transversal of } \mathcal{F}^s\}$$

is a transverse invariant mesaure for $\mathcal{F}^s$. Since $f_*(\bar{\mu}^s) = \lambda^s \bar{\mu}^s$, we see that $f_*(\mu^s) = \lambda^s \mu^s$. For $\sigma=u$ we obtain the same one. Clearly every finite Borel measure of μ^s and of μ^u is non-atomic and positive on all non-empty open sets.

Since f preserves $\mathcal{F}^s$ and $\mathcal{F}^u$, we can show that $\lambda^s \lambda^u = 1$. Indeed, let $\mathcal{R}$ be the family of $R \subset M$ with the following property: there is a C^0 embedding H_R: $[0, 1] \times [0, 1] \to M$ with $H_R([0, 1] \times [0, 1]) = R$ such that
(1) if $L^s \in \mathcal{F}^s$ then $H_R^{-1}(L^s) = [0, 1] \times A$ for some $A \subset [0, 1]$,
(2) if $L^u \in \mathcal{F}^u$ then $H_R^{-1}(L^u) = B \times [0, 1]$ for some $B \subset [0, 1]$. Since $\mathcal{F}^s$ and $\mathcal{F}^u$ are transverse, it is easily checked that $\mathcal{R}$ generates the Borel σ-field of M. For $R \in \mathcal{R}$ we let

$$R^s = H_R([0, 1] \times \{0\}) , \quad R^u = H_R(\{0\} \times [0, 1])$$

and define $\mu: \mathcal{R} \to \boldsymbol{R}$ by

$$\mu(R) = \mu^s(R^u) \cdot \mu^u(R^s) .$$

Then μ is extended to a finite Borel measure μ on M. Obviously μ is positive on all non-empty open sets. Since f preserves $\mathcal{F}^s$ and $\mathcal{F}^u$, $f(R) \in \mathcal{R}$ for all $R \in \mathcal{R}$, and hence

$$\begin{aligned}\mu(f(R)) &= \mu^s((f(R))^u)\ \mu^u((f(R))^s) \\ &= \mu^s(f(R^u))\ \mu^u(f(R^s)) \\ &= \lambda^s\ \mu^s(R^u)\ \lambda^u\ \mu^u(R^s) \\ &= \lambda^s\ \lambda^u\ \mu(R)\,.\end{aligned}$$

Therefore $\mu\circ f=\lambda^s\lambda^u\ \mu$ on Borel σ-field. Since μ is finite, we have $\lambda^s\lambda^u=1$. The proof of Proposition B is completed.

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