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PROPAGATION OF SINGULARITIES FOR HYPERBOLIC OPERATORS WITH TRANSVERSE PROPAGATION CONE

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1. Introduction

In this note, we study the propagation of singularities for hyperbolic pseudo-differential operators with multiple characteristics. It is possible in general that two principal symbols have the same multiple characteristic set but the bicharacteristics behave quite differently. Thus, it is natural to impose conditions not only on the cahracteristic set but also on the behavior of bicharacteristics when we study the propagation of singularities.

Given a multiple characteristic ρ, we can consider the localization of the principal symbol at ρ, which is a hyperbolic polynomial defined *via* the Taylor expansion. The propagation cone of the localization is a "*minimal*" cone including the tangents of the bicharacteristics with the limit point ρ (see Subsection 2.1).

As mentioned above, we impose a condition on the bicharacteristics in terms of the propagation cone of the localization: the propagation cone of the localization is transversal to the multiple characteristic set. This condition may realize some typical situations. When ρ is a double characteristic, this condition is valid if and only if the principal symbol is effectively hyperbolic at ρ, where the smoothness of the multiple characteristics et is always assumed. In case the multiplicity exceeds 2, we assume the Levi conditions on the lower order terms.

Our first result is concerned with an operator such that the localization at ρ (with multiplicity r) is strictly hyperbolic on the normal bundle of Σ_r—the set of characteristics of order r. We prove that, if there are no singularities on the backward bicharacteristics with the limit point ρ, then there is no singularity at ρ (Theorem 2.1).

Our second result is concerned with an operator of which the characteristic set is the union of r hypersurfaces through ρ with linearly independent normals. Hence the multiple characteristic set is the union of each intersection of the hypersurfaces. We show that, if there are no singularities on the backward characteristic curves passing through ρ of these hypersurfaces, then there is none at ρ (Theorem 2.2) assuming in addition that the rank of the symplectic form, restricted to the symplectic dual of the tangent space of Σ_r at ρ, does not exceed 2. In this case, we can get a result asserting the propagation of the singularities actually occurs through ρ (Corollary 2.1). Weaker results are found in our previous works [17] and [18].

In Subsection 2.1, we recall the definition of the propagation cone of the localization and the time function. The main results, Theorems 2.1 and 2.2, are stated in Subsections 2.2 and 2.3, and proved in Sections 3 and 4, respectively. Corollary 2.1 is stated and proved in Subsection 2.3.

For studies on the propagation of sungularities in the case the propagation cone is not transversal to the tangent space of the multiple characterisitc set, we refer, for example, to Lascar [11], Uhlmann [20], [21], Ivrii [10], Melrose-Uhlmann [13], Sjöstrand [19].

2. Statement of the results

2.1. Preliminaries

Let $P(x, D)$ be a classical pseudo-differential operator of order m in an open set $\Omega\subset \boldsymbol{R}^{d+1}$ with *real* principal symbol $p(x, \xi)\in C^\infty(T^*\Omega\backslash 0)$ where $T^*\Omega$ is the contangent bundle over Ω.

Let $\rho\in T^*\Omega\backslash 0$ be a characteristic of $p(x, \xi)$ of order r:

$$d^jp(\rho) = 0 \quad \text{for} \quad 0\leqq j\leqq r-1\,,$$

where d^jp is the j^{th} differential of p. We study the propagation of wave front sets near ρ of solutions of the equation $Pu=f$ when ρ does not belong to the wave front set of f. For this purpose it is necessary to observe the Taylor expansion of $p(x, \xi)$ at ρ. Let us define $p_\rho(X)$, which is a homogeneous polynomial of degree r in $X\in T_\rho(T^*\Omega)$ (the tangent space of $T^*\Omega$ at ρ), by

$$p_\rho(X) = d^rp(\rho; X, \cdots, X)/r!\,, \quad X\in T_\rho(T^*\Omega)\,.$$

Recall that $p_\rho(X)$ is called the localization of p at ρ (see Hörmander [4], Atiyah-Bott-Gårding [2]). Throughout this note we assume that $p_\rho(X)$ is hyperbolic with respect to some $\theta\in T_\rho(T^*\Omega)$. Note that this assumption is implied by the hyperbolicity of $p(x, \xi)$ (see Ivrii-Petkov [8], Hörmander [6] for details). Naturally we are led to consider the hyperbolic cone $\Gamma(p_\rho, \theta)$ of p_ρ defined by the connected component of the set $\{X\in T_\rho(T^*\Omega);\ p_\rho(X)\neq 0\}$ containing θ and the propagation cone $C(p_\rho, \theta)$ of p_ρ which is defined by

$$C(p_\rho, \theta) = \{X\in T_\rho(T^*\Omega);\ \sigma(X, Y)\leqq 0 \quad \text{for any} \quad Y\in\Gamma(p_\rho, \theta)\}\,,$$

where σ is the symplectic 2 form given by

$$\sigma = \sum_{j=0}^{d} d\xi_j \wedge dx_j \,,$$

with natural coordinates $(x, \xi)=(x_0, \cdots, x_d, \xi_0, \cdots, \xi_d)$ on $T^*\Omega$. Note that if $r=1$, then the propagation cone $C(p_\rho, \theta)$ is the half line spanned by the Hamilton vector field $H_p(\rho)$ of p at ρ since $p_\rho(X)=\sigma(X, H_p(\rho))$. We say that $t(x, \xi)$ is a time function near ρ with respect to $\Gamma(p_\rho, \theta)$ if

$$-H_t(\rho)\in\Gamma(p_\rho, \theta)\,, \quad t(\rho)=0\,,$$

that is, the tangent space at ρ of the surface $t(x, \xi)=0$ is transversal to $C(p_\rho, \theta)$. We may assume that $t(x, \xi)$ is homogeneous of degree 0 in ξ.

Given a linear subspace W of $T_\rho(T^*\Omega)$, we denote by W^σ the annihilator with respect to σ:

$$W^\sigma = \{X\in T_\rho(T^*\Omega);\ \sigma(X, Y)=0 \quad \text{for every} \quad Y\in W\}\ .$$

2.2. Operators with strictly hyperbolic localizations

Our first result is on operators with strictly hyperbolic localizations. Denoting by Σ_r the set of characteristics of $p(x, \xi)$ of order r, we assume that

(2.1) there is a conic neighborhood V of ρ such that $V\cap\Sigma_r$ is a C^∞ manifold.

It then follows that

$$p_\rho(X+tY)=p_\rho(X) \quad \text{for} \quad t\in\boldsymbol{R},\ Y\in T_\rho\Sigma_r,\ X\in T_\rho(T^*\Omega)$$

so that we may regard $p_\rho(X)$ as a polynomial on the quotient space $N_{\Sigma_r}(T^*\Omega)_\rho= T_\rho(T^*\Omega)/T_\rho\Sigma_r$ (see Hörmander [4], Atiyah-Bott-Gårding [2]). Denoting by $[X]$ the equivalence class of $X\in T_\rho(T^*\Omega)$, we assume that

$$\text{(2.2)} \qquad p_\rho([X]) \quad \text{is strictly hyperbolic with respect to} \quad [\theta]\in N_{\Sigma_r}(T^*\Omega)_\rho\,,$$

and that $C(p_\rho, \theta)$ is transversal to Σ_r at ρ:

$$\text{(2.3)} \qquad C(p_\rho, \theta)\cap T_\rho\Sigma_r=\{0\}\ .$$

Clearly, (2.2) implies the hyperbolicity of $p_\rho(X)$ with respect to θ. In case $r\geqq 3$, we assume an additional hypothesis on lower order terms of P. Let $P(x, \xi)$ denote the total symbol of $P(x, D)$ and hence to be asymptotic to the sum $p(x, \xi)+p_{m-1}(x, \xi)+\cdots+p_i(x, \xi)+\cdots$ where $p_i(x, \xi)$ is homogeneous of degree i with respect to ξ. We assume

$$\text{(2.4)} \qquad p_{m-j}(x, \xi) \quad \text{vanishes of order } r-2j \text{ on } \Sigma_r \text{ near } \rho \text{ with} \quad r-2j>0\,.$$

Let γ denote the union of bicharacteristics of $p(x, \xi)$ with the limit point ρ along which $t(x, \xi)$ is increasing. Then we have

Theorem 2.1. *Let $p(x, D)$ be a classical pseudo-differential operator with real principal symbol $p(x, \xi)$ and let ρ be a characteristic of order r of $p(x, \xi)$. Assume that* (2.1)$\sim$(2.4) *are satisfied and that $t(x, \xi)$ is a time function near ρ with respect to $\Gamma(p_\rho, \theta)$. If $u \in \mathcal{D}'(\Omega)$ and*

$$WF(u) \cap \gamma \cap \{t(x, \xi) = -\kappa\} = \emptyset,\ \rho \notin WF(Pu)$$

with a sufficiently small $\kappa > 0$ then we have

$$\rho \notin WF(u)\,.$$

Note that under the hypothesis (2.1), the assumption (2.2) is always valid when $r=2$ except for a special case $\dim N_{\Sigma_r}(T^*\Omega)_\rho = 1$, that is, $p_\rho([X])$ is a polynomial of one variable.

It is clear that (2.4) is invariant under conjugation by Fourier integral operators. Furthermore, assuming (2.1), the condition (2.4) is actually a necessary one for the Cauchy problem of P to be well posed in C^∞ (see Ivrii-Petkov [8] for more details).

Here we note that, assuming (2.1), $p(x, \xi)$ is effectively hyperbolic at ρ if and only if $r=2$ and (2.2), (2.3) are satisfied (see Hörmander [6], Nishitani [15]). Then in this case the result is contained in Melrose [12], Nishitani [14].

REMARK 2.1. As will be proved in the proof of Lemma 3.2 below, there are *at least* r different bicharacteristics of $p(x, \xi)$ having ρ as the limit point along which a time function $t(x, \xi)$ is increasing.

EXAMPLE 2.1. Let $q(\zeta)$ be a homogeneous polynomial of degree r in $\zeta = (\zeta_0, \cdots, \zeta_k)$ which is strictly hyperbolic with respect to $\Theta \in \boldsymbol{R}^{k+1}$. Let $\varphi_j(x, \xi)$ $(j=0, \cdots, k)$ be real valued, homogeneous of degree 1 in ξ and C^∞ in a conic neighborhood of ρ. Assume that $\varphi_j(\rho)=0$ and $d\varphi_j(\rho)$ are linearly independent. We set

$$p(x, \xi) = q(\varphi_0(x, \xi), \cdots, \varphi_k(x, \xi))\,,$$

then $p(x, \xi)$ satisfies (2.1), (2.2) with $[\theta] = \Theta$, where we identify $N_{\Sigma_r}(T^*\Omega)_\rho$ with $\boldsymbol{R}^{k+1}$ by taking a basis $[X_i]$ such that $d\varphi_i(X_j) = \delta_{ij}$. Denoting by $\{\varphi_i, \varphi_j\}$ the Poisson bracket, we introduce a $(k+1)\times(k+1)$ matrix $A = (\{\varphi_i, \varphi_j\}(\rho))$. Then (2.3) is satisfied if

$$A(\boldsymbol{R}^{k+1}) \cap \Gamma(q, \Theta) \neq \emptyset\,.$$

In particular, if A is nonsingular, that is, if the tangent space of the surface $\{\varphi_j(x, \xi)=0,\ j=0, \cdots, k\}$ at ρ is symplectic, (2.3) is always satisfied.

2.3. Operators with normally intersecting characteristics

We next consider the case in which the characteristic set, $\Sigma = \{(x, \xi) \in T^*\Omega\backslash 0;\ p(x, \xi)=0\}$ of $p(x, \xi)$, is the union of r hypersurfaces S_i normally intersecting at ρ:

$$\Sigma = \bigcup_{i=1}^{r} S_i,\ S_i = \{(x, \xi) \in T^*\Omega \backslash 0;\ q_i(x, \xi) = 0\}\ . \tag{2.5}$$

Here $q_i(x, \xi)$ are real valued, vanishing at ρ, homogeneous of degree 1 in ξ and C^∞ in a conic neighborhood of ρ with linearly independent differentials at ρ.

For a subset I of $\{1, \cdots, r\}$, we denote by $|I|$ the number of indices of I and set

$$S_I = \bigcap_{i \in I} S_i\,, \quad S = \bigcap_{i=1}^{r} S_i\,.$$

We then assume that

$$\operatorname{rank}(\sigma \mid T_\rho^\sigma S) \leqq 2\,, \quad \text{where} \quad T_\rho^\sigma S = (T_\rho S)^\sigma\,, \tag{2.6}$$

$$C(p_\rho, \theta) \cap T_\rho S_I = \{0\} \quad \text{for every } I \text{ with } \quad |I| = 2\,. \tag{2.7}$$

In case $r \geqq 3$, we again assume an additional hypothesis on lower order terms of P:

$$\begin{aligned} &p_{m-j}(x, \xi) \text{ vanishes of order } |I| - 2j \text{ on } S_I \text{ near } \rho \\ &\text{for every } I, j \text{ with } |I| - 2j > 0\,. \end{aligned} \tag{2.8}$$

Let γ_j denote the bicharacteristic for $q_j(x, \xi)$ (that is, a characteristic curve of S_j) through ρ and denote by γ their union. We choose and fix, near ρ, a time function $t(x, \xi)$ with respect to $\Gamma(p_\rho, \theta)$. Then we have

Theorem 2.2. *Let $P(x, D)$ be a classical pseudo-differential operator with real principal symbol $p(x, \xi)$ and ρ be a characteristic of order r of $p(x, \xi)$. Suppose that* (2.5)$\sim$(2.8) *hold and that $t(x, \xi)$, γ are as above. If $u \in \mathcal{D}'(\Omega)$ and*

$$WF(u) \cap \gamma \cap \{t(x, \xi) = -\kappa\} = \emptyset,\ \rho \notin WF(Pu)$$

with a sufficiently small $\kappa > 0$ then it follows that

$$\rho \notin WF(u)\,.$$

REMARK 2.2. This result is a conformally invariant version of Theorem 2.2 in [16] (see Lemma 4.1 below). When $r=2$ more precise results were obtained, see Alinhac [1], Hanges [3], Ivrii [9] and the references given there.

Note that (2.8) is a necessary condition for the Cauchy problem of P to be well posed in C^∞ under the assumption (2.5) (see Ivrii-Petkov [8] for more details) and that (2.8) is invariant under conjugation of Fourier integral operators. In case $r=2$ or $r=3$ the condition (2.6) is automatically satisfied since σ is skew symmetric.

It is clear that near ρ

$$p(x, \xi) = q(x, \xi) \prod_{j=1}^{r} q_j(x, \xi), \quad p_\rho(X) = q(\rho) \prod_{j=1}^{r} q_{j\rho}(X),$$

where $q(x, \xi)$ is homogeneous of degree $m-r$ and $q(\rho) \neq 0$. From this fact, it is also clear that $p_\rho(X)$ is hyperbolic with respect to θ whenever $q_{j\rho}(\theta) \neq 0$, $j=1, \cdots, r$. In view of $q_{j\rho}(X) = \sigma(X, H_{q_j}(\rho))$, we see that

$$\Gamma(p_\rho, \theta) = \{X \in T_\rho(T^*\Omega);\ q_{j\rho}(\theta)\, q_{j\rho}(X) > 0, j = 1, \cdots, r\},$$

$$C(p_\rho, \theta) = \{X \in T_\rho(T^*\Omega);\ X = \sum_{j=1}^{r} \alpha_j\, q_{j\rho}(\theta)\, H_{q_j}(\rho), \alpha_j \geqq 0\}.$$

We remark that the condition (2.6) is independent of the choice of a hyperbolic direction θ although (2.7) *depends* on θ.

Let $\tilde{\theta}$ be another hyperbolic direction of $p_\rho(X)$ and assume that

$$C(p_\rho, \tilde{\theta}) \cap T_\rho S_I = \{0\} \quad \text{for every } I \text{ with } |I| = 2. \tag{2.9}$$

Corollary 2.1. *Assume the same conditions as in Theorem* 2.2 *and* (2.9). *If* $u \in \mathcal{D}'(\Omega)$ *and*

$$WF(u) \cap (\bigcup_{j\in I^+} \gamma_j) \cap \{t(x, \xi) = -\kappa\} = \emptyset,$$

$$WF(u) \cap (\bigcup_{j\in I^-} \gamma_j) \cap \{t(x, \xi) = -\kappa\} \neq \emptyset, \quad \rho \notin WF(Pu)$$

with a sufficiently small $\kappa > 0$ *then we have*

$$WF(u) \cap (\bigcup_{j\in I^-} \gamma_j) \cap \{t(x, \xi) = \kappa\} \neq \emptyset$$

where $I^+ = \{j \in \{1, \cdots, r\};\ q_{j\rho}(\theta)\, q_{j\rho}(\tilde{\theta}) > 0\}$, $I^- = \{1, \cdots, r\} \backslash I^+$.

REMARK 2.3. When $r=2$ this corollary reduces to Theorem 1 in Hanges [3]. See also Theorem 0.3 in Ivrii [9].

Proof. We take a time function $\tilde{t}(x, \xi)$ with respect to $\Gamma(p_\rho, \tilde{\theta})$ and hence $d\tilde{t}(q_{j\rho}(\tilde{\theta})\, H_{q_j}(\rho)) > 0$, $1 \leqq j \leqq r$. Assume that the assertion was false: $WF(u) \cap (\bigcup_{j\in I^-} \gamma_j) \cap \{t(x, \xi) = \kappa\} = \emptyset$. Then it is clear that

$$WF(u) \cap \gamma \cap \{\tilde{t}(x, \xi) = -\tilde{\kappa}\} = \emptyset$$

with a small $\tilde{\kappa} > 0$ since $d\tilde{t}(q_{j\rho}(\theta)\, H_{q_j}(\rho)) > 0, j \in I^+$ and $d\tilde{t}(q_{j\rho}(\theta)\, H_{q_j}(\rho)) < 0$, $j \in I^-$. Then Theorem 2.2 would give $\rho \notin WF(u)$. On the other hand the second condition of the corollary shows $\rho \in WF(u)$ since $WF(u)$ is closed. This contradiction proves the assertion.

EXAMPLE 2.2. Let us consider the symbol

$$p(x, \xi) = \prod_{j=1}^{r} q_j(x, \xi), \quad q_j(x, \xi) = \xi_0 - a_j(x)\, b_j(\xi'),$$

where $\xi'=(\xi_1, \cdots, \xi_d)$, $a_j(x)$ are real valued, C^∞ near $\hat{x}$ vanish at $\hat{x}$ and $b_j(\xi')$ are real valued homogeneous of degree 1 in ξ', C^∞ in a conic neighborhood of $\hat{\xi}'$. We assume that

$$\partial_{x_0}a_i(\hat{x})\, b_i(\hat{\xi}') \neq \partial_{x_0}a_j(\hat{x})\, b_j(\hat{\xi}') \quad \text{for any } i, j \text{ with } \quad i \neq j\,,$$

where $\partial_{x_0}a(x)$ is the derivative with respect to x_0. Then $p(x, \xi)$ satisfies (2.5)$\sim$(2.7) with $\rho=(\hat{x}, 0, \hat{\xi}')$ if we assume that $dq_j(\rho)$ are linearly independent.

3. Proof of Theorem 2.1

For $X\in T_\rho(T^*\Omega)$ we denote by $\langle X\rangle$ the line spanned by X. To prove Theorem 2.1 we first choose a homogeneous symplectic coordinates near ρ so that $p(x, \xi)$ takes a convenient form in order to apply our previous results in [17].

Lemma 3.1. *Assume* (2.1)$\sim$(2.3). *Then we can choose a homogeneous symplectic coordinates near* ρ *so that* $\rho=(0, e_d)$, $e_d=(0, \cdots, 0, 1)\in \boldsymbol{R}^{d+1}$ *and*

$$p(x, \xi) = e(x, \xi)\, (\xi_0^r + a_2(x, \xi')\, \xi_0^{r-2} + \cdots + a_r(x, \xi')) = e(x, \xi)\, q(x, \xi)$$

with $e(\rho)\neq 0$. *Here* $a_j(x, \xi')$ *are real valued, homogeneous of degree* j *in* $\xi'=(\xi_1, \cdots, \xi_d)$, C^∞ *in a conic neighborhood of* $(0, e_d')$, $e_d'=(0, \cdots, 1)\in \boldsymbol{R}^d$. *Moreover* $a_j(x, \xi')$ *vanish at* $(0, e_d')$. *Furthermore*

(3.1) $q_\rho([X])$ *is strictly hyperbolic with sespect to* $[H_{x_0}]$ *in* $N_{\Sigma_r}(T^*\Omega)_\rho$,

(3.2) $$\langle H_{x_0}\rangle^\sigma \supset T_\rho \Sigma_r \cap T_\rho^\sigma \Sigma_r\,,$$

where Σ_r *is the set of characteristics of order* r *of* q.

Proof. We repeat similar arguments to those in the proof of Theorem 1.3 in [17]. Under the notations in §2, let $V\cap \Sigma_r$ be given by the equations

$$b_0(x, \xi) = \cdots = b_k(x, \xi) = 0\,,$$

where $b_j(x, \xi)$ are homogeneous of degree 1 in ξ, C^∞ in a conic neighborhood of ρ with linearly independent differentials at ρ. Without loss of generality we may assume that $\rho=(0, e_d)$. Note that (2.3) is equivalent to

$$\Gamma(p_\rho, \theta) \cap T_\rho^\sigma \Sigma_r \neq \emptyset\,,$$

hence we can take $Z\neq 0$ in $\Gamma(p_\rho, \theta)\cap T_\rho^\sigma \Sigma_r$. Since $T_\rho^\sigma \Sigma_r$ is spanned by the $H_{b_j}(\rho)$, Z is a linear combination of $H_{b_j}(\rho)$ with non negative coefficients α_j. Set

$$\varphi(x, \xi) = \sum_{j=0}^{k} \alpha_j\, b_j(x, \xi)$$

so that $Z=H_\varphi(\rho)$. In view of $H_\varphi(\rho)\in\Gamma(p_\rho, \theta)$ we see that $p_\rho(H_\varphi(\rho))\neq 0$ and hence

$$(H_{\varphi}^{r}p)(\rho)\neq 0\,. \tag{3.3}$$

by the definition of localization. Set $y_0=\varphi(x,\xi)$ and note that $H_{\varphi}(\rho)$ and the radial vector field at ρ are linearly independent because the latter is in $T_\rho\Sigma_r$ and $p_\rho(T_\rho\Sigma_r)=0$. Thus one can extend y_0 to a full homogeneous symplectic coordinates (y_j,η_j) near ρ so that $(y,\eta)(0,e_d)=(0,e_d)$ (see, for example, Theorem 21.1.9 in Hörmander [7]). To simplify notation we write (x,ξ) instead of (y,η). Taking into account that $H_{x_0}^{j}p(\rho)=0$ for $0\leqq j\leqq r-1$ and (3.3), the Malgrange preparation theorem gives a factorization of p asserted in the lemma apart from the (possible) presence of a term $a_1(x,\xi')\,\xi_0^{r-1}$ in $q(x,\xi)$. Clearly this term is removed by taking a new homogeneous symplectic coordinates preserving the x_0 coordinate and ρ. This gives a desired factorization of p.

Since H_{x_0} belongs to the hyperbolic cone of p_ρ (2.2) implies (3.1). Noticing that Σ_r is contained in the surface $x_0=0$ we see that $\langle H_{x_0}\rangle^{\sigma}\supset T_\rho\Sigma_r$ and hence (3.2). This completes the proof.

From this lemma, a pseudo-differential analogue of Malgrange's division theorem shows that

$$P(x,D)\equiv E(x,D)\,\{D_0^r+A_1(x,D')\,D_0^{r-1}+\cdots+A_r(x,D')\}\;=E(x,D)\,Q(x,D)\,,$$

modulo a smoothing operator near ρ where $E(x,D)$, $Q(x,D)$ have the principal symbols $e(x,\xi)$, $q(x,\xi)$ respectively. We take an elliptic pseudo-differential operator $F(x,D)$ of order $-m+r$ so that $\rho\in WF(FE-I)$. Multiplication of operator P by F reduces the proof of Thoerem 2.1 to the case of operator Q. Denote by $Q(x,\xi)$ the total symbol of Q which is asymptotic to the sum $q(x,\xi)+q_{r-1}(x,\xi)+\cdots+q_i(x,\xi)+\cdots$. From the formula of asymptotic expansion for a product of pseudo-differential operators it is easy to see that the condition (2.4) for p_j implies:

(3.4) $q_{r-j}(x,\xi)$ vanishes of order $r-2j$ on Σ_r near ρ with $r-2j>0$.

With $x'=(x_1,\cdots,x_d)$ set

$$\Delta_b=\{(x,\xi)\in T^*\Omega\backslash 0;\ -bx_0>|(x',\xi|\xi'|^{-1})-(0,e_d)|,\ \xi'\neq 0\}\,,$$

where b is a positive parameter. Denoting by π the projection: $(x,\xi)\to(x',\xi')$, we recall a result which follows easily from Proposition 6.2 in [17],

Proposition 3.1. *Assume that* (3.1), (3.2) *and* (3.4) *hold. Then there is a constant $\beta>0$ with the following property: let $u\in C^{r-1}(I,H^s(\boldsymbol{R}^d))$ with some $s\in\boldsymbol{R}$ and an open interval I containing $x_0=0$. If*

$$WF(D_0^j u(-\kappa,\cdot))\cap\pi(\Delta_\beta\cap\{x_0=-\kappa\})=\emptyset\quad for\quad 0\leqq j\leqq r-1\,,$$

with a sufficiently small $\kappa>0$ *and*

$$(0, e_d') \notin WF(Qu(x_0, \cdot)),\ e_d' = (0, \cdots, 0, 1) \in \boldsymbol{R}^d,$$

uniformly in x_0 *near* $x_0=0$, *then it follows that*

$$(0, e_d') \notin WF(u(x_0, \cdot)),$$

uniformly in x_0 *near* $x_0=0$.

Now we discuss the singularities of u as a distribution in $\boldsymbol{R}^{d+1}$ instead of those of u for fixed x_0.

Proposition 3.2. *Assume that* (3.1), (3.2) *and* (3.4) *are satisfied. Then there is a constant* $b>0$ *with the following property*: *if* $u \in \mathcal{D}'(\Omega)$ *and*

$$WF(u) \cap \Delta_b \cap \{x_0 = -\kappa\} = \emptyset, \quad \rho \notin WF(Qu)$$

with a sufficiently small $\kappa>0$ *then we have*

$$\rho \notin WF(u).$$

We postpone the proof of Proposition 3.2. Theorem 2.1 will be proved by combining Proposition 3.2 and abstract results on generalized flow in [22]. We prefer to give a rather straightforward proof togehter with that of Remark 2.1 applying this proposition.

We first make some observations on behaviors of bicharacteristics of $p(x, \xi)$ following Melrose [12], Nishitani [14]. By Lemma 3.1 it can be assumed that Σ_r is given by $f_0(x, \xi)=\xi_0=0$, $f_j(x, \xi')=0$, $j=1, \cdots, k$. Take $X_j \in T_\rho(T^*\Omega)$, $j=1, \cdots, k$ and $X_0=-H_{x_0}$ so that $df_i(X_j)=\delta_{ij}$ and $Y_j \in T_\rho(T^*\Omega)$, $j=1, \cdots, 2d+1-k$ so that Y_j form a basis for $T_\rho \Sigma_r$. We define a polynomial $s(z)$ by

$$q_\rho(\Sigma z_j X_j + \Sigma w_j Y_j) = q_\rho(\Sigma z_j X_j) = s(z).$$

Note that (3.1) means that $s(z)$ is strictly hyperbolic with respect to $(1, 0, \cdots, 0) \in \boldsymbol{R}^{k+1}$. It is clear that $q_\rho(X)=s(df_0(X), \cdots, df_k(X))$ and hence we can write

$$q(x, \xi) = s(\xi_0, f(x, \xi')) + \sum_{i \leq r-2, i+|\alpha|=r} a_{i\alpha}(x, \xi')\, \xi_0^i f(x, \xi')^\alpha$$

with the notation $f(x, \xi')=(f_1(x, \xi'), \cdots, f_k(x, \xi'))$. Note that $a_{i\alpha}(0, e_d')=0$. We define $\tilde{q}(z; x, \xi')$ by replacing $(\xi_0, f(x, \xi'))$ by $z=(z_0, z')$ in the above expression. Since the zeros z_0 of $s(z)$ are real distinct and $a_{i\alpha}(x, \xi')$ are real valued, it follows from Rouché's theorem

$$\tilde{q}(z; x, \xi') = \prod_{j=1}^{r} (z_0 - \lambda_j(z'; x, \xi')), \tag{3.5}$$

where $\lambda_j(z'; x, \xi')$ are C^∞ in $(\boldsymbol{R}^k \backslash 0) \times W$, homogeneous of degree 1 and 0 in z', ξ' respectively and W is a conic neighborhood of $(0, e_d')$. By the homogeneity with respect to z', ξ' shrinking W if necessary, that

$$|\partial_{z'}^{\alpha}\partial_{x}^{\beta}\partial_{\xi'}^{\gamma}\lambda_j(z'; x, \xi')| \leqq C_{\alpha\beta\gamma}|z'|^{1-|\alpha|}|\xi'|^{-|\gamma|} \tag{3.6}$$

in $(\boldsymbol{R}^k\backslash 0)\times W$. Substituting $(\xi_0, f(x, \xi'))$ into $z=(z_0, z')$ in (3.5) we obtain with $\tilde{\lambda}_j(x, \xi')=\lambda_j(f(x, \xi'); x, \xi')$ that

$$q(x, \xi) = \prod_{j=1}^{r} \tilde{q}_j(x, \xi), \quad \tilde{q}_j(x, \xi) = \xi_0 - \tilde{\lambda}_j(x, \xi').$$

Note that this expression is valid if $x_0 \neq 0$, $(x, \xi')\in W$ since we can assume that $\{f(x, \xi')=0\}$ is contained in the surface $\{x_0=0\}$ (see the proof of Lemma 3.1). It follows from (3.6) that

$$|\partial_{\xi_i}\tilde{\lambda}_j(x, \xi')| \leqq C, \quad |\partial_{x_i}\tilde{\lambda}_j(x, \xi')| \leqq C|\xi'| \quad (x, \xi')\in W, x_0 \neq 0 \tag{3.7}$$

for any i, j.

We shall now be working in a neighborhood of ρ which is not conic. Note that near ρ with $x_0 \neq 0$ a bicharacteristic of $q(x, \xi)$ is any one of $\tilde{q}_j(x, \xi)$ and hence by (3.7) the tangent of such a curve is in the cone $\cup C^{\pm}$, $C^+=\{C_1 x_0 \geqq |(x', \xi)|\}$, $C^-=-C^+$.

Denote by S_ε the hyperplane $x_0=-\varepsilon$ and by B_δ a box in S_δ with sides a, $B_\delta=\{|\xi-e_d|<a, |x'|<a, x_0=-\delta\}$. We introduce a map from B_δ into S_ε, $F_{j\varepsilon}^{\delta}$: $(y', \eta)\rightarrow(\tilde{y}', \tilde{\eta})$ where $(-\delta, y', \eta)$ and $(-\varepsilon, \tilde{y}', \tilde{\eta})$ lie on the same integral curve of $H_{\tilde{q}_j}$. Since near ρ with $x_0 \neq 0$ the tangent of such integral curve is controlled by the cone $\cup C^{\pm}$, taking a, δ sufficiently small, the map $F_{j\varepsilon}^{\delta}$ is well defined for any $0<\varepsilon(\leqq\delta)$. By (3.7) it is easy to see that

$$|F_{j\varepsilon_1}^{\delta}(y', \eta)-F_{j\varepsilon_2}^{\delta}(y', \eta)| \leqq B|\varepsilon_1-\varepsilon_2| \quad \text{near} \quad (0, e_d) \tag{3.8}$$

with a constant $B>0$ independent of (y', η) and ε_i. This allows us to define a continuous map from B_δ into S_0; $F_j^{\delta}(y', \eta)=\lim_{\varepsilon\downarrow 0} F_{j\varepsilon}^{\delta}(y', \eta)$. Take $\delta>0$ sufficiently small so that $C^-\cap S_\delta \Subset B_\delta$. Let K_j be the inverse image of the point ρ by F_j^{δ} which is a compact set in S_δ and so is K, the union of K_j. Here we note that the intersection of γ and S_δ is just K.

Let $u\in\mathcal{D}'(\Omega)$ and suppose that

$$WF(u)\cap K = \emptyset, \ \rho\notin WF(Qu). \tag{3.9}$$

With $B_{\mu\nu}=\{(x, \xi)\in T^*\Omega\backslash 0; -\nu\leqq x_0<0, |(x', \xi)-(0, e_d)|\leqq\mu\}$ we have

Lemma 3.2. *Let Q be as above and assume* (3.9). *Then*

$$B_{\mu\nu}\cap WF(u) = \emptyset$$

for sufficiently small μ, ν.

Proof. We fix a compact set $K'\subset S_\delta$ so that $B_\delta\supset K'\supset C^-\cap S_\delta$ and take an open set O in S_δ with $K'\supset O\supset K$, $O\cap WF(u)\cap S_\delta=\emptyset$. Let M_j be the image of $K'\backslash O$ by F_j^δ and M be the union of M_j. It is obvious that M is compact and $\rho\notin M$. Then one can choose μ, ν so that

$$(B_{\mu\nu}+C^+)\cap M=\emptyset\,,\quad (\bar{B}_{\mu\nu}+C^-)\cap S_\delta\subset K'\,, \tag{3.10}$$

where $\bar{B}_{\mu\nu}$ is the closure of $B_{\mu\nu}$. Suppose that $B_{\mu\nu}\cap WF(u)$ would contain $(y,\eta)=(-\varepsilon,y',\eta)$, $0<\varepsilon<\delta$. From (3.9) we may assume $(y,\eta)\notin WF(Qu)$ taking μ, ν small. In what follows we fix these μ, ν. Then it follows from Theorem 2.2.2 in Hörmander [5] that $q(y,\eta)=0$ and hence $\tilde{q}_j(y,\eta)=0$ with some j. With $(\tilde{y},\tilde{\eta})=F_{j\varepsilon}^\delta(y,\eta)$, Theorem 3.2.1 in [5] shows that $(\tilde{y},\tilde{\eta})\in WF(u)$ and hence $(\tilde{y},\tilde{\eta})\in K'\backslash O$ by the second condition of (3.10). This would give a contradiction to the first condition of (3.10) since $F_j^\delta(\tilde{y},\tilde{\eta})\in M_j\subset M$ and $F_j^\delta(\tilde{y},\tilde{\eta})=F_{j\varepsilon}^\delta(y,\eta)\in\{(y,\eta)+C^+\}\cap S_0$. This proves the lemma.

A similar argument shows that F_j^δ is surjective. Thus there are at least r different bicharacteristics of q having the limit point ρ along which x_0 is increasing and this shows Remark 2.2.

Proof of Proposition 3.2. Let V be a conic neighborhood of ρ which does not contain the ξ_0 axis. We choose $a\in S^0(\boldsymbol{R}^{d+1}\times\boldsymbol{R}^{d+1})$ equal to 1 in a conic neighborhood of ρ and supported in V. Set $v=a(x,D)u\in\mathcal{E}'(\Omega)$, $g=Q(x,D)v$. Since $q(x,\xi)\neq 0$ when $(x,\xi)\notin F=\{|\xi_0||\xi'|^{-1}\leq C_1|(x,\xi'|\xi'|^{-1})-\rho'|,\xi'\neq 0\}$ by (3.7), then it follows from Theorem 2.2.2 in [5] that $WF(v)\subset F\cap V$. Thus, noting that $a=1$ near ρ, we can easily examine that

$$(0,e_d')\notin WF(g(x_0,\cdot))\,,$$

uniformly in x_0 near $x_0=0$. For any given $\beta>0$, one can take $b>0$ so that

$$V\cap(\Delta_b\cup F^c)\supset V\cap(\Delta_\beta+\langle H_{x_0}\rangle)\,,$$

where F^c is the complement of F in which $\xi'\neq 0$. Since $WF(v)\subset F\cap V$ the assumption of Proposition 3.1 means that

$$WF(v)\cap(\Delta_\beta+\langle H_{x_0}\rangle)\cap\{x_0=-\kappa\}=\emptyset$$

for a sufficiently small $\kappa>0$. Now Proposition 3.2 shows that $(0,e_d')\notin WF(v(x_0,\cdot))$ uniformly in x_0 near $x_0=0$. Hence $(0,e_d)\notin WF(v)$ which completes the proof.

Proof of Theorem 2.1. We shall examine that the hypothesis of Theorem 2.1 implies that of Proposition 3.2. Let $b>0$ be a positive constant in Proposition 3.2. If we take $\kappa>0$ sufficiently small, it is clear that the intersection of the conic hull of $B_{\mu\nu}$ and $\{x_0=-\kappa\}$ contains $\Delta_b\cap\{x_0=-\kappa\}$. Then from Lemma 3.2 it follows that $WF(u)\cap\Delta_b\cap\{x_0=-\kappa\}=\emptyset$. This is the desired assertion.

4. Proof of Theorem 2.2

First we rewrite the hypotheses (2.6), (2.7) in a more convenient form to the proof. Under the notations of §1, we set

$$a_{ij} = q_{i\rho}(\theta)\, q_{j\rho}(\theta) \{q_i, q_j\}(\rho)\,,$$
$$\omega = \sum a_{ij}\, dq_i \wedge dq_j\,.$$

Note that both σ and ω, restricted to $T_\rho^\sigma S/(T_\rho^\sigma S \cap T_\rho S)$, have the same rank.

Lemma 4.1. *The conditions* (2.6), (2.7) *are equivalent to*

(4.1) $$a_{ij} \neq 0 \quad \textit{for any pair } i, j \textit{ with } \quad i \neq j\,,$$

(4.2) *there are positive constants c_i such that*
$$c_i\, c_j\, a_{ij} + c_j\, c_k\, a_{jk} + c_k\, c_i\, a_{ki} = 0 \quad \textit{for any triplet } i, j, k.$$

Proof. It is convenient first to show that (4.1), (4.2) are equivalent to (2.7) and (4.3) below,

(2.7) $$C(p_\rho, \theta) \cap T_\rho S_I = \{0\} \quad \text{for all } I \text{ with } \quad |I| = 2\,,$$

(4.3) $$\omega = \omega_1 \wedge \omega_2 \quad \text{with some one forms} \quad \omega_i$$

and after that we prove the equivalence between (2.6) and (4.3) assuming (2.7). Here note that (4.3) is equivalent to the Plüker relations:

(4.4) $$a_{ij}\, a_{kl} + a_{jk}\, a_{il} + a_{ki}\, a_{jl} = 0 \quad \text{for all} \quad i, j, k, l\,.$$

We first show that (4.1), (4.2) imply (2.7), (4.3). Set $b_{ij} = c_i\, c_j\, a_{ij}$ then b_{ij} verify the conditions of cocycles by (4.2). Then there are constants $\tilde{b}_i$ such that $b_{ij} = \tilde{b}_i - \tilde{b}_j$. With $b_i = c_i^{-1} \tilde{b}_i$ it follows that

(4.5) $$a_{ij} = c_j^{-1}\, b_i - c_i^{-1}\, b_j\,.$$

This proves (4.3) with $\omega_1 = \sum b_j\, dq_j$, $\omega_2 = \sum c_j^{-1}\, dq_j$. Let $X \in C(p_\rho, \theta)$, which is a linear combination of $q_{k\rho}(\theta)\, H_{q_k}(\rho)$ with non negative coefficients α_k. From (4.5) it follows immediately that $(c_i\, q_{i\rho}(\theta)\, dq_i - c_j\, q_{j\rho}(\theta)\, dq_j)(X)$ is equal to $c_i\, c_j\, a_{ij} \sum \alpha_k\, c_k^{-1}$. Since $c_k > 0$ one has $\alpha_k = 0$ if $dq_i(X) = dq_j(X) = 0$. Thus (2.7) is obtained. Now we prove that (2.7), (4.3) imply (4.1), (4.2). If $a_{ij} = 0$ then $H_{q_i}(\rho) + H_{q_j}(\rho)$ belongs to $C(p_\rho, \theta) \cap T_\rho S_I$ with $I = \{i, j\}$ which would contradict to (2.7) and thus (4.1) follows obviously. Next note that for $I = \{i, j, k\}$, $T_\rho S_I \cap T_\rho^\sigma S_I$ is spanned by $Z_I = a_{ij}\, q_{k\rho}(\theta)\, H_{q_k}(\rho) + a_{jk}\, q_{i\rho}(\theta)\, H_{q_i}(\rho) + a_{ki}\, q_{j\rho}(\theta)\, H_{q_j}(\rho)$ and that for any J, $J \subset I$, $|J| = 2$ one has $T_\rho S_I \cap T_\rho^\sigma S_I = T_\rho S_J \cap T_\rho^\sigma S_J$. This implies in virtue of (2.7) that $Z_I \notin C(p_\rho, \theta)$. Using this fact, renumbering q_i if necessary, we may assume that $a_{12} > 0$, $a_{im} < 0$ for $i = 1, 2, \cdots, m-1$. Once more renumbering q_i one can suppose that $a_{12} > 0$, $a_{3j} > 0$ for any $j, j \neq 3$. Define c_i by

$$c_i = -a_{12}\,a_{23}\,a_{31}(a_{12}\,a_{3i}+\varepsilon a_{31}\,a_{i2})^{-1} \quad \text{for} \quad i=1, 2, \cdots, m\,,$$

where ε is taken sufficiently small so that $c_i>0$ for $i=4, 5, \cdots, m$, which is possible since $a_{12}\,a_{23}\,a_{31}<0$, $a_{12}\,a_{3i}>0$. By (4.4) it is easy to examine that c_i satisfy (4.2) and this proves the assertion.

We next prove the equivalence between (4.3) and (2.6) assuming (2.7). As noted above ω, restricted to $T_\rho^\sigma S/(T_\rho^\sigma S \cap T_\rho S)$, is non degenerate. Then it follows from (4.3) that $\dim(T_\rho^\sigma S/(T_\rho^\sigma S \cap T_\rho S))=2$ hence $\dim(T_\rho^\sigma S \cap T_\rho S)=r-2$. This implies (2.6). Conversely (2.6) implies that $\dim(T_\rho S_J \cap T_\rho^\sigma S_J)=4-\operatorname{rank}(a_{ij})_{ij\in J}=2$ for any J with $|J|=4$ since $a_{ij}\neq 0\ (i\neq j)$. Recalling that $T_\rho S_I \cap T_\rho^\sigma S_I$ is spanned by Z_I for $|I|=3$ we see that $T_\rho S_I \cap T_\rho^\sigma S_I \subset T_\rho S_J \cap T_\rho^\sigma S_J$ for any $J(\supset I)$ with $|J|=4$. Since $J(\supset I)$ is arbitraly we get $T_\rho S_I \cap T_\rho^\sigma S_I \subset T_\rho S$. This implies that $dq_l(Z_I)=0$ for all l and hence (4.4).

Before reducing the proof of Theorem 2.2 to the case of a second order system we make similar observations to those in §3. Under the notations in §1 we recall that $p(x, \xi)=q(x, \xi)\prod_{j=1}^{r} q_j(x, \xi)$. Since $q(\rho)\neq 0$ and $q_{j\rho}(\theta)\neq 0$, by a similar argument after the proof of Lemma 3.1, we may suppose that $P(x, D)$ is of order r with principal symbol $p(x, \xi)$ which is the product of $q_j(x, \xi)$ with $q_{j\rho}(\theta)=1$. Moreover the hypothesis (2.8) can be verified with $m=r$. The conditions (4.1) and (4.2) are invariant by multiplication of q_i by positive constants c_i then we may assume that $c_i=1$ in (4.2). Also we may assume that $\rho=(0, e_d)$, $e_d=(0, \cdots, 0, 1)\in \boldsymbol{R}^{d+1}$ as in §3. Then (4.2) means that

$$\{q_i-q_j,\, q_k-q_l\}\,(\rho)=0 \quad \text{for any } i, j, k, l\,. \tag{4.6}$$

Set $y_0=(q_1-q_2)$ if $\{q_1, q_2\}\,(\rho)<0$ and $y_0=-(q_1-q_2)$ if $\{q_1, q_2\}\,(\rho)>0$. From (4.6) it follows, in both cases, that

$$dq_1(H_{y_0})=\cdots=dq_r(H_{y_0})=\pm a_{12}<0\,. \tag{4.7}$$

Since H_{y_0} and the radial vector field at ρ are linearly independent we can extend y_0 to a full homogeneous symplectic coordinates (y_j, η_j) near ρ so that $(y, \eta)(0, e_d)=(0, e_d)$. For the sake of simplicity we write (x, ξ) instead of (y, η). Then by (4.7) one can write $q_i(x, \xi)=e_i(x, \xi)\,(\xi_0-a_i(x, \xi'))$ with $e_i(\rho)>0$ where $a_i(x, \xi')$ are real valued homogeneous of degree 1 in ξ', C^∞ in a conic neighborhood of ρ'. From the same arguments as in §3 one can assume that

$$p(x, \xi)=\prod_{j=1}^{r} q_j(x, \xi)\,, \quad q_j(x, \xi)=\xi_0-a_j(x, \xi')\,.$$

Note that (2.8) with $m=r$ implies that near ρ, $p_{r-j}(x, \xi)$ is a linear combination of $q_I(x, \xi)$, $|I|=r-2j$, with coefficients which are homogeneous of degree j in ξ, C^∞ in a conic neighborhood of ρ, where $q_I(x, \xi)$ stands for the product of $q_j(x, \xi)$ over $j\in I$. This enables us to transform the equation $Pu=f$ to a second order system. Indeed, taking $\{\langle D'\rangle^{j-1} q_I(x, D')u, 0<|I|=m-2j<m, \langle D'\rangle^{[m/2]-1} u\}$ as new unknowns, the equation is reduced to a second order $N\times N$ system

$$LU = F \tag{4.8}$$

with a diagonal principal symbol whose entries consist of $q_i(x, \xi)\, q_j(x, \xi)$ with $i\neq j$ apart from repetition. Here $\langle\xi'\rangle^2=1+|\xi'|^2$ and $[m/2]$ denotes the integer part of $m/2$. Since the components of F consist of f and 0 it is obvious that $\rho\notin WF(F)$ (resp. $\rho\notin WF(U)$) implies $\rho\notin WF(f)$ (resp. $\rho\notin WF(u)$) and *vice versa*. For $I=\{i, j\}$ we set $K_I=\{(x, \xi)\in T^*\Omega\backslash 0;\ q_i(x, \xi)-q_j(x, \xi)=0\}$. Obviously K_I contains the ξ_0 axis and then

$$T_\rho K_I \supset T_\rho S_I + \langle H_{x_0}\rangle \quad \text{for any } I \text{ with } \quad |I| = 2\,. \tag{4.9}$$

Note that (4.6) implies that

$$\sigma(T^\sigma_\rho K_I,\ T^\sigma_\rho K_J) = 0 \quad \text{for any } I, J \text{ with } \quad |I|=|J| = 2\,. \tag{4.10}$$

Also from (4.6) we have

$$C(p_\rho, \theta)\cap T_\rho K_I = \{0\} \quad \text{for any } I \text{ with } \quad |I| = 2\,. \tag{4.11}$$

In fact (4.6) shows that $\{q_k, q_i-q_j\}(\rho)=\{q_l, q_i-q_j\}(\rho)=a_{ji}\neq 0$ for any i, j, k, l and thus $(dq_i-dq_j)(X)=0$, $X\in C(p_\rho, \theta)$ imply $X=0$ since

$$a_{ji}(dq_i-dq_j)(X) = a_{ji}^2 \textstyle\sum \alpha_k \quad \text{for} \quad X = \sum \alpha_k H_{q_k}(\rho)\,. \tag{4.12}$$

Setting $C'=-C(p_\rho, \theta)+\langle H_{x_0}\rangle+\rho$, we recall a result which follows easily from Proposition 8.1 in [18],

Proposition 4.1. *Assume that* (4.9)~(4.11) *are satisfied and let* $U\in C^1(I, (H^s(\boldsymbol{R}^d))^N)$ *with some* $s\in\boldsymbol{R}$ *and an open interval* I *containing* $x_0=0$. *If*

$$WF(D_0^j U(-\kappa, \cdot))\cap\pi(C'\cap\{x_0=-\kappa\}) = \emptyset \quad \textit{for} \quad 0\leqq j\leqq 1$$

with a sufficiently small $\kappa>0$ *and*

$$(0, e_d')\notin WF(LU(x_0, \cdot))\,,$$

uniformly in x_0 *near* $x_0=0$, *then it follows that*

$$(0, e_d')\notin WF(U(x_0, \cdot))\,,$$

uniformly in x_0 *near* $x_0=0$.

The same argument as in the proof of Proposition 3.2 proves with $C_b'=C'\cap\Delta_b$ that

Proposition 4.2. *Assume that* (4.9)~(4.11) *hold. Then there is a con-*

stant $b>0$ *with the following property: if* $U\in(\mathcal{D}'(\Omega))^N$ *and*

$$WF(U)\cap C'_b\cap\{x_0=-\kappa\}=\emptyset\,,\quad \rho\notin WF(LU)$$

with a sufficiently small $\kappa>0$ *then we have*

$$\rho\notin WF(U)\,.$$

Proof of Theorem 2.2. Assume that

$$WF(u)\cap\gamma\cap\{t(x,\xi)=-\kappa\}=\emptyset\,,\quad \rho\notin WF(Pu) \tag{4.13}$$

with a small $\kappa>0$. Set $\Lambda=\cap\{(x,\xi)\in T^*\Omega\backslash 0;\ a_{ji}(q_i-q_j)>0\}$ where the intersection is taken over all pairs i, j with $i\neq j$. We show that there is a $\varepsilon>0$ such that

$$WF(u)\cap\Lambda_b\cap\{x_0=-\varepsilon\}=\emptyset\quad\text{where}\quad \Lambda_b=\Lambda\cap\Delta_b\,. \tag{4.14}$$

Suppose for a moment that (4.14) is proved. It then follows from $\Lambda\supset\Lambda+\langle H_{x_0}\rangle$ and (4.12) that $\Lambda_b\cap\{x_0=-\varepsilon\}\supset C'_b\cap\{x_0=-\varepsilon\}$. Then Proposition 4.2 shows $\rho\notin WF(U)$ and hence $\rho\notin WF(u)$.

Assume that (4.14) were not true. Then there are $\rho_\varepsilon\in WF(u)\cap\Lambda_b\cap\{x_0=-\varepsilon\}$ for any $\varepsilon>0$. From (4.13) one may assume that $\rho_\varepsilon\notin WF(Pu)$ for sufficiently small ε. Then Theorem 2.2.2 in Hörmander [5] shows that $p(\rho_\varepsilon)=0$, that is, $q_j(\rho_\varepsilon)=0$ for some $j=j(\varepsilon)$. From this it is clear that $\rho_\varepsilon\to\rho$ when $\varepsilon\to 0$. On the other hand by the definition of Λ, Λ is contained in the set of simple characteristics of $p(x,\xi)$ then the part of a bicharacteristic in Λ of $q_j(x,\xi)$ $(j=j(\varepsilon))$ through ρ_ε is in $WF(u)$ by Theorem 3.2.1 in [5]. Letting ρ_ε tend to ρ such a bicharacteristic is as close to $\gamma_j(j=j(\varepsilon))$ as we please but this would contradict to the first hypothesis in (4.13). This proves (4.14). Then the proof is complete.

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