Popov, G. and Rangelov, Ts.
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EXPONENTIAL GROWTH OF THE LOCAL ENERGY FOR MOVING OBSTACLES

GEORGI POPOV and TSVIATKO RANGELOV

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1. Introduction

In this paper we study the wave equation in the exterior of a moving obstacle. Our body may move or change its shape smoothly, as long as it remains in a fixed sphere and moves slower than the wave speed. The scattering theory for such obstacles is not highly developed yet. One of the reasons is that we do not know much about the behaviour in time of the local energy. We do not know in general when the global energy is bounded.

The decay of the local energy and the boundedness of the global one for the wave equation in the presence of a moving obstacle $O(t)$ have been studied by J. Cooper [2], J. Cooper and W. Strauss [4, 5], etc. In their works the obstacle $O(t)$ has been assumed star-shaped with respect to the origin. H. Tamura [12] improved their results allowing $O(t)$ to be star-shaped with respect to a center $a(t)$ moving slower than the wave speed. V. Georgiev and V. Petkov [6] investigated the decay of the local energy for non-trapping obstacles provided the global energy is bounded. Recently, J. Cooper [3] constructed an example of two periodically mowing obstacles for which the local energy is not uniformly bounded in time.

We turn now to an outline of our results. Let Ω be a domain in $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$, $n\geqslant 2$, with a smooth boundary Σ. Denote by $\Omega_t=\{x\in\boldsymbol{R}^n;\,(t,x)\in\Omega\}$ the (open) cross-section of Ω at time t, $\Sigma_t=\{x\in\boldsymbol{R}^n;\ (t,x)\in\Sigma\}$. We assume that the obstacle $O(t)=\boldsymbol{R}^n\backslash\Omega_t$ remains in a fixed bounded set

$$(1.1)\qquad O(t)\subset B_{R_0}=\{x\in\boldsymbol{R}^n;\ |x|\leqslant R_0\}\ .$$

Denote by $\nu=(\nu_t,\nu_x)$ the unit normal vector to Σ pointing into Ω. We assume that the speed of motion of $O(t)$ is less than one, which means that

$$(1.2)\qquad |\nu_t|<|\nu_x|\quad\text{for}\quad (t,x)\in\Sigma\ .$$

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For a pair of functions $f=(f_1, f_2)$ with $f_1, f_2 \in C_0^\infty(\Omega_t)$ the local energy in a domain $K \subset \Omega_t$ is defined by

$$\|f\|_K^2 = \frac{1}{2}\int_K (|\nabla f_1|^2 + |f_2|^2)\, dx$$

and $\|f\|_{\Omega_t}$ is the energy norm of f. Let $\mathcal{H}_t$ be the completion of $C_0^\infty(\Omega_t)\times C_0^\infty(\Omega_t)$ in the energy norm. Consider the mixed problem

$$\begin{aligned} &u_{tt}-\Delta u = 0 \quad \text{in} \quad \Omega \\ (1.3) \qquad &u = 0 \quad \text{on} \quad \Sigma \\ &u(s, x) = f_1(x),\ u_t(s, x) = f_2(x),\ x \in \Omega_s \end{aligned}$$

According to [4] there exists a unique solution $u(t, x)$ of (1.3) for initial data $f \in \mathcal{H}_s$. Moreover, the two-parameter family of operators

$$U(t, s)\colon \mathcal{H}_s \ni f \to (u(t, \cdot), u_t(t, \cdot)) \in \mathcal{H}_t$$

is continuous with respect to (s, t).

In this paper we investigate the local energy $\|U(t, s)f\|_{\Omega_t \cap B_R}$, $R > R_0$, for $f \in \mathcal{H}_s$ when $t \to \infty$. For periodically moving obstacles we find some conditions on the boundary Σ which guarantee the existence of initial data $f \in \mathcal{H}_s$ such that the local energy of $U(t, s)f$ grows exponentially as $t \to \infty$. We also consider motions of Σ which are not periodic in time. If the body expands whenever a fixed trapping ray hits the boundary we prove that the local energy is not bounded in time.

In Section 2 we describe the broken rays for moving obstacles emphasizing the change of the variable τ dual to the time t when a reflection at the boundary occurs. If $\tau^-(\tau^+)$ is the value of τ on the incoming (outgoing) ray hitting the boundary at a point $\rho \in T^*\Sigma$, then $\tau^+ = \mu\tau^-$ where μ depends only on the velocity of the moving boundary Σ and the direction of the incoming ray at ρ. In this section we formulate our main result-Theorem 1.

In Section 3 we construct some global in time asymptotic solutions of (1.3) which are concentrated near a broken ray. For this purpose we use Maslov's canonical operators. In Section 4 we evaluate the local energy of these solutions. We suppose that a broken ray γ hits $T^*\Sigma$ at infinitely many points ρ_j and $\mu_j \geqslant C > 1$, thus $\tau_j^+ \to \infty$. We show that the principal symbols of the corresponding asymptotic solutions are proportional to $|\tau_j|^{1/2}$ near ρ_j. This enables us to prove that the local energy grows exponentially in time. In Section 5 we present two examples of periodically moving obstacles which illustrate Theorem 1. We first consider a convex body $O_2(t)$ moving periodically backwards and forwards another fixed convex body O_1. If the distance between o_1 and $o_2(t)$ varies near $T/2$, T being the period of motion, we prove that the local energy grows exponentially.

A similar example was studied by J. Cooper in [3] who used the geometric optics procedure of J. Ralston [11] to construct approximate solutions of (1.3) whose local energy is not uniformly bounded in time. The boundary Σ_t of the obstacle $O(t)=O_1\cup O_2(t)$ in [3] is planar in a neighbourhood of each point of contact with a fixed trapping ray which simplifies the construction of the asymptotic solutions.

As a second example we consider a periodically moving body $O(t)$ with a time-like boundary which is star-shaped for each $t\in\mathcal{R}^1$ with respect to a moving center $a(t)\in O(t)$. Allowing $a(t)$ to move faster than the sound we prove that the local energy grows exponentially in time.

2. Broken rays and increasing of the local energy

Our main assumption on Ω concerns the existence of a suitable trapping generalized bicharacteristic arc in $T^*\Omega$ which hits the boundary $T^*\Sigma$ transversally. Let γ be a broken bicharacteristic arc of the operator $\square=\partial_t^2-\triangle$ issuing from a point $\rho=(s, y, \tau, \eta)\in T^*\Omega$ (for definition see [9], sect. 24.2). For the sake of convenience we shall parametrize γ by the time t. More precisely, denote by $B=\{t_j(\rho); j\in Z_+\}$ the sequence of times at which γ hits the boundary $T^*\Sigma$ and set $t_0(\rho)=0$, $I=[0, \infty)$. The broken ray γ consist of linear segments in $T^*\Omega$

$$I\backslash B\ni t\to\Phi^t(\rho)=(s+t, x^t(\rho), \tau^t(\rho), \xi^t(\rho))\in T^*\Omega \tag{2.1}$$

defined by

$$x^t(\rho)=x_j(\rho)-(t-t_j)\,\xi_j/\tau_j,\ \tau^t(\rho)=\tau_j,\ \xi^t(\rho)=\xi_j$$

for $t\in I_j=(t_j(\rho), t_{j+1}(\rho)), j=0, 1, \cdots$, where $\Phi^0(\rho)=\rho$,

$$x^{t_j+0}(\rho)=x^{t_j-0}(\rho)=x_j(\rho)$$

while the codirections $\zeta_j=(\tau_j, \xi_j)$ are constant vectors in I_j, $\zeta_0=(\tau, \eta)$, and

$$\xi_j^2=\tau_j^2,\ j=0, 1, \cdots \tag{2.2}$$

According to ([9], sect. 24), the codirections ζ_{j-1} and ζ_j of the incident and the reflected ray are determined at $z_j=(t_j(\rho), x_j(\rho))\in\Sigma, j\in Z_+$, by

$$\zeta_{j-1|T_{z_j}\Sigma}=\zeta_{j|T_{z_j}\Sigma}$$

Identifying $T^*_{z_j}(\boldsymbol{R}^{n+1})$ and $T_{z_j}(\boldsymbol{R}^{n+1})$ via the Euclidean metric in $\boldsymbol{R}^{n+1}$ we obtain

$$\zeta_j-\zeta_{j-1}=\alpha_j(\rho)\,\nu(z_j), j\in Z_+ \tag{2.3}$$

for some $\alpha_j(\rho)\neq 0$. Taking the scalar prodict of (2.3) by $(\tau_j+\tau_{j-1}, -\xi_j-\xi_{j-1})$ and using (2.2) we have

$$\nu_t(\tau_j+\tau_{j-1})-\langle\nu_x, \xi_j+\xi_{j-1}\rangle=0\,. \tag{2.4}$$

On the other hand,

$$\nu_t(\tau_j-\tau_{j-1})-\langle\nu_x,\xi_j-\xi_{j-1}\rangle=\alpha_j(\rho)\,(\nu_t^2-\nu_x^2)$$

which implies

$$\alpha_j(\rho)=2(\nu_t(z_j)\,\tau_{j-1}-\langle\nu_x(z_j),\xi_{j-1}\rangle)\,(\nu_x(z_j)^2-\nu_t(z_j)^2)^{-1}\,. \tag{2.5}$$

Thus (2.1), (2.3), (2.5) determine the broken rays in $T^*\Omega$ completely.

Following [1], we denote by $v_j(\rho)$ the velocity-vector of Σ at the point $z_j(\rho)$, i.e.

$$v_j(\rho)=-\frac{\nu_t(z_j)}{|\nu_x(z_j)|}\frac{\nu_x(z_j)}{|\nu_x(z_j)|}\,.$$

Let $\varphi_j(\rho)$ be the angle between $v_j(\rho)$ and $-\xi_{j-1}/\tau_{j-1}$ at $z_j(\rho)$, $\varphi_j=0$ if $v_j=0$ and $0\leqslant\varphi_j\leqslant\pi$. Now (2.3), (2.5) yield

$$\begin{aligned}\tau_j&=\mu_j(\rho)\,\tau_{j-1}\,,\\ \mu_j(\rho)&=(1-2|v_j(\rho)|\cos\varphi_j+|v_j(\rho)|^2)\,(1-|v_j|^2)^{-1}\,.\end{aligned} \tag{2.6}$$

Note that $|v_j(\rho)|<1$ and $\mu_j(\rho)>0$.

Theorem 1. *Suppose that Ω_t is periodically moving, i.e. $\Omega_{t+T}=\Omega_t$ for any $t\in\boldsymbol{R}^1$ and some $T>0$. Let the broken ray* (2.1) *issuing from $\rho=(s,y,\tau,\eta)\in T^*\Omega$ intersect Σ at infinitely many points, $t_j(\rho)\to\infty$ as $j\to\infty$, and*

$$\prod_{\{j\,;\,t_j(\rho)\leqslant t\}}\mu_j(\rho)\geqslant Ce^{\delta t},\ t\in[0,\infty] \tag{2.7}$$

for some $C>0$, $\delta>0$. Then for each neighbourhood K of y in Ω_s and $\varepsilon<\delta/2$ there exists $f=(f_1,f_2)\in\mathcal{H}_s$ with supp $f\subset K$ *such that*

$$||U(t+s,s)f||_{\Omega_{t+s}\cap B_R}\geqslant C_1\,e^{\varepsilon t},\ t\in[s,\infty) \tag{2.8}$$

where $R\geqslant R_0+T$ and $C_1=C_1(\varepsilon,s,f)$.

Now for *n-odd* and $a>R$ denote by $Z^a(t,s)=P_+^a\,U(t,s)\,P_-^a$ the local evolution operator [5]. Then we have

$$Z^a(t+s,s)f=P^aU(t+s,s)f=U(t+s,s)f$$

where $t\geqslant0$ and f is given by Theorem 1 with supp $f\subset K\subset B_R$. Thus $||Z^a(T+s,s)^m f||_{\Omega_s}\geqslant||Z^a(mT+s,s)f||_{\Omega_s\cap B_R}=||U(mT+s,s)f||_{\Omega_s\cap B_R}\geqslant Ce^{m\varepsilon T}$ Therefore the spectral radius of $Z^a(T+s,s)$ is greater than 1 and spec $Z^a(T+s,s)\cap\{z\in C;\ |z|>1\}\neq\phi$ (see also [3]). It is interesting to know if $Z^a(T+s,s)$ has eigenvalues in $\{z\in C;\ |z|>1\}$ since they correspond to the resonances in $\{z\in C;\ |z|>1\}$ (see [5]). Probably there exist infinitely many resonances z with $\mathrm{Im}\,z<0$ in the circumstances of theorem 1.

3. Construction of asymptotic solutions near trapped rays

Let $u(t, x)$ be the solution of (1.3) with initial data $f\in\mathcal{H}_s$. For any open $K\subset\Omega_s$ and any $R\geq R_0$ denote by $|||U(t, s)|||_{K,R}$ the local energy norm

$$|||U(t, s)|||_{\Omega,R} = \sup \{\frac{||U(t, s)f||_{\Omega_t\cap B_R}}{||f||_{\Omega_s}}; f\in\mathcal{H}_s, \operatorname{supp} f\subset K, ||f||_{\Omega_s}\neq 0\}$$

Theorem 1 follows immediately from the next.

Theorem 2. *Let* (1.1), (1.2) *be fulfilled. Suppose that the broken ray* $\Phi^t(\rho^0)$ *intersects* $T^*{}_\Sigma \boldsymbol{R}^{n+1}$ *infinitely many times for some* $\rho^0=(s, y, \tau, \eta)\in T^*\Omega_s$. *Then*

$$|||U(t+s, s)|||^2_{K,R}\geq \prod_{t_j\leq t} \mu_j(\rho^0) \tag{3.1}$$

for each $t\geq 0$, *and each neighborhood* $K\subset\Omega_s$ *of* y.

Proof of theorem 1. Let $\varphi\in C_0^\infty(\boldsymbol{R}^n)$, $\varphi(x)=0$ outside B_R, $R\geq R_0$, and $\psi\in C_0^\infty(K)$, $\psi(x)=1$ in a neighborhood of y in Ω_s. Then

$$A_m = e^{-\varepsilon mT}\varphi U(mT+s, s)\psi,\ 0<\varepsilon<\delta/2,\ m\in Z_+\,,$$

form a family of bounded operators in $\mathcal{H}_s$ which are not uniformly bounded with respect to $m\in\boldsymbol{Z}_+$ in view of theorem 2. Then there exists $g\in\mathcal{H}_s$ such that $||A_m g||_{\Omega_s}\to\infty$ as $m\to\infty$ according to Banach-Steinhaus theorem. Moreover,

$$||A_m g||_{\Omega_s}\leqslant Ce^{-\varepsilon mT}||U(mT+s, s)||_{\Omega\cap B_R}$$

where $f=\Psi g$. This proves theorem 1 since for $t\in[(m-1)\,T, mT]$ we have

$$||U(mT+s, s)f||_{\Omega_s\cap B_R}\leqslant|||U(s, s+t-mT)|||_{Q,R}\,||U(t, s)f||_Q,\ Q = \Omega_{t+s}\cap B_{R+T}$$

To prove theorem 2 we need some preliminaries. Denote

$$\Phi_j^t(\rho) = (s+t, x_j^t(\rho), \tau_j, \xi_j),\ x_j^t(\rho) = x_j-(t-t_j)\,\xi_j/\tau_j,\ t\in(t_j-\varepsilon, t_{j+1}+\varepsilon)$$

and

$$\Lambda_j = \{\Phi_j^t(\rho);\ \rho = (s, x, \tau, \eta),\ t\in(t_j-\varepsilon, t_{j+1}+\varepsilon),\ x\in\Gamma\subset\Omega_s\} \tag{3.2}$$

where Γ is a neighbourhood of y, $\varepsilon>0$ and $0\leqslant j\leqslant J$. For the sake of simplicity set $s=0$. We can assume that the curve $(t_j-\varepsilon, t_{j+1}+\varepsilon)\ni t\to\Phi_j^t(\rho)$, $0\leqslant j\leqslant J$, intersects $T^*\boldsymbol{R}^{n+1}|_\Sigma$ only for $t=t_j, t_{j+1}$ and the intersection is transversal. Then Λ_j is a Lagrangian submanifold of $T^*\boldsymbol{R}^{n+1}$ with respect to the symplectic form $d\tau\wedge dt+\sum_{j=1}^n d\xi_j\wedge dx_j$.

We shall construct an asymptotic solution of (1.3) concentrated near the ray $[0, t_J]\ni t\to\Phi^t(\rho)$, $\rho=(0, y, \tau, \eta)$, $|\eta|=-\tau=1$, given as a sum of global oscillatory functions associated with Λ_j (see [8]). Let $O_{j,p}$, $p=1, \cdots, p_j$ be a suitable open covering of $\{(t, x_j^t(\rho));\ t\in(t_j-\varepsilon, t_{j+1}+\varepsilon)\}$, $j=0, 1, \cdots, J$. We look for an asymptotic solution of the form

$$V(t, x, k) = \sum_{l=0}^{J} V_j(t, x, k) \tag{3.3}$$

where

$$V_j(t, x, k) = \sum_{p=1}^{p_j} V_{j,p}(t, x, k)\,,$$

$$V_{j,p}(t, x, k) = \left(\frac{k}{2\pi}\right)^{q_{j,p}/2} \int_{R^{q_{j,p}}} \exp(ik\Phi_{j,p}(t, x, \theta))\, a_{j,p}(t, x, \theta, k)\, d\theta$$

Here $\Phi_{j,p}$ is a smooth, real-valued and non-degenerate phase function defined in $O_{j,p}\times\Gamma_{j,p}$, $\Gamma_{j,p}\subset \boldsymbol{R}^{q_{j,p}}$, $q_{j,p}\geqslant 0$, is open and relatively compact and $\Phi_{j,p}$ defines Λ_j over $O_{j,p}$ (see [8]). The amplitudes $a_{j,p}$ are given by

$$a_{j,p}(t, x, \theta, k) = \sum_{l=1}^{N} a_{j,p,l}(t, x, \theta)\, k^{-l}$$

where $N>(n+3)/2$ and $a_{j,p,l}\in C_0^\infty(O_{j,p}\times\Gamma_{j,p})$. We look for $V(t, x, k)$ such that

$$\Box V(t, x, k) = O(k^{-N+(n+2)/2}) \quad \text{for} \quad x\in\Omega_t,\ t\in[0, t_J]\,, \tag{3.4}$$

$$V|_\Sigma = O(k^{-N+n/2}) \tag{3.5}$$

$$\begin{aligned} V(0, x, k) &= k^{-1} e^{ik\langle x,\eta\rangle}\,\varphi(x)+O(k^{-N+n/2}) \\ V_t(0, x, k) &= -ie^{ik\langle x,\eta\rangle}\,\varphi(x)+O(k^{-N+(n+1)/2}) \end{aligned} \tag{3.6}$$

for some $\varphi\in C_0^\infty(\Gamma)$ and

$$\operatorname{supp}_x V\subset B_R\,. \tag{3.7}$$

It is convenient to consider $V(t, x, k)$ as a half-density in Ω rather than as a function, which can be achieved by multiplying the scalar functions by the standard half-density $|dt|^{1/2}\otimes|dx|^{1/2}$ in $\boldsymbol{R}^{n+1}$. The principal symbol of $V_j(t, x, k)$ (which is of order -1) has the form (see [8])

$$\sum_{p=1}^{p_j} e^{ik\Phi_{j,p}(\lambda)}\, A_{j,p}(\lambda)\,, \quad \lambda\in\Lambda_j\,,$$

where $A_{j,p}$, $p=1, \cdots, p_j$, form a section A_j in $L(\Lambda_j)\otimes\Omega_{1/2}(\Lambda_j)$, $L(\Lambda_j)$ is Keller-Maslov line bundle and $\Omega_{1/2}(\Lambda_j)$ is the half-density bundle on Λ_j. The density part σ_j of A_j can be described as in [9], sect. 25.1. Let $(\lambda_1, \cdots, \lambda_{n+1})$ be some local coordinates in

$$C_{j,p} = \{(t, x, \theta)\in O_{j,p}\times\Gamma_{j,p};\ d_\theta\,\Phi_{j,p}(t, x, \theta) = 0\}$$

and denote

$$d_C = |D(\lambda, d_\theta \Phi_{j,p})/D(x, \theta)|^{-1}|d\lambda|$$

where $|d\lambda|$ is the Lebesgue density of $C_{j,p}$. Translating $(a_{j,p,l}|_{C_{j,p}})\sqrt{d_C}$ via the map

$$C_{j,p}(t, x, \theta) \xrightarrow{\pi} (t, x, d_t \Phi_{j,p}, d_x \Phi_{j,p}) \in \Lambda_j$$

we obtain the half-density part α_j of A_j on $\pi(C_{j,p}) \subset \Lambda_j$. Let Σ' be a hypersurface in a neighbourhood of a point $z^0 \in \boldsymbol{R}^{n+1}$ given by $\Sigma' = \{F(z)=0\}$, $F(z^0)=0$, $\nabla F(z^0) \neq 0$, and $\Lambda_j^0 = \Lambda_j \cap (T^*\boldsymbol{R}^{n+1}|_\Sigma)$. For any $n+1-$ form ω on Λ_j denote by $|\omega|^\alpha$ the corresponding α-density. Denote $\omega_{|\Lambda_j^0} = i^*\omega$ where i: $\Lambda_j^0 \to \Lambda_j$ is the inclusion map.

Lemma 1· *Let Λ_j be transversal to $T^*\boldsymbol{R}^{n+1}|_\Sigma$, at $\rho^0 = (z^0, \zeta^0) \in \Lambda_j^0$. Then Λ_j^0 is a Lagrangian submanifold of $T^*\Sigma'$ near ρ^0 and $V_{j|\Sigma'}$ is a global oscillatory function of order -1 associated to Λ_j^0. Suppose that $\sigma_j = a|\omega \wedge d(F_{|\Lambda_j})|^{1/2}$ for some $a \in C^\infty(\Lambda_j)$ and an n-form ω on Λ_j, such that $\omega \wedge d(F_{|\Lambda_j})$ does not vanish near ρ^0. Then the half-density part σ_j^0 of the principal symbol of $V_{j|\Sigma'}$ equals*

$$\sigma_j^0 = (a_{|\Lambda_j^0})|\omega_{|\Lambda_j^0}|^{1/2}.$$

REMARK. This lemma can easily be derived from [10] (see also [7] sect. 7). For the sake of completeness we prove it directly using the arguments in [10].

Proof of Lemma 1. First note that σ_j^0 is uniquely determined. Suppose that $\sigma_j = a|\omega \wedge d(F_{|\Lambda_j})|^{1/2} = \tilde{a}|\tilde{\omega} \wedge d(F_{|\Lambda_j})|^{1/2}$, then arg $a(\rho)$=arg $\tilde{a}(\rho)$ and $|a|^2\omega = \alpha|\tilde{a}|\tilde{\omega} + \omega_1 \wedge d(F_{|\Lambda_j})$ near ρ^0 for some $\alpha(\rho) \in C$, $|\alpha|=1$, and an n-form ω_1, which proves the uniqueness.

There exists local coordinates in $\boldsymbol{R}^{n+1}$ near z^0 such that the projection $\Lambda_j \ni (z, \zeta) \to \zeta$ is a diffeomorphism near $\rho^0 = (z^0, \zeta^0)$ (see [9], sect. 21). In these coordinates Σ' is given by $z_{n+1} = f(z')$, $z' = (z_1, \cdots, z_n)$. Let $y(z) = (z', z_{n+1} - f(z'))$, $\eta = ({}^tDy(z))^{-1}\zeta$. The projection $\Lambda_j \ni (y, \eta) \to \eta$ is still a diffeomorphism near ρ^0, thus $\Lambda_j = \{(d_\eta h_1(\eta), \eta)\}$ for some smooth h_1 and $\Sigma' = \{y_{n+1}=0\}$ near $\rho^0 = (y^0, \eta^0)$. Moreover, $\partial^2 h_1/\partial\eta_{n+1}^2(\eta^0) \neq 0$ in view of the transversality condition. Thus there exists a smooth function $\eta_{n+1} = \eta_{n+1}(\eta')$, $\eta' = (\eta_1, \cdots, \eta_n)$, such that $\partial h_1/\partial\eta_{n+1}(\eta', \eta_{n+1}(\eta'))=0$. Perform a new sympletic change of the variables $\xi' = \eta'$, $\xi_{n+1} = \eta_{n+1} - \eta_{n+1}(\eta')$, $x = ({}^tD\xi(\eta))^{-1}y$ and set $h(\xi) = h_1(\eta(\xi))$. In these coordinates Σ' is still given by $\{x_{n+1}=0\}$, $\Lambda_j = \{(d_\xi h(\xi), \xi)\}$ while $\partial h/\partial\xi_{n+1}(\xi', 0)=0$ and $\Lambda_j^0 = \{(d_\xi h(\xi', 0), 0, \xi', 0)\}$. The phase functions

$$\Phi_{j,p}(x, \xi) = \langle x, \xi\rangle - h(\xi),\ \Phi_{j,p}^0(x', \xi') = \langle x', \xi'\rangle - h(\xi', 0)$$

generate Λ_j and Λ_j^0 respectivily. The stationary phase method yields

$$V_j(x', 0, k) = (k/2\pi)^{(n+1)/2} \int_{R^{n+1}} \exp(ik\Phi_{j,p}(x', 0, \xi))\, a_{j,p}(\xi, k)\, d\xi$$
$$= (k/2\pi)^{n/2} \int_{R^n} \exp(ik\Phi^0_{j,p}(x', \xi'))\, a^0_{j,p}(\xi', k)\, d\xi'$$

where the principal part $a^0_{j,p,1}(\xi')$ of $a^0_{j,p}$ equals $a(\xi', 0)$ and $a(\xi)$ is given by

$$a(\xi) = a_{j,p,1}(\xi)\, |\partial^2 h/\partial\xi^2_{n+1}(\xi)|^{-1/2} .$$

Set $\omega = d\xi_1 \wedge \cdots \wedge d\xi_n$. Then

$$\sigma_j = a_{j,p,1}(\xi) \sqrt{d_{\Lambda_j}} = a_{j,p,1}(\xi)\, |\omega \wedge d\xi_{n+1}|^{1/2}$$
$$= a(\xi)\, |\omega \wedge d(x_{n+1|\Lambda_j})|^{1/2}$$

while $\sigma^0_j = a(\xi', 0)\, |\omega_{|\Lambda^0_j}|^{1/2}$. Going back to the old coordinates we prove the lemma.

We are going to solve (3.4)—(3.6) modulo some oscillatory functions of order -2. Since the Maslov factors are constants locally, (3.4) yields

$$L_{H_r} \sigma_j = 0\,, \quad j = 0, 1, \cdots, J\,, \tag{3.8}$$

where L_{H_r} is the Lie derivative along the Hamilton vector field H_r, $r = \tau^2 - \xi^2$ (see [8]). First we construct some half-densities on Λ_j invariant with respect to L_{H_r}. Define the functions f^j_l, $l = 1, \cdots, n$ by $(\Phi^t_j)^* f^j_l = x_l$ and set $\omega_j = df^j_1 \wedge \cdots \wedge df^j_n$. Then $dt \wedge \omega_j$ is a volume form on Λ_j, so $\sigma_j = a_j\, |dt \wedge \omega_j|^{1/2}$ for some smooth $a_j \in C^\infty(\Lambda_j)$. Moreover,

$$L_{H_r} \sigma_j = (H_r a_j)\, |dt \wedge \omega_j|^{1/2} + 2\tau_j a_j \frac{d}{ds} (\Phi^s_j)^* |dt \wedge \omega_j|^{1/2}{}_{|s=0}$$
$$= (H_r a_j)\, |dt \wedge \omega_j|^{1/2}$$

and (3.8) yields

$$H_r a_j = 0 \tag{3.9}$$

Shrinking Γ if necessary and choosing ε small enough, we can arrange $V_j = O(k^{-M+n/2})$ near $\{t=0\}$ for $j = 1, 2, \cdots, J$. Then (3.6) and lemma 1 imply

$$a_{0|t=0} = \varphi(x) \tag{3.10}$$

Similarly $V_{l|\Sigma} = O(k^{-N+n/2})$ near $z_j(\rho)$ for $l \neq j-1, j$ and (3.5) yields

$$(V_j + V_{j-1})_{|\Sigma} = O(k^{-N+n/2}) \tag{3.11}$$

in a neighborhood Σ_j of $z_j(\rho)$.

We are going to determine the amplitude a_j in the density part σ_j of the principal symbol of V_j. We fix some $j \geq 1$ and suppose that $\Sigma_j = \{F(z) = 0\}$ near

z_j where $\nabla F(z_j)\neq 0$. To find a_j we write the volume form $dt\wedge\omega_j$ on Λ_j near (z_j, ζ_j) as $\omega_j\wedge dF$ times a smooth function and use lemma 1 as well as (3.9)-(3.11).

Denote as before $\rho_j^+=(z_j(\rho), \zeta_j(\rho))$, $\rho_j^-=(z_j(\rho), \zeta_{j-1}(\rho))$ and $\rho_j=(z_j(\rho), \zeta_j^0(\rho))$ where $\zeta_j^0(\rho)$ is the restriction of $\zeta_j(\rho)$ on $T_{z_j}\Sigma_j$. We have

$$d(F_{|\Lambda_j}) = \alpha_j\, d(t_{|\Lambda_j}) + \sum_{l=1}^{n} \beta_{lj}\, d(f^j_{l|\Lambda_j})$$

near ρ_j. Applying H_r to the both sides of the equality we obtain $\{r, F\}\,(\rho)= 2\,\tau_j\,\alpha_j(\rho)$ for ρ in Λ_j since the functions f_l^j are invariant under the flow of H_r. Moreover, the Poisson brackets $\{r, F\}\,(\rho)\neq 0$ at ρ_j^+ since H_r is transversal to the hypersurface $\{F(z)=0\}$ in $T^*\boldsymbol{R}^{n+1}$. Therefore,

$$\sigma_j = a_j\,|2\,\tau_j\{r, F\}^{-1}(\rho)|^{1/2}\,|\omega_j\wedge d(F_{|\Lambda_j})|^{1/2}$$

over any $\rho\in\Lambda_j$ and in view of lemma 1 the half-density part σ_j^+ of the principal symbol of the operator $V_{j|\Sigma_j}$ equals

$$\sigma_j^+ = a_j(\rho_j^+)\,|2\,\tau_j\{r, F\}^{-1}(\rho_j^+)|^{1/2}\,|\omega_{j|\Lambda_j^0}|^{1/2}$$

over ρ_j where

$$\Lambda_j^0 = \Lambda_j\cap(T^*\boldsymbol{R}^{n+1}{}_{|\Sigma_j})$$

Analogously, for the half-density part σ_j^- of the principal symbol of the operator $V_{j-1|\Sigma_j}$ we obtain

$$\sigma_{j-1}^- = a_{j-1}(\rho_j^-)\,|2\,\tau_{j-1}\{r, F\}^{-1}(\rho_j^-)|^{1/2}\,|\omega_{j-1|\Lambda_j^0}|^{1/2}$$

Since the restrictions of ω_j and ω_{j-1} coincide on Λ_j^0 we obtain using (3.11) the equality

$$a_{j-1}(\rho_j^-)\,|2\,\tau_{j-1}\{r, F\}^{-1}(\rho_j^-)|^{1/2}+a_j(\rho_j^+)\,|2\,\tau_j\{r, F\}^{-1}(\rho_j^+)|^{1/2} = 0$$

On the other hand (2.4) yields

$$\text{(3.13)}\qquad \{r, F\}(\rho_j^-) = 2(\tau_{j-1}\,v_t-\langle\xi_{j-1}, v_x\rangle)\,|\nabla F(z_j)| = -\{r, F\}(\rho_j^+)$$

Now, using (3.9), (3.10), (3.12) and (3.13) we obtain

$$a_j(\Phi_j^t(\rho)) = (-1)^j\,|\tau_j(\rho)|^{-1/2}\,\varphi(x),\ \rho = (0, x, \tau, \eta)\,,\quad \text{for}\quad t_j-\varepsilon<t<t_j+\varepsilon\,,$$

since $\tau(\rho)=1$. Therefore, the half-density part σ_j of the principal symbol of V_j equals

$$\text{(3.14)}\qquad \sigma^j = (-1)^j(\Phi_j^{-t})^*(|\tau_j|^{-1/2}\,\varphi)\,|dt\wedge\omega_j|^{1/2}$$

Multiplying (3.14) by the corresponding Maslov's factors and by $\exp(i\Phi_{j,p})$ we obtain the principal symbol of V_j which determines V_j uniquely modulo some oscillatory integrals of order -2. Repeating this procedure we obtain some $V_j(t, x, k)$, $1 \leq j \leq J$, whose sum V satisfies (3.4)—(3.7) for $0 \leq t \leq t_J$. Adding a function $\tilde{V}(t, x, k)$ such that

$$D_t^\alpha D_x^\beta \tilde{V} = O(k^{-M+n/2+\alpha+|\beta|}),\ \alpha+|\beta| \leq N, \tag{3.15}$$

we can satisfy (3.5) and (3.6) exactly.

4. Estimates of the asymptotic solutions

We are going to evaluate the energy norm of $(\nabla_z V)_{|\Omega_t}$, $z=(t, x)$, for any t fixed, $t_m \leq t \leq t_{m+1}$, $0 \leq m \leq J-1$. First we write $(\nabla_z V_j)_{|\Omega_t}$, $0 \leq j \leq J$, as a sum of oscillating integrals of the form

$$\begin{aligned}(\nabla_z V_{j,p})_{|\Omega_t}(x, k) &= ik\left(\frac{k}{2\pi}\right)^{q_{j,p}} \int_{\Gamma_{j,p}} \exp(ik\Phi_{j,p}(t, x, \theta)) \\ &\times (\nabla_z \Phi_{j,p})\,(t, x, \theta)\, a_{j,p,1}(t, x, \theta)\, d\theta\end{aligned}$$

modulo lower order terms for $1 \leq p \leq p_j$. The oscillatory integral $(\nabla_z V_j)_{|\Omega_t}$ of order zero is associated with the Lagrangean manifold

$$\Lambda_j(t) = \{(x, \xi) \in T^*\Omega_t;\ (t, x, -|\xi|, \xi) \in \Lambda_j\}$$

The additional factor $\nabla_z \Phi_{j,p}$ is transformed to (τ_j, ξ_j) under the map $\pi\colon C_{j,p} \to \Lambda_j$ described in §3 thus the principal symbol of the oscillatory integral $\nabla_z V_j$ equals

$$(-1)^j (\Phi_j^{-t})^* (\varphi |\tau_j|^{-1/2} (\tau_j, \xi_j)) \,|dt \wedge \omega_j|^{1/2}$$

Using lemma 1 and the equality $(\Phi^t)^*(\omega_{|\Lambda_j(t)}) = dy$ it is easy to see that the half-density part of the principal symbol of $(\nabla_z V_j)|_{\Omega_t}$ equals

$$\begin{aligned}\sigma_j^t &= i(-1)^j\, (\Phi_j^{-t})^* (\varphi |\tau_j|^{-1/2} (\tau_j, \xi_j))\, |\omega_{|\Lambda_j(t)}|^{1/2} \\ &= i(-1)^j\, (\Phi_j^{-t})^* (\varphi |\tau_j|^{-1/2} (\tau_j, \xi_j)\, |dy|^{1/2})\end{aligned} \tag{4.1}$$

if $\Lambda_j(t) \neq 0$ and it equals 0 otherwise. Fix some $\varepsilon, \varepsilon_1 > 0$. Shrinking Γ if necessary we can suppose that $\Lambda_j(t) = \phi$ for any $j, m \leq J$, $j \neq m$, when $t_{m-1} + \varepsilon < t < t_m - \varepsilon$. Therefore $\nabla V_j(t, x, k) = O_J(k^{-N})$ for any $t \in (t_{m-1} + \varepsilon, t_m - \varepsilon)$ and any $j \neq m$, $j, m \leq J$. Moreover, we suppose that

$$|\tau_m(0, x, 1, \eta) - \tau_m(0, y, 1, \eta)| \leq \varepsilon_1/2 \tag{4.2}$$

for any $x \in \Gamma$ and $m \leq J$. To evaluate the L^2-norm of the function $(\nabla_z V_m)_{|\Omega_t}$ we use formula (1.3.15) from [8] as well as (2.6), (4.1) and the relation

$$\Phi_m^{-t}(\Lambda_m(t)) = \Lambda_0(0) = \{(x, \eta);\ x \in \Gamma\}$$

We have

$$\frac{1}{2}\int_{\Omega_t\cap B_R}|\nabla_x V_m(t,x,k)|^2\,dx=\int_{\Lambda_m(t)}\sigma_m^t\,\overline{\sigma_m^t}+O_J(k^{-1})$$
$$=\int_{R^n}|\tau_m(0,x,-1,\eta)|\,|\varphi(x)|^2\,dx+O_J(k^{-1})$$
$$=\int_{R^n}(\prod_{\{j\,;\,t_j\le t\}}\mu_j(\rho)|\varphi(x)|^2\,dx+O_J(k^{-1})$$

where $\rho=(O,x,-1,\eta)$. Now using (4.2) we get from the last estimate

$$\frac{1}{2}\|(\nabla_x V_{|\Omega_t})\|^2_{L^2(\Omega_t\cap B_R)}\ge(\prod_{t_j\le t}\mu_j(\rho^0)-\frac{\mathcal{E}_1}{2})\int_{\Omega_0}|\varphi(x)|^2\,dx+O_J(k^{-1})\tag{4.3}$$

$\rho^0=(0,y,-1,\eta)$ for any t in the complement $I_{J,\varepsilon}$ of the union of the intervals $(t_{m-1}-\mathcal{E},t_m+\mathcal{E})$, $m\le J$, in $[0,t_J)$. The estimate (4.3) holds for the function $u(t,x,k)=V(t,x,k)+\tilde{V}(t,x,k)$ too in view of (3.15). Let $u_1(t,x,k)$ solve

$$\begin{aligned}\Box u_1&=\Box u\quad\text{in}\quad\Omega\\ u_1|_\Sigma&=0\\ u_1|_{t=0}&=(u_1)_t|_{t=0}=0\,.\end{aligned}$$

Using Duhamel's formula we obtain

$$u_1(t,x,k)=\int_0^t U(t,s)\,\Box u(s,x,k)\,ds$$

and it is easy to see that

$$\|(\nabla_x u_{1|\Omega_t})\|^2_{L^2(\Omega_t\cap B_R)}=O_J(k^{-1})\quad\text{for}\quad t\in I_{J,\varepsilon}\tag{4.4}$$

The function $u(t,x,k)=u(t,x,k)-u_1(t,x,k)$ solves (1.3) with initial data $f_k=(f_{1k},f_{2k})$,

$$f_{1k}(x)=k^{-1}\exp(ik\langle x,\eta\rangle)\,\varphi(x)\,,\quad f_{2k}(x)=-i\exp(ik\langle x,\eta\rangle)\,\varphi(x)\,.$$

Suppose that the L^2-norm of the function φ equals 1. Then the energy norm of f_k equals $1+O_J(k^{-1})$ and according to (4.3) and (4.4) we have

$$\|U(t,0)f_k\|^2_{\Omega_t\cap B_R}\ge\prod_{t_j\le t}\mu_j(\rho^0)-\frac{\mathcal{E}_1}{2}+O_J(k^{-1})$$

for any $t\in I_{J,\varepsilon}$. Choose k so that

$$\|U(t,0)f_k\|^2_{\Omega_t\cap B_R}\ge(\prod_{t_j\le t}\mu_j(\rho^0)-\mathcal{E}_1)\|f_k\|^2_{\Omega_0}$$

for any $t\in I_{J,\varepsilon}$. This proves (3.1), since the positive constants $\mathcal{E}$, $\mathcal{E}_1$ and J can be chosen arbitrary.

5. Examples

We are going to consider two examples illustrating theorem 1. The first example studies the local energy for two periodically moving bodies which are assumed convex for simplicity.

Example 1. Consider a convex body moving back and fort another fixed convex body. More precisely, let $\boldsymbol{R}^n \backslash \Omega_t = O_1(t) \cup O_2(t)$ where the body $O_1(t) = O_1(0) = O_1$ has a stationary boundary ∂O_1 and $O_2(t+T) = O_2(t)$ for each $t \geq 0$ and $O_1 \cap O_2(t) = \phi$. Denote $d(t) = \text{dist}\,(O_1, O_2(t))$ and set $d_1 = \min d(t)$, $d_2 = \max d(t)$. Assume that

(i) $d_1 < T/2 < d_2$

(ii) there exists $y^1 \in \partial O_1$ such that $d(t) = \text{dist}\,(y^1, y^2(t))$ for some $y^2\,(t) \in \partial O_2(t)$ and each $t \geqslant 0$.

(iii) the velocity $v(y^2(t))$ does not vanish unless $d(t) = d_1$ or $d(t) = d_2$.

Lemma 2. *Suppose* (i)—(iii) *are fulfilled. Then there exists* $\rho = (x, y, \tau, \eta) \in T^*\Omega$, $|\tau| = |\eta| = 1$ *for which the conditions of theorem* 1 *are fulfilled.*

Proof. Let $\omega(t)$ be the unit vector $\omega(t) = (y^2(t) - y^1)/|y^2(t) - y^1|^{-1}$. According to ($ii$) $w(t)$ is always orthogonal to $T_{y_1}(\partial O_1)$, thus $w(t) = w(0) = w$ for $t \geqslant 0$. It is easy to see that

$$\nu(t, y^2(t)) = (d'(t), -\omega)\,(1 + d'(t)^2)^{-1/2}, t \geqslant 0\,. \tag{5.1}$$

Then $|d'(t)| < 1$ in view of (1.2).

Consider the broken ray $\Phi^t(\rho)$, $\rho = (s, y, -1, \omega)$, $t \geqslant 0$ where y is an arbitrary point of the linear segment $[y^1, y^2(s)] \subset \Omega_s$. Then

$$x^t(\rho) = x_j(\rho) - (t - t_j)\,\xi_j(\rho)/\tau_j(\rho)\,, \quad \text{for} \quad t \in (t_j, t_{j+1})$$

and $\xi_j(\rho)/\tau_j(\rho)$ is always colinear to $\pm\omega$. Therefore $x^t(\rho) \in [y^1, y^2(t)]$ for $t \leqslant 0$. Using (i), (ii), we can find some $s_0 > 0$ so that

$$d(s_0) = T/2\,, \quad d'(s_0) < 0\,. \tag{5.2}$$

Choose some $s < s_0$ closed to s_0 and set $\rho^0 = (s, y, -1, \omega)$ where $y = y^2(s_0) - (s_0 - s)\,\omega$. Then $t_1(\rho^0) = s_0 - s$ and

$$\phi_0^{t_1+0}(\rho^0) = (s_0, y^2(s_0), -1, \omega)\,.$$

Denote $g(t) = \text{dist}\,(y^1, x^t(\rho^0), t \geqslant 0$. The graph of the function $g(t)$ consists of linear segments defined in $t \in [t_j(\rho), t_{j+1}(\rho)]$ and making an angle $\pi/4$ with the t-axis since $|\omega| = 1$. Therefore

$$t_2(\rho^0) = t_1(\rho^0) + g(t_1(\rho^0)) = t_1(\rho^0) + T/2\,.$$

Since $|d'(t)|<1$, $t\in\mathcal{R}^1$, there exists at least one point $t=t_3(\rho^0)\in\mathcal{R}^1$ such that dist $|y^1-x_2^t(\rho^0)|=d(t)$, $x_2^t(\rho^0)=x_2(\rho^0)-(t-t_2(\rho^0))\,\xi_2/\tau_2$ and in view of $d(t_2(\rho^0)+T/2)=d(t_1(\rho^0))=T/2=\text{dist}\,|y^1-x_2^{t_2(\rho^0)+T/2}(\rho^0)|$ we obtain $t_3(\rho^0)=t_1(\rho^0)+T$. Therefore the ray $t\to\Phi^t(\rho^0)$ hits the boundary at infinitely many points, $t_j(\rho^0)=t_1(\rho^0)+(j-1)\,T/2$, $x_{2j+1}(\rho^0)=y^2(t_1+jT)=y^2(t_1)$, $x_{2j}(\rho^0)=y^1$, $\xi_{2j}/\tau_{2j}=-\omega$, $\xi_{2j+1}/\tau_{2j+1}=\omega$.

According to (5.1) and (5.2) we have $v_{2j+1}(\rho^0)=d'(t_1)$, $v_{2j}(\rho^0)=0$ and $\varphi_{2j+1}=\pi$. Therefore

$$\mu_{2j}(\rho^0)=1,\ \mu_{2j+1}(\rho^0)=(1+|d'(t_1)|)(1-|d'(t_1)|)^{-1}>1$$

Now we have

$$\prod_{\{j\,;\,t_j<t\}}\mu_j(\rho)\geqslant Ce^{\delta t},\ t\geqslant 0\,,$$

with

$$\delta=\frac{1}{T}(\ln(1+|d'(t_1)|)-\ln(1-|d'(t_1)|)|,\ C>0\,.$$

EXAMPLE 2. We consider a periodically moving body in $\boldsymbol{R}^n$ which is star-shaped with respect to a point $a(t)\in\boldsymbol{R}^n$. H. Tamura proved in [12] that the local energy of the solutions of (1.3) decays exponentially in time when the speed of $a(t)$ is less than one. In this case no trapped rays occur. We shall construct a periodically moving star-shaped obstacle which traps some rays allowing the point $a(t)$, internal for the body $O(t)$, to move faster than the sound.

Let $O_0\subset\boldsymbol{R}^n$ with boundary $\Gamma_0=\{f(x);\,x\in S^{n-1}\}$, $f\in C^\infty(S^{n-1})$, $S^{n-1}=\{x\in\boldsymbol{R}^n;\,|x|=1\}$, be bounded, non-convex, and star-shaped with respect to any point of an open convex set $U\subset O_0$, i.e.

$$\langle\nu_x(y),\,y-a\rangle<0,\ a\in U,\ y\in\Gamma_0\,.$$

Choose some $y_j\in\Gamma_0$, $i=1,2$, so that the linear segment (y_1,y_2) does not intersect Γ_0 and set $\omega=(y_2-y_1)/|y_2-y_1|$. Suppose that

$$\{y_j+y;\,y\in\boldsymbol{R}^n,\,\langle y,\omega\rangle=0\}\cap U\neq\phi,\ j=1,2\,. \tag{5.3}$$

Let $f_j\in C^\infty(S^{n-1})$, $0\leqslant f_1(x)\leqslant\varepsilon$, $-\varepsilon\leqslant f_2(x)\leqslant 0$, $\operatorname{supp}f_j\subset B_\delta(y_j)=\{x\in\boldsymbol{R}^n;\,|x-y_j|\leqslant\delta\}$ and

$$f_1(x)=(-1)^{1+j}\,\varepsilon\quad\text{only for}\quad x=x_j,\,j=1,2\quad\text{where}\quad\varepsilon>0,\ \delta>0\,. \tag{5.4}$$

The boundary $\Gamma_{j,s}=\{f(x)+sf_j(x)\,\omega,\,x\in S^{u-1}\}$ is star-shaped for any $\delta\in[0,1]$ provided ε is small enough. In view of (5.3), (5.4) we can assume that ω is orthogonal to $T_{y_j(s)}(\Gamma_{j,s})$ for $|s-1|<\varepsilon/4$ where $y_j(s)=f(x_j)+sf_j(x_j)\,\omega$. Set $z_j=f(x_j)+(1-\varepsilon/8)f_j(x_j)$, $T=|z_1-z_2|$. Let $\varphi_j\in C_0^\infty(\boldsymbol{R}^1)$, $j=1,2$ be such that $\varphi_j(t+T)=\varphi_j(t)$, $0\leqslant\varphi_j\leqslant 1$, and

$$\operatorname{supp}\varphi_1\cap[0, T] = [T/8, 3T/8]\,, \quad \operatorname{supp}\varphi_2\cap[0, T] = [5T/8, 7T/8]$$

$$\varphi_1(T/4) = \varphi_2(3T/4) = (1-\varepsilon/8),\ \varphi_1'(T/4) = \varphi_2'(3T/4)>0\,. \tag{5.5}$$

Consider the obstacle $O(t)$ with boundary

$$\Gamma(t) = \{f(x)+\varphi_1(t)f_1(x)\,\omega+\varphi_2(t)f_2(x)\,\omega;\ x\in S^{n-1}\}\,.$$

The body $O(t)$ is star-shaped for any $t\in\boldsymbol{R}^1$, since $\operatorname{supp}\varphi_1\cap\operatorname{supp}\varphi_2=\phi$ and it moves with a period T. Choosing T/ε large we obtain $\sup|\varphi_j'(t)|$ small enough, thus $\Gamma(t)$ moves with a speed less than 1.

Moreover, the broken ray issued from the point $\rho^0=(0, (z_1+z_2)/2, -1, \omega)$ is periodic, $\gamma(t+T)=\gamma(t)$ and $\gamma(T/4)=z_1$, $\gamma(3T/4)=z_2$. Arguing as in lemma 2 and using (5.4), (5.5), it is easy to prove that

$$\prod_{\{j\,;\,t_j(\rho^0)\leq t\}}\mu_j(\rho^0)\geqslant Ce^{\delta t}$$

for some $C>0$, $\delta>0$.

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Institute of Mathematics
Bulgarian Academy of Science
1090 Sofia
Bulgaria