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A GÅRDING INEQUALITV FOR CERTAIN ANISOTROPIC PSEUDO DIFFERENTIAL OPERATORS WITH NON-SMOOTH SYMBOLS

NIELS JACOB

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Let $L(x, D)=\sum_{i,j=1}^{m} P_i(D)(a_{ij}(x)Q_j(D))$ be a pseudo differential operator and denote by B the sesquilinear form

$$B(u, v):=\sum_{i,j=1}^{m}\int_{\boldsymbol{R}^n} \overline{a_{ij}(x)}\,\overline{Q_j(D)u(x)}P_i(D)v(x)\,dx$$

defined on $C_0^\infty(\Omega)$, where $\Omega\subset\boldsymbol{R}^n$ is an open set. Furthermore let $H_P^{0,1}(\Omega)\subset L^2(\Omega)$ be a Hilbert space which contains $C_0^\infty(\Omega)$ as a dense subspace. In addition, suppose that B is bounded on $H_P^{0,1}(\Omega)$, i.e. that

$$|B(u, v)|\leq c\|u\|_{P,1}\|v\|_{P,1}$$

holds for all $u, v\in H_P^{0,1}(\Omega)$ (or, equivalently, for all $u, v\in C_0^\infty(\Omega)$). It is well-known that the representation problem:

Find all $u\in H_P^{0,1}(\Omega)$ such that for a given $f\in L^2(\Omega)$

$$B(u, \varphi)=\int_{\Omega}\overline{f(x)}\varphi(x)\,dx$$

holds for all $\varphi\in C_0^\infty(\Omega)$

is a resaonable generalization of the Dirichlet problem. (This formulation of the Dirichlet problem is essentially the same as that given by H. Kumano-go and C. Tsutsumi in [13].)

Assume that $H_P^{0,1}(\Omega)$ is compactly embedded into $L^2(\Omega)$. Then one can prove Fredholm's alternative to hold for the representation problem, provided that B satisfies a Gårding-type inequality, i.e.

$$\operatorname{Re} B(u, u)\geq c_0\|u\|_{P,1}^2-c_1\|u\|_0^2$$

for all $u\in H_P^{0,1}(\Omega)$ (or, equivalently, for all $u\in C_0^\infty(\Omega)$).

In this paper we will consider a class of anisotropic pseudo differential operators in generalized divergence form with non-smooth symbols. With these operators we can associate a continuous sesquilinear form defined on a certain anisotropic Sobolev space. We will prove a Gårding-type inequality for this sesquilinear form.

The symbol class under consideration is not contained in any of the classical classes and it is impossible to apply some symbolic calculus to the operators in this paper. Our proof of Gårding's inequality follows essentially the original proof of L. Gårding [5]. However, as in the case of differential operators we have to distinguish two cases depending on whether a partition of the unity may be used or not (see [10] and [12]). Moreover we have to handle the non-local character of the pseudo differential operators involved.

In [11] we pointed out that pseudo differential operators with negative definite functions as symbols arise very naturally in the theory of Dirichlet spaces. The symbol class considered in this paper is large enough to include the negative definite functions occuring in the theory of Dirichlet spaces.

The reader is referred to [8], chap. XXII, where lower bounds (Gårding's inequality) for pseudo differential operators with C^∞-coefficients are treated, especially the results of A. Melin are discussed and compared with a sharp form of Gårding's inequality.

Our notations are essentially standard, see [6]—[8] (or [11]). Whenever we use Plancherel's theorem, we normalize the Lebsegue measure in such a way that constants do not appear in the formula. For any distribution u we denote by $\hat{u}$ or Fu its Fourier transform (provided that it is defined).

1. Auxiliary Propositions

Denote by $\lambda^{(n)}$ the n-dimensional Lebesgue measure and let $P\colon \boldsymbol{R}^n \to \boldsymbol{R}$ be a continuous function satisfying the following conditions:

P.1.: For all $\xi \in \boldsymbol{R}^n$ we have $P(\xi) \geq 0$ and $\lambda^{(n)}\{\xi \in \boldsymbol{R}^n,\ P(\xi)=0\}=0$.

P.2.: There exists a constant $c \geq 0$ and a real number $r \geq 0$ such that

$$P(\xi) \leq c(1+|\xi|^2)^{r/2} \qquad \text{holds for all } \xi \in \boldsymbol{R}^n . \tag{1.1}$$

The set of all functions satisfying P.1 and P.2 is denoted by $\underline{P}$. Let $P \in \underline{P}$ and $s \in \boldsymbol{R}$, $s \geq 0$. We define the norm $\|\cdot\|_{P,s}$ by

$$\|\varphi\|_{P,s}^2 = \int_{\boldsymbol{R}^n} (1+P^2(\xi))^s |F\varphi(\xi)|^2 d\xi \tag{1.2}$$

for $\varphi \in C_0^\infty(\boldsymbol{R}^n)$. The completion of $C_0^\infty(\boldsymbol{R}^n)$ with respect to the norm (1.2) is the Hilbert-space $H_P^s(\boldsymbol{R}^n)$. In particular, let $\Lambda\colon \boldsymbol{R}^n \to \boldsymbol{R}$ be the function defined by $\Lambda(\xi)=|\xi|$. For some $t \in \boldsymbol{R}$, $t \geq 0$, the corresponding norm $\|\cdot\|_{\Lambda,t}$ is the usual Sobolev-space norm and denoted by $\|\cdot\|_t$. Moreover, instead of $H_\Lambda^s(\boldsymbol{R}^n)$ we write $H^s(\boldsymbol{R}^n)$. For any open set $\Omega \subset \boldsymbol{R}^n$ the space $C_0^\infty(\Omega)$ consists of all elements $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ with compact support supp $\varphi \subset \Omega$, i.e. functions of $C_0^\infty(\Omega)$ are difined on the whole space $\boldsymbol{R}^n$. By definition $H_P^{0,s}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm (1.2). Again, the space $H_\Lambda^{0,s}(\Omega)$ is denoted by $H^{0,s}(\Omega)$.

Obviuosly we have

Proposition 1.1. *Let $P_1, P_2 \in \underline{P}$ and suppose that for two constants $c \geq 0$ and $\rho \geq 0$ the estimate*

$$P_1(\xi) \leq c P_2(\xi) \tag{1.3}$$

holds for all $\xi \in \boldsymbol{R}^n$, $|\xi| \geq \rho$. Then for each $s \geq 0$ the space $H_{P_2}^s(\boldsymbol{R}^n)$ is continuously embedded in the space $H_{P_1}^s(\boldsymbol{R}^n)$.

Corollary 1.1. *Let P_1 and P_2 satisfy the assumption of Proposition 1.1. Then for any open set $\Omega \subset \boldsymbol{R}^n$ and $s \geq 0$ the space $H_{P_2}^{0,s}(\Omega)$ is continuously embedded in the space $H_{P_1}^{0,s}(\Omega)$.*

The following proposition determines the dual space of the space $H_P^s(\boldsymbol{R}^n)$.

Proposition 1.2. *Let $s \geq 0$ and $P \in \underline{P}$. Then the dual space of $H_P^s(\boldsymbol{R}^n)$ is the completion of $L^2(\boldsymbol{R}^n)$ with respect to the norm*

$$||f||_{P,-s} := \sup_{0 \neq u \in H_P^s(\boldsymbol{R}^n)} \frac{|(f, u)_0|}{||u||_{P,s}} . \tag{1.4}$$

Moreover, for $f \in L^2(\boldsymbol{R}^n)$ we have

$$||f||_{P,-s} = \int_{\boldsymbol{R}^n} (1+P^2(\xi))^{-s} |Ff(\xi)|^2 d\xi . \tag{1.5}$$

Since $L^2(\boldsymbol{R}^n)$ is dense in $[H_P^s(\boldsymbol{R}^n)]^*$ with respect to the norm $||\cdot||_{P,-s}$, we have $[H_P^s(\boldsymbol{R}^n)]^* = H_P^{-s}(\boldsymbol{R}^n)$.

In the case of the usual Sobolev space, i.e. $P = \Lambda$, the result can be found in [14], p. 31. The proof of Proposition 1.2 follows essentially the lines of the considerations in [2], p. 201–203, where the assertion is proved for periodic functions but general elements $P \in \underline{P}$, and it is left to the reader.

Let us recall Proposition 1.4 from [3]:

Proposition 1.3. (*Poincaré's inequality*) *Let $P \in \underline{P}$ and suppose that the embedding of $H_P^{0,s}(\Omega)$ into $L^2(\Omega)$ is compact, where $\Omega \subset \boldsymbol{R}^n$ is an open set. Then the estimate*

$$\int_\Omega |u(x)|^2 dx \leq c \int_{\boldsymbol{R}^n} |P^2(\xi)|^s |Fu(\xi)|^2 d\xi \tag{1.6}$$

holds for all $u \in H_P^{0,s}(\Omega)$.

DEFINITION 1.1. A continuous function a: $\boldsymbol{R}^n \to \boldsymbol{R}$ is said to belong to the class $\Sigma(P, s)$, $P \in \underline{P}$, $s \in \boldsymbol{R}$, if there exists a constant $c_a \geq 0$ such that

$$|a(\xi)| \leq c_a(1+P^2(\xi))^{s/2} \tag{1.7}$$

holds for all $\xi \in \boldsymbol{R}^n$.

Let $a \in \Sigma(P, s)$. On $C_0^\infty(\boldsymbol{R}^n)$ we define the operator $a(D)$ by

$$a(D)u(x) = \int_{\boldsymbol{R}^n} e^{i(x,\xi)} a(\xi)\hat{u}(\xi)d\xi . \tag{1.8}$$

Propostion 1.4. *Let* $a \in \Sigma(P, s)$ *and define* $a(D)$ *as in* (1.8). *Then* $a(D)$ *is a continuous operator from* $C_0^\infty(\boldsymbol{R}^n)$ *into* $C^\infty(\boldsymbol{R}^n)$. *Moreover, for each* $t \in \boldsymbol{R}$ *the operator* $a(D)$ *has a continuous extension (again denoted by* $a(D)$*) from* $H_P^{t+s}(\boldsymbol{R}^n)$ *into* $H_P^t(\boldsymbol{R}^n)$, provided $s \geq 0$.

Proof. Let $(u_\nu)_{\nu \in N}$ be a sequence of elements $u_\nu \in C_0^\infty(\boldsymbol{R}^n)$ converging in the topology of $C_0^\infty(\boldsymbol{R}^n)$ to an element $u \in C_0^\infty(\boldsymbol{R}^n)$. Then $(u_\nu)_{\nu \in N}$ converges to u in each of the spaces $H^t(\boldsymbol{R}^n)$, $t \geq 0$. Now, for $\alpha \in \boldsymbol{N}_0^n$ and any compact set $K \subset \boldsymbol{R}^n$ we have

$$\sup_{x \in K} |D_x^\alpha(a(D)(u_\nu - u))(x)| \leq \sup_{x \in \boldsymbol{R}^n} |D_x^\alpha(a(D)(u_\nu - u))(x)|$$

and it follows with some appropriate constant $r \in \boldsymbol{R}$, $r \geq 0$, (see P.2) that

$$\begin{aligned}
|D_x^\alpha(a(D)(u_\nu - u))(x)| &= \Big|\int_{\boldsymbol{R}^n} D_x^\alpha(e^{i(x,\xi)} a(\xi)(\hat{u}_\nu(\xi) - \hat{u}(\xi))d\xi\Big| \\
&= \Big|\int_{\boldsymbol{R}^n} \xi^\alpha a(\xi)(\hat{u}_\nu(\xi) - \hat{u}(\xi))e^{i(x,\xi)}d\xi\Big| \\
&\leq c\int_{\boldsymbol{R}^n} (1+|\xi|^2)^{|\alpha|/2}(1+|\xi|^2)^{r/2}|\hat{u}_\nu(\xi) - \hat{u}(\xi)|\,d\xi \\
&\leq c\int_{\boldsymbol{R}^n} (1+|\xi|^2)^{-(n+1)/2}(1+|\xi|^2)^{(|\alpha|+r+n+1)/2}|\hat{u}_\nu(\xi) - \hat{u}(\xi)|\,d\xi \\
&\leq c_n\|u_\nu - u\|_{|\alpha|+r+n-1} ,
\end{aligned}$$

hence the operator $a(D)$ is continuous from $C_0^\infty(\boldsymbol{R}^n)$ into $C^\infty(\boldsymbol{R}^n)$. Now let $t \in \boldsymbol{R}$, and $u \in C_0^\infty(\boldsymbol{R}^n)$. We find

$$\begin{aligned}
\|a(D)u\|_{P,t}^2 &= \int_{\boldsymbol{R}^n} (1+P^2(\xi))^t |a(\xi)\hat{u}(\xi)|^2 d\xi \\
&\leq c\int_{\boldsymbol{R}^n} (1+P^2(\xi))^t(1+P^2(\xi))^s |\hat{u}(\xi)|^2 d\xi = c\|u\|_{P,t+s}^2 ,
\end{aligned}$$

which implies the second statement of the proposition.

DEFINITION 1.2. We say that a continuous function a: $\boldsymbol{R}^n \to \boldsymbol{R}$ belongs to the class $\Sigma_0(P, s)$, $P \in \underline{P}$, $s \geq 0$, if

$$\lim_{|\xi| \to \infty} \frac{|a(\xi)|}{(1+P^2(\xi))^{s/2}} = 0$$

holds.
Obviously we have $\Sigma_0(P, s) \subset \Sigma(P, s)$.

Proposition 1.5. *A. Let $a \in \Sigma(P, s)$, then for all $u \in C_0^\infty(\boldsymbol{R}^n)$ we have*

$$\begin{aligned} ||a(D)u||_0^2 &\leq c||(1+P^2(D))^{s/2}u||_0^2 \\ &\leq c[||P^s(D)u||_0^2 + ||u||_0^2] . \end{aligned} \tag{1.9}$$

B. Let $a \in \Sigma_0(P, s)$, then for each $\varepsilon > 0$ there exists a constant $c(\varepsilon) \geq 0$ such that

$$||a(D)u||_0^2 \leq \varepsilon ||P^s(D)u||_0^2 + c(\varepsilon)||u||_0^2 \tag{1.10}$$

holds for all $u \in C_0^\infty(\boldsymbol{R}^n)$.

Proof. A. Let $u \in C_0^\infty(\boldsymbol{R}^n)$, we find

$$\begin{aligned} ||a(D)u||_0^2 &= \int_{\boldsymbol{R}^n} |a(\xi)|^2 |\hat{u}(\xi)|^2 d\xi \\ &\leq c_a \int_{\boldsymbol{R}^n} (1+P^2(\xi))^s |\hat{u}(\xi)|^2 d\xi , \end{aligned}$$

which implies the first inequality. The second inequality is an immediate consequence of the estimate

$$c \leq \frac{1+P^{2s}(\xi)}{(1+P^2(\xi))^s} \leq c' , \tag{1.11}$$

which holds for all $s \geq 0$ and all $\xi \in \boldsymbol{R}^n$.

B. Since $a \in \Sigma_0(P, s)$, it follows that for each $\varepsilon > 0$ there exists a constant $\rho(\varepsilon) \geq 0$ such that

$$|a(\xi)|^2 \leq \varepsilon (1+P^2(\xi))^s \tag{1.12}$$

holds for all $\xi \in \boldsymbol{R}^n$, $|\xi| \geq \rho(\varepsilon)$. Using (1.12) we find

$$\begin{aligned} \int_{\boldsymbol{R}^n} |a(\xi)|^2 |\hat{u}(\xi)|^2 d\xi &= \int_{|\xi| \geqq \rho(\varepsilon)} |a(\xi)|^2 |\hat{u}(\xi)|^2 d\xi + \int_{|\xi| \leq \rho(\varepsilon)} |a(\xi)|^2 |\hat{u}(\xi)|^2 d\xi \\ &\leq \varepsilon \int_{|\xi| \geqq \rho(\varepsilon)} (1+P^2(\xi))^s |\hat{u}(\xi)|^2 d\xi + \int_{|\xi| \leq \rho(\varepsilon)} |a(\xi)|^2 |\hat{u}(\xi)|^2 d\xi \end{aligned}$$

$$\leq \varepsilon \|(1+P^2(D))^{s/2}u\|_0^2 + \max_{|\xi|\leq \rho(\varepsilon)} |a(\xi)|^2 \|u\|_0^2 ,$$

which implies (1.10).

For later purposes we need an estimate for the commutator of an operator $a(D)$ with a smooth function φ.

DEFINITION 1.3. Let $a\in\Sigma(P, 1)$ and $\varphi\in C^\infty(\boldsymbol{R}^n)$. We denote by $[a(D), \varphi] = -[\varphi, a(D)]$ the operator defined on $C_0^\infty(\boldsymbol{R}^n)$ by

$$[a(D), \varphi]u(x) := a(D)(\varphi u)(x) - \varphi(x)a(D)u(x) . \tag{1.13}$$

Proposition 1.6. *Let $a\in\Sigma(P, 1)$ and assume for $\varphi\in C^\infty(\boldsymbol{R}^n)\cap L^\infty(\boldsymbol{R}^n)$ that*

$$\|[a(D), \varphi]u\|_0^2 \leq c_1\|P(D)u\|_0^2 + c_2\|u\|_0^2 \tag{1.14}$$

holds for all $u\in C_0^\infty(\boldsymbol{R}^n)$. Then we have with suitable constants c_1' and c_2'

$$\|a(D)(\varphi u)\|_0^2 \leq c_1'\|P(D)u\|_0^2 + c_2'\|u\|_0^2 . \tag{1.15}$$

Proof. Let $u\in C_0^\infty(\boldsymbol{R}^n)$, then it follows that

$$\begin{aligned}\|a(D)(\varphi u)\|_0^2 &= \|\varphi a(D)u + [a(D), \varphi]u\|_0^2 \\ &\leq 2[\|\varphi a(D)u\|_0^2 + \|[a(D), \varphi]u\|_0^2] ,\end{aligned}$$

which proves the proposition.

Finally, let us remark that in general for $P\in\underline{P}$, $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ and $u\in H_P^s(\boldsymbol{R}^n)$, $s>0$, it does not follow that $\varphi u\in H_P^s(\boldsymbol{R}^n)$ and that the estimate

$$\|\varphi u\|_{P,s} \leq c(\varphi)\|u\|_{P,s}$$

holds. This estimate does not hold even in the case where P is a polynomial. For example take $P(\xi, \eta) = \xi^2\eta^2$, $(\xi, \eta)\in\boldsymbol{R}^2$, and apply Theorem 2 in [1], p. 212.

2. Pseudo Differential Operators in Generalized Divergence Form

We want to consider pseudo differential operators in generalized divergence form and related sesquilinear forms. Let $L(x, D)$ be given by

$$L(x, D) = \sum_{i,j=1}^{m_0} P_i(D)a_{i,j}(x)Q_j(D) \tag{2.1}$$

where $m_0\in\boldsymbol{N}$. We pose the following conditions on $L(x, D)$.
Let $P\in\underline{P}$ be fixed, then we assume:
L.1.: $P_i, Q_j\in\Sigma(P, 1)$ for all $1\leq i, j\leq m_0$;
L.2.: $a_i\in L^\infty(\boldsymbol{R}^n)$ for all $1\leq i, j\leq m_0$.
First we prove

Proposition 2.1. *Let $L(x, D)$ be a pseudo-differential operator in generalized divergence form* (2.1) *satisfying condition L.1—L.2. Then $L(x, D)$ is continuous from $H_P^1(\boldsymbol{R}^n)$ into $H_P^{-1}(\boldsymbol{R}^n)$.*

Proof. For $u\in C_0^\infty(\boldsymbol{R}^n)$ we have

$$||L(\cdot, D)u||_{P,-1}^2 = \int_{\boldsymbol{R}^n} (1+P^2(\xi))^{-1}|(L(\cdot, D)u)^\wedge(\xi)|^2 d\xi\,.$$

Now we find

$$\begin{aligned}(L(\cdot, D)u)^\wedge(\xi) &= [\sum_{i,j=1}^{m_0} P_i(D)(a_{ij}(\cdot)Q_j(D)u)]^\wedge(\xi)\\ &= \sum_{i,j=1}^{m_0} P_i(\xi)(a_{ij}(\cdot)Q_j(D)u)^\wedge(\xi)\end{aligned}$$

and it follws that

$$|(L(\cdot, D)u)^\wedge(\xi)|^2\le 4|\sum_{i,j=1}^{m_0} P_i(\xi)(a_{ij}(\cdot)Q_j(D)u)^\wedge(\xi)|^2$$

Hence, using $L.1$ we get

$$\begin{aligned}&\int_{\boldsymbol{R}^n} (1+P^2(\xi))^{-1}|(L(\cdot, D)u)^\wedge(\xi)|^2 d\xi\\ &\le 4c\int_{\boldsymbol{R}^n}(1+P^2(\xi))^{-1}(1+P^2(\xi))\{\sum_{i,j=1}^{m_0}|(a_{ij}(\cdot)Q_j(D)u)^\wedge(\xi)|^2\}\, d\xi\\ &= 4c\int_{\boldsymbol{R}^n}\{\sum_{i,j=1}^{m_0}|(a_{ij}(\cdot)Q_j(D)u)^\wedge(\xi)|^2\}\, d\xi\end{aligned}$$

and by Plancherel's theorem we have

$$\begin{aligned}&\int_{\boldsymbol{R}^n} (1+P^2(\xi))^{-1}|(L(\cdot, D)u)^\wedge(\xi)|^2 d\xi\\ &\le 4c\int_{\boldsymbol{R}^n}\sum_{i,j=1}^{m_0}|a_{ij}(x)Q_j(D)u(x)|^2 dx\\ &\le c'\int_{\boldsymbol{R}^n}\sum_{j=1}^{m_0}|Q_j(D)u(x)|^2 dx = c'\int_{\boldsymbol{R}^n}\sum_{j=1}^{m_0}|Q_j(\xi)\hat{u}(\xi)|^2 d\xi\\ &\le \hat{c}\int_{\boldsymbol{R}^n}(1+P^2(\xi))|\hat{u}(\xi)|^2 d\xi = \hat{c}||u||_{P,1}^2\,,\end{aligned}$$

which proves the proposition.

We want to define a sesquilinear form determined by a pseudo differential operator in generalized divergence form satisfying L.1—L.2. First let φ, $\psi\in C_0^\infty(\boldsymbol{R}^n)$. We set

$$(2.2) \qquad B(\varphi, \psi) := (L(x, D)\varphi, \psi)_0 .$$

Using Plancherel's theorem we find the following representation of B (see [11], section 5, for the calculation):

$$(2.3) \qquad B(\varphi, \psi) = \sum_{i,j=1}^{m_0} (a_{ij}(\cdot)Q_j(D)\varphi, P_i(D)\psi)_0$$

Proposition 2.2. *Let $L(x, D)$ be a pseudo differential operator in generalized divergence form satisfying L.1—L.2 and let B be the sesquilinear form corresponding to $L(x, D)$. Then B is continuous on $H^1_P(\boldsymbol{R}^n)$, i.e. there exists a constant $c \geq 0$ such that for all $u, v \in H^1_P(\boldsymbol{R}^n)$ we have the estimate*

$$(2.4) \qquad |B(u, v)| \leq c\|u\|_{P,1}\|v\|_{P,1} .$$

Proof. Since $C_0^\infty(\boldsymbol{R}^n)$ is dense in $H^1_P(\boldsymbol{R}^n)$ it is sufficient to prove (2.4) for all $\varphi, \psi \in C_0^\infty(\boldsymbol{R}^n)$. By (2.3) we have

$$|B(\varphi, \psi)| \leq \sum_{i,j=1}^{m_0} |(a_{ij}(\cdot)Q_j(D)\varphi, P_i(D)\psi)_0|$$

using L.2 and the Cauchy-Schwarz inequality it follows that

$$|B(\varphi, \psi)| \leq c' \sum_{i,j=1}^{m_0} \|Q_j(D)\varphi\|_0 \|P_i(D)\psi\|_0$$

and by L.1 and Proposition 1.4 it follows that

$$|B(\varphi, \psi)| \leq c\|\varphi\|_{P,1}\|\psi\|_{P,1}$$

holds for all $\varphi, \psi \in C_0^\infty(\boldsymbol{R}^n)$, which implies the assertion of the proposition.

From Proposition 2.2 we get immediately

Corollary 2.1. The sesquilinear form B is continuous on $H_P^{0,1}(\Omega)$ for any open set $\Omega \subset \boldsymbol{R}^n$.

In the next section we will use the following formula

Proposition 2.3. *Let $Q_j(D)$ and $P_i(D)$, $1 \leq i, j \leq m_0$, be operators satisfying L.1 and let $a_{ij} \in \boldsymbol{C}$. Furthermore assume that for a real valued function $\varphi \in C^\infty(\boldsymbol{R}^n)$ we have for all $u \in C_0^\infty(\boldsymbol{R}^n)$*

$$(2.5) \qquad \|[Q_j(D), \varphi]u\|_0^2 \leq c(\|P(D)u\|_0^2 + \|u\|_0^2)$$

and

$$(2.6) \qquad \|[P_i(D), \varphi]u\|_0^2 \leq c(\|P(D)u\|_0^2 + \|u\|_0^2) .$$

Given an open set $\Omega \subset \boldsymbol{R}^n$, then we have for all $u \in H_P^{0,1}(\Omega)$

$$\int_{\operatorname{supp}\varphi} \overline{a_{ij}}\,\overline{Q_j(D)(\varphi u)(x)}P_i(D)(\varphi u)(x)dx$$
$$= \int_{R^n} \overline{a_{ij}}\,\overline{Q_j(D)(\varphi u)(x)}P_i(D)(\varphi u)(x)\,dx$$
$$- \int_{R^n\text{-}\operatorname{supp}\varphi} \overline{a_{ij}}\,\overline{[Q_j(D),\ \varphi]u(x)}\,[P_i(D),\ \varphi]u(x)dx\,.$$

Proof. For $u \in H_P^{0,1}(\Omega)$ it follows that

$$\int_{\operatorname{supp}\varphi} \overline{a_{ij}}\,\overline{Q_j(D)(\varphi u)(x)}P_i(D)(\varphi u)(x)dx$$
$$= \int_{R^n} \overline{a_{ij}}\,\overline{Q_j(D)(\varphi u)(x)}P_i(D)(\varphi u)(x)dx$$
$$- \int_{R^n\text{-}\operatorname{supp}\varphi} \overline{a_{ij}}\,\overline{Q_j(D)(\varphi u)(x)}P_i(D)(\varphi u)(x)dx$$
$$= \int_{R^n} \overline{a_{ij}}\,\overline{Q_j(D)(\varphi u)(x)}P_i(D)(\varphi u)(x)dx$$
$$- \int_{R^n\text{-}\operatorname{supp}\varphi} \overline{a_{ij}}\,[\overline{[Q_j(D),\ \varphi]u(x)} - \overline{\varphi(x)Q_j(D)u(x)}]\,[[P_i(D),\ \varphi]u(x)$$
$$-\varphi(x)P_i(D)u(x)]dx$$
$$= \int_{R^n} \overline{a_{ij}}\,\overline{Q_j(D)(\varphi u)(x)}P_i(D)(\varphi u)(x)dx$$
$$- \int_{R^n\text{-}\operatorname{supp}\varphi} \overline{a_{ij}}\,\overline{[Q_j(D),\ \varphi]u(x)}\,[P_i(D),\ \varphi]\,u(x)dx\,,$$

where the last line follows from the fact that

$$0 = \int_{R^n\text{-}\operatorname{supp}\varphi} \varphi(x)\overline{a_{ij}}\,\overline{Q_j(D)u(x)}\,[P_i(D),\ \varphi]u(x)dx$$
$$= \int_{R^n\text{-}\operatorname{supp}\varphi} \varphi(x)\overline{a_{ij}}\,\overline{[Q_j(D),\ \varphi]\,u(x)}P_i(D)\,u(x)\,dx$$
$$= \int_{R^n\text{-}\operatorname{supp}\varphi} \varphi^2(x)\overline{a_{ij}}\,\overline{Q_j(D)u(x)}P_i(D)u(x)dx\,.$$

3. A Gårding Inequality

Let $L(x, D)$ be a pseudo differential operator in generalized divergence form satisfying L.1—L.2. In the last section we proved that there exists a continuous sesquilinear form B on $H_P^1(\boldsymbol{R}^n)$ (or $H_P^{0,1}(\Omega)$) generated by $L(x, D)$.

The purpose of this section is to prove a (generalized) Gårding inequality for B under suitable additional assumptions on $L(x, D)$. As already mentioned in the introduction we have to distinguish two cases.

First let us assume the following conditions:

L.3.: There is a point $x_0 \in \boldsymbol{R}^n$ such that with two constants $c_0 > 0$ and $\rho_0 \geq 0$ the estimate

$$\operatorname{Re} \sum_{i,j=1}^{m_0} a_{ij}(x_0) P_i(\xi) Q_j(\xi) \geq c_0 P^2(\xi) \tag{3.1}$$

holds for all $\xi \in \boldsymbol{R}^n$ with $|\xi| \geq \rho_0$.

L.4.: Let c_0 be the constant in (3.1) then we suppose for some σ, $0 < \sigma < 1$,

$$\max_{1 \leq i,j \leq m_0} \sup_{x \in \boldsymbol{R}^n} |a_{ij}(x) - a_{ij}(x_0)| \leq \sigma \frac{c_0}{(\hat{c} m_0)^2} \tag{3.2}$$

to be satisfied, where $\hat{c}$ is a constant such that

$$\|Q_j(D)u\|_0 \leq \hat{c}[\|P(D)u\|_0 + \|u\|_0]$$

and

$$\|P_i(D)u\|_0 \leq \hat{c}[\|P(D)u\|_0 + \|u\|_0]$$

holds for all $u \in C_0^\infty(\boldsymbol{R}^n)$ and $1 \leq i, j \leq m_0$. (Note, that by Proposition 1.5 such a constant does exist!)

Now we can proev

Theorem 3.1. *Let $L(x, D)$ be a pseudo differential operator in generalized divergence form satisfying L.1—L.4. Then we have*

$$\operatorname{Re} B(u, u) \geq (1-\sigma) c_0 \|P(D)u\|_0^2 - c(\sigma) \|u\|_0^2 \tag{3.3}$$

for all $u \in C_0^\infty(\boldsymbol{R}^n)$.

Proof. Using (2.3) we find

$$\operatorname{Re} B(u, u) = \operatorname{Re} \sum_{i,j=1}^{m_0} \int_{\boldsymbol{R}^n} \overline{a_{ij}(x)}\, \overline{Q_j(D)u(x)} P_i(D)u(x)\, dx = J_1 .$$

We estimate J_1:

$$\begin{aligned} &\operatorname{Re} \sum_{i,j=1}^{m_0} \int_{\boldsymbol{R}^n} \overline{a_{ij}(x)}\, \overline{Q_j(D)u(x)} P_i(D)u(x)\, dx \\ &= \operatorname{Re} \sum_{i,j=1}^{m_0} \int_{\boldsymbol{R}^n} \overline{a_{ij}(x_0)}\, \overline{Q_j(D)u(x)} P_i(D)u(x) dx \end{aligned}$$

$$+\mathrm{Re}\sum_{i,j=1}^{m_0}\int_{\boldsymbol{R}^n}\overline{(a_{ij}(x)-a_{ij}(x_0))}\,\overline{Q_j(D)u(x)}P_i(D)u(x)dx$$

$$\geq\int_{\boldsymbol{R}^n}\mathrm{Re}\sum_{i,j=1}^{m_0}a_{ij}(x_0)Q_j(\xi)P_i(\xi)|Fu(\xi)|^2d\xi$$

$$-\sum_{i,j=1}^{m_0}\int_{\boldsymbol{R}^n}|a_{ij}(x_0)-a_{ij}(x)|\,|Q_j(D)u(x)|\,|P_i(D)u(x)|dx$$

$$\geq c_0\int_{\boldsymbol{R}^n}P^2(\xi)|Fu(\xi)|^2d\xi$$

$$-\sup_{|\xi|\leq\rho_0}\Big[\sum_{i,j=1}^{m_0}|a_{ij}(x_0)|\,|Q_j(\xi)|\,|P_i(\xi)|+c_0|P(\xi)|^2\Big]||u||_0^2$$

$$-\sum_{i,j=1}^{m_0}\sup_{x\in\boldsymbol{R}^n}|a_{ij}(x)-a_{ij}(x_0)|\,||Q_j(D)u(x)||_0||P_i(D)u(x)||_0$$

$$\geq c_0||P(D)u||_0^2-\sigma c_0||P(D)u||_0^2-c_1||u||_0^2\,,$$

where we used in the last line assumption L.4 and Proposition 1.5. The constant c_1 depends on σ, m_0 and $\hat{c}$. Hence we have

$$J_1\geq c_0(1-\sigma)||P(D)u||_0^2-c_1||u||_0^2\,. \tag{3.4}$$

In order to solve a generalized Dirichlet problem we give a formulation of Theorem 3.1 in the case of an open set $\Omega\subset\boldsymbol{R}^n$. For this it is suitable to assume instead of L.4 the following condition

L.4'.: Let c_0 be the constant in (3.1), then we suppose for some $\sigma\in(0, 1)$

$$\max_{1\leq i,j\leq m_0}\sup_{x\in\Omega}|a_{ij}(x)-a_{ij}(x_0)|\leq\sigma\frac{c_0}{(\hat{c}m_0)^2}\,, \tag{3.5}$$

where $\hat{c}$ is the same constant as in L.4.

Corollary 3.1. *Assume that $L(x, D)$ is a pseudo differential operator satisfying L.1–L.3 and L.4'. Furthermore assume $a_{ij}(x)=a_{ij}(x_0)$ for all $x\in\boldsymbol{R}^n-\Omega$, where $x_0\in\Omega$ is the point mentioned in L.3. Then there exists a constant $c(\sigma)\geq 0$ such that*

$$\mathrm{Re}\,B(u, u)\geq(1-\sigma)||P(D)u||_0^2-c(\sigma)||u||_0^2 \tag{3.6}$$

holds for all $u\in C_0^\infty(\Omega)$.
(Remember that elements of $C_0^\infty(\Omega)$ are defined on $\boldsymbol{R}^n$!)

The proof of Corollary 3.1 is just the same as that of Theorem 3.1, but by our assumptions we have now

$$\operatorname{Re}\sum_{i,j=1}\int_{\boldsymbol{R}^n}\overline{(a_{ij}(x)-a_{ij}(x_0))}\,\overline{Q_j(D)u(x)}P_i(D)u(x)\,dx$$

$$=\operatorname{Re}\sum_{i,j=1}\int_{\Omega}\overline{(a_{ij}(x)-a_{ij}(x_0))}\,\overline{Q_j(D)u(x)}P_i(D)u(x)\,dx\,.$$

In order to handle the second case, we need the follwoing

Lemma 3.1. *Let $\Omega\subset\boldsymbol{R}^n$ be an open bounded set and assume $\Omega\subset B_r(0)=\{x\in\boldsymbol{R}^n,\ |x|<r\}$ for some $r>0$. Furthermore let $\{\Omega_i,\ 1\leq i\leq N-1\}$ be an open covering of $\overline{B_{2r}(0)}$. Then there exist functions $\varphi_i\in C_0^\infty(\Omega_i)$, $1\leq i\leq N-1$, and a function $\varphi_N\in C^\infty(\boldsymbol{R}^n)$ with the following properties:*

i) $0\leq\varphi_i(x)\leq 1$ *for* $1\leq i\leq N$;

ii) $\varphi_N(x)>0$ *for* $x\in\boldsymbol{R}^n-\overline{B_r(0)}$;

iii) $\operatorname{supp}\varphi_N\subset\boldsymbol{R}^n-B_r(0)$;

iv) $\varphi_i^{1/2}\in C^\infty(\boldsymbol{R}^n)$ *for* $1\leq i\leq N$;

v) $\sum_{k=1}^{N}\varphi_i^2(x)=1$ *for* $x\in\boldsymbol{R}^n$.

We call $(\varphi_i)_{1\leq i\leq N}$ a partition of unity subordinated to the covering $\{\Omega_i,\ 1\leq i\leq N\}\cup\boldsymbol{R}^n-\overline{B_r(0)}$. The proof of Lemma 3.1 is an obvious modification of Lemma 9.16 in [17].

Instead of L.3 and L.4 let us now assume

L.5.: Let $\Omega\subset B_r(0)$ be a bounded open set in $\boldsymbol{R}^n$. Assume that there are two constants $c_0>0$ and $\rho_0\geq 0$ such that

$$\operatorname{Re}\sum_{i,j=1}^{m_0}a_{ij}(x)P_i(\xi)Q_j(\xi)\geq c_0P(\xi)^2$$

holds for all $x\in B_{2r}(0)$ and all $\xi\in\boldsymbol{R}^n$, $|\xi|\geq\rho_0$.

L.6.: For $1\leq i,j\leq m_0$ suppose that

$$|a_{ij}(x)-a_{ij}(y)|\leq g(|x-y|)$$

holds for all $x,y\in B_{3r}(0)$, where $g\colon\boldsymbol{R}^+\to\boldsymbol{R}^+$ is a function satisfying $\lim_{t\to 0}g(t)=0$.

L.7.: Let $\varphi\in C^\infty(\boldsymbol{R}^n)$ such that $D^\alpha\varphi\in C_0^\infty(\boldsymbol{R}^n)$ for all $\alpha\in\boldsymbol{N}_0^n$, $\alpha\neq 0$. It is assumed that for any $\eta>0$, we can find a constant $c(\eta)\geq 0$ such that for all $u\in C_0^\infty(\Omega)$, where Ω is the set considered in L.5, we have the estimates

$$\|[P(D),\varphi]u\|_0^2\leq\eta\|P(D)u\|_0^2+c(\eta)\|u\|_0^2\,, \tag{3.7}$$

$$\|[Q_j(D),\varphi]u\|_0^2\leq\eta\|P(D)u\|_0^2+c(\eta)\|u\|_0^2\,,\qquad 1\leq j\leq m_0\,, \tag{3.8}$$

and

$$\|[P_i(D),\varphi]u\|_0^2\leq\eta\|P(D)u\|_0^2+c(\eta)\|u\|_0^2\,,\qquad 1\leq i\leq m_0\,. \tag{3.9}$$

Theorem 3.2. *Let $\Omega\subset\subset\boldsymbol{R}^n$ be an open set and let $L(x, D)$ be a pseudo differential operator in generalized divergence form satisfying L.1—L.2 and L.5—L.7. Then for each $\varepsilon\in(0, c_0/2)$ we have*

$$\operatorname{Re} B(u, u)\geq(1/2)(c_0-\varepsilon)\|P(D)u\|_0^2-c(\varepsilon)\|u\|_0^2 \tag{3.10}$$

for all $u\in C_0^\infty(\Omega)$.

Proof. Again, by (2.3) we have

$$\operatorname{Re} B(u, u)=\operatorname{Re}\sum_{i,j=1}^{m_0}\int_{\boldsymbol{R}^n}\overline{a_{ij}(x)}\,\overline{Q_j(D)u(x)}P_i(D)u(x)dx=J_1\,.$$

We have to prove

$$J_1\geq(1/2)(c_0-\varepsilon)\|P(D)u\|_0^2-c'\|u\|_0^2\,.$$

Now, let $\{\Omega_k,\ 1\leq k\leq N-1\}$ a finite open covering of $\overline{B_{2r}(0)}$, note that $\Omega\subset B_r(0)$, such that

$$\max_{1\leq i,j\leq m_0}\max_{x,y\in\Omega_k}|a_{ij}(x)-a_{ij}(y)|\leq\frac{c_0}{2c'm_0^2}, \tag{3.11}$$

where c' is a constant such that for $1\leq i,\ j\leq m_0$

$$\|Q_j(D)u\|_0^2\leq c'[\|P(D)u\|_0^2+\|u\|_0^2]$$

and

$$\|P_i(D)u\|_0^2\leq c'[\|P(D)u\|_0^2+\|u\|_0^2]$$

hold. (Note that such a covering always exists by L.6.) Furthermore, let $(\varphi_k)_{1\leq k\leq N}$ be a partition of unity subordinated to the covering $\{\Omega_k,\ 1\leq k\leq N-1\}$ $\cup\,\boldsymbol{R}^n-\overline{B_r(0)}$ having the properties stated in Lemma 3.1 (in the following we set $\Omega_N:=\boldsymbol{R}^n-\overline{B_r(0)}$). It follows that

$$\begin{aligned}
&\operatorname{Re}\sum_{i,j=1}^{m_0}\int_{\boldsymbol{R}^n}\overline{a_{ij}(x)}\,\overline{Q_j(D)u(x)}P_i(D)u(x)\,dx\\
&=\operatorname{Re}\sum_{k=1}^{N}\sum_{i,j=1}^{m_0}\int_{\operatorname{supp}\varphi_k}\varphi_k^2(x)\overline{a_{ij}(x)}\,\overline{Q_j(D)u(x)}P_i(D)u(x)dx\\
&=\operatorname{Re}\sum_{k=1}^{N}\sum_{i,j=1}^{m_0}\int_{\operatorname{supp}\varphi_k}\overline{a_{ij}(x)}\{\overline{Q_j(D)(\varphi_k u)(x)}-\overline{[Q_j(D),\ \varphi_k]u(x)}\}\\
&\qquad\qquad\{P_i(D)(\varphi_k u)(x)-[P_i(D),\ \varphi_k]u(x)\}\,dx\\
&=A_1+A_2+A_3+A_4\geq A_1-|A_2|-|A_3|-|A_4|,
\end{aligned}$$

where

$$A_1 = \operatorname{Re} \sum_{k=1}^{N} \sum_{i,j=1}^{m_0} \int_{\operatorname{supp}\varphi_k} \overline{a_{ij}(x)}\, \overline{Q_j(D)(\varphi_k u)(x)} P_i(D)(\varphi_k u)(x) dx$$

$$A_2 = -\operatorname{Re} \sum_{k=1}^{N} \sum_{i,j=1}^{m_0} \int_{\operatorname{supp}\varphi_k} \overline{a_{ij}(x)}\, \overline{Q_j(D)(\varphi_k u)(x)} [P_i(D), \varphi_k] u(x) dx$$

$$A_3 = -\operatorname{Re} \sum_{k=1}^{N} \sum_{i,j=1}^{m_0} \int_{\operatorname{supp}\varphi_k} \overline{a_{ij}(x)}\, \overline{[Q_j(D), \varphi_k] u(x)} P_i(D)(\varphi_k u)(x) dx$$

and

$$A_4 = \operatorname{Re} \sum_{k=1}^{N} \sum_{i,j=1}^{m_0} \int_{\operatorname{supp}\varphi_k} \overline{a_{ij}(x)}\, \overline{[Q_j(D), \varphi_k] u(x)} [P_i(D), \varphi_k] u(x) dx .$$

Let us estimate A_2, A_3 and A_4. First we find

$$\begin{aligned} |A_2| &\leq \sum_{k=1}^{N} \sum_{i,j=1}^{m_0} \int_{R^n} |a_{ij}(x)|\, |Q_j(D)(\varphi_k u)(x)|\, |[P_i(D), \varphi_k] u(x)| dx \\ &\leq c \sum_{k=1}^{N} \sum_{i,j=1}^{m_0} ||Q_j(D)(\varphi_k u)||_0 ||[P_i(D), \varphi_k] u||_0 \\ &\leq c \sum_{k=1}^{N} \sum_{i,j=1}^{m_0} \tau ||Q_j(D)(\varphi_k u)||_0^2 + c(\tau) ||[P_i(D), \varphi_k] u||_0^2 , \end{aligned}$$

where $\tau > 0$ is an arbitrarily chosen non-negative number. By Proposition 1.6 we find

$$(3.12) \qquad ||Q_j(D)(\varphi_k u)||_0^2 \leq c'[||P(D)u||_0^2 + ||u||_0^2]$$

and by assumption L.7 for each $\eta > 0$ the estimate (3.13) follows:

$$(3.13) \qquad ||[P_i(D), \varphi_k] u||_0^2 \leq \eta ||P(D)u||_0^2 + c_{ik}(\eta) ||u||_0^2 .$$

Thus we obtain

$$\begin{aligned} |A_2| &\leq c m_0^2 N[\tau c'(||P(D)u||_0^2 + ||u||_0^2) + c(\tau)(\eta ||P(D)u||_0^2 + c(\eta)||u||_0^2)] \\ &= c m_0^2 N[(\tau c' + c(\tau)\eta) ||P(D)u||_0^2 + (\tau c' + c(\tau) c(\eta)) ||u||_0^2 . \end{aligned}$$

Now, given $\varepsilon > 0$ and choose η such that $c(\tau)\eta = \tau$, then it follows with $\tau = \varepsilon(8 c m_0^2 N(c'+1))^{-1}$ that

$$(3.14) \qquad |A_2| \leq (\varepsilon/8) ||P(D)u||_0^2 + c_1(\varepsilon) ||u||_0^2 .$$

Analogously we find

$$\begin{aligned} |A_3| &\leq c \sum_{k=1}^{N} \sum_{i,j=1}^{m_0} ||[Q_j(D), \varphi_k] u||_0 ||P_i(D)(\varphi_k u)||_0 \\ &\leq c \sum_{k=1}^{N} \sum_{i,j=1}^{m_0} \tau ||P_i(D)(\varphi_k u)||_0^2 + c(\tau) ||[Q_j(D), \varphi_k] u||_0^2 , \end{aligned}$$

which gives again for any $\varepsilon>0$

$$|A_3|\leq(\varepsilon/8)||P(D)u||_0^2+c_2(\varepsilon)||u||_0^2\,. \tag{3.15}$$

Moreover, it follows that

$$|A_4|\leq\sum_{k=1}^{N}\sum_{i,j=1}^{m_0}\int_{R^n}|a_{ij}(x)|\,|[Q_j(D),\,\varphi_k]u(x)|\,|[P_i(D),\,\varphi_k]u(x)|\,dx$$
$$\leq c\sum_{k=1}^{N}\sum_{i,j=1}^{m_0}||[Q_j(D),\,\varphi_k]u||_0||[P_i(D),\,\varphi_k]u||_0\,.$$

Since by our assumption L.7 for each $\eta>0$ we have the estimates

$$||[Q_j(D),\,\varphi_k]u||_0^2\leq\eta||P(D)u||_0^2+c_{jk}(\eta)||u||_0^2$$

and

$$||[P_i(D),\,\varphi_k]u||_0^2\leq\eta||P(D)u||_0^2+c_{ik}(\eta)||u||_0^2$$

we find

$$|A_4|\leq cm_0^2N[\eta||P(D)u||_0^2+c'(\eta)||u||_0^2]\,,$$

which gives with $\eta=\varepsilon(8cm_0^2N)^{-1}$

$$|A_4|\leq(\varepsilon/8)||P(D)u||_0^2+c_3(\varepsilon)||u||_0^2\,. \tag{3.16}$$

So far we have proved

$$\begin{aligned}\operatorname{Re}\sum_{i,j=1}^{m_0}\int_{R^n}\overline{a_{ij}(x)}\;\overline{Q_j(D)u(x)}P_i(D)u(x)dx&\\ \geq A_1-(3\varepsilon/8)||P(D)u||_0^2-c'(\varepsilon)||u||_0^2\,,&\end{aligned} \tag{3.17}$$

where $c'(\varepsilon)=\sum_{i=1}^{3}c_i(\varepsilon)$.

Now let us consider A_1. For $1\leq k\leq N$ we have by Proposition 2.3 with $x_k\in\Omega_k$

$$\int_{\operatorname{supp}\varphi_k}\overline{a_{ij}(x)}\;\overline{Q_j(D)(\varphi_k u)(x)}P_i(D)(\varphi_k u)(x)dx$$
$$=\int_{\operatorname{supp}\varphi_k}\overline{a_{ij}(x_k)}\;\overline{Q_j(D)(\varphi_k u)(x)}P_i(D)(\varphi_k u)(x)dx$$
$$-\int_{\operatorname{supp}\varphi_k}\overline{(a_{ij}(x_k)-a_{ij}(x))}\;\overline{Q_j(D)(\varphi_k u)(x)}P_i(D)(\varphi_k u)(x)dx$$
$$=\int_{R^n}\overline{a_{ij}(x_k)}\;\overline{Q_j(D)(\varphi_k u)(x)}P_i(D)(\varphi_k u)(x)dx$$
$$-\int_{R^n\text{-}\operatorname{supp}\varphi_k}\overline{a_{ij}(x_k)}\;\overline{[Q_j(D),\,\varphi_k]u(x)}[P_i(D),\,\varphi_k]u(x)dx$$

$$-\int_{\operatorname{supp}\varphi_k} \overline{(a_{ij}(x_k)-a_{ij}(x))}\;\overline{Q_j(D)(\varphi_k u)(x)} P_i(D)(\varphi_k u)(x)dx$$
$$\geq \int_{R^n} \overline{a_{ij}(x_k)} Q_j(\xi) P_i(\xi) |F(\varphi_k u)(\xi)|^2 d\xi$$
$$-c_{ij}\|[Q_j(D),\ \varphi_k]u\|_0\|[P_i(D),\ \varphi_k]u\|_0$$
$$-\sup_{x\in\operatorname{supp}\varphi_k}|a_{ij}(x_k)-a_{ij}(x)|\,\|Q_j(D)(\varphi_k u)\|_0\|P_i(D)(\varphi_k u)\|_0\,.$$

Now we get

$$A_1 \geq \operatorname{Re}\sum_{k=1}^{N}\sum_{i,j=1}^{m_0}\int_{R^n} \overline{a_{ij}(x_k)} Q_j(\xi) P_i(\xi) |F(\varphi_k u)(\xi)|^2 d\xi \tag{3.18}$$
$$-\operatorname{Re}\sum_{k=1}^{N}\sum_{i,j=1}^{m_0} c_{ij}\|[Q_j(D),\ \varphi_k]u\|_0\|[P_i(D),\ \varphi_k]u\|_0$$
$$-\operatorname{Re}\sum_{k=1}^{N}\sum_{i,j=1}^{m_0}\sup_{x\in\operatorname{supp}\varphi_k}|a_{ij}(x_k)-a_{ij}(x)|\,\|Q_j(D)(\varphi_k u)\|_0\|P_i(D)(\varphi_k u)\|_0\,.$$

By L.5 we find

$$\operatorname{Re}\sum_{k=1}^{N}\sum_{i,j=1}^{m_0}\int_{R^n} \overline{a_{ij}(x_k)} Q_j(\xi) P_i(\xi) |F(\varphi_k u)(\xi)|^2 d\xi \geq c_0 \sum_{k=1}^{N}\|P(D)(\varphi_k u)\|_0^2\,.$$

Moreover, using L.7 it follows that for any $\varepsilon>0$

$$\operatorname{Re}\sum_{k=1}^{N}\sum_{i,j=1}^{m_0} c_{ij}\|[Q_j(D),\ \varphi_k]u\|_0\|[P_i(D),\ \varphi_k]u\|_0 \leq (\varepsilon/16)\|P(D)u\|_0^2+c'(\varepsilon)\|u\|_0^2$$

holds.

In order to estimate the last term in (3.17), note that $\operatorname{supp} u\cap\operatorname{supp}\varphi_N=\phi$ and therefore $R(D)(\varphi_N u)=0$ for any pseudo differential operator. Further by our assumptions on the support of φ_k, $1\leq k\leq N-1$, we have

$$\sup_{x\in\operatorname{supp}\varphi_k}|a_{ij}(x_k)-a_{ij}(x)|\leq c_0(2m_0^2c')^{-1}\,,$$

which implies that

$$\sum_{k=1}^{N}\sum_{i,j=1}^{m_0}\sup_{x\in\operatorname{supp}\varphi_k}|a_{ij}(x_k)-a_{ij}(x)|\,\|Q_j(D)(\varphi_k u)\|_0\|P_i(D)(\varphi_k u)\|_0$$
$$=\sum_{k=1}^{N-1}\sum_{i,j=1}^{m_0}\sup_{x\in\operatorname{supp}\varphi_k}|a_{ij}(x_k)-a_{ij}(x)|\,\|Q_j(D)(\varphi_k u)\|_0\|P_i(D)(\varphi_k u)\|_0$$
$$\leq (c_0/2)\sum_{k=1}^{N-1}\|P(D)(\varphi_k u)\|_0^2+c''\|u\|_0^2$$
$$=(c_0/2)\sum_{k=1}^{N}\|P(D)(\varphi_k u)\|_0^2+c''\|u\|_0^2\,,$$

where we have used the estimates for $Q_j(D)$ and $P_i(D)$ given subsequent to (3.11), noting that $\varphi_k u \in C_0^\infty(\Omega)$ for $1 \leq k \leq N-1$. Now, we have proved

$$A_1 \geq (c_0/2) \sum_{k=1}^{N} \|P(D)(\varphi_k u)\|_0^2 - (\varepsilon/16)\|P(D)u\|_0^2 - \bar{c}\|u\|_0^2 .$$

Finally consider the term $\|P(D)(\varphi_k u)\|_0^2$. By (3.7) we find for any $\varepsilon > 0$

$$\begin{aligned}
\|P(D)(\varphi_k u)\|_0^2 &= \int_{R^n} |(\varphi_k P(D)u + [P(D), \varphi_k]u(x)|^2 dx \\
&\geq \int_{R^n} \varphi_k^2(x)|P(D)u(x)|^2 dx - 2\int_{R^n} |\varphi_k(x)|\,|P(D)u(x)|\,|[P(D), \varphi_k]u(x)|\,dx \\
&\quad - \|[P(D), \varphi_k]u\|_0^2 \\
&\geq \|\varphi_k P(D)u\|_0^2 - ((\varepsilon/8Nc_0)\|P(D)u\|_0^2 + c'(\varepsilon)\|u\|_0^2)
\end{aligned}$$

and we obtain

$$\begin{aligned}
A_1 &\geq (c_0/2) \sum_{k=1}^{N} [\|\varphi_k P(D)u\|_0^2 - (\varepsilon_0/8Nc_0)\|P(D)u\|_0^2 - c'(\varepsilon)\|u\|_0^2] \\
&\quad - (\varepsilon/16)\|P(D)u\|_0^2 - \bar{c}\|u\|_0^2 \\
&= ((c_0/2) - (\varepsilon/8))\|P(D)u\|_0^2 - c\|u\|_0^2 .
\end{aligned}$$

Using (3.14), (3.15) and (3.16) the last estimate yields

$$\operatorname{Re} \sum_{i,j=1}^{m_0} \int_{R^n} \overline{a_{ij}(x)}\; \overline{Q_j(D)u(x)} P_i(D)u(x)dx \geq ((c_0/2) - (1/2)\varepsilon)\|P(D)u\|_0^2 - c\|u\|_0^2$$

which proves the theorem.

4. On the Commutator $[Q(D), \varphi]$

In the last section we proved in Theorem 3.2 a generalized Garding inequality for pseudo differential operators

$$L(x, D) = \sum_{i,j=1}^{m_0} P_i(D)a_{ij}(x)Q_j(D)$$

satisfying L.1—L.2 and L.5—L.7. In particular the estimates (3.7)—(3.9) have been of greater importance. In this section we want to give a sufficient condition in order that for a symbol $Q \in \Sigma(P, 1)$ we have the estimate

$$\|[Q(D), \varphi]\|_0^2 \leq \varepsilon \|P(D)u\|_0^2 + c(\varepsilon)\|u\|_0^2$$

for a suitable class of functions $\varphi \in C^\infty(\boldsymbol{R}^n)$, see L.7. The proof of the following theorem requires some lemmas:

Lemma 4.1. ([16], *Lemma* 2.2.4) *Let* $K\in L^1(\boldsymbol{R}^n)$, *then we have for all* $u, v\in L^2(\boldsymbol{R}^n)$

$$|\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n} K(\xi-\eta)u(\eta)v(\xi)d\eta d\xi|\leq ||K||_{L^1(\boldsymbol{R}^n)}||u||_0||v||_0\,. \tag{4.1}$$

Lemma 4.2. *A*. ([16], *Lemma* 2.2.1) *For any* $q\in\boldsymbol{R}$ *and all* $\xi, \eta\in\boldsymbol{R}^n$ *the inqeuality*

$$(1+|\xi|^2)^q(1+|\eta|^2)^q\leq 2^{|q|}(1+|\xi-\eta|^2)^{|q|} \tag{4.2}$$

holds.

B. ([16], *Lemma* 2.2.2) *For* $\theta\in[0, 1]$, $q\in\boldsymbol{R}$ *and any* $\xi, \eta\in\boldsymbol{R}^n$ *we have the inequality*

$$(1+|\xi+\theta(\eta-\xi)|^2)^{|q|}\leq c[(1+|\eta|^2)^{|q|}+(1+|\xi|^2)^{|q|}]\,. \tag{4.3}$$

Now we can prove

Theorem 4.1. *Let* $\varphi\in C^\infty(\boldsymbol{R}^n)$ *such that for* $1\leq i\leq n$ *we have* $\partial_i\varphi\in C_0^\infty(\boldsymbol{R}^n)$. *Furthermore, let* $Q\in\Sigma(P, 1)\cap C^1(\boldsymbol{R}^n)$ *and assume*

$$|\mathrm{grad}\, Q(\xi)|\leq c(1+|\xi|^2)^{q/2} \tag{4.4}$$

for some $q\in\boldsymbol{R}$, $q\geq 0$, *and all* $\xi\in\boldsymbol{R}^n$. *In addition suppose*

$$\lim_{|\xi|\to\infty}\frac{(1+|\xi|^2)^{q/2}}{|P(\xi)|}=0\,, \tag{4.5}$$

i.e. $(1+|\cdot|^2)^{q/2}\in\Sigma_0(P, 1)$. *Then for each* $\varepsilon>0$ *the estimate*

$$||[Q(D), \varphi]u||_0^2\leq\varepsilon||P(D)u||_0^2+c(\varepsilon)||u||_0^2 \tag{4.6}$$

holds for all $u\in H_P^1(\boldsymbol{R}^n)$.

Proof. By our assumptions we have $\partial_i\varphi\in C_0^\infty(\boldsymbol{R}^n)$, $1\leq i\leq n$, which implies $\varphi_{|\boldsymbol{R}^n-B_r(0)}=M$, $M\in\boldsymbol{R}$, where $B_r(0)\subset\boldsymbol{R}^n$ contains $\bigcup_{i=1}^n \mathrm{supp}\,\partial_i\varphi$. Hence, we find $\varphi-M\in C_0^\infty(\boldsymbol{R}^n)$ and $\mathrm{supp}\,(\varphi-M)\subset B_r(0)$. For any $u\in C_0^\infty(\boldsymbol{R}^n)$ we have

$$\begin{aligned}&Q(D)(\varphi u)(x)-\varphi(x)Q(D)u(x)\\&\quad=Q(D)(((\varphi-M)+M)u)(x)-((\varphi(x)-M)+M)Q(D)u(x)\\&\quad=Q(D)((\varphi-M)u)(x)-(\varphi(x)-M)Q(D)u(x)\,.\end{aligned}$$

Therefore the theorem is proved, if we have shown (4.6) for any function $\psi\in C_0^\infty(\boldsymbol{R}^n)$. Now, using

$$\psi(x)Q(D)u(x) = \psi(x) \int_{R^n} e^{i(x,\xi)} Q(\xi)\hat{u}(\xi)d\xi ,$$

$$Q(D)(\psi u)(x) = \int_{R^n} e^{i(x,\xi)} Q(\xi)(\psi u)^\wedge \xi()d\xi$$

and

$$(Q(D)u)^\wedge(\xi) = Q(\xi)\hat{u}(\xi)$$

we find

$$\begin{aligned}([Q(D), \psi]u)^\wedge(\xi) &= (Q(D)(\psi u)^\wedge(\xi) - (\psi Q(D)u)^\wedge(\xi) \\ &= Q(\xi)(\psi u)^\wedge(\xi) - \int_{R^n} (\hat{\psi}\xi - \eta)(Q(D)u)^\wedge(\eta)d\eta \\ &= \int_{R^n} [Q(\xi)\hat{\psi}(\xi-\eta)\hat{u}(\eta) - \hat{\psi}(\xi-\eta)Q(\eta)\hat{u}(\eta)]d\eta \\ &= \int_{R^n} \hat{\psi}(\xi-\eta)(Q(\xi)-Q(\eta))\hat{u}(\eta)d\eta .\end{aligned}$$

Furthermore, for any $v \in L^2(\boldsymbol{R}^n)$ we have

$$\begin{aligned}&\int_{R^n} \left\{ \int_{R^n} \overline{(\hat{\psi}\xi-\eta)}(Q(\xi)-Q(\eta))\overline{\hat{u}(\eta)}\hat{v}(\xi)d\eta \right\} d\xi \\ &= \int_{R^n} \left\{ \int_{R^n} \overline{(\hat{\psi}\xi-\eta)} \frac{Q(\xi)-Q(\eta)}{(1+|\eta|^2)^{q/2}} (1+|\eta|^2)^{q/2} \overline{\hat{u}(\eta)}\hat{v}(\xi)d\eta \right\} d\xi .\end{aligned}$$

Consider the expression

$$\overline{\hat{\psi}(\xi-\eta)} \frac{Q(\xi)-Q(\eta)}{(1+|\eta|^2)^{q/2}} .$$

In order to apply Lemma 4.1 we want to show

$$\left| (\hat{\psi}\xi-\eta) \frac{Q(\xi)-Q(\eta)}{(1+|\eta|^2)^{q/2}} \right| \leq c \frac{1}{(1+|\xi-\eta|^2)^{(n+1)/2}} .$$

Using the mean value theorem we find

$$Q(\xi)-Q(\eta) = (\text{grad } Q(\xi+\theta(\eta-\xi)), \xi-\eta)_0$$

which implies

$$\begin{aligned}\hat{\psi}(\xi-\eta)(Q(\xi)-Q(\eta)) &= \sum_{i=1}^n \hat{\psi}(\xi-\eta)(\xi-\eta)_i \frac{\partial}{\partial\tau_i} Q(\tau)_{|\tau=\xi+\theta(\eta-\xi)} \cdot \\ &= \sum_{i=1}^n (D_i\psi)^\wedge(\xi-\eta) \frac{\partial}{\partial\tau_i} Q(\tau)_{|\tau=\xi+\theta(\eta-\xi)} \cdot\end{aligned}$$

Since by our assumption $D_i\psi\in C_0^\infty(\boldsymbol{R}^n)$, $1\leq i\leq n$, we find (see [18], p. 146),

$$|(D_i\psi)^\wedge(\xi-\eta)|\leq c\frac{1}{(1+|\xi-\eta|^2)^{m/2}}$$

for any $m\geq 0$. This implies together with (4.4)

$$\left|\hat{\psi}(\xi-\eta)\frac{Q(\xi)-Q(\eta)}{(1+|\eta|^2)^{q/2}}\right|\leq c\frac{1}{(1+|\xi-\eta|^2)^{m/2}}\frac{(1+|\xi+\theta(\eta-\xi)|^2)^{q/2}}{(1+|\eta|^2)^{q/2}}.$$

Now, by (4.3) we have

$$(1+|\xi+\theta(\eta-\xi)|^2)^{q/2}\leq c[(1+|\eta|^2)^{q/2}+(1+|\xi|^2)^{q/2}]$$

and by (4.2) we obtain

$$\frac{(1+|\xi+\theta(\eta-\xi)|^2)^{q/2}}{(1+|\eta|^2)^{q/2}}\leq c\left[1+\frac{(1+|\xi|^2)^{q/2}}{(1+|\eta|^2)^{q/2}}\right]\leq c(1+(1+|\xi-\eta|^2)^{q/2}).$$

Finally we get

$$\left|\hat{\psi}(\xi-\eta)\frac{Q(\xi)-Q(\eta)}{(1+|\eta|^2)^{q/2}}\right|\leq c\frac{1}{(1+|\xi-\eta|^2)^{m/2}}(1+(1+|\xi-\eta|^2)^{q/2}).$$

Taking $m=q+n+1$ it follows by Lemma 4.1 that

$$\begin{aligned}&\left|\int_{\boldsymbol{R}^n}\left\{\int_{\boldsymbol{R}^n}\overline{\hat{\psi}(\xi-\eta)}(Q(\xi)-Q(\eta))\overline{\hat{u}(\eta)}\hat{v}(\xi)d\eta\right\}d\xi\right|\\&\leq c\int_{\boldsymbol{R}^n}\left\{\int_{\boldsymbol{R}^n}\frac{1}{(1+|\xi-\eta|^2)^{(n+1)/2}}(1+|\eta|^2)^{q/2}|\hat{u}(\eta)|\,|\hat{v}(\xi)|\,d\eta\right\}d\xi\\&\leq c||u||_q||v||_0,\end{aligned}$$

or

$$|([Q(D),\psi]u,v)_0|\leq c||u||_q||v||_0,$$

which gives

$$||[Q(D),\psi]u||_0^2\leq c||u||_q^2.$$

By (4.5) we conclude that $(1+|\cdot|^2)^{q/2}\in\Sigma_0(P,1)$ and using Proposition 1.5.B it follows that

$$||u||_q^2=||(1-\Delta)^{q/2}u||_0^2\leq\varepsilon||P(D)u||_0^2+c(\varepsilon)||u||_0^2$$

holds for any $\varepsilon>0$. This implies the theorem.

5. The Dirichlet Problem

We want to solve a generalized Dirichlet problem and later an associated boundary value problem for a pseudo differential operator $L(x, D)$ of the initial form given in (2.1). First let us formulate the weak Dirichlet problem:

PROBLEM 5.1. Let $\Omega\subset\boldsymbol{R}^n$ be an open set and $L(x, D)$ be a pseudo differential operator of form (2.1). Moreover let $f\in L^2(\Omega)$ be a given function. Find all elements $u\in H_P^{0,1}(\Omega)$ such that

$$B(u, \varphi)=(f, \varphi)_0 \tag{5.1}$$

holds for all $\varphi\in C_0^\infty(\Omega)$, where B is the sesquilinear form (2.3).

Note, that for a certain class of pseudo differential operators with smooth symbols belonging to some class $S^t_{\rho,\delta}(\Omega)$, Problem 5.1 is the formulation of the weak Dirichlet problem given by H. Kumano-go and C. Tsutsumi in [13], p. 165. For translation invariant pseudo differential operators the weak Dirichlet-problem was handled in [3] and in a special case in [9]. In [11] we handled Problem 5.1 for pseudo differential operators the symbols of which are negative definite functions.

Theorem 5.1. *Let $\Omega\subset\boldsymbol{R}^n$ be an open set, $f\in L^2(\Omega)$ and $L(x, D)$ a pseudo differential operator satisfying L.1—L.2. Moreover, assume that with two constants $c_0>0$ and $c_1\geq 0$ the inequality*

$$\operatorname{Re} B(u, u)\geq c_0\|u\|_P^2-c_1\|u\|_0^2 \tag{5.2}$$

holds for all $u\in H_P^{0,1}(\Omega)$.

A. If $c_1=0$, then Problem 5.1 has a unique solution for all $f\in L^2(\Omega)$.

B. If $c_1>0$ and if the embedding of $H_P^{0,1}(\Omega)$ into $L^2(\Omega)$ is compact, then for Problem 5.1 Fredholm's alternative holds.

Proof. *A.* Since B is continuous on $H_P^{0,1}(\Omega)$, the statement is nothing but the statement of the Lax-Milgram theorem (see [18], p. 92).

B. This part of the theorem follows using the continuity of B on $H_P^{0,1}(\Omega)$, Garding's inequality (5.2) and the compactness of the embedding of $H_P^{0,1}(\Omega)$ into $L^2(\Omega)$ by the same arguments as Theorem I.14.6 in [4].

By Corollary 3.1 and Theorem 3.2 we get

Corollary 5.1. *Suppose that $H_P^{0,1}(\Omega)$ is compactly embedded into $L^2(\Omega)$ and that the operator $L(x, D)$ fulfills either L.3 and L.4′ or L.5 to L.7. Then for the weak Dirichlet problem Fredholm's alternative holds.*

Let us also mention

Corollary 5.2. *Let $P\in\underline{P}$ and suppose that the embedding of $H_P^{0,1}(\Omega)$ into $L^2(\Omega)$ is compact. Then the weak Dirichlet problem posed for the operator $P^2(D)$ has a unique solution.*

This follows from Proposition 1.3 and the Lax-Milgram theorem. For general $P \in \underline{P}$ we cannot decided whether or whether not the embedding of $H_P^{0,1}(\Omega)$ into $L^2(\Omega)$, $\Omega \subset\subset \boldsymbol{R}^n$, is comapct. However there exist several sufficient conditions, see [7], Theorem 10.1.10 and for polynomials [12], Theorem 4. An obvious criterion is

Proposition 5.1. *Suppose that $H_P^{0,1}(\Omega)$ is continuosly embedded into the space $H^{0,t}(\Omega)$ for some $t>0$. Then the embedding of $H_P^{0,1}(\Omega)$ into $L^2(\Omega)$ is compact, provided $\lambda^{(n)}(\Omega)<\infty$.*

The proof of Proposition 5.1 follows from the fact that for $\lambda^{(n)}(\Omega)<\infty$ the space $H^{0,t}(\Omega)$, $t>0$, is compactly embedded into $L^2(\Omega)$.

REMARK. Note that all considerations in this paper remains true if $L(x, D)$ is substituted by the operator

$$K(x, D) = L(x, D)+\sum_{\nu=1}^{3} \sum_{l,k=1}^{m_\nu} R_k^{(\nu)}(D) b_{kl}^{(\nu)}(x) S_l^{(\nu)}(D)$$

where $m_\mu \in \boldsymbol{N}$ for $\mu=1, 2, 3$, and we pose the following conditions on $K(x, D)$.

Let $P \in \underline{P}$ be fixed, then we assume:
K.0.: L(x, D) satisfies the condition L.1 and L.2;
K.1.: $R_k^{(1)}, S_l^{(2)} \in \Sigma(P, 1)$ for all $1 \leq k \leq m_1$ and $1 \leq l \leq m_2$;
K.2.: $R_k^{(2)}, R_{k_3}^{(3)}, S_l^{(1)}, S_{l_3}^{(3)} \in \Sigma_0(P, 1)$ for all $1 \leq l \leq m_1$, $1 \leq k \leq m_2$ and $1 \leq k_3$, $l_3 \leq m_3$;
K.3.: $b_{k_\nu l_\nu}^{(\nu)} \in L^\infty(\boldsymbol{R}^n)$ for all $1 \leq k_\nu$, $l_\nu \leq m_\nu$ for $\nu=1, 2, 3$.
In that case $L(x, D)$ becomes a generalized principal part of $K(x, D)$.

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Niels Jacob
Mathematisches Institut
Universität Erlangen Nürnberg
Bismarckstraße 1 1/2
D-8520 Erlangen
BRDeutschland