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Osaka J. Math.
26 (1989), 759–773

CODEGREE OF SIMPLE LIE GROUPS

Dedicated to Professor Shôrô Araki on his 60th birthday

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(Received December 13, 1988)

1. Introduction

Let G be a connected Lie group such that $\pi_3(G)\cong \boldsymbol{Z}$. Connected simple Lie groups satisfy the condition. Let $f\colon S^3\to G$ be a map generating $\pi_3(G)$. The purpose of this note is to study the image of

$$f^*\colon \pi_s^3(G) \to \pi_s^3(S^3)\cong \boldsymbol{Z}$$

where $\pi_s^*(\ )$ is the reduced stable cohomotopy. The order of the cokernel of f^* which is finite by Proposition 2.5 below is called the codegree of G and denoted by $cd(G)$. Since G and its maximal compact subgroup have the same homotopy type, it suffices to study the case that G is compact. We denote by $cd_p(G)$ the exponent of the prime number p in the prime decomposition of $cd(G)$. Our result is

Theorem. *Let G be a compact simply connected simple Lie group. Then*
(1) $cd_p(G)=0$ *if and only if G is p-regular in the sense of Serre* [22];
(2) $cd_p(G)$ *can be determined except for the following cases*:

$$\begin{array}{ll} E_8,\ E_7,\ E_6,\ F_4,\ Spin(n)\ (n\geqq 9) & \text{for } p=2;\\ E_8,\ E_7,\ E_6,\ F_4 & \text{for } p=3^{*)};\\ E_8 & \text{for } p=5\,. \end{array}$$

More precise statement and upper bounds of the excluded cases in (2) can be seen in 3.3, 4.1, and 4.2 below. For non simply connected groups, see 2.7 below.

In §2, we prove general properties of codegrees, though some of them are not used in this note. In §3, for classical groups G, we prove that $cd(G)$ is equal to the codegree of some canonical vector bundle. As a consequence, $cd(SU(n))$ can be determined from Crabb-Knapp [4] and then $cd(Sp(n))$ and odd components of $cd(Spin(n))$ can also be determined (see Proposition 3.3).

*) A. Kono informed me that we could prove $cd_3(E_6)=cd_3(F_4)=2$ by using Harper's mod 3 decomposition of F_4.

In §4, we study relations of $cd(G)$ and p-regularity [13, 22], quasi p-regularity [15], and mod p decomposability [17] of G.

In this note all cohomology theories are reduced.

I am indebted to H. Minami for useful conversation and suggestions on Lemma 4.4; to A. Kono for informing me that any map $E_8 \to U(240)$ factorizes through $U(124)$; and to M. Mimura for information on the topology of E_6 and F_4.

2. Codegree of a map

Let $f: S^n \to X$ be a based map (or more generally stable map), where X is a connected finite CW-complex having a base point. Let E be a generalized reduced cohomology theory, that is, E is a spectrum, and let $m \in \boldsymbol{Z}$. The E^m-codegree of f, $cd(f; E^m)$, is defined to be 0 or the cardinal number of the cokernel of

$$E^{m+n}(X) \xrightarrow{f^*} E^{m+n}(S^n) \xrightarrow{q} E^{m+n}(S^n)/\mathrm{Tor} \cong E^m(S^0)/\mathrm{Tor}$$

according as $q \circ f^*$ is 0 or not, where Tor denotes the torsion subgroup and q is the quotient homomorphism. We set $cd(f) = cd(f; \pi_s^0)$ which is most interesting, because, by 2.3(2) below, $cd(f; E^m) \mid cd(f)$ provided there exists a natural transformation $\pi_s^* \to E^{*+m}$ inducing an isomorphism $\pi_s^0(S^0) \cong E^m(S^0)/\mathrm{Tor}$. Here $a \mid b$ means that an integer b is an integral multiple of an integer a.

Remark 2.1. Our codegree of a map $f: S^n \to X$ is a multiple of Gottlieb's codegree of $f_*: H_*(S^n; \boldsymbol{Z}) \to H_*(X; \boldsymbol{Z})$ and degree of $f^*: H^*(X; \boldsymbol{Z}) \to H^*(S^n; \boldsymbol{Z})$ [7].

Proposition 2.2 ([21; 2.1]). *For any integer k we have*

(1) *If $f^* = 0: H^n(X; \boldsymbol{Z}) \to H^n(S^n; \boldsymbol{Z})$, then the image of $f^*: E^k(X) \to E^k(S^n)$ is contained in* $\mathrm{Tor}(E^k(S^n))$, *the torsion subgroup of $E^k(S^n)$.*

(2) *If $E^k(S^n) \otimes \boldsymbol{Q} \neq 0$, then the converse of* (1) *holds.*

Proposition 2.3. (1) *$cd(f; E^m) = cd(f; E[0, \infty)^m)$ if $m \leqq n$ and X is $(m-1)$-connected, where $E[0, \infty) \to E$ is the (-1)-connective covering.*

(2) *If $\theta: E^* \to F^{*+k}$ is a natural transformation of cohomology theories of degree k such that $\theta_*: E^m(S^0)/\mathrm{Tor} \cong F^{m+k}(S^0)/\mathrm{Tor}$, then $cd(f; F^{m+k}) \mid cd(f; E^m)$.*

(3) *Suppose that there exist stable maps $g: X \to X'$ and $f': S^n \to X'$ such that $g \circ f = f' \circ k\iota_n$, where X' is a connected finite based CW-complex, $\iota_n: S^n \to S^n$ is the identity map, and $k \in \boldsymbol{Z}$. Then $cd(f) \mid k \cdot cd(f')$. If $k = \pm 1$, then $cd(f; E^m) \mid cd(f'; E^m)$, in particular, $cd(f; E^m) \mid cd(g \circ f; E^m)$.*

(4) *Suppose that $E^m(S^0) \otimes \boldsymbol{Q} \neq 0$. Then $cd(f; E^m) = 0$ if and only if $f^* = 0$: $H^n(X; \boldsymbol{Z}) \to H^n(S^n; \boldsymbol{Z})$.*

Proof. We have (4) by 2.2. Others can be easily proved. We omit the details.

We call $f\colon S^n \to X$ orientable if $f^*\colon H^n(X;\boldsymbol{Z})/\mathrm{Tor} \cong H^n(S^n;\boldsymbol{Z})$, and call f quasi-orientable if $f^*\colon H^n(X;\boldsymbol{Z})/\mathrm{Tor} \to H^n(S^n;\boldsymbol{Z})$ is a non trivial injection. In both cases an element $U(f) \in H^n(X;\boldsymbol{Z})$ which generates $H^n(X;\boldsymbol{Z})/\mathrm{Tor}$ is called an orientation class of f.

EXAMPLE 2.4. Let V be an n-dimensional vector bundle over a connected finite CW-complex Y, and let $f\colon S^n = T(V;\{y\}) \to T(V;Y)$ be the inclusion of Thom spaces for some point y of Y. Then f is quasi-orientable if and only if f is orientable if and only if V is orientable. Put $cd(V) = cd(f)$. See [20].

Proposition 2.5. *Let G be a compact connected Lie group such that $\pi_3(G) \cong \boldsymbol{Z}$ and let $f\colon S^3 \to G$ be a generator of $\pi_3(G)$. Then $cd(G) \neq 0$. Also f is quasi-orientable if and only if* dim $Z(G) \leqq 2$, *where $Z(G)$ is the centre of G. Moreover f is orientable if G is simply connected.*

Proof. As is well-known, G has a covering group $q\colon T^k \times G' \to G$ whose kernel is a central finite subgroup of $T^k \times G'$, where G' is a compact simply connected simple Lie group, and $T^k = (S^1)^k$ is the k-dimensional toral group with $k =$ dim $Z(G)$. Let $i\colon G' = \{1\} \times G' \to T^k \times G'$ be the inclusion and let $f'\colon S^3 \to G'$ be a map such that $q \circ i \circ f' \simeq f$. Then $q^*\colon H^*(G;\boldsymbol{Q}) \cong H^*(T^k \times G';\boldsymbol{Q})$ and $f'^*\colon H^3(G';\boldsymbol{Z}) \cong H^3(S^3;\boldsymbol{Z})$. Hence $f^* \neq 0\colon H^3(G;\boldsymbol{Q}) \to H^3(S^3;\boldsymbol{Q})$, therefore $cd(G) \neq 0$ by 2.3 (4). Also we have

$$\begin{aligned} f \text{ is quasi-orientable } &\Leftrightarrow f^*\colon H^3(G;\boldsymbol{Q}) \cong H^3(S^3;\boldsymbol{Q}) \\ &\Leftrightarrow i^*\colon H^3(T^k \times G';\boldsymbol{Q}) \cong H^3(G';\boldsymbol{Q}) \\ &\Leftrightarrow k \leqq 2\,. \end{aligned}$$

The last assertion of 2.5 follows from the fact that if G is simply connected then G is 2-connected. This completes the proof.

Proposition 2.6. *Let $g\colon X \to Y$ be a map between connected finite CW-complexes. Suppose that $H^n(Y;\boldsymbol{Q}) \cong \boldsymbol{Q}$ and $f\colon S^n \to X$ $(n \geqq 1)$ is quasi-orientable, and that there exists a stable map $\tau\colon Y_+ \to X_+$ (Z_+ is the union of Z and the disjoint base point) such that $\tau^* g^*\colon H^n(Y_+;\boldsymbol{Z}) \to H^n(Y_+;\boldsymbol{Z})$ is the multiplication by an integer χ. Then $cd(g \circ f) \mid \chi \cdot cd(f)$.*

Proof. Consider the following commutative diagram in which h are the Hurewicz homomorphisms.

$$\begin{tikzcd} \pi_s^n(S^n) \arrow[r, phantom, "\cong"] \arrow[d, "h"', "\cong"] & \pi_s^n(S^n_+) \arrow[d, "h"', "\cong"] & \pi_s^n(X_+) \arrow[l, "f^*"'] \arrow[d, "h"] \arrow[r, shift right, "\tau^*"'] & \pi_s^n(Y_+) \arrow[l, shift right, "g^*"'] \arrow[d, "h"] \\ H^n(S^n;\boldsymbol{Z}) \arrow[r, phantom, "\cong"] & H^n(S^n_+;\boldsymbol{Z}) & H^n(X_+;\boldsymbol{Z}) \arrow[l, "f^*"'] \arrow[r, shift right, "\tau^*"'] & H^n(Y_+;\boldsymbol{Z}) \arrow[l, shift right, "g^*"'] \end{tikzcd}$$

Let $a\in H^n(X_+; \boldsymbol{Z})$, $b\in H^n(Y_+; \boldsymbol{Z})$ be generators of respective free parts. There exist integers k, l, m such that

$$f^*(a) = k[S^n],$$
$$g^*(b) \equiv la \bmod \mathrm{Tor}(H^n(X_+; \boldsymbol{Z})),$$
$$\tau^*(a) \equiv mb \bmod \mathrm{Tor}(H^n(Y_+; \boldsymbol{Z}))$$

where $[S^n]$ is a generator of $H^n(S^n; \boldsymbol{Z})$. We have $\chi b=\tau^* g^*(b)\equiv lmb$ mod $\mathrm{Tor}(H^n(Y_+; \boldsymbol{Z}))$, hence $\chi=lm$. By definition, there exists $\beta\in\pi_s^n(X_+)$ with $f^*h(\beta)=h f^*(\beta)=cd(f)[S^n]$. Hence $h(\beta)\equiv(cd(f)/k)a$ mod $\mathrm{Tor}(H^n(X_+; \boldsymbol{Z}))$, and so $h\tau^*(\beta)=\tau^*h(\beta)\equiv(m\cdot cd(f)/k)b$ mod $\mathrm{Tor}(H^n(Y_+; \boldsymbol{Z}))$, and therefore $hf^* g^* \tau^*(\beta)=f^* g^* h\tau^*(\beta)=f^* g^*((m\cdot cd(f)/k)\, b)=lm\cdot cd(f)\,[S^n]=\chi\cdot cd(f)\,[S^n]$. Thus $cd(g\circ f)\,|\,\chi\cdot cd(f)$ as desired.

Corollary 2.7. *Let $q\colon G'\to G$ be a covering group such that G' and G are compact connected Lie groups, $\pi_3(G)\cong\boldsymbol{Z}$, and* $\dim Z(G)\leqq 2$. *Then*

$$cd(G')\,|\,cd(G)\,|\,m\cdot cd(G')$$

where m is the order of the kernel of q.

Proof. Note that m is finite and $\dim Z(G')=\dim Z(G)$. By [2], there exists a transfer map $\tau\colon G_+\to G'_+$ whic is a stable map such that $\tau^* q^*\colon H^*(G_+; \boldsymbol{Z})\to H^*(G_+; \boldsymbol{Z})$ is the multiplication by m. Then the result follows from 2.5 and 2.6.

We recall K-theory codegree from Crabb-Knapp [3]. Let KF denote the complex K-theory K ($F=\boldsymbol{C}$) and the real K-theory KO ($F=\boldsymbol{R}$). If $f\colon S^n\to X$ is quasi-orientable, then we define

$$cd^{KF}(f) = \mathrm{Min}\,\{m>0;\ m\cdot U(f)\in\mathrm{Im}(ch_F\colon KF^n(X)\to H^*(X; \boldsymbol{Q}))\}$$

where $ch_{\boldsymbol{C}}=ch$, the Chern character, $ch_{\boldsymbol{R}}$ is the composition of the complexification and ch, and Min and Im denote the minimum and the image, respectively. If f is non quasi-orientable, then we define $cd^{KF}(f)=0$.

Proposition 2.8 ([20; §2]). *If $f\colon S^n\to X$ is quasi-orientable, then $cd^{KF}(f)$ is well-defined and satisfies*

$$cd(f; KF^0)\,|\,cd(f; H^0)\, cd^{KF}(f) \quad \text{and} \quad cd^{KF}(f)\,|\,cd(f)\,.$$

Next we consider J-theory codegrees

$$cd^{jF'}(f) = \Pi_p\, cd(f; jF'\langle p\rangle^0)\,,$$
$$cd^{jF}(f) = \Pi_p\, cd(f; jF\langle p\rangle^0)\,.$$

See [3], [4] or [20] for their definitions. We denote the exponents of p in the prime decompositions of $cd(f)$, $cd^{KF}(f)$, $cd^{jF}(f)$, $\cdots$ by $cd_p(f)$, $cd_p^{KF}(f)$, $cd_p^{jF}(f)$, $\cdots$, respectively. We use also the notations $cd(G; E^m)$, $cd^{KF}(G)$, $cd_p^{KF}(G)$, $\cdots$ in canonical sense. Notice that the p-components of $cd^{jF'}(f)$ and $cd^{jF}(f)$ are $cd(f; jF'\langle p\rangle^0)$ and $cd(f; jF\langle p\rangle^0)$ respectively.

The following theorem can be proved by modifying the proof of Theorem 3.3 of [20]. So we omit the details.

Theorem 2.9. *Suppose that* $f\colon S^n\to X$ *is quasi-orientable.*

(1) $cd^K(f)\,|\,cd^{KO}(f)\,|\,2cd^K(f)$.

(2) $cd^{jF'}(f)\,|\,cd^{jF}(f)\,|\,cd(f)$.

(3) *If f is orientable, then* $cd^{KF}(f)\,|\,cd^{jF'}(f)$.

(4) *If* $H^{n-1}(X; \boldsymbol{Z}_{(p)})=0$, *then* $cd_p^{j\boldsymbol{C}'}(f)=cd_p^{j\boldsymbol{C}}(f)$.

(5) $cd^{j\boldsymbol{C}'}(f)\,|\,cd^{j\boldsymbol{R}'}(f)$.

(6) $p>2$.

(6–1) $cd_p^{j\boldsymbol{C}'}(f)=cd_p^{j\boldsymbol{R}'}(f)$, $cd_p^{j\boldsymbol{C}}(f)=cd_p^{j\boldsymbol{R}}(f)$, *and they are equal if X is $(n-1)$-connected.*

(6–2) *If $H^*(X; \boldsymbol{Z})$ has no p-torsion and f is orientable, then* $cd_p^{KF}(f)=cd_p^{jF'}(f)$.

(7) $p=2$.

(7–1) *If $H^*(X; \boldsymbol{Z})$ has no 2-torsion and f is orientable, then* $cd_2^K(f)=cd_2^{j\boldsymbol{C}'}(f)$ *and* $cd_2^{j\boldsymbol{R}'}(f)\leqq 1+cd_2^{KO}(f)$.

(7–2) *If $H^*(X; \boldsymbol{Z})$ and $KO^n(X)$ have no 2-torsion and f is orientable, then* $cd_2^{j\boldsymbol{R}'}(f)=cd_2^{KO}(f)$.

Given $f\colon S^n\to X$ with $n\geqq 1$, we define abelian groups $A_1, \cdots, A_{11}$ by using (f, X) instead of (i, T) in [20; Lemma 4.7]. For example $A_1=\mathrm{Ker}(\gamma_*\colon KO[4, \infty)^n(X)\to KO[0, \infty)^n(X))$, where $\gamma\colon KO[4, \infty)\to KO[0, \infty)$ is the connective fibering. Then the following lemma can be proved by almost the same proof as [20; Lemma 4.7]. We omit the details.

Lemma 2.10. (1) $A_1\cong A_2\cong A_3\supset A_5\supset A_6\cong\cdots\cong A_{10}$.

(2) *If* $H^{n-1}(X; \boldsymbol{Z})=0$, *then* $A_3=A_5$.

(3) *If X is $(n-1)$-connected, then* $A_3\cong A_4\cong A_5$.

(4) *If* $H^n(X; \boldsymbol{Z}_2)\cong\boldsymbol{Z}_2$ *and Sq^2 is non trivial on* $H^n(X; \boldsymbol{Z}_2)$, *then* $A_5=A_6$.

(5) *If* $f^*\colon H^n(X; \boldsymbol{Z}_2)\cong H^n(S^n; \boldsymbol{Z}_2)$ *and* $f^*\colon KO[1, \infty)^{n-1}(X)\to KO[1, \infty)^{n-1}(S^n)$ *is surjective, then Sq^2 is trivial on* $H^n(X; \boldsymbol{Z}_2)$ *and* $b_{1*}\colon KO[2, \infty)^n(X)\to KO[1, \infty)^n(X)$ *is injective, where* $b_1\colon KO[2, \infty)\to KO[1, \infty)$ *is the connective fibering. If moreover X is $(n-1)$-connected, then* $A_1=A_{10}\cong A_{11}$, *so* $A_3=A_5=A_6$.

(6) *If* $i^*\colon H^{n+1}(X/X^n; \boldsymbol{Z}_2)\cong H^{n+1}(X^{n+1}/X^n; \boldsymbol{Z}_2)$, *then*
$A_8\cong\mathrm{Coker}(i^*\colon KO^{n-1}(X/X^n)\to KO^{n-1}(X^{n+1}/X^n))$.

Proposition 2.11. *If G is a compact simply connected simple Lie group, then* $\gamma_*\colon KO[4, \infty)^3(G)\to KO[0, \infty)^3(G)$ *is injective, hence* $cd^{j\boldsymbol{R}'}(G)=cd^{j\boldsymbol{R}}(G)$.

Proof. We have $A_1 \cong \cdots \cong A_5 \supset A_6 \cong \cdots \cong A_{10}$ by 2.10, since G is 2-connected. As is well-known, $H^4(G; \boldsymbol{Z}_2)=0$, hence $A_{10}=0$, and if $G \neq Sp(n)$ for every n then Sq^2 is non trivial on $H^3(G; \boldsymbol{Z}_2)$, hence $A_5=A_6$ by 2.10 (4), so $A_1 \cong A_{10}=0$ and γ_* is injective on $KO[4, \infty)^3(G)$. On the other hand, we have the following commutative diagram, where f: $S^3=Sp(1) \subset Sp(n)$.

$$\begin{array}{ccccccc}
KO[1, \infty)^2(Sp(n)) & = & KO^2(Sp(n)) & = & [Sp(n), Sp/U] & \xleftarrow{q_*} & [Sp(n), Sp] \\
\downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\
KO[1, \infty)^2(S^3) & = & KO^2(S^3) & = & [Sp(1), Sp/U] & \underset{q_*}{\leftarrow} & [Sp(1), Sp]
\end{array}$$

Since $\pi_2(U)=0$, the above lower q_* is surjective. Let i: $Sp(n) \subset Sp$ be the inclusion. Then $f^*(i)$ is a generator of $\pi_3(Sp) \cong \boldsymbol{Z}$. Hence the fourth f^* is surjective, and so is the first f^*. Thus we have $A_1 \cong A_{10}$ by 2.10(5), so $A_1=0$ and γ_* is injective on $KO[4, \infty)^3(Sp(n))$. This completes the proof.

Recall from Serre [22] that a connected finite CW-complex Y is p-regular if there exists a map g: $\Pi_i S^{n(i)} \rightarrow Y$ such that g_*: $\pi_k(\Pi S^{n(i)}) \rightarrow \pi_k(Y)$ has the kernel and the cokernel which are finite and have orders prime to p for every k, that is, g is a p-equivalence. If moreover Y is a simply connected p-regualr H-space, then there exists a p-equivalence n: $Y \rightarrow \Pi S^{n(i)}$ [16; Theorem 1.7].

Proposition 2.12. *If a compact simply connected simple Lie group G is p-regular, then $cd_p(G)=0$.*

Proof. Suppose that a compact simply connected simple Lie group G is p-regular. Then there exists a p-equivalence g: $G \rightarrow \Pi_i S^{n(i)}$. Hence there is only one $n(i)$ with $n(i)=3$. Let q: $\Pi S^{n(i)} \rightarrow S^3$ be the projection and f: $S^3 \rightarrow G$ be a generator of $\pi_3(G)$. Then the degree of $q \circ g \circ f$ is prime to p. Hence $cd_p(G)=0$.

3. $cd(G)$ for classical groups

Proposition 3.1. (1) *$cd(U(n))=cd(SU(n))=cd(\xi_{n-1})$ $(n \geqq 2)$, where ξ_{n-1} is the Hopf bundle over the complex projective $(n-2)$-space $P(\boldsymbol{C}^{n-1})$.*

(2) *$cd(Sp(n))=cd(\zeta_n)$ $(n \geqq 1)$, where ζ_n is the vector bundle over the quaternionic projective $(n-1)$-space $P(\boldsymbol{H}^n)$ associated with the canonical principal S^3-bundle $S^{4n-1} \rightarrow P(\boldsymbol{H}^n)$ and the adjoint representation of S^3 on its Lie algebra.*

(3) *$cd(SO(n))$ $(n \geqq 5)$ is the order of the cokernel of i^*: $\pi_s^3(SO(n)) \rightarrow \pi_s^3(SO(3))$, where i: $SO(3) \subset SO(n)$.*

(4) *$cd(SO(n))=cd(ad_2 \oplus can_2)$ $(n \geqq 5)$, where ad_2 (resp., can_2) is 1 (resp., 2) dimensional vector bundle associated with the principal $O(2)$-bundle $O(n-1)/O(n-3) \rightarrow O(n-1)/(O(2) \times O(n-3))$ and the non-trivial representation $O(2) \rightarrow O(1)$ (resp., the canonical representation $O(2)$ on* $\mathrm{Hom}_{\boldsymbol{R}}(\boldsymbol{R}^2, \boldsymbol{R})$).

Proof. We have $cd(U(n))=cd(SU(n))$, since $U(n)$ is homeomorphic to $S^1\times SU(n)$. By James [12; (7.11)], the inclusions $\Sigma P(\boldsymbol{C}^n)=\Sigma T(\xi_{n-1}; P(\boldsymbol{C}^{n-1}))\subset SU(n)$ and $Q_n=T(\zeta_n; P(\boldsymbol{H}^n))\subset Sp(n)$ are stable retracts with which the compositions of $S^3=\Sigma P(\boldsymbol{C}^2)\subset\Sigma P(\boldsymbol{C}^n)$ and $S^3=Q_1\subset Q_n$ generate $\pi_3(SU(n))$ and $\pi_3(Sp(n))$ respectively. We then obtain (1) and (2).

To prve (3), let $n\geqq 5$ and i': $S^3=Spin(3)\to Spin(n)$ be the inclusion. Then $\mathrm{Im}(i'_*: \pi_3(S^3)\to\pi_3(Spin(n)))=2\pi_3(Spin(n))$, since $\pi_3(Spin(n)/Spin(3))\cong\boldsymbol{Z}_2$. Hence there exists f': $S^3\to Spin(n)$ which generates $\pi_3(Spin(n))$ and $i'\simeq f'\circ 2\iota_3$. Let q_m: $Spin(m)\to SO(m)$ be the double covering and set $f=q_n\circ f'$: $S^3\to SO(n)$. Then f generates $\pi_3(SO(n))$, and $f\circ 2\iota_3=q_n\circ f'\circ 2\iota_3\simeq q_n\circ i'=i\circ q_3$, hence $2\cdot cd(SO(n))=cd(i\circ q_3)$. On the other hand, we have $\pi_s^3(SO(3))\cong\boldsymbol{Z}$ and $\mathrm{Im}(q_3^*: \pi_s^3(SO(3))\to\pi_s^3(S^3))=2\pi_s^3(S^3)$. Hence $cd(SO(n))$ is the order of the cokernel of i^*: $\pi_s^3(SO(n))\to\pi_s^3(SO(3))$ and this proves (3).

To prove (4), we prepare some notations. Let ad_k denote the adjoint representation of $O(k)$ on its Lie algebra, can_k denote the canonical representation of $O(k)$ on $\mathrm{Hom}_{\boldsymbol{R}}(\boldsymbol{R}^k, \boldsymbol{R})$, and $V_{n,k}=O(n)/O(n-k)\to G_{n,k}=O(n)/(O(k)\times O(n-k))$ denote the canonical principal $O(k)$-bundle. We may form the associated vector bundles over $G_{n,k}$ for ad_k and can_k; we denote them by ad_k and can_k for saving of notations. Note that dim $ad_k=k(k-1)/2$ and dim $can_k=k$.

By Miller [14], there exists a stable homotopy equivalence which are natural with respect to n:

$$SO(n)\simeq \vee_{k=1}^{n-1} T_k$$

where $T_k=T(ad_k\oplus can_k; G_{n-1,k})$. As is easily seen, we have $T_1=P(\boldsymbol{R}^n)$, the real projective $(n-1)$-space, the 3-skeleton of T_2 is $S^3=T(ad_2\oplus can_2; G_{2,2})$, and T_k $(k\geqq 3)$ is 5-connected. The inclusion i: $SO(3)\to SO(n)$ corresponds under the above equivalence to

$$P(\boldsymbol{R}^3)\vee S^3\subset P(\boldsymbol{R}^n)\vee T_2\subset \vee_{k=1}^{n-1} T_k\,.$$

Hence $\mathrm{Im}(i^*: \pi_s^3(SO(n))\to\pi_s^3(SO(3)))\cong\mathrm{Im}(i^*: \pi_s^3(T_2)\to\pi_s^3(S^3))$, where i is the inclusion, thus $cd(ad_2\oplus can_2)$ is equal to the order of the cokernel of i^*: $\pi_s^3(SO(n))\to\pi_s^3(SO(3))$. Therefore we obtain (4) from (3). This completes the proof.

REMARK 3.2. Of course we can prove (1) and (2) of 3.1 by using Miller's stable splitting [14] of $SU(n)$ and $Sp(n)$.

Proposition 3.3. (1) *For $n\geqq 3$, we have*

$$cd_p(SU(n))=\begin{cases}\mathrm{Max}\,\{i\mid p^i<n\} & p>2\\ \mathrm{Max}\,\{i\mid 2^i\leqq n\} & p=2\,.\end{cases}$$

(2) *For $p>2$, we have*

$$cd_p(SO(2n+1)) = cd_p(Spin(2n+1)) = cd_p(Sp(n)) \quad (n \geqq 1),$$
$$cd_p(SO(2n)) = cd_p(Spin(2n)) = cd_p(Spin(2n-1)) \quad (n \geqq 3).$$

(3) $cd_2(SU([n/2])) \leqq cd_2(SO(n)) \leqq cd_2(SU([n/2]))+2$ ($n \geqq 6$), *where* $[n/2]$ *is the largest integer smaller than or equal to* $n/2$.

(4) $cd(Sp(n))=cd(SU(2n))$.

Proof. By 3.1(1), $cd(U(n))=cd(SU(n))=cd(\xi_{n-1})$ for $n \geqq 2$. Crabb-Knapp [4; 5.16] has determined $cd(\xi_{n-1})$ as in (1). Moreover they proved that $cd(\xi_m)=cd^{jR}(\xi_m)$. Notice from Theorem 1.1 of [20] and 2.9 that $cd^{jR}(\xi_m)=cd^{KO}(\xi_m)$.

To prove (2), let p be an odd prime. Then $cd_p(SO(n))=cd_p(Spin(n))$ by 2.7, and

$$Spin(2n+1) \simeq_p Sp(n),$$
$$Spin(2n) \simeq_p Spin(2n-1) \times S^{2n-1}$$

by Harris [8]. Hence (2) follows by the technique which will be used in the proof of Theorem 4.1 below.

Concerning the natural inclusions we have

$$\mathrm{Coker}(\pi_3(SO(n)) \to \pi_3(U(n))) \cong \pi_3(U/O) \cong \boldsymbol{Z}_2 \quad (n \geqq 5),$$
$$\mathrm{Coker}(\pi_3(U(n)) \to \pi_3(SO(2n))) \cong \pi_3(O/U) = 0 \quad (n \geqq 3).$$

It then follows from 2.3(3) that

$$cd(SO(n)) \mid 2 \cdot cd(U(n)) \quad (n \geqq 5),$$
$$cd(U(n)) \mid cd(SO(2n)) \quad (n \geqq 3)$$

and so we obtain (3).

The rest of this section is devoted to the proof of (4). We use Thom classes which are defined by Atiyah, Bott and Shapiro [1]. Let $g\colon P(\boldsymbol{C}^{2n}) \to P(\boldsymbol{H}^n)$ be the usual fibration, $h\colon T(\xi_{2n}) \to T(\xi_{2n}^2)$ be the map induced by the fibre preserving map $\xi_{2n} \to \xi_{2n}^2 = \xi_{2n} \otimes_c \xi_{2n}$ which sends z into z^2. Then $g^* \zeta_n \cong \xi_{2n}^2 \oplus 1$ [26] and the degree of h on fibres is 2. Let γ be the canonical quaternionic line bundle over $P(\boldsymbol{H}^n)$, $\beta \in K(S^2)$ be a generator, and $r\colon K \to KO$ be the real restriction. Set $\theta_c = \gamma - 2 \in K(P(\boldsymbol{H}^n))$, $\mu = \xi_{2n+1} - 1 \in K(P(\boldsymbol{C}^{2n+1}))$, $\mu_3 = r(\beta^3 \mu) \in KO^{-6}(P(\boldsymbol{C}^{2n+1}))$, $\mu_0 = r(\mu) \in KO(P(\boldsymbol{C}^{2n+1}))$, and $t = c_1(\xi_{2n+1}) \in H^2(P(\boldsymbol{C}^{2n+1}); \boldsymbol{Z})$. Recall from [24] that $K(P(\boldsymbol{H}^n)_+) = \boldsymbol{Z}[\theta_C]/(\theta_C^n)$ and the complexification $c\colon KO(P(\boldsymbol{H}^n)_+) \to K(P(\boldsymbol{H}^n)_+)$ is injective and has the image generated by $e_i\, \theta_C^i$ $(0 \leqq i \leqq n-1)$, where e_i is equal to 1 if i is even and equal to 2 if i is odd. Let $\theta_R^i \in KO(P(\boldsymbol{H}^n)_+)$ be the element with $c(\theta_R^i) = e_i\, \theta_C^i$. Notice that θ_R^i does not mean a power of some element ($i \geqq 2$). Recall from [6] that $KO^{-6}(P(\boldsymbol{C}^{2n+1}))$ is free module with basis $\mu_3, \mu_3 \mu_0, \cdots, \mu_3 \mu_0^{n-1}$. Note that $cd^{KO}(\xi_{2n})$ is equal to

$$\mathrm{Min}\{m>0 \mid mt \in \mathrm{Im}(ch_R\colon KO^{-6}(P(\boldsymbol{C}^{2n+1})) \to H^*(P(\boldsymbol{C}^{2n+1}); \boldsymbol{Q}))\}$$

and that $ch_{\boldsymbol{R}}(\mu_3\,\mu_0^i)=(e^t-e^{-t})\,(e^t+e^{-t}-2)^i$.

Consider the commutative diagram

$$\begin{tikzcd}
 & & KF(P(\boldsymbol{C}^{2n})_+) \arrow[d, "\cong"'] & KF(P(\boldsymbol{H}^n)_+) \arrow[l, "g^*"'] \arrow[d, "\cong"', "\phi_{KF}"] \\
KF^{-6}(P(\boldsymbol{C}^{2n+1})) \arrow[d, "ch_F"] & KF(T(\xi_{2n}\oplus 6)) \arrow[l, "\sigma"', "\cong"] \arrow[d, "ch_F"] & KF(T(\xi_{2n}^2\oplus 6)) \arrow[l, "h^*"'] \arrow[d, "ch_F"] & KF(T(\zeta_n\oplus 5)) \arrow[l, "g^*"'] \arrow[d, "ch_F"] \\
H^{*-6}(P(C^{2n+1});\boldsymbol{Q}) & H^*(T(\xi_{2n}\oplus 6);\boldsymbol{Q}) \arrow[l, "\sigma"', "\cong"] & H^*(T(\xi_{2n}^2\oplus 6);\boldsymbol{Q}) \arrow[l, "h^*"'] & H^*(T(\zeta\oplus 5);\boldsymbol{Q}) \arrow[l, "g^*"'] \\
 & & H^{*-8}(P(C^{2n})_+;\boldsymbol{Q}) \arrow[u, "\cong", "\phi_H"'] & H^{*-8}(P(H^n)_+;\boldsymbol{Q}) \arrow[l, "g^*"'] \arrow[u, "\cong", "\phi_H"']
\end{tikzcd}$$

where vertical isomorphisms are Thom isomorphisms and $\oplus k$ means the whitney sum with the real k-dimensional trivial bundle. Computations show that

$$\begin{aligned}
&\sigma\, h^*\, g^*\, \phi_K(\textstyle\sum_{i=1}^{n-1} a_i\, \theta_C^i) \\
&\quad = \beta^3(\{(1+\mu)-(1+\mu)^{-1}\}\, \textstyle\sum_{i=1}^{n-1} a_i\{(1+\mu)+(1+\mu)^{-1}-2\}^i); \\
&\sigma\, h^*\, \phi_H(x) = 2tx,\ x\in H^*(P(\boldsymbol{C}^{2n})_+;\boldsymbol{Q})
\end{aligned}$$

where we consider $H^*(P(\boldsymbol{C}^{2n});\boldsymbol{Q})$ as a submodule of $H^*(P(\boldsymbol{C}^{2n+1});\boldsymbol{Q})$ naturally. Hence we have

$$\sigma\, h^*\, g^*\, ch_F\, \phi_{KF}(\textstyle\sum_{i=0}^{n-1} a_i\, \theta_F^i) = (e^t-e^{-t})\, \textstyle\sum_{i=0}^{n-1} a_i\, e_i^F(e^t+e^{-t}-2)^i$$

where $e_i^C=1$ and $e_i^R=e_i$. By definition there exist integers $a_0^F, \cdots, a_{n-1}^F$ such that $cd^{KF}(\zeta_n)\cdot 1=\phi_H^{-1}\, ch_F\, \phi_{KF}(\sum_{i=0}^{n-1} a_i^F\, \theta_F^i)$. In this case we have

$$2\cdot cd^{KF}(\zeta_n)\, t = (e^t-e^{-t})\, \textstyle\sum_{i=0}^{n-1} a_i^F\, e_i^F(e^t+e^{-t}-2)^i\,.$$

Hence $a_0^F=cd^{KF}(\zeta_n)$, and also

$$2\cdot cd^K(\zeta_n) = cd^{KO}(\xi_{2n})\,.$$

By [19] we have

$$i!\, a_0^C = (-1)^i\, 2^i\cdot 3\cdot\cdots\cdot(2i+1)\cdot a_i^C \quad \text{for} \quad 1\leqq i\leqq n-1$$

hence

$$\nu_2(a_0^C) = \alpha(i)+\nu_2(a_i^C)$$

since $\nu_2(i\,!)=i-\alpha(i)$, where $\nu_2(m)$ is the exponent of 2 in the prime decomposition of m, and $\alpha(i)$ is the number of 1's in the dyadic expansion of i. It follows that

$$cd_2^K(\zeta_n) = \operatorname{Max}\{\alpha(i) \mid 1 \leqq i \leqq n-1\}$$

and from the similar reason that

$$\begin{aligned} cd_2^{KO}(\zeta_n) &= \operatorname{Max}\{\alpha(i)+\nu_2(e_i) \mid 1 \leqq i \leqq n-1\} \\ &= 1+cd_2^K(\zeta_n) \end{aligned}$$

so that

$$cd^{KO}(\zeta_n) = 2 \cdot cd^K(\zeta_n)$$

by 2.9(1). We then have

$$cd^{KO}(\zeta_n) = cd(\zeta_n) = cd(U(2n)) \quad (n \geqq 2)$$

from the equations

$$\begin{aligned} &cd^{KO}(\zeta_n) \mid cd(\zeta_n) = cd(Sp(n)), \\ &cd(Sp(n)) \mid cd(U(2n)) = cd(U(2n+1)) = cd^{KO}(\xi_{2n}) = cd^{KO}(\zeta_n). \end{aligned}$$

This proves (4) and completes the proof of Proposition 3.3.

4. Mod p decomposition of Lie groups

The result of this section is

Theorem 4.1. *Let G be a compact simly connected simple Lie group. Then the following statements hold.*

(1) $cd_p(G)=0$ *if and only if G is p-regular.*
(2) $cd_p(G)=1$ *if G is not p-regular but quasi p-regular.*
(3) $cd_7(E_8)=cd_7(E_7)=cd_5(E_7)=1$.
(4) $cd(G_2)=cd(Spin(7))=cd(Spin(8))=cd(SO(7))=2\cdot cd(SU(7))=2^3\cdot 3\cdot 5$.
(5) $2^4\cdot 3^2\cdot 5\cdot 7\cdot 11 \mid cd(G)$ *for* $G=E_8, E_7, E_6, F_4$.
(6) $cd_2(E_8)\leqq 8$; $cd_2(E_7)\leqq 7$; $cd_2(E_6)\leqq 5$; $cd_2(F_4)\leqq 5$;
$cd_3(E_8)\leqq 5$; $cd_3(E_7)\leqq 4$; $cd_3(E_6)=cd_3(F_4)\leqq 3$;
$cd_5(E_8)\leqq 3$.

From 4.1 and Table (4.3) below, we have

Corollary 4.2. *For simply connected exceptional groups, $cd_p(G)=1$ if (and only if provided $G \neq E_8$) G is not p-regular and has no p-torsion in $H^*(G; \boldsymbol{Z})$.*

Precise values of p in 4.1 and 4.2 can be read from the following Table (4.3).

Table (4.3)

G	p-torsion	quasi p-regular	p-regular	$cd_p(G)=1$
$SU(n)$ $(2\leqq n)$	NO	$n/2<p$	$n\leqq p$	$p<n\leqq p^2$ $(p>2)$ $n=3$ $(p=2)$
$Sp(n)$ $(2\leqq n)$	NO	$n<p$	$2n\leqq p$	$p<2n<p^2$ $(p>2)$ NO $(p=2)$
$Spin(n)$ $(7\leqq n)$	2	$(n-1)/2<p$	$n-1\leqq p$	$p<n-1\leqq p^2$ $(p<2)$ NO $(p=2)$
G_2	2	$5\leqq p$	$7\leqq p$	$2<p<7$
F_4	2, 3	$5\leqq p$	$13\leqq p$	$3<p<13$
E_6	2, 3	$5\leqq p$	$13\leqq p$	$3<p<13$
E_7	2, 3	$11\leqq p$	$19\leqq p$	$3<p<19$
E_8	2, 3, 5	$11\leqq p$	$31\leqq p$	

Lemma 4.4. *We have* $cd^K(Sp(2))=2\cdot 3$; $cd^K(Spin(7))=2^3\cdot 3\cdot 5$; $cd^K(G_2)=2^2\cdot 3\cdot 5$; $cd^K(F_4)=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11$; $cd^{KO}(G_2)/cd^K(G_2)=cd^{KO}(F_4)/cd^K(F_4)=2$.

Proof. The computation of $cd^K(G)$ can be easily done by using Watanabe [27]. We omit the details.

Suppose that $G=G_2$ or F_4. Set $r=\mathrm{rank}$ G. Consider the commutative diagram:

$$\begin{tikzcd} RO(G) \arrow[r, "\beta"] \arrow[d, "c"'] & KO^{-1}(G_+) \arrow[d, "c"] \\ R(G) \arrow[r, "\beta"'] & K^{-1}(G_+) \end{tikzcd}$$

where $RO(G)$, $R(G)$ are real, complex representation rings of G, and β is the natural homomorphism [11, 23], and c is the complexification. By Yokota [28, 29], c: $RO(G)\cong R(G)$. Let $\rho_1^C, \cdots, \rho_r^C \in R(G)$ be basic representations [5, 11] and $\rho_1^R, \cdots, \rho_r^R \in RO(G)$ be representations such that $c(\rho_k^R)=\rho_k^C$. It follows from Seymour [23; 5.6] that $KO^*(G_+)$ is a free $KO^*(S^0)$-module and is generated multiplicatively by $\beta(\rho_1^R), \cdots, \beta(\rho_r^R)$. Let A^F denote the $KF^*(S^0)$-submodule of $KF^*(G_+)$ generated by decomposable elements with respect to $\{\beta(\rho_1^F), \cdots, \beta(\rho_r^F)\}$. Then

$$KO^3(G)=KO^3(G_+)\equiv KO^4(S^0)\ \{\beta(\rho_1^R), \cdots, \beta(\rho_r^R)\} \bmod A^R$$

and so

$$c(KO^3(G_+))\equiv 2K^4(S^0)\ \{\beta(\rho_1^C), \cdots, \beta(\rho_r^C)\} \bmod A^C .$$

Therefore $cd^{KO}(G)=2\cdot cd^{K}(G)$ and this completes the proof.

REMARK 4.5. $cd(Sp(2))=cd(\zeta_2)=cd^{KO}(\zeta_2)=12$.

Proof of Theorem 4.1 (1), (2), (5). If G is p-regular then $cd_p(G)=0$ by 2.12.

By 3.3 or [19], $cd_p(SU(n))=0$ if and only if $n\leqq p$. Hence (1) holds for $SU(n)$.

By 3.3, 4.3, and 2.3(3), we have $cd_p(Sp(n))\neq 0$ for $2\leqq n$ and $p\leqq 2n$. Hence (1) hols for $Sp(n)$.

By 4.4 and 2.3(3), we have $3\leqq cd_2(Spin(n))$ for $n\geqq 7$. It follows from 3.3 that if $cd_p(Spin(n))=0$ then $n-1\leqq p$ and so $Spin(n)$ is p-regular. Hence (1) holds for $Spin(n)$ $(n\geqq 7)$.

It follows from 4.4 that if $cd_p(G_2)=0$ then $p\geqq 7$ and so G_2 is p-regular, and that if $cd_p(F_4)=0$ then $p\geqq 13$ and so F_4 is p-regular. Hence (1) holds for G_2 and F_4.

Since there exist embeddings $F_4\subset E_6\subset E_7\subset E_8$ which induce isomorphisms of the third homotopy groups, we have $cd^{KO}(F_4)\,|\,cd^{KO}(G)\,|\,cd(G)$ for $G=E_6$, E_7, E_8 by 2.3(3). It follows from 4.4 that (5) holds and that if $cd_p(E_k)=0$ then $p\geqq 13$ and so (1) holds for E_6.

By [15], if G is quasi p-regular, then there exists a p-equivalence u: $B_1(p)\times Y\to G$ where $B_1(p)=S^3\cup e^{2p+1}\cup e^{2p+4}$ is a S^3-bundle over S^{2p+1} such that $H^*(B_1(p);\boldsymbol{Z}_p)=\wedge(x_3, \mathcal{P}^1 x_3)$, and Y is a 4-connected finite CW-complex. There exists a p-equivalence v: $G\to B_1(p)\times Y$, since G is p-universal [16]. Let f: $S^3\to G$ represent a generator of $\pi_3(G)$, and f': $S^3\subset B_1(p)$ be the inclusion. Then f' generates $\pi_3(B_1(p))\cong\boldsymbol{Z}$. Since f'^*: $H^3(B_1(p);\boldsymbol{Z})\cong H^3(S^3;\boldsymbol{Z})$, it follows that $cd(f')\neq 0$ by 2.3(4). Let q: $B_1(p)\times Y\to B_1(p)$ be the projection, and i: $B_1(p)=B_1(p)\times\{*\}\subset B_1(p)\times Y$ be the inclusion. Since q, i induce isomorphisms on $\pi_3(\ \ )$, and since u, v are p-equivalences, there exist integers m, n such that m, n are prime to p, $q\circ v\circ f\simeq f'\circ m\iota_3$, and $u\circ i\circ f'\simeq f\circ n\iota_3$. It follows from 2.3(3) that $cd(f)\,|\,m\cdot cd(f')$ and $cd(f')\,|\,n\cdot cd(f)$, so that $cd_p(f')=cd_p(f)=cd_p(G)$. On the other hand, since the attaching map a: $S^{2p}\to S^3$ of $(2p+1)$-cell of $B_1(p)$ is a generator of $\pi_{2p}(S^3)_{(p)}\cong\pi^s_{2p}(S^3)_{(p)}\cong\boldsymbol{Z}_p$ (see [15, 25]), it follows that $cd_p(f')\geqq 1$ and that the map $p\iota_3\circ a$ extends stably to a': $S^3\cup e^{2p+1}\to S^3$. Since $\pi^s_{2p+3}(S^3)_{(p)}=0$ (see [25]), a' extends stably to a'': $B_1(p)\to S^3$ such that $a''\circ f'=p\iota_3$, hence $cd_p(f')\leqq 1$, so $cd_p(G)=cd_p(f')=1$, and this proves (2). In particular we have $cd_p(E_7)=1$ for $11\leqq p\leqq 17$ and $cd_p(E_8)=1$ for $11\leqq p\leqq 29$. This completes the proof of (1).

To prove Theorem 4.1(3), we need the following theorem. See [17] for notations.

Theorem 4.6 ([17; 8.1]). (1) $E_7\simeq_5 B^5_1(5)\times B^2_7(5)$.

(2) $E_7\simeq_7 B^3_1(7)\times B^3_5(7)\times S^{19}$.

(3) $E_8\simeq_7 B^4_1(7)\times B^4_{11}(7)$.

Proof of Theorem 4.1(3). We will prove only that $cd_7(E_8)=1$, because other cases can be proved by the same method. Note that the 16-skeleton of $B_{\mathrm{I}}^{4}(7)$ is S^3, and that $B_{\mathrm{II}}^{4}(7)$ is 22-connected. Let i: $S^3 \to B_{\mathrm{I}}^{4}(7)$ be the inclusion. By the same method as the proof of (1), we have 7-equivalences u: $B_{\mathrm{I}}^{4}(7) \times B_{\mathrm{II}}^{4}(7) \to E_8$, v: $E_8 \to B_{\mathrm{I}}^{4}(7) \times B_{\mathrm{II}}^{4}(7)$, and $cd_7(E_8) = cd_7(i)$. By definition (see [17]), there exists a map h: $B_{\mathrm{I}}^{4}(7) \to SU(20)$ such that h^*: $H^*(SU(20); \boldsymbol{Z}_7) \to H^*(B_{\mathrm{I}}^{4}(7); \boldsymbol{Z}_7)$ is a surjection. In particular the index of the image of $h_*: \pi_3(B_{\mathrm{I}}^{4}(7)) \to \pi_3(SU(20)) \cong \boldsymbol{Z}$ is prime to 7. That is, $h \circ i \simeq f \circ m\iota_3$ for some integer m which is prime to 7, where f: $S^3 \to SU(20)$ represents a generator of $\pi_3(SU(20))$. By 2.3(3), we have $cd(i) \mid m \cdot cd(SU(20))$, so $cd_7(i) \leqq cd_7(SU(20)) = 1$ by 3.3, hense $cd_7(E_8) = cd_2(i) = 1$ by (5).

Proof of Theorem 4.1(4). There exist homomorphisms $G_2 \subset Spin(7) \to SO(7)$ which induce isomorphisms on $\pi_3(\ \ )$. Hence

$$cd(G_2) \mid cd(Spin(7)) \mid cd(SO(7)) \mid 2 \cdot cd(SU(7))$$

where the last divisibility was proved in the proof of 3.3(3). Since $cd^{KO}(G_2) = 2 \cdot cd(SU(7)) = 120$ by 3.3(1) and 4.4, the above four codegrees are the same. We have also $cd(Spin(7)) = cd(Spin(8))$, since $Spin(8)$ is homeomorphic to $Spin(7) \times S^7$.

REMARK 4.7. We can prove $cd(G_2) = 120$ by constructing a stable map g: $G_2 \to S^3$ having the degree 120 on S^3.

Proof of Theorem 4.1(6). Let $\{w_1, \cdots, w_r\}$ be a system of fundamental weights with respect to a system of simple roots of G, $\rho(w_i)$ be the irreducible representation with highest weight w_i $(1 \leqq i \leqq r)$, β be a root of G of maximal length, and set $\delta = \Sigma_i w_i$. Then, from an observation of Harris [10] (cf., Naylor [18]), we know that the cokernel of $\rho(w_i)_*: \pi_3(G) \to \pi_3(U(\dim \rho(w_i)))$ is a cyclic group of order

$$n_i = \frac{2(w_i, w_i + 2\delta)}{(\beta, \beta)} \cdot \frac{\dim \rho(w_i)}{\dim G}$$

where (,) is the Killing form. The number n_i was called by Dynkin [5] the index of the representation $\rho(w_i)$. It follows from 2.3(3) that

$$cd(G) \mid n_i \cdot cd(U(\dim \rho(w_i))) . \tag{4.8}$$

From Tables 5 and 41 of Dynkin [5] (cf., Harris [10]), we know that there exist irreducible representations of minimal dimension

$$\begin{aligned} G_2 &\to U(7), & n &= 2, \\ F_4 &\to U(26), & n &= 6, \\ E_6 &\to U(27), & n &= 6, \\ E_7 &\to U(56), & n &= 12, \\ E_8 &\to U(240), & n &= 60. \end{aligned}$$

Hence we obtain the desired upper bounds except $cd_p(E_8)$ for $p=2$, 5 by (4.8) and 3.3(1). Since the complex Stiefel manifold $U(240)/U(124)$ is 248-connected and the dimension of E_8 is 248, the representation $E_8 \to U(240)$ factorizes up to homotopy $E_8 \to U(124) \subset U(240)$ as an unstable map, it follows that $cd(E_8) \mid 60 \cdot cd(U(124))$ so that $cd_2(E_8) \leqq 8$ and $cd_5(E_8) \leqq 3$.

By [9; Proposition 1], we have $E_6 \simeq_3 F_4 \times E_6/F_4$ so that $cd_3(E_6) = cd_3(F_4)$, since E_6/F_4 is 8-connected. This proves (6) and completes the proof of Theorem 4.1.

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