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ALMOST IDENTICAL IMITATIONS OF (3, 1)-DIMENSIONAL MANIFOLD PAIRS

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Dedicated to Professor Fujitsugu Hosokawa on his 60th birthday

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By a 3-*manifold* M, *we mean a compact connected oriented* 3-*manifold throughout this paper.* Let $\partial_0 M$ be the union of torus components of ∂M and $\partial_1 M = \partial M - \partial_0 M$. In the case that $\partial_1 M = \emptyset$, if Int M has a complete Riemannian structure with constant curvature -1 and with finite volume, then we say that M is *hyperbolic* and we denote its volume by Vol M. Next we consider the case that $\partial_1 M \neq \emptyset$. Then the double, $D_1 M$, of M pasting two copies of M along $\partial_1 M$ has $\partial_1 D_1 M = \emptyset$. If $D_1 M$ is hyperbolic in the sense stated above, then we say that M is *hyperbolic* and we define the *volume*, Vol M, of this M by Vol M = Vol $D_1 M/2$. In this latter case, M is usually said to be *hyperbolic with* $\partial_1 M$ *tatally geodesic* (cf. **[T-1]**), but we use this simple terminology throughout this paper. When M is hyperbolic, ∂M has no 2-sphere components and by Mostow rigidity theorem (cf. **[T-2]**, **[T-3]**), Vol M is a topological invariant of M. By a 1-*manifold in* M, we mean a compact smooth 1-submanifold L of M with $\partial L = L \cap \partial M$ and the pair (M, L) is simply called a (3,1)-*manifold pair*. A 1-manifold L in M is called a *link* if $\partial L = \emptyset$, a *tangle* if L has no loop components, and a *good* 1-manifold if $|L \cap S^2| \geq 3$ for any 2-sphere component S^2 of ∂M. A (3,1)-manifold pair (M, L) is also said to be *good* if L is a good 1-manifold in M. In **[Kw-1]**, we defined the notions of imitation, pure imitation and normal imitation for any general manifold pair. In Section 1 we shall define a notion which we call an *almost identical imitation* (M, L^*) of (M, L), for any good (3,1)-manifold pair (M, L). Roughly speaking, this imitation is a normal imitation with a special property that if q: $(M, L^*) \to (M, L)$ is the imitaiton map, then $q|(M, L^* - a^*)$: $(M, L^* - a^*) \to (M, L - a)$ is ∂-relatively homotopic[1] to a diffeomorphism for any connected components a^*, a of L^*, L with $qa^* = a$. Let P be a polyhedron in a 3-manifold M. For a regular neighborhood N_P of P in M (meeting ∂M regularly), the diffeomorphism type of $E(P, M) = \mathrm{cl}_M(M - N_P)$ is uniquely determined by the topological type of the

[1] This homotopy can be taken as a one-parameter family of normal imitation maps.

pair (M,P) and we call $E(P, M)$ the *exterior* of P in M. Then our main result of this paper, stated in Theorem 1.1 precisely, asserts the *existence of an infinite family of almost identical imitations (M, L^*) of every good (3,1)-manifold pair (M, L) such that the exterior $E(L^*, M)$ of L^* in M is hyperbolic.*

The proof of Theorem 1.1 will be given in Section 5. Several applications to spatial graphs, links and 3-manifolds are given throughout Sections 2–4. In Section 2, we prove the existence of an almost trivial spatial $\hat{\Gamma}$-graph, for every planar graph $\hat{\Gamma}$ without vertices of degrees ≤ 1, affirming a conjecture of Simon and Wolcott. In Section 3, we show a construction of a non-trivial fusion band family from a trivial link to a trivial knot, and a construction of a tangle with hyperbolic exterior in any link. In Section 4, we show that if a closed 3-manifold M is obtained from a link L with two or more components by Dehn's surgery, then M is also obtained from a hyperbolic link L^*, which is a normal link-imitation of L, by Dehn's surgery with the same surgery coefficient data, and that every 3-manifold without 2-sphere boundary component has a hyperbolic 3-manifold as a normal imitation.

This paper is a revised version of a main part of [**Kw-0**] and a prelude to the principal theorem of [**Kw-2**] where furhter consequences are announced.

1. An almost identical imitation of a good (3,1)-manifold pair. Let $I=[-1, 1]$. For a (3, 1)-manofold pair (M, L) we call an element $\alpha \in \text{Diff}((M, L)\times I)$ a *reflection* in $(M, L)\times I$ if $\alpha^2=1$, $\alpha(M\times 1)=M\times(-1)$ and $\text{Fix}(\alpha, M\times I)$ is a 3-manifold. In this case, $\text{Fix}(\alpha, (M, L)\times I)$ is a (3, 1)-manifold pair in our sense (See [**Kw-1**]). We say that a reflection α in $(M, L)\times I$ is *standard* if $\alpha(x, t)=(x,-t)$ for all $(x, t)\in M\times I$, and *normal* if $\alpha(x, t)=(x, -t)$ for all $\alpha(x, t)\in\partial(M\times I)\cup U_L\times I$, with U_L a neighborhood of L in M. A reflection α in $(M, L)\times I$ is said to be *isotopically standard* if $h\alpha h^{-1}$ is the standard reflection in $(M, L)\times I$ for an $h\in \text{Diff}_0((M, L)\times I, \text{rel}\ \partial((M, L)\times I))$[2]. For a good (3, 1)-manifold pair (M, L) a reflection α in $(M, L)\times I$ is *isotopically almost standard* if ϕ is isotopically standard in $(M, L-a)\times I$ for each connected component a of L. A smooth embedding ϕ from a (3,1)-manifold pair (M^*, L^*) to $(M, L)\times I$ with $\phi(M^*, L^*)=\text{Fix}(\alpha, (M, L)\times I)$ is called a *reflector* of a reflection in $(M, L)\times I$. Let p_1: $(M, L)\times I\to(M, L)$ be the projection to the first factor. In [**Kw-1**], we defined that (M^*, L^*) is an *imitation* (or a *normal imitation*, respectively) of (M, L), if there is a reflector ϕ: $(M^*, L^*)\to(M, L)\times I$ of a reflection (or normal reflection, respectively) α in $(M, L)\times I$, and the composite $q=p_1\phi: (M^*, L^*)\to(M, L)$ is the *imitation map.*

DEFINITION. A (3,1)-manifold pair (M^*, L^*) is an *almost identical imitation*

[2] Diff_0 denotes the path connected component of the topological diffeomorphism group Diff containing 1(cf. [**Kw-1**]).

of a good (3, 1)-manifold pair (M, L) if there is a reflector $\phi: (M^*, L^*) \to (M, L) \times I$ of an isotopically almost standard normal reflection α in $(M, L) \times I$, and the composite $q = p_1 \phi: (M^*, L^*) \to (M, L)$ is the *imitation map*.

In this definition, (M^*, L^*) is also a good (3, 1)-manifold pair and $q|L^*: L^* \to L$ is a diffeomorphism and $q|(M^*, L^*-a^*): (M^*, L^*-a^*) \to (M, L-a)$ is ∂-relatively homotopic to a diffeomorphism. We identify M^* with M so that $q|\partial M$ is the identity on ∂M. We may write any almost identical imitation of (M, L) as (M, L^*). We state here our main theorem.

Theorem 1.1. *For any number $K > 0$ and any good (3,1)-manifold pair (M, L) there are a number $K^+ > K$ and an infinite family of almost identical imitations (M, L^*) of (M, L) such that the exterior $E(L^*, M)$ of L^* in M is hyperbolic with* $\operatorname{Vol} E(L^*, M) < K^+$ *and* $\operatorname{Sup}_{L^*} \operatorname{Vol} E(L^*, M) = K^+$.

2. An almost identical spatial graph imitation. Let (M^0, L) be a good (3,1)-manifold pair such that ∂M^0 has at least one 2-sphere component. For some 2-sphere components $S_1, S_2, \cdots, S_r$ of ∂M^0, let (M^0_+, L_+) be a pair obtained from (M^0, L) by taking a cone over $(S_i, S_i \cap L)$ for each i. Then note that M^0_+ is a 3-manifold and L_+ is a finite graph which we may consider to be smoothly embedded in M^0_+ except the vertices of degrees ≥ 3. We call this pair (M^0_+, L_+) the *spherical completion* of (M^0, L) *associated with* the 2-spheres $S_1, S_2, \cdots, S_r$. A graph Γ embedded in a 3-manifold M is said to be *good* if (M, Γ) is diffeomorphic to the spherical completion (M^0_+, L_+) of a good (3,1)-manifold pair (M^0, L) associated with some 2-sphere components of ∂M^0.

DEFINITION. For good graphs Γ^*, Γ in a 3-manifold M the pair (M, Γ^*) is an *almost identical imitation* of the pair (M, Γ) if there are a good (3,1)-manifold pair (M^0, L) and some 2-sphere components $S_1, S_2, \cdots, S_r$ of ∂M^0 and an almost identical imitation (M^0, L^*) of (M^0, L) such that the spherical completions (M^0_+, L^*_+) and (M^0_+, L_+) of (M^0, L^*) and (M^0, L) associated with the 2-spheres $S_1, S_2, \cdots, S_r$ are diffeomorphic to (M, Γ^*) and (M, Γ), respectively.

Note that there is a map $q: (M, \Gamma^*) \to (M, \Gamma)$ uniquely determined by the imitation map $q^0: (M^0, L^*) \to (M^0, L)$. We also call this map q the *imitation map of the almost identical imitation* (M, Γ^*) *of* (M, Γ). Since, in this definition, the exterior $E(\Gamma^*, M)$ of Γ^* in M is diffeomorphic to $E(L^*, M^0)$, the following theorem follows directly from Theorem 1.1:

Theorem 2.1. *For each good graph Γ in a 3-manifold M and a positive number K, there are a number $K^+ > K$ and an infinite family of almost identical imitations (M, Γ^*) of (M, Γ) such that $E(\Gamma^*, M)$ is hyperbolic with* $\operatorname{Vol} E(\Gamma^*, M) < K^+$ *and* $\operatorname{Sup}_{\Gamma^*} \operatorname{Vol} E(\Gamma^*, M) = K^+$.

Let $\tilde{\Gamma}$ be a finite graph without vertices of degrees ≤ 1. If a good graph Γ in the 3-sphere S^3 is obtained by an embedding of $\tilde{\Gamma}$, then we call this Γ a *spatial* $\tilde{\Gamma}$*-graph*. Two spatial $\tilde{\Gamma}$-graphs Γ', Γ'' are *equivalent* if there is an orientation-preserving diffeomorphism $h: S^3 \rightarrow S^3$ with $h(\Gamma')=\Gamma''$. The occurring equivalence classes of spatial $\tilde{\Gamma}$-graphs are called the *knot types* of spatial $\tilde{\Gamma}$-graphs. These knot types were studied by Kinoshita, Suzuki (cf. [**Su-1**]) as a generalization of the usual knot theory and are now studied in a connection with the synthetic study in molecular chemistry by, for example, Walba [**Wa**], Simon [**Si**], Sumners [**Sum**]. We say that a finite graph in S^3 is *trivial* if it is on a 2-sphere smoothly embedded in S^3. A spatial $\tilde{\Gamma}$-graph Γ is said to belong to an *almost trivial knot type*, if Γ is not trivial but the graph in S^3 resulting from Γ by removing any open arc is necessarily trivial. Simon and Wolcott (cf. [**Si**]) conjectured that *for every planar graph* $\tilde{\Gamma}$ *without vertices of degrees* ≤ 1, *there exists a spatial* $\tilde{\Gamma}$*-graph belonging to an almost trivial knot type*. Several examples supporting this conjecture were given by Kinoshita [**Ki**], Suzuki [**Su-2**], M. Hara(unpublished) and Wolcott [**Wo**]. Theorem 2.1 solves this conjecture affirmatively. In fact, we have the following stronger result:

Corollary 2.2. *For every planar graph* $\tilde{\Gamma}$ *without vertices of degrees* ≤ 1 *and any number* $K>0$, *there are a number* $K^+>K$ *and an infinite family of spatial* $\tilde{\Gamma}$*-graphs* Γ^* *belonging to infinitely many almost trivial knot types such that* $E(\Gamma^*, S^3)$ *is hyperbolic with* $\mathrm{Vol}\, E(\Gamma^*, S^3)<K^+$ *and* $\mathrm{Sup}_{\Gamma^*} \mathrm{Vol}\, E(\Gamma^*, S^3)=K^+$ *and the quotient group* $\bar{\pi}_1(E(\Gamma^*, S^3)$ *of* $\pi_1(E(\Gamma^*, S^3))$ *by the intersection of the derived series of* $\pi_1(E(\Gamma^*, S^3))$ *is a free group of rank* $\beta_1(\Gamma^*)$ *with a basis represented by meridians of* Γ^* *in* S^3, *where* $\beta_1(\Gamma^*)$ *denotes the first Betti number of* Γ^*.

Proof. Let Γ be a trivial spatial $\tilde{\Gamma}$-graph. By Theorem 2.1, there are a number $K^+>K$ and an infinite family of almost identical imitations (S^3, Γ^*) of (S^3, Γ) such that $E(\Gamma^*, S^3)$ is hyperbolic with $\mathrm{Vol}\, E(\Gamma^*, S^3)<K^+$ and $\mathrm{Sup}_{\Gamma^*} \mathrm{Vol}$ $E(\Gamma^*, S^3)=K^+$. Clearly, this Γ^* belongs to an almost trivial knot type. If q: $(S^3, \Gamma^*) \rightarrow (S^3, \Gamma)$ is the imitation map, then q induces a meridian-preserving isomorphism $\bar{\pi}_1(S^3-\Gamma^*) \cong \bar{\pi}_1(S^3-\Gamma)$ (See [**Kw-1**]). Since $\pi_1(S^3-\Gamma)$ is a free group of rank $\beta_1(\Gamma)$ with a basis represented by meridians of Γ in S^3, we see from [**L-S**, p. 14] that $\bar{\pi}(S^3-\Gamma)=\pi_1(S^3-\Gamma)$, so that $\bar{\pi}_1(E(\Gamma^*, S^3)) \cong \bar{\pi}_1(S^3-\Gamma^*)$ is a free group with a desired property. This completes the proof.

3. Applications to links. We discuss here two applications to links. One concerns a construction of a non-trivial fusion band family from a trivial link to a trivial knot and the other, a construction of a tangle with the exterior hyperbolic in any link. We say that a mutually disjoint band family $\{B_1^0, B_2^0, \cdots, B_l^0\}$ in S^3 spanning a trivial link L_0 (as 1-handles) is *trivial* if the union $L_0 \cup B_1^0 \cup B_2^0 \cup \cdots \cup B_l^0$ is on a 2-sphere smoothly embedded in S^3. Let a trivial link L_0

have $r+1$ components. We consider mutually disjoint r bands $B_1, B_2, \cdots, B_r$ in S^3 which give a fusion from L_0 to a trivial knot (that is to say, which span L_0 and along which the surgery of L_0 produces a trivial knot). We say that this family $\{B_1, B_2, \cdots, B_r\}$ is a *fusion band family* from L_0 to a trivial knot. For $r=1$, Scharlemann [**Sc**] proved that any fusion band family $\{B_1\}$ is necessarily trivial. For $r=2$, Howie and Short [**H-S**] gave an example of a non-trivial fusion band family $\{B_1, B_2\}$ (cf. [**Kw-2**, Figure 4]). In their example, the exteroir $E=E(L_0 \cup B_1 \cup B_2, S^3)$ is easily seen to have a solid torus as a disk summand and hence it is not hyperbolic. As a corollary to Theorem 2.1, we have an infinite family of non-trivial fusion band families with such exteriors hyperbolic.

Corollary 3.1. *For any number $K>0$ and any integer $r\geq 2$, there are a number $K^+>K$ and an infinite family of non-trivial fusion band families $\beta^*=\{B_1^*, B_2^*, \cdots, B_r^*\}$ from an $(r+1)$-component trivial link L_0 to a trivial knot such that the exterior $E_{\beta^*}=E(L_0 \cup B_1^* \cup B_2^* \cup \cdots \cup B_r^*, S^3)$ is hyperbolic with $\mathrm{Vol}\, E_{\beta^*}<K^+$ and $\mathrm{Sup}_{\beta^*} \mathrm{Vol}\, E_{\beta^*}=K^+$ and $\bar{\pi}_1(E_{\beta^*})$ is a free group of rank $r+1$ with a basis represented by meridians of L_0.*

Proof. Consider a trivial fusion band family $\{B_1, B_2, \cdots, B_r\}$ from L_0 to a trivial knot. Let L_0' be an r-component trivial link obtained from L_0 by surgery along B_r. When we regard the band B_r as a band spanning L_0', we denote it by B_r'. Note that a spine $\Gamma=L_0' \cup b_1 \cup b_2 \cup \cdots \cup b_r'$ of $L_0' \cup B_1 \cup B_2 \cup \cdots \cup B_r'$ is a good planar graph in S^3. By Theorem 2.1, we have a number $K^+>K$ and an infinite family of almost identical imitations $q\colon (S^3, \Gamma^*) \to (S^3, \Gamma)$ such that $\mathrm{Vol}\, E(\Gamma^*, S^3) <K^+$ and $\mathrm{Sup}_{\Gamma^*} \mathrm{Vol}\, E(\Gamma^*, S^3)=K^+$. Regard the bands $B_1, B_2, \cdots, B_r'$ as very narrow bands. Then since $r\geq 2$ and q is an almost identical imitation map, we may consider that q defines a map $(S^3, L_0' \cup B_1^* \cup \cdots \cup B_{r-1}^* \cup B_r') \to ((S^3, L_0' \cup B_1 \cup \cdots \cup B_{r-1} \cup B_r')$, where B_i^* denotes a band given by $B_i^*=q^{-1}B_i$ for each $i\leq r-1$. Then we see that the bands $B_1^*, B_2^*, \cdots, B_r^*$ with $B_r^*=B_r$ form a fusion band family from L_0 to a trivial knot. Clearly, the exterior E of $L_0 \cup B_1^* \cup B_2^* \cup \cdots \cup B_r^*$ in S^3 is diffeomorphic to $E(\Gamma^*)$. By the proof of Corollary 2.2, $\bar{\pi}(E)$ is seen to be a desired free group. This completes the proof of Corollary 3.1.

REMARK 3.2. In the above proof, we can see that the band family $\{B_1^*, \cdots, B_{i-1}^*, B_{i+1}^*, \cdots, B_r^*\}$ spanning L_0 is trivial for each i with $1\leq i\leq r-1$. In particular, if $r\geq 3$, then each band $B_i^*(1\leq i\leq r)$ spans L_0 trivially.

As another application, we shall show the following:

Corollary 3.3. *For any link L in S^3 we take 3-balls B, B' in S^3 so that $B'=S^3-\mathrm{Int}\, B$ and $T=B\cap L$ is a trivial tangle with 2 or more strings in B and $T'=B'\cap L$ is a good 1-manifold in B'. Then for any number $K>0$, there are a number $K^+>K$ and an infinite family of almost identical imitations (B', T'^*)*

of (B', T') such that the exterior $E(T'^*, B')$ is hyperbolic with $\mathrm{Vol}\,E(T'^*, B')<K^+$ *and* $\mathrm{Sup}_{T'^*}\,\mathrm{Vol}\,E(T'^*, B')=K^+$*, and the extension q'^+: $(S^3, L^*)\to(S^3, L)$ of the imitation map q': $(B', T'^*)\to(B', T')$ by the identity on (B, T) is homotopic to a diffeomorphism.*

Proof. Let T be a good tree graph in B obtained by joining the components of $\hat{T}$ by arcs so that B collapses to $\hat{T}$, and Γ the union of $\hat{T}$ and T' which is a good graph in S^3. By Theorem 2.1 we have a number $K^+>K$ and an infinite family of almost identical imitations (S^3, Γ^*) of (S^3, Γ) such that the exterior $E(\Gamma^*, S^3)$ is hyperbolic with $\mathrm{Vol}\,E(\Gamma^*, S^3)<K^+$ and $\mathrm{Sup}_{\Gamma^*}\,\mathrm{Vol}\,E(\Gamma^*, S^3)=K^+$. By replacing B by a slender regular neighborhood of $\hat{T}$ in B, we can consider that the almost identical imitation map q: $(S^3, \Gamma^*)\to(S^3, \Gamma)$ induces the identity on B and the restriction $q'=q|B'$ induces an almost identical imitation map $(B', T'^*)\to(B', T')$ with $T'^*=q'^{-1}T'$. Moreover, we see that the extension q'^+: $(S^3, L^*)\to(S^3, L)$ of q' by the identity on (B, T) is homotopic to a diffeomorphism. Noting that $E(T'^*, B')$ is diffeomorphic to $E(\Gamma^*, S^3)$, we complete the proof of Corollary 3.3.

This corollary includes a hyperbolic version of Nakanishi's result [**N**], telling that every link is splittable by a 2-sphere into a prime 1-manifold and a trivial two-string tangle.

4. Applications to 3-manifolds. Let T_i, $i=1, 2, \cdots, r$, be mutually disjoint tubular neighborhoods of the components k_i, $i=1, 2, \cdots, r$ of a link L in S^3. Remove $\mathrm{Int}\,T_i$ from S^3 for each i and then attach T_i again by using an $h_i\in\mathrm{Diff}\,\partial T_i$ for each i. By this operation, we obtain from S^3 a closed 3-manifold M. Let m_i be a meridian of T_i, and l_i a longitude of T_i determined by $T_i\subset S^3$. Write $h_{i*}[m_i]=a_i[m_i]+b_i[l_i]$ in $H_i(\partial T_i; Z)$ with integers a_i, b_i. Then we see that the diffeomorphism type of M depends only on the pairs (k_i, c_i) with $c_i=a_i/b_i\in Q\cup\{\infty\}$, $i=1, 2, \cdots, r$, and we say that *M is obtained from S^3 by Dehn's surgery along the knots k_i with coefficients $c_i (i=1, 2, \cdots, r)$* or that *M has a surgery description* $(S^3; (k_1, c_1), (k_2, c_2), \cdots, (k_r, c_r))$. It is well known that every closed connected orientable 3-manifold M has a surgery description $(S^3; (k_1, c_1), (k_2, c_2), \cdots, (k_r, c_r))$ (cf. [**We**], [**L**]). We obtain from Theorem 1.1 the following:

Corollary 4.1. *For any number $K>0$ and any surgery description $(S^3; (k_1, c_1), (k_2, c_2), \cdots, (k_r, c_r))$ of any closed 3-manifold M with $r\geq 2$, there are a number $K^+>K$ and an infinite family of normal imitations (S^3, L^*) of (S^3, L) such that the exterior $E(L^*, S^3)$ is hyperbolic with* $\mathrm{Vol}\,E(L^*, S^3)<K^+$ *and* $\mathrm{Sup}_{L^*}\,\mathrm{Vol}\,E(L^*, S^3)=K^+$ *and $(S^3; (k_1^*, c_1), (k_2^*, c_2), \cdots, (k_r^*, c_r))$ is a surgery description of M with $k_i^*=q^{-1}k_i$, $i=1, 2, \cdots, r$ for the imitation map q: $(S^3, L^*)\to(S^3, L)$.*

Proof. Let M' be the manifold with surgery description $(S^3; (k_r, c_r))$. Let

k'_r be a core of the solid torus in M' resulting from the Dehn surgery. Regard that $k_1, k_2, \cdots, k_{r-1}$ are in M'. Let $L'=k_1 \cup \cdots \cup k_{r-1} \cup k'_r$. By Theorem 1.1, we have a number $K^+ > K$ and an infinite family of almost identical imitations (M', L'^*) of (M', L') such that $E(L'^*, M')$ is hyperbolic with $\mathrm{Vol}\, E(L'^*, M') < K^+$ and $\mathrm{Sup}_{L'^*} \mathrm{Vol}\, E(L'^*, M') = K^+$. Let $k_i^* = q'^{-1} k_i$, $i=1, \cdots, r-1$, and $k_r'^* = q'^{-1} k'_r$ for the imitation map $q' : (M', L'^*) \to (M', L')$. Since q' is an almost identical imitation map, we may consider that $k_r'^* = k'_r$, so that q' induces a normal imitation map $q: (S^3, L^*) \to (S^3, L)$ with $L^* = k_1^* \cup \cdots \cup k_{r-1}^* \cup k_r^* \subset S^3$ and $k_r^* = k_r$ such that $(S^3; (k_1^*, c_1), \cdots, (k_{r-1}^*, c_{r-1}), (k_r, r_r))$ is a surgery description of M. Since $E(L^*, S^3)$ is diffeomorphic to $E(L'^*, M')$, we complete the proof of Corollary 4.1.

REMARK 4.2. In the above proof, the restriction $q | (S^3, L^* - k_i^*): (S^3, L^* - k_i^*) \to (S^3, L - k_i)$ is homotopic to a diffeomorphism for each i, $1 \leq i \leq r-1$. In particular, if $r \geq 3$, then k_i^* and k_i belong to the same knot type for all i, $1 \leq i \leq r$.

As a final application, we have the following:

Corollary 4.3. *For any number $K>0$ and any 3-manifold M such that ∂M has no 2-sphere components, there are a number $K^+ > K$ and an infinite family of normal imitations M^* of M such that M^* is hyperbolic with* $\mathrm{Vol}\, M^* < K^+$ *and* $\mathrm{Sup}_{M^*} \mathrm{Vol}\, M^* = K^+$.

Proof. For a trivial knot O in $\mathrm{Int}\, M$, we obtain from Theorem 1.1 an almost identical imitation (M, O^*) of the good pair (M, O) such that $E(O^*, M)$ is hyperbolic with $\mathrm{Vol}\, E(O^*, M) > K$. For an integer $n \neq 0$, let M_n^* be a 3-manifold obtained from M by Dehn surgery along O^* with coefficient $1/n$. Since the diffeomorphism type of M is unaffected by Dehn surgery along O with coefficient $1/n$, the imitation map $q: (M, O^*) \to (M, O)$ induces a normal imitation map $q_n^*: M_n^* \to M$. Let $K^+ = \mathrm{Vol}\, E(O^*, M)$. By Thurston's theorem on hyperbolic Dehn surgery **[T-2]**, **[T-3]**, there is an integer $N>0$ such that M_n^* is hyperbolic for all n with $|n| \geq N$, and for all such n, $\mathrm{Vol}\, M_n^* < K^+$ and $\mathrm{Sup}_n \mathrm{Vol}\, M_n^* = K^+$. This completes the proof.

5. Proof of Theorem 1.1. We first show that Theorem 1.1 is obtained from the following:

Lemma 5.1. *For any good (3,1)-manifold pair (M, L), there is an almost identical imitation (M, L^*) of (M, L) such that $E(L^*, M)$ is hyperbolic.*

Proof of Theorem 1.1 assuming Lemma 5.1. We can see from Jφrgensen's theorem (cf. **[T-2]**,**[T-3]**) that for any number $K>0$ there is an integer $N'>0$ such that every hyperbolic 3-manifold M' with $\mathrm{Vol}\, M' \leq K$ has the homology group $H_1(M'; Z)$ generated by at most N' elements. Let $L^+=L\cup L_0$ with L_0 an N'-component trivial link in $\mathrm{Int}(M-L)$. By Lemma 5.1, there is an almost identical imitation map $q: (M, L^{+*})\to(M, L^+)$ such that $E(L^{+*}, M)$ is hyperbolic. Let $K^+=\mathrm{Vol}\,E(L^{+*}, M)$. Since

$$H_1(E(L^{+*}, M); Z)\cong H_1(E(L^+, M); Z)\cong H_1(E(L, M); Z)\oplus_{N'}Z$$

(cf. [**Kw-1**]), we see that $H_1(E(L^{+*}, M); Z)$ can not be generated by N' elements, so that $K^+>K$. Let $L^*=q^{-1}L$ and $L_0^*=q^{-1}L_0$. Note that L_0^* is a trivial link in $\mathrm{Int}\,M$. For an integer $n\neq 0$, let (M, L_n^*) be a good (3,1)-manifold pair obtained from (M, L^*) by Dehn surgery of M along each component of L_0^* with coefficient $1/n$. Then q induces an almost identical imitation map $q_n: (M, L_n^*)\to (M, L)$. By Thurston's theorem on hyperbolic Dehn surgery [**T-2**], [**T-3**], there is an integer $N>0$ such that $E(L_n^*, M)$ is hyperbolic for all n with $|n|\geq N$ and, for all such n, $\mathrm{Vol}\,E(L_n^*, M)<K^+$ and $\mathrm{Sup}_n \mathrm{Vol}\,E(L_n^*, M)=K^+$. This completes the proof of Theorem 1.1 assuming Lemma 5.1.

We say that a tangle T in a 3-ball B is *trivial* if T is on a disk smoothly and properly embedded in B.

Proof of Lemma 5.1. We can see from arguments on Heegaard splitting of M and on isotopic deformation of L that M is splitted by a compact connected surface F with $\partial F\cap L=\emptyset$ into two handlebodies H_i, $i=1, 2$, of the same genus, say g, such that

(1) $F_i^c=\partial H_i-\mathrm{Int}\,F$ is a planar surface with the same component number as ∂M,
(2) Each component of L meets F transversely,
(3) Each disk component of F_i^c meets L,
(4) There is a 3-ball $B_i\subset H_i$ separated by a proper disk D_i such that $T_i=L\cap H_i$ is a trivial tangle of s_i strings in B_i where $s_i\geq 1$ and $g+s_i\geq 3$.

Our desired situation is illustrated in Figure 1. This situation is made up by the following procedure: When $\partial M=\emptyset$, we take any Heegaard splitting $(H_1, H_2; F)$ of M. When $\partial M\neq\emptyset$, we split M by a connected surface F_M into two 3-submanifolds M_i, $i=1, 2$, such that ∂M_i is connected and ∂M_i-$\mathrm{Int}\,F_M$ is a planar surface with the same component number as ∂M. Then note that ∂M_i, $i=1, 2$ have the same genus. We obtain a Heegaard splitting $(H_1, H_2; F)$ of M with condition (1) from $(M_1, M_2; F_M)$ by boring along 1-handles in M_i attaching to F_M. Next, we deform L so that L is disjoint from ∂F and has (2), (3) by an isotopic deformation of L in M. Finally, we deform L so that L has (4) by an isotopic deformation of L in M keeping ∂M fixed and increasing the geometric intersection number with F. We proceed to the proof of Lemma 5.1 by assuming the following lemma:

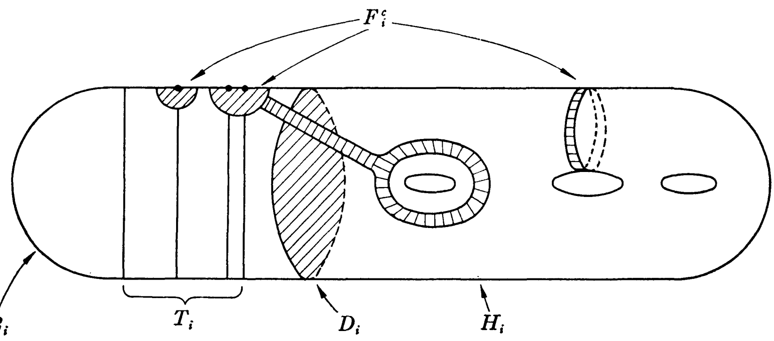


Figure 1

Lemma 5.2. *For any integer $r\geq 3$ let T be a trivial tangle of r strings in a 3-ball B. Then there is an almost identical imitation (B, T^*) of (B, T) such that $E(T^*, B)$ is hyperbolic.*

Since H_i is the exterior of a trivial g-tangle in a 3-ball and $g+s_i\geq 3$, we obtain from Lemma 5.2 an almost identical imitation map q_i: $(H_i, T_i^*)\rightarrow(H_i, T_i)$ such that $E(T_i^*, H_i)$ is hyperbolic. Let U_L be a tubular neighborhood of L in $M-\partial F$ meeting ∂H_i regularly. We can assume that $U_i=U_L\cap H_i$ is a tubular neighborhood of T_i in B_i-D_i and $E(L, M)=\mathrm{cl}_M(M-U_L)$ and $E(T_i, H_i)=\mathrm{cl}_{H_i}(H_i-U_i)$ and $E(T_i^*, H_i)=q_i^{-1}E(T_i, H_i)$. Clearly, q_1 and q_2 define an almost identical imitation map q: $(M, L^*)\rightarrow(M, L)$ with $L^*=T_1^*\cup T_2^*$. Note that $E(L^*, M)=q^{-1}E(L, M)$ is a union of $E(T_1^*, H_1)$ and $E(T_2^*, H_2)$ pasting along a surface $F^E=\mathrm{cl}_F(F-F\cap U_L)$. Then we see from the following lemma that $E(L^*, M)$ is hyperbolic:

Lemma 5.3. *Let a 3-manifold M be splitted into two 3-submanifolds M_i, $i=1, 2$, by a proper surface F. If the following conditions are all satisfied, then M is hyperbolic*:

(1) *M_1 and M_2 are hyperbolic,*
(2) *F has no disk, annulus, torus components,*
(3) *$F_i^c=\partial M_i-\mathrm{Int}\, F$ has no disk components.*

This lemma is a direct consequence of Myers' lemmas (Lemmas 2.4, 2.5) in **[My]** and Thurston's hyperbolization theorem in **[T-3]**, **[Mo]**. We complete the proof of Lemma 5.1, assuming Lemma 5.2.

Proof of Lemma 5.2. We construct a pure r-braid σ with strings $b_1, b_2, \cdots, b_r$ in the 3-cube I^3 as follows (cf. Kanenobu **[Kn]**): Take $b_1\cup b_2\cup\cdots\cup b_{r-1}$ to be a trivial $(r-1)$-braid. Then take b_r so that b_r represents the $(r-2)$th commutator $[x_1, x_2, \cdots, x_{r-1}]$ in the free group $\pi=\pi_1(S^3-\hat{b}_1\cup\hat{b}_2\cup\cdots\cup\hat{b}_{r-1}, *)$ with a basis $x_1, x_2, \cdots, x_{r-1}$ represented by meridians of $\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_{r-1}$, for the closure link

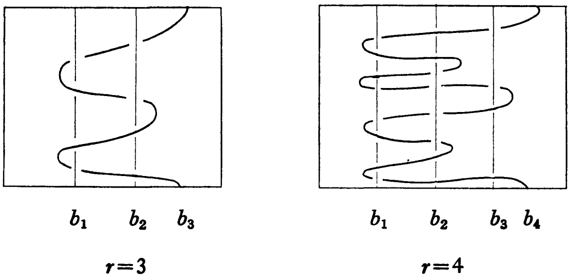


Figure 2

$\hat{\sigma}=\hat{b}_1\cup\hat{b}_2\cup\cdots\cup\hat{b}_r$ in S^3. For $r=3, 4$, we illustrate σ in Figure 2. Note that this r-braid σ has the following important property: That is, if we drop any one string b_i from σ, then the resulting $(r-1)$-braid is a trivial braid. The link $\hat{\sigma}$ is a typical example of a *link with Brunnian property* (cf. Rolfsen [**R**]), or in other words, an *almost trivial link* (cf. Milnor [**Mi**]). From this r-braid $\sigma\subset I^3$ and any two-string tangle $T\subset B$, we construct a new r-string tangle $T^{\oplus}\subset B^{\oplus}$ as it is illustrated in Figure 3.

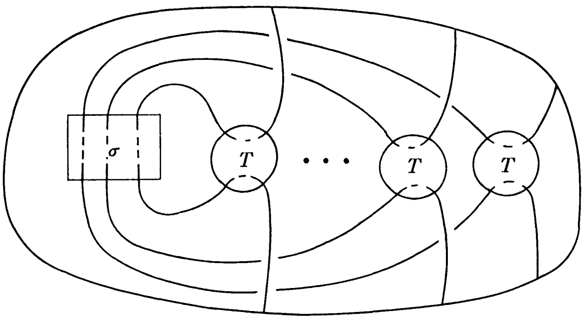


Figure 3

This construction has been suggested by Kanenobu [**Kn**, Figure 7]. A two-string tangle $T\subset B$ is said to be *simple*, if it is a prime tangle and the exterior $E(T, B)$ has no incompressible torus (cf. [**So**]) [*Note*: $E(T, B)$ may have an essential annulus as we observe in Remark 5.6]. The following lemma is obtained from Kanenobu's results in [**Kn**, Theorem 3 and Proposition 4] and Thurston's hyperbolization theorem [**T-3**], [**Mo**]:

Lemma 5.4. *If a two-string tangle* $T\subset B$ *is simple, then the exterior* $E(T^{\oplus}, B^{\oplus})$ *of the resulting new tangle* $T^{\oplus}\subset B^{\oplus}$ *is hyperbolic.*

Let $T^{\wedge}\subset B^{\wedge}$ be a one-string tangle obtained from a two-string tangle $T\subset B$ by adding a trivial one-string tangle $a_0\subset B_0$ as it is illustrated in Figure 4(1).

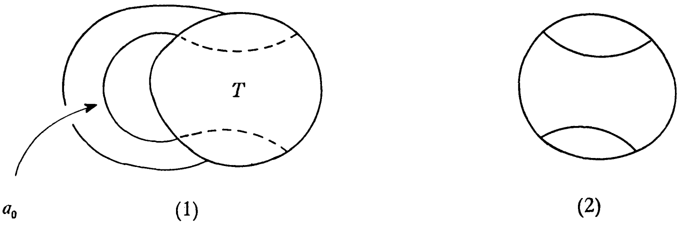


Figure 4

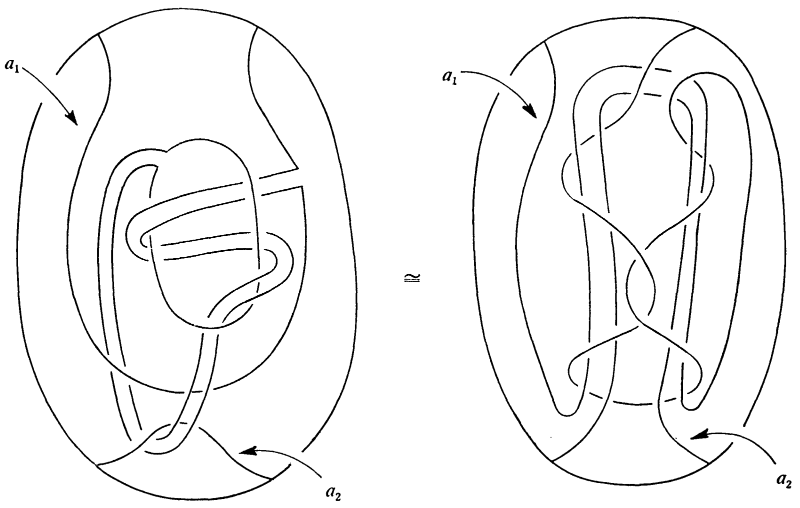


Figure 5

Let $T_0 \subset B$ be a trivial two-string tangle illustrated in Figure 4(2). Assume that there is a normal reflection α in $(B, T_0)\times I$ such that $\mathrm{Fix}(\alpha,(B, T_0)\times I)\cong(B, T)$. Let $\alpha^\wedge$ be the normal reflector in $(B^\wedge, T_0^\wedge)\times I$, extending α naturally, so that $\mathrm{Fix}(\alpha^\wedge, (B^\wedge, T_0^\wedge)\times I)\cong(B^\wedge, T^\wedge)$. If $\alpha^\wedge$ is isotopically standard, then we would have an almost identical imitation map $q\colon (B^\oplus, T^\oplus) \to (B^\oplus, T_0^\oplus)$. Since $(B^\oplus, T_0^\oplus)$ is a trivial tangle, we complete the proof of Lemma 5.2 when we assume the following lemma:

Lemma 5.5. *There are a simple two-string tangle $T\subset B$ and a normal reflection α in $(B, T_0)\times I$ with $T_0\subset B$ a trivial two-string tangle such that $(B, T)\cong$ $\mathrm{Fix}(\alpha, (B, T_0)\times I)$ and the extending normal reflection $\alpha^\wedge$ in $(B^\wedge, T_0^\wedge)\times I$ is isotopically standard.*

Proof of Lemma 5.5. Consider a two-string tangle $T=a_1\cup a_2\subset B$ illustrated in Figure 5. Since a_1 is a non-trivial arc in B[In fact, $E(a_1, B)$ is diffeomorphic to the exterior of the 11-crossing Kinoshita-Terasaka knot (cf. **[K-T]**, **[Kw-1]**)] and the one-string tangle $T^\wedge\subset B^\wedge$ is trivial, it follows from a result of Nakanishi

[**N**, Lemma 5.4] that (B, T) is a prime tangle. This tangle $T\subset B$ can be obtained from the Kinoshita-Terasaka tangle $T'=a_1'\cup a_2'\subset B$, illustrated in Figure 6, by sliding a boundary point of a_1' along ∂B and a_2'.

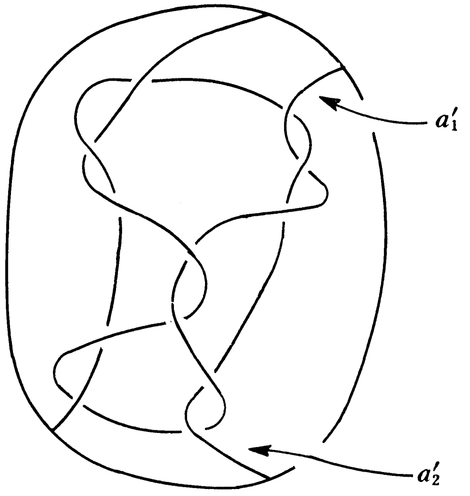


Figure 6

This means that $E(T, B)\cong E(T', B)$, so that $T\subset B$ is a simple tangle, because $T'\subset B$ is known to be simple (cf. Soma [**So**]). Let F be a union of two proper disks in $B\times I$ illustrated in Figure 7 by the motion picture method (cf. [**K-S-S**]). We denote by α_0 the standard reflection in $B\times I$ and by $\alpha_0^\wedge$ the extension to $B^\wedge\times I$. Let G be a 1-manifold with a band in B given by $(B, G)\times(1/4)=(B\times I, F)\cap B\times(1/4)$. We take annuli A, A' in the figure of $G\subset B$ as we illustrate in Figure 8. In Figure 8, $\{C_1, C_2\}$, $\{C_1', C_2'\}$ denote the boundary components of A, A' and the intersections $A\cap G$, $A'\cap G$ denote disks attaching to the circles C_1, C_1', respectively. Let $(B^\wedge\times I, F^\wedge)$ be a (4,2)-disk pair obtained from $(B\times I, F)$ by adding $(B_0, a_0)\times I$ with (B_0, a_0) in Figure 4(1). Note that C_2, C_2' bound disjoint disks D, D' in $B^\wedge - G^\wedge$ (where $G^\wedge = G\cup a_0$) so that $\bar{A}=A\cup D$, $\bar{A}'=A'\cup D'$ are disjoint disks in $B^\wedge$ with $\partial\bar{A}=C_1$, $\partial\bar{A}'=C_1'$. Let F' be a union of two proper disks in $B\times I$ illustrated in Figure 9, and $(B^\wedge\times I, F'^\wedge)$

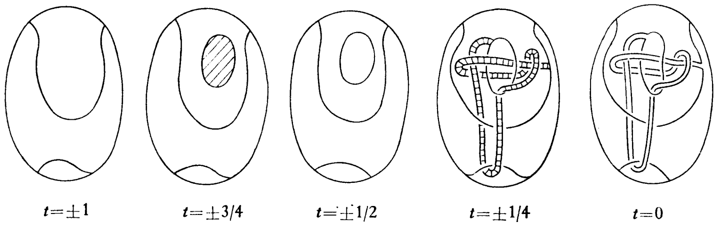


Figure 7

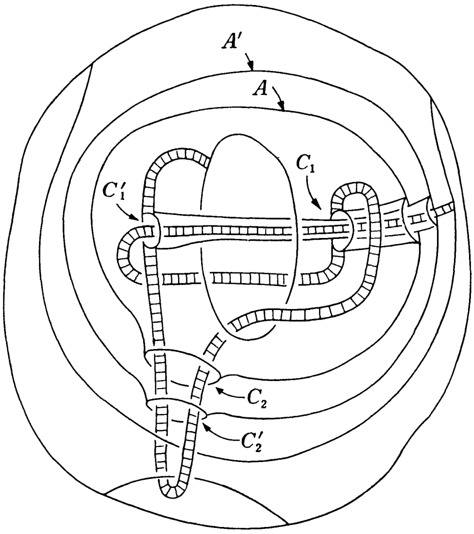
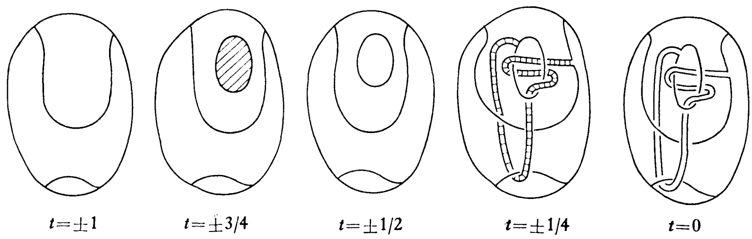


Figure 8

a (4, 2)-disk pair obtained from $(B\times I, F')$ by adding $(B_0, a_0)\times I$. Let G' be a 1-manifold with a band in B given by $(B, G')\times(1/4)=(B\times I, F')\cap B\times(1/4)$. Note that there is an $f\in \mathrm{Diff}_0(B^\wedge, \mathrm{rel}\,(B^\wedge - R))$ with $f(G^\wedge)=G'^\wedge$ for a regular neighborhood R of $\bar{A}\cup\bar{A}'$ in Int $B^\wedge$ by sliding the disks $\bar{A}\cap G^\wedge$, $\bar{A}'\cap G^\wedge$ along the disks $\bar{A}$, $\bar{A}'$. This means that there is an $\bar{f}\in \mathrm{Diff}(B^\wedge\times I, \mathrm{rel}\,(B^\wedge\times I - R\times I'))$ with $I'=[-1/2, 1/2]$ such that $\bar{f}$ is $\alpha_0^\wedge$-invariant and $\bar{f}(F^\wedge)=F'^\wedge$. Next, note that there is a $g\in \mathrm{Diff}_0(B^\wedge\times I, \mathrm{rel}\,\partial(B^\wedge\times I)\cup F^\wedge\cup F'^\wedge)$ such that $g((\bar{A}\cup\bar{A}')\times I')\subset B\times I$ by pushing $D\times I'$, $D'\times I'$ into $B\times(1/2, 3/4)$.



$t=\pm1$ $t=\pm3/4$ $t=\pm1/2$ $t=\pm1/4$ $t=0$

Figure 9

Then we may consider that $g(R\times I')\subset B\times I$. Let $h=g\bar{f}g^{-1}\in \mathrm{Diff}(B^\wedge\times I, \mathrm{rel}\,\partial(B^\wedge\times I))$. Then since $h(B\times I)=B\times I$, we can define an $h'\in \mathrm{Diff}(B\times I, \mathrm{rel}\,\partial(B\times I))$ by $h'=h\,|\,B\times I$. Note that $h'(F)=F'$. Since the bands appearing in Figure 7 are untied, we see that there is a $d\in \mathrm{Diff}(B\times I, \mathrm{rel}\,\partial(B\times I))$ such that d is α_0-invariant and $d(F)=T_0\times I$, where T_0 is a trivial two-string tangle in B determined by $T_0\times 1=F\cap B\times 1$. Let $\alpha_1=dh'^{-1}\alpha_0 h'd^{-1}$. Then α_1 defines a reflection in $(B, T_0)\times I$ with $\mathrm{Fix}(\alpha_1, (B, T_0)\times I)\cong(B, T)$. Further, we can find an $e\in\mathrm{Diff}_0((B, T_0)\times I, \mathrm{rel}\,\partial(B\times I))$ such that $\alpha=e\alpha_1 e^{-1}$ is a normal reflection in $(B, T_0)\times I$ by the fact that $\mathrm{Diff}(D, \mathrm{rel}\,\partial D)=\mathrm{Diff}_0(D, \mathrm{rel}\,\partial D)$ for a 2-disk D and the isotopy extension theorem and the uniqueness of tubular neighborhoods. Then

$$\mathrm{Fix}(\alpha, (B, T_0)\times I)\cong(B, T)$$

and

$$\alpha^\wedge = e^\wedge d^\wedge h^{-1}\alpha_0^\wedge h(d^\wedge)^{-1}(e^\wedge)^{-1},$$

where $d^\wedge$ and $e^\wedge$ denote the extension of d and e to $B^\wedge\times I$ by the identity, respectively. Let

$$h^* = e^\wedge d^\wedge h^{-1}\bar{f}(d^\wedge)^{-1}.$$

Then

$$h^* = e^\wedge d^\wedge g\bar{f}^{-1}g^{-1}\bar{f}(d^\wedge)^{-1}\in\mathrm{Diff}_0((B^\wedge, T_0^\wedge)\times I, \mathrm{rel}\,\partial(B^\wedge\times I)),$$

because $g\in\mathrm{Diff}_0(B^\wedge\times I, \mathrm{rel}\,\partial(B^\wedge\times I)\cup F^\wedge\cup F'^\wedge)$, and

$$h^{*-1}\alpha^\wedge h^* = d^\wedge\bar{f}^{-1}\alpha_0^\wedge\bar{f}(d^\wedge)^{-1} = \alpha_0^\wedge$$

because $\bar{f}$ and $d^\wedge$ are $\alpha_0^\wedge$-invariant. Hence $\alpha^\wedge$ is isotopically standard. This completes the proof of Lemma 5.5.

Therefore, we complete the proof of Theorem 1.1.

Remark 5.6. The exterior of the tangle $T\subset B$ in Figure 5, that is, the exterior of the Kinoshita-Terasaka tangle $T'\subset B$ in Figure 6 has an essential annulus, as it is illustrated in Figure 10. Hence it is not hyperbolic in our sense.

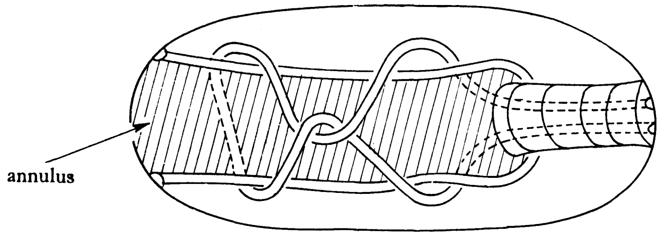


Figure 10

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