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A GENERALIZED LOCAL LIMIT THEOREM FOR LASOTA-YORKE TRANSFORMATIONS

Dedicated to Professor N. Ikeda on his sixtieth birthday

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0. Introduction

Let T be a Lasota-Yorke transformation of the unit interval $I=[0, 1]$. In virtue of the results in [7], we know that T has an m-absolutely continuous invariant probability measure $\mu=h_0 m$ with $h_0 \in BV$ where m denotes the Lebesgue measure on I and BV denotes the totality of functions of bounded variation on I. Hofbauer and Keller [3] investigate the ergodic properties of the dynamical system (T, μ). By use of the results Rousseau-Egele studies the limiting behavior of the distribution of the sum $S_n f=\sum_{j=0}^{n-1} f\circ T^j$ and proves a local limit theorem for a certain class of $f\in BV$ in [9]. The methods of those papers are based on the spectral analysis of the Perron-Frobenius operator (P-Foperator) $\mathcal{L}$: $L^1(m)\to L^1(m)$ and its perturbed operator $\mathcal{L}(it)$: $L^1(m)\to L^1(m)$ which are defined by $\mathcal{L}g=\frac{d}{dm}\int_{T^{-1}(\cdot)} g\,dm$ and $\mathcal{L}(it)g=\mathcal{L}(e^{itf}g)$ for $g\in L^1(m)$ respectively. We notice that Rousseau-Egele's method is quite similar to Nagaev's method in [8].

In this paper we shall investigate more detailed spectral properties of the perturbed operator $\mathcal{L}(it)$ and classify the elements in $BV_0=\{f\in BV_0(I\to \boldsymbol{R});$ $\int_I f\,d\mu=0\}$ into six types in Section 3. After the classification we shall prove the main theorem which asserts that the local limit theorem can be expressed in a quite general form in term of Schwartz distributions, for any $f\in BV_0\,(I\to \boldsymbol{R})$ with non-degenerate vairance. More precisely, imposing the mixing conditon (M) on T (see Section 2) we can prove:

Theorem (Theorem 4.1 in Section 4). Assume that $f\in BV_0$ with $\sigma^2=\lim_{n\to\infty}\frac{1}{n}\int (S_n f)^2 d\mu>0$. Then, there exist a $a>0$, $\lambda\in S^1$, and an S^1-valued measurable fun function h such that

$$\lim_{n\to\infty}\sup_{z\in R}\left|\sqrt{n}\int_I u(S_nf+z)g\,dm-\Phi_{g,z,n}(u)\frac{1}{\sqrt{2\pi}\sigma}\exp\left[\frac{-z^2}{2n\sigma^2}\right]\right|=0$$

holds for any $g\in BV$ and for any rapidly decreasing function u on R, where $\{\Phi_{g,z,n}\}$ is a bounded family of elements in $\mathcal{S}'$ defined by

$$\Phi_{g,z,n}(u)=\sum_{k=-\infty}^{\infty}\hat{u}(ka)e^{ikaz}\lambda^{kn}\int_I \bar{h}^k g\,dm\int_I h^k h_0\,dm$$

for any rapodly decreasing function u on $\boldsymbol{R}$.

In Section 1, we shall give a complete proof a Lasota-Yorke type inequality which will play important roles in our argument. Rousseau-Egele's proof of the local limit theorem also depends on an inequality of the same type but one may find that his proof of the inequality is not complete. In Section 2, we shall investigate the spectral properties of the perturbed P-F operators and we shall classify the elements in BV_0 in Section 3. Section 4 is devoted to the proof of the main theorem. In the last section we shall discuss about typical examples.

1. Preliminaries

First of all, we define the Lasota-Yorke transformation.

DEFINITION 1.1. A transformation T from the unit interval $I=[0, 1]$ into itself is called a *Lasota-Yorke transformation* or an *L-Y transformation* if the follwoing conditions (1), (2), and (3) are satisfied:

(1) There is a partition $\{I_j\}_j$ of I consisting of non-empty intervals such that (i) $T|\operatorname{Int} I_j$ is monotonic for each j, (ii) $T|\operatorname{Int} I_j$ is of class C^2 and can be extended to the closed interval $\bar{I}_j$ for each j, and (iii) $T(\operatorname{Int} I_j)=(0, 1)$ except for a finite number of j.

(2) (Renyi's condition).

$$\sup_x \frac{|T''(x)|}{|T'(x)|^2}<\infty\,, \tag{1, .1}$$

where $\sup_x$ is taken over all x at which T is twice differentiable.

(3) There is a positive integer N such that

$$\inf_x |(T^N)'(x)|\geqq 1/c \qquad \text{for some} \quad 0<c<1\,, \tag{1.2}$$

where inf is taken over all x at which T is differentiable.

We call a partition $P=\{I_j\}_{j'}$ a *defining partition* of T if it satisfies the condition (1) and is minimal in the folloaing sense: If $Q=\{I'_k\}_k$ is another partition satisfying the condition (1), then for each k, we can find $j=j(k)$ with $\mathrm{Int}\, I'_k \subset \mathrm{Int}\, I_j$.

We call T an *L-Y transformation of type I* if its defining partition is finite. We call T an *L-Y transformation of type II* if its defining partition is infinite.

REMARK 1.1. One can easily show that if T is an L-Y transfomation, then so is $T^n = \overbrace{T \circ \cdots \circ T}^{n}$ for any $n \in \boldsymbol{N}$.

Throughout the paper functions are assumed to be complex valued unless otherwise stated. For a measure μ on I $L^1(\mu)$ denotes the usual L^1-space with L^1-norm $\|\cdot\|_{1,\mu}$. We denote by BV the totality of elements in $L^1(m)$ which have versions of bounded variation. BV turns out to be a Banach space with Banach norm $\|g\|_{BV} = Vg + \|g\|_{1,m}$, where Vg denotes the infimum of total variations of all versions of $g \in BV$. BV_0 denotes the subspace of $BV(I \to R)$ whose elements satisfy $\int f\, d\mu = 0$. $\mathcal{S}$, $\mathcal{D}$, and $\mathcal{D}_N$ denote the spaces of rapidly decreasing functions on $\boldsymbol{R}$, smooth functions with compact support, and smooth functions on $(-N, N)$ with compact support respectively.

Next we define the P-F operators.

DEFINITION 1.2. Let T be an L-Y transformation. Let m be the Lebesgue measure on I. The *Perron-Frobenius operator* or the *P-F operator* $\mathcal{L}$ of T with respect to m is defined by

$$Lg = \frac{d}{dm} \int_{T^{-1}(\cdot)} g\, dm \qquad \text{for} \quad g \in L^1(m)\,. \tag{1.3}$$

For a real valued measurable function f on I and $t \in \boldsymbol{R}$, the perturbed P-F operator $\mathcal{L}(it) = \mathcal{L}(itf)$ of L is defined by

$$\mathcal{L}(it)g = \mathcal{L}(e^{itf} g) \qquad \text{for} \quad g \in L^1(m)\,. \tag{1.4}$$

REMARK 1.2. (1) For $g \in L^1(m)$, $\mathcal{L}g = g$ if and only if the complex measure gm is T-invariant.
(2) For any $n \in \boldsymbol{N}$, $\mathcal{L}(it)^n g = \mathcal{L}^n((\exp[itS_n f])g)$ where $S_n f = \sum_{j=0}^{n-1} f \circ T^j$. In particular $\int \mathcal{L}(it)^n g\, dm = \int (\exp[itS_n f]) g\, dm$. Therefore the asymptotic behavior of the distribution of $S_n f$ can be expressed in terms of the perturbed P-F opertors $L(it)$.

From the definition of L it is easy to show:

Proposition 1.1. *Let $\mathcal{L}$ be the P-F operator of T with respect to m. Let*

$\mu=h_0m$ be an m-absolutely continuous invariant probability measure with density h_0. Consider the operator $\mathcal{L}_\mu$ (the P-F operator of T with respect to μ) which is defined by $L_\mu g=\frac{d}{d\mu}\int_{T^{-1}(\cdot)}gd\mu$ for $g\in L^1(\mu)$. Then, for $g\in L^1(\mu)$ and an S^1-valued function φ the following are equivalent: (1) *$\mathcal{L}(\phi gh_0)=gh_0$ in $L^1(m)$*, (2) *$L_\mu(\phi g)=g$ in $L^1(\mu)$, and* (3) *$g\circ T=\phi g$ in $L^1(\mu)$.*

Proof. We know that the proposition is ture if φ is a constant function (see Ishitani [6]). In the present case, one can prove the proposition in the same way as in [6] except for the assertion that (2) implies (3). Therefore we restrict ourselves to prove this implication. Assume that $\mathcal{L}_\mu(\varphi g)=g$ in $L^1(\mu)$. Then it is not hard to see that $\mathcal{L}_\mu|g|=|g|$ in $L^1(\mu)$. Thus we have $|g|\circ T=|g|$ and $I_A\circ T=I_A$ in $L^1(\mu)$ where $A=\{x;\ |g|(x)\neq 0\}$. Since $\mathcal{L}_\mu$ preserves the value of the integration, we have

$$\int I_A\frac{\varphi g}{g\circ T}d\mu=\int I_A\circ T\frac{\varphi g}{g\circ T}d\mu=\int\mathcal{L}_\mu\Big(I_A\circ T\frac{\varphi g}{g\circ T}\Big)d\mu=I_A\frac{1}{g}\mathcal{L}_\mu(\varphi g)d\mu=\mu(A)\,.$$

On the other hand $\left|\frac{\varphi g}{g\circ T}\right|=|\varphi|=1$ *μ a.e.* on A. Hence we can conslude that *$g\circ T=\varphi g$ μ–a.e.*

In the rest of this section we prove the basic inequality in our argument.

Propositon 1.2 (Lasota-Yorke type inequality). *Let T be an L-Y transformation which satisfies the expanding condition* (1.2) *for $N=1$. Let $\mathcal{L}$ be the P-F operator of T with respect to m. Then, for any $n\in N$ and $f_0,f_1,\cdots,f_{n-1}\in BV(I\to S^1)$ we have*

$$V(\mathcal{L}^n((\prod_{k=0}^{n-1}f_k\circ T^k)g)\leqq(2+\sum_{k=0}^{n-1}Vf_k)\,[c^nVg+2(l_n^{-1}+R_n(T))\|g\|_{l,m}] \tag{1.5}$$

where $l_n=\min\{1, m(J_j);\ J_j$ is the element of a defining partition of T^n such that $T(Int\,J_j)\neq(0, 1)\}$ and

$$R_n(T)=\sup_x\frac{|(T^n)''(x)|}{|(T^n)'(x)|^2}\,.$$

Proof. Let $\{J_j\}_j$ be a defining partition of T^n. Notice that $S_j=T^n|Int\,J_j$ is a homeomorphism from $Int\,J_j$ onto its image for each j. We have

$$\begin{aligned}
&V\mathcal{L}^n((\prod_{k=0}^{n-1}f_k\circ T^k)g)\\
&=V[\sum\chi_{T^nJ_j}(S_j^{-1})|(T^n)'(S_j^{-1})|^{-1}\prod_{k=0}^{n-1}f_k(T^kS_j^{-1})g(S_j^{-1})]\\
&\leqq\sum_j\underset{j}{V}[|(T^n)|^{-1}\prod_{k=0}^{n-1}f_k\text{-}T^k)g]+\sum_j{}'\sup_j|(T^n)'|^{-1}[|g(a_j)|+|g(b_j)|]\\
&=\sum_j I_j+\sum_j{}'\,II_j,
\end{aligned}$$

where $\underset{j}{V}$ ednotes the total variation on J_j $\sup_j$ is taken over all $x \in Int\, J_j$, the summation $\sum_j{}'$ is taken over all j such that $T^n(Int\, J_j) \neq (0, 1)$, $a_j = \inf J_j$, and $b_j = \sup J_j$.

Before estimating I_j and II_j we claim that $\sup_j |(T^n)'|^{-1} d_j^{-1} \leqq l_n^{-1} + R_n(T)$, where $d_j = m(J_j)$. In fact

$$\begin{aligned} |(T^n)'(x)|^{-1} &\leqq |(T^n)'(x)^{-1} - (T^n)'(y)^{-1}| + |(T^n)'(y)|^{-1} \\ &\leqq R_n(T) d_j + |(T^n)'(y)|^{-1} \qquad \text{for any} \quad x, y \in Int\, J_j . \end{aligned}$$

Therefore we have $|(T^n)'(x)|^{-1} d_j^{-1} \leqq l_n^{-1} + R_n(T)$.

Using the claim and the inequality $V(g_1 g_2) \leqq \sup |g_1| V g_2 + \sup |g_2| V g_1$, we have

$$\begin{aligned} I_j &\leqq \Big(\sum_{k=0}^{n-1} V f_k\Big) \sup_j \big(|(T^n)'|^{-j} |g|\big) + \underset{j}{V}\big(|(T^n)'|^{-1} g\big) \\ &\leqq \Big(\sum_{k=0}^{n-1} V f_k\Big) \Big[\big(\sup_j |(T^n)'|^{-1}\big)\Big(d_j^{-1} \int_{J_j} |g|\, dm + \underset{j}{V} g\Big)\Big] \\ &\qquad + \big(\sup_j |(T^n)'|^{-1}\big) \underset{j}{V} g + \int_{J_j} \frac{|(T^n)''|}{|(T^n)'|^2} |g|\, dm \\ &\leqq \Big(1 + \sum_{k=0}^{n-1} V f_k\Big) \Big[c^n \underset{j}{V} g + (l_n^{-j} + R_n(T)) \int_{J_j} |g|\, dm\Big] . \end{aligned}$$

On the other hand we have

$$\begin{aligned} II_j &\leqq \sup_j |(T^n)'|^{-1} \Big(\underset{j}{V} g + 2 d_j^{-1} \int_{J_j} |g|\, dm\Big) \\ &\leqq c^n \underset{j}{V} g + 2(l_n^{-1} + R_n(T)) \int_{J_j} |g|\, dm] . \end{aligned}$$

since $|g(a_j)| + |g(b_j)| \leqq |g(a_j) - g(x)| + |g(x) - g(b_j)| + 2|g(x)|$ for any $x \in Int\, J_j$. Combining these estimates we obtain the inequality (1.5). //

REMARK 1.3. Since $\mathcal{L}(it)^n g = \mathcal{L}^n((\exp[itS_n f])g) = \mathcal{L}^n(e^{itf} e^{itf(T)} \cdots e^{itf(T^{n-1})} g)$, we can apply the inequality (1.5) to $\mathcal{L}(it)^n g$ if $f \in BV(I \to \boldsymbol{R})$. Therefore we can justify Proposition 5 in [9] which asserts that $\mathcal{L}(it)$ satisfies the conditions of Ionescu Tulcea and Marinescu Theorem (see [5] and [9]).

3. Spectral decomposition of perturbed *P-F* operators

From now on we impose the following mixing condition (M) on T.

(M) T has a unique m-absolutely continuous probability measure $\mu = h_0 m$ with support I and the dynamical system (T, μ) is mixing (see Bowen [1, Theorem 2]).

In what follows, T denotes an L-Y transformation which satisfies the condition (M), unless otherwise stated.

Lemma 2.1. *For $f\in BV(I\to\boldsymbol{R})$, and $t\in\boldsymbol{R}$ define $U(it)$: $L^1(\mu)\to L^1(\mu)$ by $U(it)g=e^{-itf}g\circ T$. Then we have the following:*

(1) *For $\lambda\in S^1$, λ is an eigenvalue of $\mathcal{L}(it)$ on $L^1(m)$ if and only if $\bar{\lambda}=\lambda^{-1}$ is an eigenvalue of $U(it)$.*

(2) *If h is an eigenvector of $U(it)$ on $L^1(\mu)$ corresponding to an eigenvalue with modulus 1, then $|h|$ is constant μ-a.e.*

(3) *Let $\lambda\in S^1$ be an eigenvalue of $U(it)$. For $h\in L^1(\mu)$, h is an eigenvector corresponding to λ if and only if hh_0 is an eigenvector of $\mathcal{L}(it)$ on $L^1(m)$.*

(4) *If λ is an eigenvalue of $U(it)$ on $L^1(\mu)$, then it is simple.*

(5) *$U(it)$ has at most one eigenvalue of modulus 1.*

Proof. (1) and (3) follows immediately from Proposition 1.1 and (2) is a direct consequence of the ergodicity of the dynamical system (T,μ). Now we prove (4). Assume that $h_j\in L^1(\mu)$ $(j=1,2)$ satisfies $h_j\circ T=\bar{\lambda}e^{itf}h_j$ for $\lambda\in S^1$. From (2) we may assume that $|h_j|=1$ μ-a.e.. Therefore $h_1h_2^{-1}\in L^1(\mu)$ and $(h_1h_2^{-1})\circ T=(h_1\circ T)(h_2\circ T)=h_1\circ T)^{-1}=h_1h_2^{-1}$. Thus $h_1h_2^{-1}=$constant μ-a.e. by the ergodicity of (T,μ). Hence λ is simple.

Next we prove (5). Assume that $h_j\in L^1(\mu)$ and $\lambda_j\in S^1$ $j=1,2$ satisfy $h_j\circ T=\lambda_je^{itf}h_j$. In the same way as in the proof of (4) we have $(h_1h_2^{-1})\circ T=\lambda_1\lambda_2^{-1}h_1h_2^{-1}$ μ-a.e. Since the dynamical system (T,μ) is mixing, $\lambda_1\lambda_2^{-1}$ must be 1. Thus $\lambda_1=\lambda_2$.

In virtue of Lemma 2.1, we may write $\lambda(it)$ to denote the eigenvelue of $\mathcal{L}(it)$ with modulus 1 if it exists.

DEFINITION 2.1. For $f\in BV(I\to\boldsymbol{R})$ define

$\Lambda(f)=\{t\in\boldsymbol{R}$; $\mathcal{L}(it)$ on $L^1(m)$ has an eigenvalue with modulus $1\}$

$G(f)=\{\lambda\in S^1;\lambda=\lambda(it)$ for some $t\in\Lambda(f)\}$

$H_0(f)=\{h\in L^1(\mu)$; h is S^1-valued and $h\circ T=\bar{\lambda}e^{itf}h$ for some $t\in\Lambda(f)$ and $\lambda\in G(f)\}$

and

$H(f)=\{(h);h\in H_0(f)\}$

where (h) denotes the equivalent class containing h under the following equivalence relation: $h_1\sim h_2$ if and only if $h_1=\kappa h_2$ for some $\kappa\in S^1$.

Lemma 2.2. *$\Lambda(f)$ is a subgroup of $\boldsymbol{R}$, $G(f)$ is a subgroup of S^1 and $H(f)$ is an abelian group under the multiplication $(h_1)(h_2)=(h_1h_2)$.*

Proof. Let $a_j\in\Lambda(f)$, $\lambda_j\in G(f)$, and $h_j\in H_0(f)$ for $j=1,2$ with $h_j\circ T=\bar{\lambda}_j(\exp[ia_jf])h_j$. Then we have $(h_1h_2)\circ T=\bar{\lambda}_1\bar{\lambda}_2(\exp[i(a_1+a_2)f])h_1h_2$. Thus

$a_1+a_2 \in \Lambda(f)$, $\lambda_1\lambda_2 \in G(f)$, and $h_1h_2 \in H(f)$. On the other hand $\bar{h}_1 \circ T = \lambda_1(\exp[-ia_1f])\bar{h}_1$ implies that a_1, λ_1, and h_1 have inverse elements $-a_1$, $\bar{\lambda}_1$, and $\bar{h}_1$ respectively. It is obvious that the group operation $(h_1)(h_2)=(h_1h_2)$ of $H(f)$ is well-defined. //

The following lemma plays important roles throughout the paper.

Lemma 2.3. *Let T be an L-Y transformation satisfying the mixing condition (M) and $f \in BV_0$. Then we have the following:*

(1) *For each $t \in \boldsymbol{R}$, the perturbed P-F operator $\mathcal{L}(it)=\mathcal{L}(itf)$ is a bounded operator on BV as well as a bounded operator on $L^1(m)$.*

(2) *If $s \notin \Lambda(f)$ the spectral radius of $\mathcal{L}(it)$ as an operator on BV is less than 1.*

(3) *If $s \in \Lambda(f)$ then for t in a heighborhood $N(s)$ of s in $\boldsymbol{C}$, $\mathcal{L}(it)$ has the spectral decomposition*

$$\mathcal{L}(it)^n = \lambda(it)^n E(it) + R(it)^n \qquad \text{for} \quad n \geqq 1 \tag{2.1}$$

as an operator on BV with the following properties:

(i) *$\lambda(it)$ is holomorphic in $N(s)$ and coincides with the eigenvalue of $\mathcal{L}(it)$ with maximal modulus. In addition we have*

$$\lambda'(is) = \left(\frac{d\lambda}{dt}\right)_{t=is} = 0\,, \tag{2.2}$$

and

$$\lambda''(is) = \left(\frac{d^2\lambda}{dt^2}\right)_{t=is} = \lim_{n\to\infty}\frac{1}{n}\int_I (S_nf)^2 d\mu \cdot \lambda(is) = \sigma(f)^2\lambda(is)\,. \tag{2.3}$$

(ii) *$E(it)$ is the projection operator onto the one-dimensional eigenspace corresponding to $\lambda(it)$ which depends holomorphically in $t \in N(s)$ and satisfies*

$$\int E(is)g\, dm = \int \bar{h}g\, dm \int hh_0\, dm \tag{2.4}$$

for any $g \in BV$, where h denotes an arbitrary eigenvector corresponding to $\lambda(is)$ with $|h|=1$ μ-a.e.

(iii) *$R(it)$ is the operator valued holomorphic function in $N(s)$ defined by the Dunford integral*

$$R(it)^n = \frac{1}{2\pi i}\int_{|\gamma|=r} \gamma^n R_\gamma(it) d\gamma \tag{2.5}$$

for some $0<r<1$, hwere $R_\gamma(it)=(\gamma I - \mathcal{L}(it))^{-1}$.

(iv) *At $t=s$, the spectral decomposition (2.1) has still a meaning in $L^1(m)$. Precisely, $E(is)$ and $R(is)$ turn out to be bounded operators on $L^1(m)$ and the range of $E(it)$ as an operator on $L^1(m)$ coincides with the range of $E(it)$ as an operator* *on BV, and $R(it)^n g \to 0$ in $L^1(m)$ as $n \to \infty$, for any $g \in L^1(m)$.*

Proof. In virtue of the Lasota-Yorke type inequality (1.5) for $\mathcal{L}(it)$, we can apply Ionescu Tulcea and Marinescu Theorem in [5] to $\mathcal{L}(it)$. On the other hand we know that $\mathcal{L}(it)$ has at most one eigenvalue of modulus 1 and if it exists, then it is simple for each $t \in \boldsymbol{R}$ from Lemma 2.1. Combining those facts with the general perturbation theorey (see Dunford and Schwartz [2, p. 584–] and Rousseau-Egele [9, Proposition 5]), we can see the lemma except for the equalities (2.2), (2.3), and (2.4). (2.2) and (2.3) can be proved in the same way as Lemma 2 and Lemma 3 in [9] (see also Lemma 5.1 and Lemma 5.2 in [6]). Therefore we restrict ourselves to give a sketch. If $t+s \in N(s)$, we have

$$\begin{aligned}\int \exp[itS_nf]\,d\mu &= \int (\exp[itS_nf])h_0\,dm \\ &= \int \bar{h}\circ T^n \lambda(is)^{-n}(\exp[i(s+t)S_nf])hh_0\,dm \\ &= \lambda(is)^{-n}\int \bar{h}\mathcal{L}^n((\exp[i(s+t)S_nf])hh_0)\,dm \\ &= \lambda(is)^{-n}\int \bar{h}\mathcal{L}(i(s+t))^n(hh_0)\,dm\end{aligned}$$

for any $h \in H_0(f) \cap E(is)(BV)$. Here we have used the identity $h\circ T^n = \lambda(is)^{-n}(\exp[isS_nf])h$. Thus we have

$$\begin{aligned}\int \exp[itS_nf]\,d\mu &= \lambda(is)^{-n}\lambda(i(t+s))^n \int \bar{h}\,E(i(t+s))\,(hh_0)\,dm \\ &\quad + \lambda(is)^{-n}\int \bar{h}\,R(i(t+s))^n(hh_0)\,dm \\ &= p(t)+r(t)\,.\end{aligned} \tag{2.6}$$

Now we have

$$\left(\frac{d}{dt}\left(\int \exp\left[it\frac{S_nf}{n}\right]d\mu\right)\right)_{t=0} = \left(\frac{dp(tn^{-1})}{dt}\right)_{t=0} + \left(\frac{dr(tn^{-1})}{dt}\right)_{t=0} \tag{2.7}$$

and

$$\left(\frac{d^2}{dt^2}\left(\int \exp\left[it\frac{S_nf}{\sqrt{n}}\right]d\mu\right)\right)_{t=0} = \left(\frac{d^2p(tn^{-1/2})}{dt^2}\right)_{t=0} + \left(\frac{d^2r(tn^{-1/2})}{dt^2}\right)_{t=0}. \tag{2.8}$$

The left hand side of (2.7) equals $i\int \frac{S_nf}{n}d\mu$ and goes to 0 as $n \to \infty$ by the ergodic theorem. The right hand side goes to $i\lambda(is)'^{-1}\lambda'(is)\int \bar{h}\,E(is)\,(hh_0)\,dm = i\lambda(is)^{-1} \times \lambda'(is)$ as $n \to \infty$, by the same way as in Lemma 2 in [9]. Note that we have used the fact that hh_0 is an eigenvector of $\mathcal{L}(it)$ corresponding to $\lambda(is)$ (see Lemma 2.1). Next the left hand side of (2.8) equals $-\frac{1}{n}(S_nf)^2d\mu$. On the other hand, the right hand side of (2.8) goes to $-\lambda(is)^{-1}\lambda''(is)$ by the same way as in Lemma 3 in [9].

Next we prove the identity (2.4). Since $h\circ T^n=\lambda(is)^{-n}(\exp[isS_nf])h$, we have

$$\int h\circ T^n\bar{h}g\,dm=\lambda(is)^{-n}\int\mathcal{L}(is)^n g\,dm\,. \tag{2.9}$$

Since the dynamical system (T,μ) is mixing, the left hand side of (2.9) goes to $\int\bar{h}g\,dm\int hh_0\,dm$. Clearly, the right hand side of (2.9) goes to $\int E(is)g\,dm$ from (2.1) and (2.5).

As a corollary to Lemma 2.3, we obtain Lemma 2.4. The proof is quite similar to the proof of Lemma 7 in [9] and Lemma 5.3 in [6].

Lemma 2.4. *Let $f\in BV_0$ with $\sigma^2=\sigma(f)^2=\lim_{n\to\infty}\frac{1}{n}\int(S_nf)^2d\mu>0$ and $g\in BV$. If $s\in\Lambda(f)$, there exist positive number A_3, A_4, A_1, A_2, and $0<\rho<1$ depending on s and g such that*

$$\begin{aligned}\Big|\int_I&(\exp[i(s+tn^{-1/2})S_nf])g\,dm\\&-\lambda(is)^n\int_I\bar{h}g\,dm\int_I hh_0\,dm\left(\exp\left[-\frac{1}{2}\sigma^2t^2\right]\right)\int_I g\,dm\\&\leqq\exp\left[-\frac{1}{4}\sigma^2t^2\right](A_1|t|^3n^{-1/2}+A_2|t|n^{-1/2})+A_3\rho^n\,,\end{aligned} \tag{2.10}$$

whenever $|t|\leqq A_4n^{1/2}$, where h is any element in $H_0(f)\cap E(is)(BV)$ and the second term of (2.10) is independent of the choice of h.

Proof. We notice that

$$\begin{aligned}&\int_I(\exp[i(s+t)S_nf])g\,dm\\&=\int_I\mathcal{L}(i(s+t))^n g\,dm\\&=\lambda(is)^n\left(\frac{\lambda(i(s+t))/^n}{\lambda(is)^n}\right)\int_I E(i(s+t))g\,dm+\int_I R(i(s+t))^n g\,dm\end{aligned}$$

for t with small modulus. Therefore we can prove the lemma by considering the Taylor expansion of the last line around s as in the proof of Lemma 7 in [9].

3. Classification of BV_0

Let T be an L-Y transformation which satisfies the condition (M).

DEFINITION 3.1. $B_0=\{f\in BV_0; \Lambda(f)=\boldsymbol{R}\}$,
$B_1=\{f\in BV_0; \Lambda(f)\simeq\boldsymbol{Z}, G(f)\simeq\boldsymbol{Z}/p\boldsymbol{Z}, \boldsymbol{H}(f)\simeq\boldsymbol{Z}/q\boldsymbol{Z}$ for some p and $q\}$,
$B_2=\{f\in BV_0; \Lambda(f)\simeq\boldsymbol{Z}, G(f)\simeq\boldsymbol{Z}/p\boldsymbol{Z}, H(f)\simeq\boldsymbol{Z}$ for some $p\}$
$B_3=\{f\in BV_0; \Lambda(f)\simeq\boldsymbol{Z}, G(f)\simeq\boldsymbol{Z}, \boldsymbol{H}(f)\simeq\boldsymbol{Z}/p\boldsymbol{Z}$ for some $p\}$
$B_4=\{f\in BV_0; \Lambda(f)\simeq\boldsymbol{Z}, G(f)\simeq\boldsymbol{Z}, H(f)\simeq\boldsymbol{Z}\}$ and
$B_4=\{f\in BV_0; \Lambda(f)=\{0\}, G(f)=\{1\}, \boldsymbol{H}(f)=\{(1)\}\}$, where $A\simeq B$ means that A and B are isomorphic as groups.

Then we have:

Theorem 3.1. *For $f\in BV_0$ we have:*

(1) *$f\in B_0$ if and only if $\sigma(f)=0$. in this case $G(f)$ is automatically $\{1\}$.*

(2) *$f\in B_1$ if and only if there exist $b>0$, and $K\in BV(I\to\boldsymbol{Z})$ such that $bf=2\pi K$, $\int Kd\mu=0$, and $\sigma(K)>0$.*

(3) *$f\in B_2$ if and only if there exist $b>0$, $K\in BV(I\to\boldsymbol{Z})$, and a real valued bounded function g such that ng is not a $\boldsymbol{Z}$-valued function for $n\in\boldsymbol{Z}-\{0\}$, $bf=2\pi(g\circ T-g+K)$, $\int Kd\mu=0$, and $\sigma(K)>0$.*

(4) *$f\in B_3$ if and only if there exist $b>0$, $\theta\in(0,1)\cap\boldsymbol{Q}^c$ and $K\in BV(I\to\boldsymbol{Z})$ such that $bf=2\pi(\theta+K)$ and $\int Kd\mu=-\theta$.*

(5) *$f\in B_4$ if and only if there exist $b>0$, $\theta\in(0,1)\cap\boldsymbol{Q}^c$, and a real valued bounded function g such that ng is not a $\boldsymbol{Z}$-valued function for $n\in\boldsymbol{Z}-\{0\}$, $bf=2\pi(g\circ T-g+K+\theta)$, and $\int Kd\mu=-\theta$.*

(6) *$BV_0=\bigcup_{j=0}^{5} B_j$ (disjoint union).*

Proof. (1) If $f\in BV_0$ with $\sigma(f)=0$, then we can write $f=g\circ T-g$ for some $g\in L^2(\mu)$ (see [4, p. 323], and [9, Lemma 6]). Therefore we have $e^{itg}\circ T=e^{itf}e^{itg}$ for any $t\in\boldsymbol{R}$. Thus we have seen that $\Lambda(f)=\boldsymbol{R}$. Conversely, if $\sigma(f)>0$, then $\lambda'(0)=0$ and $\lambda''(0)>0$ (i.e. $\left(\frac{d^2\lambda(it)}{dt^2}\right)_{t=0}=-\lambda''(0)<0$). Therefore in a neighborhood of 0 in $\boldsymbol{R}$, $|\lambda(it)|<1$ if $t\neq0$. This implies $\Lambda(f)\neq\boldsymbol{R}$. Hence $f\in B_0$ implies $\sigma(f)=0$. Next we prove the assertion (6). For $f\in BV_0-B_0$, define $a=\inf\{t>0; t\in\Lambda(f)\}$ if the set is not empty, $a=\infty$ otherwise. If $a=\infty$, it is obvious that $f\in B_5$. If $a<\infty$, then we can show $a\in\Lambda(f)$. In fact if $t_n\in\Lambda(f)$, $n=1,2,\cdots$ $t_n\downarrow a$ $(n\to\infty)$, and $h_n\circ T=\bar{\lambda}_n(\exp[it_nf])h_n$ for $h_n\in H_0(f)$ and $\lambda_n\in G(f)$, then there exists a constant $C>0$ such that $V(h_0h_n)\leqq C$ for any n, in virtue of the Lasota-Yorke type inequality (1.5) and (iv) of (3) in Lemma 2.3. Therefore we can choose a subsequence $\{h_{n'}\}$ of $\{h_n\}$, an S^1-valued measurable function h and $\lambda\in S^1$ such that $h_{n'}\to h$ μ-a.e. $(n'\to\infty)$ and $\lambda_{n'}\to\lambda$. Therefore $h\circ T=\bar{\lambda}(\exp[iaf])h$. Thus we have seen $a\in\Lambda(f)$. Moreover, it is not hard to see that $\Lambda(f)=a\boldsymbol{Z}$, $G(f)=\langle\lambda\rangle=\{\lambda^n; n\in\boldsymbol{Z}\}$ and $H(f)=\langle(h)\rangle=\{(h^n); n\in\boldsymbol{Z}\}$. Hence $BV_0-B_0=\bigcup_{j=1}^{5}B_j$. The proofs of (2), (3), (4), and (5) are quite similar to one another. So we prove (5) only. If $f\in B_5$, then we have $\Lambda(f)=a\boldsymbol{Z}$, $G(f)=\langle\lambda\rangle=\{e^{2\pi i\theta n}; \theta\in(0,1)\cap\boldsymbol{Q}^c, n\in Z\}$ and $H(f)=\langle(h); h^n$ is not constant function $h\in\boldsymbol{H}_0(f)\rangle$. Putting $g=arg[h]/2\pi$, we have $af=2\pi(g\circ T-g+\theta+K)$ for some Z-valued function K. Since $hh_0\in BV$ and $h_0\in BV$ we can see that g has a version without discontinuities of the second kind. Therefore K has also a version without discontinuities of the second kind. Thus K is in BV since it is Z-valued. Conversely, assume that $bf=2\pi(g\circ T-g+\theta+K)$ for some $b>0$, g, θ and K which satisfy the conditions in the assertion (5). Then we have $\sigma(f)>0$ from the assertion (1). Therefore $b=ja$ for some $j\in N$, where $a=\inf\{t>0; t\in\Lambda(f)\}$. From these fact it is not hard to see that $\Lambda(f)=aZ$. $G(f)=\langle e^{2\pi i\theta/j}\rangle\simeq\boldsymbol{Z}$, and $H(f)=\langle e^{2\pi ig/j}\rangle$.

4. Generalized local limit theorem

In this section we prove the main theorem.

Theorem 4.1. *Let $f\in BV_0-B_0$. Assume that $a>0$, $\lambda\in S^1$ $h\in H_0(f)$ satisfy $\Lambda(f)=a\boldsymbol{Z}$, $G(f)=\langle\lambda\rangle$, $H(f)=\langle(h)\rangle$, and $h\circ T=\bar{\lambda}(\exp[iaf])h$. Then for any $u\in\mathcal{S}$, and for any $g\in BV$ we have*

(4.1)

$$\lim_{n\to\infty}\sup_{z\in R}|\sqrt{n}\int_I u(S_nf(x)+z)g(x)m(dx)-\Phi_{g,z,n}(u)\frac{1}{\sqrt{2\pi}\sigma(f)}\exp[\left[\frac{-z^2}{2n\sigma(f)^2}\right]|=0\,.$$

Here for any $g\in BV$, $\{\Phi_{g,z,n}\}_{z,n'}$ is a bounded set in $\mathcal{S}'$ defined by

$$\Phi_{g,z,n}(u)=\sum_{K=-\infty}^{\infty}\hat{u}(ka)e^{ikaz}\lambda^{kn}\int_I\bar{h}^kg\ dm\int_I h^kh_0\,dm \tag{4.2}$$

for any $u\in S$, where $\hat{u}(t)=\int_{-\infty}^{\infty}e^{-itx}u(\mathrm{t}x)\,dx$.

Proof. It suffices to prove the theorem for $g\in BV$ with $g\geqq 0$, and $\int_I g\,dm=1$. First of all, we consider the case $\hat{u}\in\mathcal{D}_N$ for some $N>0$. It is easy to see

$$\begin{aligned}\sqrt{n}\int_I u(S_nf+z)g\,dm&=\frac{\sqrt{n}}{2\pi}\int_R\hat{u}(t)\phi_n(t)e^{itz}dt\\&=\sum_{k=-\infty}^{\infty}\frac{\sqrt{n}}{2\pi}\int_{(-1/2)a}^{(1/2)a}\hat{u}(ka+t)\phi_n(ka+t)e^{i(ka+t)z}dt\end{aligned}$$

where $\phi_n(t)=\int(\exp[itS_nf])g\,dm$.

Fix $k\in Z$ for a while.

$$\frac{\sqrt{n}}{2\pi}\int_{(-1/2)a}^{(1/2)a}\hat{u}(ka+t)\phi_n(ka+t)e^{i(ka+t)z}\,dt \\ -\hat{u}(ka)e^{ikaz}\alpha_k(n)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{z^2}{2n\sigma^2}\right] \\ =R_2(n)+R_1(n)+R_3(n)+R_4(n)\,, \tag{4.3}$$

where $\alpha_k(n)=\lambda^{kn}\int\bar{h}g\,dm\int hh_0\,dm$,

$$R_1(n)=\frac{\sqrt{n}}{2\pi}\int_{\varepsilon_n\le|t|<(1/2)a}\hat{u}(ka+t)\phi_n(ka+t)e^{i(ka+t)z}\,dt\,,$$

$$R_2(n)=\frac{1}{2\pi}\int_{|t|\le\varepsilon_n\sqrt{n}}\hat{u}((ka+t/\sqrt{n})-\hat{u}(ka))\phi_n(ka+t/\sqrt{n})e^{i(ka+t/\sqrt{n})z}dt\,,$$

$$R_3(n)=\frac{1}{2\pi}\int_{|t|\le\varepsilon_n\sqrt{n}}(\phi_n(ka+t/\sqrt{n})-\alpha_k(n)\exp[-\frac{1}{2}\sigma^2t^2])\hat{u}(ka)e^{i(ka+t/\sqrt{n})z}\,dt\,,$$

and

$$R_4(n)=-\frac{1}{2\pi}\int_{|t|\ge\varepsilon_n\sqrt{n}}\exp[-\frac{1}{2}\sigma^2t^2+itz/\sqrt{n}]\,dt\alpha_k(n)\hat{u}(ka)e^{ikaz}\,.$$

The number ε_n will be determined later. Since $\left(\frac{d\lambda(it)}{dt}\right)_{t=ka}=0$, $\left(\frac{d^2\lambda(it)}{dt^2}\right)_{t=ka}=-\lambda(ika)\sigma^2=-\lambda^k\sigma^2$, and the spectral radius of $L(i(ka+t))$ is less than 1 for $\varepsilon_n\le|t|<\frac{1}{2}a$, we have

$$|R_1(n)|\le\frac{\sqrt{n}}{2\pi}\sup|\hat{u}|\int_{\varepsilon_n\le|t|<(1/2)a}\|\mathcal{L}(i(ka+t))^n g\|_{BV}\,dt \\ \le\frac{\sqrt{n}}{2\pi}\sup|\hat{u}|C_k\int_{\varepsilon_n\le|t|<(1/2)a}(1-\frac{1}{4}\sigma^2t^2)^n dt\|g\|_{BV} \tag{4.4}$$

in virtue of the spectral decomposition (2.1). Here C_k is a positive constant depending only on k. In virtue of the mean value theorem we have

$$|R_2(n)|\le\frac{1}{2\pi}\varepsilon_n^2\sqrt{n}\,\sup|(\hat{u})'|\,. \tag{4.5}$$

From Lemma 2.4 we have

$$|R_3(n)|\le\frac{1}{2\pi}C_k'\int_{|t|\le\varepsilon_n\sqrt{n}}(|t|^3/\sqrt{n}+|t|/\sqrt{n}\,\rho_k^n)dt(\sup|\hat{u}|) \\ \le\frac{1}{\pi}C_k''(\varepsilon_n^4n^{3/2}+\varepsilon_n^2n^{1/2})\,(\sup|\hat{u}|)\,, \tag{4.6}$$

where C_k' and $\rho_k<1$ are positive constants depending only on k. Clearly we obtain

$$(4.7)\qquad |R_4(n)|\leqq\frac{1}{2\pi}\int_{|t|\geqq\varepsilon_n\sqrt{n}}\exp\left[-\frac{1}{4}\sigma^2t^2\right]dt\,|\alpha_k(n)|(\sup|\hat{u}|)\,.$$

If we take ε_n so that $\varepsilon_n\downarrow 0$, $\varepsilon_n n^{1/2}\uparrow\infty$, and $\varepsilon_n^4 n^{3/2}\downarrow 0$ as $n\uparrow\infty$, then we can find a sequence $\{\gamma_n\}_n$ with $\gamma_n\to 0$ as $n\uparrow\infty$ such that

$$(4.8)\qquad |R_1(n)+R_2(n)+R_3(n)+R_4(n)|\leqq C_N\gamma_n(\sup|\hat{u}|+\sup|(\hat{u})'|)$$

in virtue of the estimates (4.4), (4.5), (4.6), and (4.7), where C_N is a positive constant depending only on g and N. We notice that (4.8) shows that the set $\{\sqrt{n}\,\phi_n(t)e^{itz}\}_{z,n}$ is a bounded set in the space $\mathcal{B}'$ of bounded distributions. In fact we have $|\phi_{g,z,n}(u)|\leqq 2\left[\frac{N}{a}\right]\sup|\hat{u}|$ for each $\hat{u}\in\mathcal{D}_N$. Therefore we obtain

$$\begin{aligned}
&\left|\int\sqrt{n}\,\phi_n(t)e^{itz}\hat{u}(t)\,dt\right|\\
&\qquad\leqq|R_1(n)+R_2(n)+R_3(n)+R_4(n)|+\left|\Phi_{g,z,n}(u)\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{z^2}{2n\sigma^2}\right]\right|\\
&\qquad\leqq C_N(\sup|\hat{u}|+\sup|(\hat{u})'|)+\frac{2}{\sqrt{2\pi}\sigma}\left[\frac{N}{a}\right]\sup|\hat{u}|
\end{aligned}$$

from the estimate (4.8).

Next we take a sequence $\{\rho_j\}_{j=1}^{\infty}$ of probability measures on R which converges weakly to δ_0 as $j\to\infty$ and $\hat{\rho}_j\in\mathcal{D}$ for every j. We write $\sqrt{n}\int u(S_nf+z)\times g\,dm$ as $\int_R u(t)\mu_{z,n}(dt)$ for convenience. Clearly the characteristic function $\hat{\mu}_{z,n}$ of $\mu_{z,n}$ is $\sqrt{n}\,\phi_n e^{itz}$. Take $u\in\mathcal{S}$ and fix it. Then we have

$$\begin{aligned}
&\left|\int_R u(t)\,(\rho_j*\mu_{z,n})\,(dt)-\int_R u(t)\mu_{z,n}(dt)\right|\\
&\leqq\int_{|s|<\delta}\rho_j(ds)\left|\int_R(u(t+s)-u(t))\mu_{z,n}(dt)\right|\\
&\qquad+\int_{|s|\geqq\delta}\rho_j(ds)\left|\int_R(u(t+s)-u(t))\mu_{z,n}(dt)\right|\\
&=I_n+II_n\,.
\end{aligned}$$

Since $\{\hat{u}_{z,n}\}_{z,n}$ is a bounded set in $\mathcal{B}'$ and $\hat{u}\in\mathcal{S}\in\mathcal{B}$ we have

$$\sup_{s\in R}\left|\int u(t+s)\mu_{z,n}(dt)\right|=\sup_{s\in R}\left|\frac{1}{2\pi}\int\hat{u}(t)\sqrt{n}\,\phi_n(t)e^{it(z+s)}dt\right|\leqq C_1(u)$$

where $C_1(u)$ is a constant depending only on u. Since the set $\{v_s(t)=s^{-1}(u(t+s)-u(t))\}_{0<|s|\leqq 1}$ is bounded in $\mathcal{S}$, we see $\{\hat{v}_s\}_{0<|s|\leqq 1}$ is a bounded set in $\mathcal{S}$ and consequently bounded set in $\mathcal{B}$. Therefore we have

$$\sup_z \sup_{|s|\leqq 1} |\int v_s(t)\mu_{z,n}(dt)| \leqq C_2(u)$$

where $C_2(u)$ is a constant depending only on u. Now we obtain

$$I_n \leqq \int_{|s|\leqq\delta} \rho_j(ds)|s|\,|\int v_s(t)_{z,n}(dt)| \leqq C_2(u)\delta \quad \text{and} \quad II_n \leqq \rho_j(|s|\geqq\delta)2C_1(u)$$

Therefore for any small $\delta>0$, thr there exists $j_0=j_0(\delta)$ such that

$$I_n+II_n \leqq C_3(u)\delta\,. \tag{4.9}$$

On the other hand for j fixed we have

$$\begin{aligned} &\int_{\boldsymbol{R}} u(t)\,(\rho_j*\mu_{z,n})\,(dt)-\Phi_{g,z,n}((\hat{u}\hat{\rho}_j)^{\check{}})\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{z^2}{2n\sigma^2}\right] \\ &= \frac{1}{2\pi}\int_{\boldsymbol{R}} \hat{u}(t)\hat{\rho}_j(t)\hat{\mu}_{z,n}(t)\ \ dt-\Phi_{g,z,n}((\hat{u}\hat{\rho}_j)^{\check{}})\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{z^2}{2n\sigma^2}\right]\to 0 \end{aligned} \tag{4.10}$$

uniformly in z from the estimate (4.8), where $(v)^{\check{}}$ denotes the inverse Fourier transform of v. In addition there exists $N_0=N_0(\delta)$ such that $\sum_{|k|>N_0}|\hat{u}(ka)|<\delta$ since $u\in\mathcal{S}$. Therefore we have

$$|\Phi_{g,z,n}((\hat{u}\hat{\rho}_j)^{\check{}})-\Phi_{g,z,n}(u)| \leqq \sum_{|k|\leqq N_0}|\hat{u}(ka)\,(\hat{\rho}_j-1)|+\delta\,. \tag{4.11}$$

The first term goes to 0 as $j\to\infty$. Combine the estimates (4.9), (4.10), and (4.11) we conclude that if n is large

$$\sup_z |\sqrt{n}\int u(S_nf+z)g\,dm-\Phi_{q,z,n}(u)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{z^2}{2n\sigma^2}\right]| \leqq C_4(u)\delta$$

where $C_4(u)$ is a constant depending only on u. Now the proof of the theorem is complete. //

REMARK 4.1. If $f\in B_1\cup B_3$, then $H(f)\simeq\boldsymbol{Z}/p\boldsymbol{Z}$. Therefore if we conbine the Poisson summation formula and Theorem 4.1, we obtain the usual local limit theorem. If $f\in B_4$ then $\Lambda(f)=\{0\}$, $G(f)=\{1\}$, and $H(f)=\langle(1)\rangle$. Thus $\Phi_{g,z,n}(u)=\hat{u}(0)\int_I g dm$ i.e., $d\Phi_{g,z,n}=\int_I g\,dm\,dt$ (See Rousseau-Egele [9]).

5. Ezamples

In Section 3 we classified the elements of BV_0. In the present section we discuss about some examples. For this purpose we need the following theorems.

Theorem 5.1. *Let T be an L-Y transformation which satisfies the mixing condition (M). Lei $\{p_1, p_2, \cdots, p_n\}$ be a periodic orbit of T with* $p_{j+1}=$

$Tp_j \pmod n$. *Assume that* T *is continuous at each* p_j. *If* $f \in BV_0$ *satisfies* $\sum_{j=1}^{n} \bar{f}(p_j) \neq 0$, *then* $\sigma(f) > 0$, *i.e.,* $f \in BV_0 - B_0$, *where* $\bar{f}$ *is the function defined by* $\bar{f}(x) = \{f(x+) + f(x-)\}/2$ *for any bounded variation version of* f.

Proof. Assume that $\sigma(f) = 0$. Then we have $f = g\circ T - g$ for some $g \in L^2(\mu)$. Thus, for any $t \in \boldsymbol{R}$, we have $\exp[itg]\circ T = \exp[itf]\exp[itg]$. Therefore $\exp[itg]$ has a version without discontinuities of the second kind since $(\exp[itg])h_0 \in BV$ and $h_0 \in BV$ in virtue of (iv) of (3) in Lemma 2.3. Let $g_t = t^{-1}\arg[\exp[itg]]$ for $t \neq 0$. Then we obtain $f = g_t\circ T - g_t + \frac{2\pi}{t} K_t$ where K_t is a $\boldsymbol{Z}$-valued function. Thus we have

$$\sum_{j=1}^{n} (f(p_j+) + f(p_j-)) - \sum_{j=1}^{n} (g_t(T(p_j+)) - g_t(p_j+) \\ + g_t(T(p_j-)) - g_t(p_j-)) \in \frac{2\pi}{t}\boldsymbol{Z} \qquad \text{for any} \quad t \in \boldsymbol{R} - \{0\} .$$

On the other hand we have

$$\sum_{j=1}^{n} (g_t(T(p_j+)) - g_t(p_j+) + g_t(T(p_j-)) - g_t(p_j-)) \\ = \sum_{j=1}^{n} (g_t(T(p_j+)) + g_t(T(p_j-)) - g_t(p_{j+1}+) - g_t(g_{j+1}-)) = 0 .$$

Hence we conclude that

$$2\sum_{j=1}^{n} \bar{f}(p_j) \in \frac{2\pi}{t}\boldsymbol{Z} \qquad \text{for any} \quad t \in \boldsymbol{R} - \{0\} .$$

This implies that $\sum_{j=1}^{n} \bar{f}(p_j) = 0$. Hence we obtain the result.

Theorem 5.2. *If* T *is an L-Y transformation of type II with the mixing condition (M), then* $B_0 = \{0\}$, *and* $BV_0 - B_0 = B_1 \cup B_3 \cup B_5$. *In particular,* $\sigma(f)$ *is positive for any non-trivial element* $f \in BV_0$.

Proof. First of all we show that $h\circ T = \lambda e^{itf}h$, $\lambda \in S^1$ implies $h =$ constant μ-a.e. We may assume that h has no discontinuity of the second kind. From the assumption there exists a sequence of intervals $I_j = (a_j, b_j)$ such that $T(\mathrm{Int}\, I_j) = (0, 1)$ and $a_j, b_j \to a (j \to \infty)$ for some $a \in I$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - a| < \delta$, and $|y - a| < \delta$ implies $|h(x) - h(y)| < \varepsilon/2$ and $|t|\,|f(x) - f(y)| < \varepsilon/2\|h\|_\infty$. Thus we have $|h(Tx) - h(Ty)| = |h(x)e^{itf(x)} - h(y)e^{itf(y)}| \leqq |h(x) - h(y)| + \|h\|_\infty |f(x) - f(y)|\,|t| < \varepsilon$ whenever $|x - a| < \delta$ and $|y - a| < \delta$. If j is large, any point in I_j satisfies $|x - a| < \delta$. Thus $|h(x) - h(y)| < \varepsilon$ for any $x, y \in I$. Hence $h =$ constant μ-a.e. Let $\theta = \arg[h]$, then we can write $tf = \theta + 2\pi K$. This implies $B_2 = B_4 = \phi$. If $\sigma(f) = 0$ we have $h\circ T = e^{itf}h$ for some $h \in L^1(\mu)$. But in the same way as above, we obtain $h =$ constant μ-a.e.

Therefore $e^{itf}=1$ for any $t\in\boldsymbol{R}$. Thus $f=0$.

EXAMPLE 5.1. A typical example of an L-Y transformation of type I is $Tx=2x \bmod 1$. In this case $\mu=m$. Define functions g, K_1, and K_2 by

$$g(x)=\cos(2\pi x)$$

$$K_1(x)=\begin{cases} 1 & 0\leqq x<\frac{1}{2} \\ -1 & \frac{1}{2}\leqq x\leqq 1 \end{cases}$$

and

$$K_2(x)=\begin{cases} -1 & 0\leqq x<a \\ 1 & a\leqq x\leqq 1 \end{cases}$$

with $\theta=2a-1\in\boldsymbol{Q}^c$.

Then we have $\sigma(g)>0$ since $g(0)\neq 0$ (Theorem 5.1), $\sigma(K_1)>0$ since $K_1(0)\neq 0$, $\int K_1\,dm=0$, $\int K_2\,dm=-\theta$, and ng can not be $\boldsymbol{Z}$-valued for any $n\in\boldsymbol{Z}-\{0\}$. Moreover, (1) $2\pi K_1\in B_1$, (2) $2\pi(g\circ T-g+K_1)\in B_2$, (3) $2\pi(K_2+\theta)\in B_3$, and (4) $2\pi(g\circ T-g+K_2+\theta)\in B_4$.

EXAMPLE 5.2. A typical example of an L-Y transformation of type II is the so-called Gauss transformation $Tx=\frac{1}{x}-\left[\frac{1}{x}\right]$. In this case $\mu=(\log 2)^{-1}\times(1+x)^{-1}m$. From Theorem 5.2 we can see that any Lipschitz function $f\neq 0$ with $\int f\,d\mu=0$ belongs to B_5. //

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