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MODULI SPACES OF YANG-MILLS CONNECTIONS OVER QUATERNIONIC KÄHLER MANIFOLDS

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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Introduction

The concept of anti-self-dual connections plays an important role in Yang-Mills theory for 4-manifolds (cf. Atiyah's monograph [1]). For instance, Atiyah, Hitchin and Singer [2] determined the moduli space of instantons on S^4 by differential geometric method, while Hartshorne [5] obtained the same result via twistor theory by showing that the moduli space of instantons over S^4 is the real part of the moduli space of null-correlation bundles over $P^3(\boldsymbol{C})$.

Now the purpose of this paper is to give a generalization of the result of Hartshorne [5] in the following way. We have the notion of B_2-connections ∇ on vector bundles over quaternionic Kähler manifolds M as higher dimensional analogue of anti-self-dual connections over 4-manifolds (cf. [3], [11], [15]). Let $p: Z \rightarrow M$ be the twistor space. Then, to each B_2-connection ∇ over M, we can associate in a unique way an Einstein-Hermitian connection $\tilde{\nabla} := p^*\nabla$ over Z. Our main result is:

Theorem. *The mapping $\nabla \mapsto \tilde{\nabla}$ natually induces an embedding of the moduli space of B_2-connections over M as a totally real submanifold of the moduli spcae of Einstein-Hermitian connections over Z.*

In a forthcoming paper, we shall give a compactification of the moduli space of Einstein-Hermitian connections for null-correlation bundles on $P^{2m+1}(\boldsymbol{C})$.

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1. Notation, conventions and preliminaries

For this section we refer to [6], [7], [8], [9], [10] and [11].

Let N be a compact complex manifold and (F, h_F) a Hermitian vector bundle over N where F is a C^∞ complex vector bundle and h_F is a Hermitian metric on F.

DEFINITION. A Hermitian connection D on (F, h_F) is said to be *integrable*, if the curvature R^D of D is an End(F)-valued (1, 1)-form. An integrable connection D on (F, h_F) is said to be *irreducible*, if the only parallel sections of End (F) are constant multiples of the identity endomorphism id_F of F.

We denote by $U(F, h_F)$ the group of unitary gauge transformations of (F, h_F) and by $C'_H(F, h_F)$ the set of all irreducible integrable connections D on (F, h_F). The set of all equivalence classes in $C'_H(F, h_F)$ modulo $U(F, h_F)$ is called the moduli space of irreducible integrable connections on (F, h_F), which we denote by $\mathcal{H}'(F, h_F)$.

Now we assume that N admits a Kähler metric with Kähler form ω_N. The mapping L: $\wedge^p T^*N \ni \eta \mapsto L(\eta) \in \wedge^{p+2} T^*N$ being defined by $L(\eta) = \omega \wedge \eta$, we denote its adjoint operator by Λ. This induces the mapping

$$id \otimes \Lambda \colon \mathrm{End}(F, h_F) \otimes \wedge^{p+2} T^*N \to \mathrm{End}(F, h_F) \otimes \wedge^p T^*N .$$

When a connection D on F is given, R^D denotes the curvature tensor of the connection D. Put $\mathrm{Ric}(D) := \sqrt{-1}(id \otimes \Lambda) R^D$, which is called the *Ricci curvature* of D.

DEFINITION. A Hermitian connection D on (F, h_F) is called an *Einstein-Hermitian connection* if the *Ricci curvature* Ric(D) of D is a constant multiple of id_F.

Let $C'_E(F, h_F)$ be the set of all irreducible Einstein-Hermitian connections on (F, h_F). The set of all equivalence classes in $C'_E(F, h_F)$ modulo the group of unitary gauge transformations $U(F, h_F)$ is called the moduli space of irreducible Einstein-Hermitian connections on (F, h_F), which we denote by $\mathcal{E}'(F, h_F)$.

Let D be an irreducible integrable connection on (F, h_F). Consider the connection, denoted also by D, on End(F) induced by D. We then have a Dolbeaut complex

$$(A_D)\colon 0 \to A^{0,0}(\mathrm{End}(F)) \to A^{0,1}(\mathrm{End}(F)) \to \cdots \to A^{0,n}(\mathrm{End}(F)) \to 0 \qquad (n = \dim_C N),$$

where $A^{0,1}(\mathrm{End}(F))$ is the space of all End(F)-valued $(0, i)$-forms on N and D'': $A^{0,1}(\mathrm{End}(F)) \to A^{0,i+1}(\mathrm{End}(F))$ is the $(0, i+1)$ part of the covariant exterior derivative d^D. Recall that the moduli space $\mathcal{H}'(F, h_F)$ adimts a non-Hausdorff complex analytic space structure (see [7; (0.2)], [8; Chapter 7, (3.35)] and [10; (2.7)]). As a neighborhood of the equivalence class $<D>$ of D, we can take an open set (centered at 0) of a slice

$$S_H = \{\alpha \in A^{0,1}(\mathrm{End}(F));\ D''\alpha \wedge \alpha = 0\,, \quad D''^*\alpha = 0\}\ .$$

For the above Dolbeault complex (A_D), we denote by G_H, K_H and H_H the Green operator, the Kuranishi map and the orthogonal projection to the space $\mathcal{H}^1(N, A_D)$ of all End(F)-valued harmonic 1-forms on N respectively. Then this open set of S_H is homeomorphic to an open set of a complex analytic space

$$O_H = \{\alpha \in \mathcal{H}^1(N, A_D);\ H_H(K_H(\alpha)\wedge K_H(\alpha)) = 0\} .$$

Let End$(F)_0$ be the subbundle $\{S\in \mathrm{End}(F) \mid \mathrm{trace}\,(S)=0\}$ of End(F). We then have the following subcomplex $(\tilde{A}_D)$ of (A_D):

$$(\tilde{A}_D)\colon 0 \to A^{0,0}(\mathrm{End}(F)_0) \to A^{0,1}(\mathrm{End}(F)_0) \to \cdots \to A^{0,n}(\mathrm{End}(F)_0) \to 0 \qquad (n = \dim_C N)\,,$$

where $A^{0,i}(\mathrm{End}(F)_0)$ is the space of all End$(F)_0$-valued $(0, i)$-forms on N. Denote by $C''_H(F, h_F)$ the set of all irreducible integrable connections D on (F, h_F) such that the second cohomology of the Dolbeaut complex $(\tilde{A}_D)$ vanishes. Then the quotient space $\mathcal{H}''(F, h_F) := C''_H(F, h_F)/G(F, h_F)$ is a (possibly non-Hausdorff) complex manifold (cf. [8]), where $G(F, h_F)$ denotes the group of automorphisms of (F, h_F) whose determinant is one at each point.

On the other hand, an irreducible Einstein-Hermitian connection D on (F, h_F) induces a connection on End(F, h_F), denoted also by D. We denote by $A^i(\mathrm{End}(F, h_F))$ the space of all End(F, h_F)-valued i-forms. Then we have the following elliptic complex (B_D) due to Kim [7]:

$$(B_D)\colon 0 \to A^0(\mathrm{End}(F, h_F)) \xrightarrow{D} A^1(\mathrm{End}(F, h_F)) \xrightarrow{D_+} A^2_+(\mathrm{End}(F, h_F)) \xrightarrow{D_2}$$
$$A^{0,3}(\mathrm{End}(F, h_F)) \xrightarrow{D''} \cdots \xrightarrow{D''} A^{0,n}(\mathrm{End}(F, h_F)) \to 0\,,$$

where $A^p(\mathrm{End}(F, h_F))$ is the space of all real C^∞ p-forms with values in End (F, h_F), $A^{p,q}(\mathrm{End}(F, h_F))$ is the space of C^∞ (p, q)-forms with values in End(F, h_F) and

$$A^2_+(\mathrm{End}(F, h_F)) =$$
$$A^2(\mathrm{End}(F, h_F)) \cap (A^{2,0}(\mathrm{End}(F, h_F)) + A^{0,2}(\mathrm{End}(F, h_F)) + A^0(\mathrm{End}(F, h_F))\otimes \omega_N)\,.$$

Moreover D_+ and D_2 are defined as $D_+ = p_+ \circ d^D$ and $D_2 = D'' \circ p^{0,2}$, where p_+ and $p^{0,2}$ are natural projections of $A^2(\mathrm{End}(F, h_F))$ onto $A^2_+(\mathrm{End}(F, h_F))$ and $A^{0,2}(\mathrm{End}(F, h_F))$, respectively. Note that the moduli space $\mathcal{E}'(F, h_F)$ is a Hausdorff real analytic space (cf. [7], [8] and [10]). We can identify a neighborhood of $<D>$ in $\mathcal{E}(F, h_F)$ with a small open subset (centered at 0) of a slice

$$S_E = \{\beta \in A^1(\mathrm{End}(F, h_F));\ D_+\beta + p_+(\beta\wedge\beta) = 0\,,\ \ D^*\beta = 0\} .$$

This open subset of S_E is homeomorphic to an open set (centered at 0) of the real analytic space

$$O_E = \{\beta \in \mathcal{H}^1(N, B_D);\ H_E(K_E(\beta) \wedge K_E(\beta)) = 0\}\ ,$$

where G_E, K_E and H_E are the operators of (B_D), corresponding respectively to the Green operator, the Kuranishi map and the orthogonal projection to the space $\mathcal{H}^1(N, B_D)$ of all $\mathrm{End}(F, h_F)$-valued harmonic 1-forms of (B_D). The moduli space $\mathcal{E}'(F, h_F)$ is naturally embedded in $\mathcal{H}'(F, h_F)$ as an open subset of $\mathcal{H}'(F, h_F)$ (cf. [7], [8] and [10]). Let $H^i(N, A_D)$ and $H^i(N, B_D)$ be the i-th cohomology groups of the complexes (A_D) and (B_D) respectively. Then $H^1(N, A_D) \simeq H^1(N, B_D)$ (cf. [7], [8] and [10]). More precisely, we have

$$\mathcal{H}^1(N, A_D) + \overline{\mathcal{H}^1(N, A_D)} = \mathcal{H}^1(N, B_D)^C\ .$$

Let $(\tilde{B}_D)$ be the subcomplex (B_D) consisting of the sections with trace 0, and let $C''_E(F, h_F)$ be the set of all irreducible Einstein-Hermitian connections D on (F, h_F) such that the second cohomology of the complex $(\tilde{B}_D)$ vanishes. We denote by $\mathcal{E}''(F, h_F)$ the quotient space $C''_E(F, h_F)/(U(F, h_F) \cap G(F, h_F))$. Then $\mathcal{E}''(F, h_F)$ has a natural structure of Kähler manifold (cf. [8] and [10]) and is holomorphically embedded in $\mathcal{H}''(F, h_F)$ as an open subset.

Let M be a compact quaternionic Kähler manifold and $p: Z \to M$ the associated twistor space. The vector bundle $\wedge^2 T^*M$ over M formed by covectors of degree 2 is expressed as a direct sum of three holonomy invariant vector subbundles A'_2, A''_2 and B_2 (cf. [14]). Fix an arbitrary C^∞ vector bundle V over M. Then a connection D on V is called a B_2-connection, if the curvature R^D of D is an $\mathrm{End}(V)$-valued B_2-form. We now assume that V is a complex vector bundle over M, and choose a Hermitian metric h_V on V. Recall that Z has a natural real structure, i.e., an involutive antiholomorphic mapping $\tau: Z \to Z$ (cf. [11; (2.8)]). Let $C_B(V, h_V)$ be the set of all Hermitian B_2-connections on (V, h_V) and let $\tilde{C}_H(p^*V, p^*h_V)$ be the set of all integrable connections on (p^*V, p^*h_V) satisfying the conditions: (a) D is trivial on each fibre $p^{-1}(x)$ $(x \in M)$, and (b) the connection form associated with D is fixed by the pull-back τ^* (for more details aee [11; Introduction]). Then we have the following:

Theorem 1.1 ([11]). *The pull-back $D \mapsto p^*D$ of connections induces a natural bijective correspondence: $C_B(V, h_V) \simeq \tilde{C}_H(p^*V, p^*h_V)$. Furthermore, if the scalar curvature σ_M of M is positive, then $\tilde{C}_H(p^*V, p^*h_V)$ is the set of all Einstein-Hermitian connections on (p^*V, p^*h_V) satisfying the conditions (a) and (b).*

2. Moduli spaces of Hermitian B_2-connections

Let $\mathrm{End}(V, h_V)_0$ be the subbundle consisting of $S \in \mathrm{End}(V, h_V)$ such that $\mathrm{trace}\,(S) = 0$. Let D be a Hermitian B_2-connection on (V, h_V). Then D induces B_2-conenction on $\mathrm{End}(V, h_V)$ and $\mathrm{End}(V, h_V)_0$, which we denote also by D. Using the B_2-connection D on $\mathrm{End}(V, h_V)$, we have an $\mathrm{End}(V, h_V)$-valued elliptic complex $C_D = \{(A^i, d_i),\ 0 \le i \le 2m\}$ $(\dim M = 4m)$ (cf. [11; (3.5.)]), where A^1

is the space of all $\mathrm{End}(V, h_V)$-valued 1-forms on M. Furthermore, the B_2-connection D on $\mathrm{End}(V, h_V)_0$ induces an $\mathrm{End}(V, h_V)_0$-valued ellipic complex $\tilde{C}_D = \{(\tilde{A}^i, \tilde{d}_i)\}$ (cf. [11; (3.5)]), where in this case $\tilde{A}^1$ is the space of all $\mathrm{End}(V, h_V)_0$-valued 1-forms on M. We denote the i-th cohomology groups of C_D and $\tilde{C}_D$ by $H^i(M, C_D)$ and $H^i(M, \tilde{C}_D)$ respectively. The spaces of the i-th harmonic elements for C_D and $\tilde{C}_D$ are denoted by $\mathcal{H}^i(M, C_D)$ and $\mathcal{H}^i(M, \tilde{C}_D)$ respectively.

Now we denote by $U(V, h_V)$ the group of unitary gauge transformations of (V, h_V). Let $C'_B(V, h_V)$ be the set of all Hermitian B_2-connections D on (V, h_V) such that $H^0(M, \tilde{C}_D) = \{0\}$, namely the set of all irreducible Hermitian B_2-connections on (V, h_V). We denote by $\mathcal{B}'(V, h_V)$ the quotient space $C'_B(V, h_V)/U(V, h_V)$, which is called the moduli space of irreducible Hermitian B_2-connections on (V, h_V). Furthermore, let $C''_B(V, h_V)$ be the set of Hermitian B_2-connections D on (V, h_V) such that $H^0(M, \tilde{C}_D) = H^2(M, \tilde{C}_D) = \{0\}$. We then put $\mathcal{B}''(V, h_V) := C''_B(V, h_V)/U(V, h_V)$. In the complex C_D, let $H_S: A^* \to \mathcal{H}^*(M, C_D)$ be the orthogonal projection to harmonic part and let G_S be the Green operator for $\Delta_S = \sum_{i=1}^{2m} (d_i \circ d_{i-1}^* + d_i^* \circ d_i)$. Note that $id = H_S + G_S \circ \Delta_S$.

Lemma 2.1. *Given a connection D in $C_B(V, h_V)$, we denote by φ_D the set of forms $\alpha \in A^1$ such that $d_1\alpha + \pi_2(\alpha \wedge \alpha) = 0$ and $d_0^*\alpha = 0$, where π_2 denotes the natural projection of $\Gamma(M, \mathrm{End}(V, h_V) \otimes \wedge^2 T^*M)$ onto A^2. Then the mapping: $\varphi_D \ni \alpha \mapsto [D+\alpha] \in \mathcal{B}'$ is a homeomorphism of an open neighborhood of the origin in φ_D to an open set in $\mathcal{B}'$ around $[D]$.*

Proof. This is proved by the same argument as in the proof of the slice lemma in [7; (1.7)].

The mapping K_S: $A^1 \ni \alpha \mapsto \alpha + (d_2^* \circ G_S \circ \pi_2)(\alpha \wedge \alpha) \in A^1$, called the Kuranish map of C_D. The restriction of K_S defines a diffeomorphism between two small open neighborhoods of the origin on A^1. Let K_S^{-1} be its inverse. Then we have:

Lemma 2.2. *Put*

$$\mathcal{V}_D = \{a \in \mathcal{H}^1(M, C_D);\ (H_S \circ \pi_2)(K_S^{-1}(\alpha) \wedge K_S^{-1}(\alpha)) = 0\}\ .$$

Then the restriction of the Kuranishi map defines a local homeomorphism between certain small neighborhoods of the origin of φ_D and $\mathcal{V}_D$.

We here observe that if $H^2(M, \tilde{C}_D) = \{0\}$, then $\mathcal{V}_D$ is equal to $\mathcal{H}^1(M, C_D)$. Now by Lemmas 2.1 and 2.2, the following theorems follows immediately:

Theorem 2.3. *The moduli space $\mathcal{B}'(V, h_V)$ of irreducible Hermitian B_2-connections has a natural real analytic structure.*

Theorem 2.4. *The quotient space $\mathcal{B}''(V, h_V)$ is a smooth manifold. The* *dimension of the connected component containing* $[D]$ *is* $\dim_{\mathbf{R}} H^1(M, C_D)$. *Moreover, by identifying the tangent space* $T_{[D]}\mathcal{B}''(V, h_V)$ *with* $\mathcal{H}^1(M, C_D)$, *the* L^2*-inner product of* $\mathcal{H}^1(M, C_D)$ *defines a Riemannian metric on* $\mathcal{B}''(V, h_V)$.

Theorems 2.3 and 2.4 are valid also for the case where the holonomy group of connections is a closed subgroup of $SO(r)$ or $U(r)$. Furthermore, by the same argument as in Kim [7], it is easily checked that *both the spaces* $\mathcal{B}'(V, h_V)$ *and* $\mathcal{B}''(V, h_V)$ *are Hausdorff.*

3. B_2-connections and Einstein-Hermitian connections

From now on, we fix a compact connected quaternionic Kähler manifold M and a Hermitian vector bundle (V, h_V) over M. In the subsequent sections we use the notations introduced in Section 2. We prove the following:

Theorem 3.1. *If* M *has positive scalar curvature,* $\mathcal{B}''(V, h_V)$ *is embedded in* $\mathcal{E}''(p^*V, p^*h_V)$ *as a totally real submanifold.*

Given a Hermitian connection D on (V, h_V), we denote by p^*D the pull-back of D by p.

Lemma 3.2. *If* $D \in C_B(V, h_V)$ *is irreducible, then so is* $p^*D \in C'_H(p^*V, p^*h_V)$. *In particular, if the scalar curvature* σ_M *of* M *is positive, then we have* $p^*(C'_B(V, h_V)) \subset C'_E(p^*V, p^*h_V)$, *where* $p^*(C'_B(V, h_V)) := \{p^*D \mid D \in C'_B(V, h_V)\}$ (*cf. Theorem* 1.1).

Proof. Fix an arbitrary $D \in C'_B(V, h_V)$ and suppose that $(p^*D)\tilde{s} = 0$ for some $\tilde{s} \in \Gamma(Z, p^* \operatorname{End}(V, h_V))$. Let $(v_1, \cdots, v_r)$ be a local unitary frame for (V, h_V) over an open set U of M. Let $\omega = (\omega_{ij})$ be the connection form of D defined by $Dv_j = \sum_{i=1}^r v_i \omega_{ij}$. Then by setting $\tilde{v}_i := p^*v_i$, we can express $\tilde{s}$ as $\tilde{s} = \sum_{1 \le i,j \le r} \tilde{s}_{ij}\, \tilde{v}_i \otimes \tilde{v}_j^*$. In terms of the frame $(\tilde{v}_1, \cdots, \tilde{v}_r)$, the assumption $(p^*D)\tilde{s} = 0$ is written as

$$(1) \qquad (d\tilde{s}_{ij}) + [p^*\omega, (\tilde{s}_{ij})] = 0 .$$

By (1), the restriction of the form $d\tilde{s}_{ij}$ to each fibre of p is zero, which means that the function $\tilde{s}_{ij}$ is constant along the fibres of p. Hence there exists a global section $s \in \Gamma(M, \operatorname{End}(V, h_V))$ such that $p^*s = \tilde{s}$. By the irreducibility of D, s is a constant multiple of id_V. Thus $\tilde{s}$ is a constant multiple of id_{p^*V}, as required.

Lemma 3.3. *Let* $D_1, D_2 \in C_B(V, h_V)$. *Then* $[D_1] = [D_2]$ *if and only if* $\langle p^*D_2 \rangle = \langle p^*D_2 \rangle$, *where* $[D_\alpha]$ (*resp.* $\langle \tilde{D}_\alpha \rangle$) $(\alpha = 1, 2)$ *denotes the equivalence class of* D_α (*resp.* $\tilde{D}_\alpha$) *modulo the unitary gauge groups on* (V, h_V) (*resp.* (p^*V, p^*h_V)).

Proof. It suffices to show $[D_1] = [D_2]$ when $\langle p^*D_1 \rangle = \langle p^*D_2 \rangle$. In this case, there exists a gauge transformation $\tilde{g}$ for (p^*V, p^*h_V) such that $p^*D_1 = \tilde{g} \cdot p^*D_2$.

Let $(v_1, \cdots, v_r)$ be a local unitary frame for (V, h_V). Each $D_\alpha (\alpha=1, 2)$ defines the connection form $\omega^{(\alpha)}=(\omega^{\alpha}_{ij})_{1\le i,j\le r}$ by $D_\alpha v_j=\sum_{i=1}^{r} v_i\, \omega^{\alpha}_{ij}$. Write $\tilde{g}$ as $\sum_{1\le i,j\le r} \tilde{g}_{ij}\, \tilde{v}_i \otimes \tilde{v}_j^*$, where $\tilde{v}_k=p^*v_k$, $1\le k\le r$. Then the condition $p^*D_1=\tilde{g}\cdot p^*D_2$ is locally expressed in the form

$$p^*\omega^{(1)} = p^*\omega^{(2)}+\tilde{G}^{-1}\, d\tilde{G}\,, \tag{2}$$

where $\tilde{G}$ denotes the $r\times r$ matrix $(\tilde{g}_{ij})$. From (3.3.1) the restriction of $d\tilde{G}$ to each fibre of p is zero, and so every $\tilde{g}_{kl}$ is constant along the fibres of p. Hence, there exists a gauge transformation g for (V, h_V) such that $\tilde{g}=p^*g$. Thus $D_1=g\cdot D_2$, i.e., $[D_1]=[D_2]$.

Theorem 3.4. *The mapping $p^*\colon C'_B(V, h_V)\to C'_H(p^*V, p^*h_V)$, induced from the projection $p\colon Z\to M$, gives rise to an injection: $\mathcal{B}'(V, h_V)\to\mathcal{H}'(p^*V, p^*h_V)$ (which is also denoted by p^*.)*

Proof. This follows immediately from Lemmas 3.2 and 3.3.

REMARK 3.5. If $\sigma_M>0$, then the image of $p^*\colon \mathcal{B}'(V, h_V)\to\mathcal{H}'(p^*V, p^*h_V)$ is contained in $\mathcal{E}'(p^*V, p^*h_V)$ (cf. Theorem 1.1).

We denote by $(\tilde{C}_D)^C$ the complexification of the elliptic complex $(\tilde{C}_D)$. Then by Carpia and Salamon [4; Theorem 3] the i-th cohomology group of the complex $(\tilde{C}_D)^C$ on M is embedded, via p^*, as a subgroup in the corresponding cohomology group of the Dolbeault complex (A_{p^*D}) on Z, and this embedding is an isomorphism for $i\ge 1$. It follows the following:

Corollary 3.6. *The mapping p^* maps $C''_B(V, h_V)$ to $C''_H(p^*V, p^*h_V)$ injectively. Moreover, this mapping induces an injection: $\mathcal{B}''(V, h_V)\to\mathcal{H}''(p^*V, p^*h_V)$ (denoted also by p^*). In particular, if $\sigma_M>0$, the image of $\mathcal{B}''(V, h_V)$ under the injection $p^*\colon \mathcal{B}''(V, h_V)\to\mathcal{H}''(p^*V, p^*h_V)$ is contained in $\mathcal{E}''(p^*V, p^*h_V)$.*

Since p^*V is trivial on each fibre of $p\colon Z\to M$, τ induces a bundle automorphism $\tau^\sharp\colon p^*V\to p^*V$ such that the following diagram is commutative:

$$\begin{array}{ccc} p^*V & \xrightarrow{\ \tau^\sharp\ } & p^*V \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\ \tau\ } & Z\,. \end{array}$$

Let $C_H(p^*V, p^*h_V)$ be the set of all Hermitian integrable connections on (p^*V, p^*h_V). Then the bundle automorphism $\tau^\sharp$ induces the mapping $\tilde{\tau}$ defined as follows:

$$C_H(p^*V, p^*h_V)\ni\tilde{D}\mapsto \tilde{\tau}(\tilde{D}) := \tau^\sharp\circ\tilde{D}\circ\tau^\sharp\in C_H(p^*V, p^*h_V)\,.$$

We shall now write $\tilde{\tau}$ explicitly in terms of local frames. Choose an open cover $\{U_\alpha\}$ of M with a local unitary frame $(v_1^\alpha, \cdots, v_r^\alpha)$ for (V, h_V) over U_α. Then $\{p^{-1}(U_\alpha)\}$ is an open cover of Z with local unitary frame $(p^*v_1^\alpha, \cdots, p^*v_r^\alpha)$ for (p^*V, p^*h_V) over $p^{-1}(U_\alpha)$. Given a Hermitian integrable connection $\tilde{D}$ on (p^*V, p^*h_V), we denote by (ω_{ij}^α) the connection form for $\tilde{D}$ on $p^{-1}(U_\alpha)$ with respect to the frame $(p^*v_1^\alpha, \cdots, p^*v_r^\alpha)$, (i.e, $\tilde{D}(p^*v_j^\alpha)=\sum(p^*v_i^\alpha)\,\omega_{ij}^\alpha$). Then $(\tau^*\omega_{ij}^\alpha)$ is just the connection form for $\tilde{\tau}(\tilde{D})$ with respect to the same frame on $p^{-1}(U_\alpha)$. Since τ is antiholomorphic, $\tilde{\tau}(\tilde{D})$ is also integrable. Note that if $\tilde{D}$ is irreducible, then $\tilde{\tau}(\tilde{D})$ is also irreducible, and that $\tilde{D}$ is fixed by $\tilde{\tau}$ if and only if $\tilde{D}$ satisfies the condition (b) in Section 1. Hence, by $\tilde{\tau}^2=id$, the mapping $\tilde{\tau}$ is a bijection of $C'_H(p^*V, p^*h_V)$ onto itself. Since τ is an isometry of Z, the same argument is applied also to $C'_E(p^*V, p^*h_V)$. Given a unitary transformation $\tilde{s}\in U(p^*V, p^*h_V)$ and an integrable connection $\tilde{D}\in C'_H(p^*V, p^*h_V)$, we have the identity

$$\tilde{s}\cdot\tilde{\tau}(\tilde{D}) = \tilde{\tau}(s'\cdot\tilde{D})\,,$$

where $s' := \tau^\sharp\cdot\tilde{s}\circ\tau^\sharp$. Hence, $\tilde{\tau}$ naturally induces a bijection of the moduli space $\mathcal{H}'(p^*V, p^*h_V)$ onto itself, denoted by $\tau'\colon \mathcal{H}'(p^*V, p^*h_V)\to\mathcal{H}'(p^*V, p^*h_V)$, and the restriction of τ' to $\mathcal{E}'$ gives a bijection of $\mathcal{E}'$ onto itself (denoted also by τ': $\mathcal{E}'(p^*V, p^*h_V)\to\mathcal{E}'(p^*V, p^*h_V)$). Recall that the complex structure of Z induces those of $\mathcal{H}'(p^*V, p^*h_V)$ and $\mathcal{E}'(p^*V, p^*h_V)$. Since τ is antiholomorphic, we have

Theorem 3.7. *Both the mappings*

$$\tau'\colon \mathcal{H}'(p^*V, p^*h_V)\to\mathcal{H}'(p^*V, p^*h_V) \quad \textit{and}$$
$$\tau'\colon \mathcal{E}'(p^*V, p^*h_V)\to\mathcal{E}'(p^*V, p^*h_V)$$

are antiholomorphic bijection. Therefore τ defines real structures of $\mathcal{H}'(p^*V, p^*h_V)$ and $\mathcal{E}'(p^*V, p^*h)$.

Given an integrable connection $\tilde{D}$ on (p^*V, p^*h_V), we obtain the elliptic complex $(\tilde{A}_{\tilde{\tau}(\tilde{D})})$ from the complex $\tau^*(\tilde{A}_{\tilde{D}})$ by taking complex conjugation. Similarly, for any Einstein-Hermitian connection $\tilde{D}$, we obtain $(\tilde{B}_{\tilde{\tau}(\tilde{D})})$ from $\tau^*(\tilde{B}_{\tilde{D}})$ by complex conjugation. Hence the restrictions of the bijections

$$\tau'\colon \mathcal{H}'(p^*V, p^*h_V)\to\mathcal{H}'(p^*V, p^*h_V) \quad \text{and}$$
$$\tau'\colon \mathcal{E}'(p^*V, p^*h_V)\to\mathcal{E}'(p^*V, p^*h_V)$$

on $\mathcal{H}''(p^*V, p^*h_V)$ and $\mathcal{E}''(p^*V, p^*h_V)$ define the bijections

$$\tau''\colon \mathcal{H}''(p^*V, p^*h_V)\to\mathcal{H}''(p^*V, p^*h_V) \quad \text{and}$$
$$\tau''\colon \mathcal{E}''(p^*V, p^*h_V)\to\mathcal{E}''(p^*V, p^*h_V)$$

respectively. The Kähler metric of $\mathcal{E}''(p^*V, p^*h_V)$ is defined by the L^2-inner product on $\mathcal{H}^1(Z, B_{\tilde{D}})$, which identified with the tangent space of $\mathcal{E}''(p^*V, p^*h_V)$

at $\langle\tilde{D}\rangle$. Since τ is isometry on Z, the real structure τ'': $\mathcal{E}''(p^*V, p^*h_V)\rightarrow\mathcal{E}''(p^*V, p^*h_V)$ is an isometry.

Now we fix an arbitrary element $\tilde{D}$ of $p^*(\mathcal{C}'_B(V, h_V))$. Put

$$\eta_H(\alpha) = H_H(K_H^{-1}(\alpha)\wedge K_H^{-1}(\alpha)) \quad \text{for} \quad \alpha\in\mathcal{H}^1(Z, A_{\tilde{D}}), \quad \text{and}$$
$$\eta_E(\beta) = H_E(K_E^{-1}(\beta)\wedge K_E^{-1}(\beta)) \quad \text{for} \quad \beta\in\mathcal{H}^1(Z, B_{\tilde{D}}).$$

Since $\tilde{D}$ is fixed by $\tilde{\tau}$ (cf. Section 1) we immediately obtain:

$$\eta_H(\tau^*\alpha) = \tau^*\eta_H(\alpha), \quad \alpha\in\mathcal{H}^1(Z, A_{\tilde{D}}), \tag{3}$$
$$\eta_E(\tau^*\beta) = \tau^*\eta_E(\beta), \quad \beta\in\mathcal{H}^1(Z, B_{\tilde{D}}). \tag{4}$$

Let $(\mathcal{H}'(p^*V, p^*h_V))_{\boldsymbol{R}}$, $(\mathcal{E}'(p^*V, p^*h_V))_{\boldsymbol{R}}$, $(\mathcal{H}''(p^*V, p^*h_V))_{\boldsymbol{R}}$, $(\mathcal{E}''(p^*V, p^*h_V))_{\boldsymbol{R}}$ be the subsets of $\mathcal{H}'(p^*V, p^*h_V)$, $\mathcal{E}'(p^*V, p^*h_V)$, $\mathcal{H}''(p^*V, p^*h_V)$, $\mathcal{E}''(p^*V, p^*h_V)$, respectively consisting of all elements fixed by the real structures defined above. Then by Theorem 1.1, $p^*(\mathcal{B}'(V, h_V))$ is embedded in $(\mathcal{E}'(p^*V, p^*h_V))_{\boldsymbol{R}}$ $(\subset(\mathcal{H}'(p^*V, p^*h_V))_{\boldsymbol{R}})$ and $p^*(\mathcal{B}''(V, h_V))\subset(\mathcal{E}''(p^*V, p^*h_V))_{\boldsymbol{R}}$ $(\subset(\mathcal{H}''(p^*V, p^*h_V))_{\boldsymbol{R}})$.

4. Proof of Theorem 3.1

Let g_M denote the g,ven metric on M and let g_Z denote the induced metric by g_M on Z. Then $g_V := g_Z - p^*g_M$ is an indefinite metric which is positive definite on each fibre of the submersion p: $(Z, g_Z)\rightarrow(M, g_M)$. Let J_Z be the complex structure on Z. We define a 2-form ω_V on Z by

$$\omega_V(v_1, v_2) := g_V(v_1, J_Z v_2), \quad v_1, v_2\in T_zZ \quad (z\in Z).$$

Recall that Salamon [14; p. 144] introduced (locally defined) vector bundles H and E on M such that the complexification T^*M^C of the cotangent bundle T^*M is nothing but $H\otimes_C E$. Let (h_1, h_2) and $(e_1, \cdots, e_{2m})$ be symplectic local frames of H and E respectively, and (z^1, z^2) the dual coordinate of H. (We follow [11; (3.2.2)] for definition of symplectic frames.) Moreover H and E have natural connections induced by Riemannian connection of M (cf. [14]). Let (ω^i_j) be the connection form on H with respect to the frame (h_1, h_2). Then ω_V is written as $c(|z^1|^2+1)^{-2}\theta\wedge\bar{\theta}$, where $\theta := dz^1 + z^1 p^*\omega^1_1 + p^*\omega^2_1 - (z^1)^2 p^*\omega^1_2 - z^1 p^*\omega^2_2$ and c is a constant depending only on the scalar curvature of M and the dimension of M (cf. [14] for more details).

Then we have

Lemma 4.1. *Put*

$$u_i = (|z^1|^2+1)^{-1/2}(z^1 p^*(e_i\otimes h_1) + p^*(e_i\otimes h_2)) \ (1\leq i\leq 2m), \quad \text{and}$$
$$\theta_V = (|z^1|^2+1)^{-1}\theta.$$

Then we have

$$d\omega_V = -2c(\textstyle\sum_{i=1}^m u_i \wedge u_{m+1} \wedge \bar{\theta}_V + \bar{u}_i \wedge \bar{u}_{m+1} \wedge \theta_V)\,.$$

Proof. $d\omega_V = c\{-2(|z^1|^2+1)^{-2}(z^1 d\bar{z}^1 + \bar{z}^1 dz^1) \wedge \theta \wedge \bar{\theta}$
$+(|z^1|^2+1)^{-2}(dz^1 \wedge p^*\omega_1^1 + z^1 p^* d\omega_1^1 + p^* d\omega_2^1 - 2z^1 dz^1 \wedge p^*\omega_1^2 - (z^1)^2 p^* d\omega_1^2$
$-dz^1 \wedge p^*\omega_2^2 - z^1 p^* d\omega_2^2) \wedge \bar{\theta} -$
$(|z^1|^2+1)^{-2}\, \theta \wedge (-d\bar{z}^1 \wedge p^*\omega_1^1 - \bar{z}^1 p^* d\omega_1^2 + 2\bar{z}^1\, d\bar{z}^1 \wedge p^*\omega_2^1 + (\bar{z}^1)^2 p^* d\omega_2^1 + d\bar{z}^1 \wedge p^*\omega_2^2$
$+\bar{z}^1 p^* d\omega_2^2)\}$
$= c(\,|z^1|^2+1)^{-2}\{z^1 p^*(d\omega_1^1 + \omega_2^1 \wedge \omega_1^2) + p^*(d\omega_2^1 + \omega_1^1 \wedge \omega_2^1 + \omega_2^1 \wedge \omega_2^2)$
$-(z^1)^2 p^*(d\omega_1^2 + \omega_1^2 \wedge \omega_1^1 + \omega_1^2 \wedge \omega_1^2) - z^1 p^*(d\omega_2^2 + \omega_1^2 \wedge \omega_2^1)\} \wedge \bar{\theta}$
$+c(|z^1|^2+1)^{-2}\, \theta \wedge \{\bar{z}^1 p^*(d\omega_1^1 + \omega_2^1 \wedge \omega_1^2) + p^*(d\omega_1^2 + \omega_1^2 \wedge \omega_1^1 + \omega_2^2 \wedge \omega_1^2)$
$-(\bar{z}^1)^2 p^*(d\omega_2^1 + \omega_1^1 \wedge \omega_2^1 + \omega_2^1 \wedge \omega_2^2) - \bar{z}^1 p^*(d\omega_2^2 + \omega_1^2 \wedge \omega_2^1)\}\,.$

We denote by (Ω_j^i) the curvature form of the vector bundle H with respect to (h_1, h_2):

$$\Omega_j^i = d\omega_j^i + \textstyle\sum_{k=1}^2 \omega_k^i \wedge \omega_j^k\,.$$

We have the following formula due to Salamon [14; Proposition 3.2].

$$\begin{aligned}
\Omega_1^1 &= -\textstyle\sum_{i=1}^m((e_i \otimes h_1) \wedge (e_{m+i} \otimes h_2) + (e_i \otimes h_2) \wedge (e_{m+i} \otimes h_1))\,,\\
\Omega_1^2 &= -2\textstyle\sum_{i=1}^m((e_i \otimes h_2) \wedge (e_{m+i} \otimes h_2))\,,\\
\Omega_2^1 &= 2\textstyle\sum_{i=1}^m((e_i \otimes h_1) \wedge (e_{m+i} \otimes h_1))\,,\\
\Omega_2^2 &= \textstyle\sum_{i=1}^m((e_i \otimes h_2) \wedge (e_{m-i} \otimes h_1) + (e_i \otimes h_1) \wedge (e_{m+i} \otimes h_2))\,.
\end{aligned}$$

Using this we get:

$$\begin{aligned}
d\omega_V &= c(|Z^1|^2+1)^{-2}\{(z^1 p^*\Omega_1^1 + p^*\Omega_2^1 - (z^1)^2 p^*\Omega_1^2 - z^1 p^*\Omega_2^2) \wedge \bar{\theta} +\\
&\quad (\bar{z}^1 p^*\Omega_1^1 + p^*\Omega_1^2 - (\bar{z}^1)^2 p^*\Omega_2^1 - \bar{z}^1 p^*\Omega_2^2) \wedge \theta\}\\
&= -2c(\textstyle\sum_{i=1}^m(u_i \wedge u_{m+i} \wedge \bar{\theta}_V + \bar{u}_i \wedge \bar{u}_{m+i} \wedge \theta_V))\,,
\end{aligned}$$

which proves Lemma 4.1.

Let D be a Hermitian B_2-connection on (V, h_V) on M. Then we have a morphism q between the complexes (C_D) and (A_{p^*D}) defined as follows:

$$C^i(\mathrm{End}(V, h_V)) \ni d \mapsto (pr^{(0,i)} \circ p^*)(d) \in A^i(\mathrm{End}(p^*V))\,,$$

where $pr^{(i,j)}\colon \Gamma(Z, \mathrm{End}(p^*V) \otimes_C \wedge^i T^*Z) \to \Gamma(Z, \mathrm{End}(p^*V) \otimes_C \wedge^{(i,j)} T^*Z)$ is the natural projection. Let $\tilde{\mathscr{D}}''$ and $\mathscr{D}_i$ be the formal adjoint of $(p^*D)''$ and d_i in the complexes A_{p^*D} and C_D respectively. Then we obtain:

Lemma 4.2. *Denoting by $*_M$ and $*_Z$ the star operators for vector bundles on M and Z, we have*

$$\tilde{\mathscr{D}}''qv = q(\mathscr{D}_{i-1}v) - (*_Z \circ pr^{(2m-1,2m)} \circ *_M)\, v \wedge (-2c \textstyle\sum_{i=1}^m u_i \wedge u_{m+i} \wedge \theta_V)$$

for all $v \in C^i(\mathrm{End}(V, h_V))$.

Proof. Write the volume forms on M and Z as dv_M and dv_Z respectively. Then $dv_Z = p^*(dv_M) \wedge \omega_V$. Hence, for any $v \in C^i(\mathrm{End}(V, h_V))$,

$$\begin{aligned}
\tilde{\mathscr{D}}''qv &= -(*_Z \circ (d^{p^*D})' \circ *_Z \circ q)(v) = -(*_Z \circ (d^{p^*D})' \circ *_Z \circ pr^{(0,1)} \circ p^*)(v) \\
&= -(*_Z \circ (d^{p^*D})' \circ pr^{(2m+1-i,2m+1)})\,(p^*(*_M v) \wedge \omega_V) \\
&= -(*_Z \circ (d^{p^*D})')\,((pr^{(2m-i,2m)}(*_M v)) \wedge \omega_V) \\
&= -*_Z\{(d^{p^*D})'\,((pr^{(2m-i,2m)}(p^*(*_M v))) \wedge \omega_V) + pr^{(2m+i,2m)}(p^*(*_M v)) \wedge d'\omega_V) \\
&= -*_Z\{(pr^{(2m-i,2m)}(p^*(d^D(*_M v)))) \wedge \omega_V - (pr^{(2m-1,2m)}(p^*(*_M v))) \wedge d'\omega_V\} \\
&= -pr^{(0,i)}((p^* \circ *_M \circ d^D \circ *_M)v) - *_Z\{(pr^{(m-i,2m)}(p^*(*_M v))) \wedge d'\omega_V\}\ .
\end{aligned}$$

By using Lemma 4.1, it follows:

$$\tilde{\mathscr{D}}''qv = -q\mathscr{D}_{i-1}v - (*_Z \circ pr^{(2m-i,2m)} \circ *_M)\, v \wedge (-2c \textstyle\sum_{i=1}^m u_i \wedge u_{m+i} \wedge \theta_V)\,,$$

which proves Lemma 4.2.

In view of Lemma 4.2, we have $q(\mathscr{H}^1(M, C_D) \subset \mathscr{H}^1(Z, A_{p^*D})$. From [4; Theorem 3], it follows that $\dim_{\boldsymbol{C}} \mathscr{H}^1(Z, A_{p^*D}) = \dim_{\boldsymbol{C}} \mathscr{H}^1(M, (C_D)^{\boldsymbol{C}}) = \dim_{\boldsymbol{R}} \mathscr{H}^1 (M, C_D)$. Together with the argument used by Kim [7; (1.3)], we have $\mathscr{H}^1 (Z, A_{p^*D}) + \overline{\mathscr{H}^1(Z, A_{p^*D})} = (\mathscr{H}^1(Z, B_{p^*D}))^{\boldsymbol{C}}$. Hence

$$(1) \qquad p^*\mathscr{H}^1(M, C_D) + J_Z\, p^*\mathscr{H}^1(M, C_D) = \mathscr{H}^1(Z, B_{p^*D})\,.$$

The tangent space of $\mathscr{B}''(V, h_V)$ at $[D]$ is $\mathscr{H}^1(M, C_D)$ and the tangent space of $\mathscr{E}''(p^*V, p^*h_V)$ at $\langle p^*D \rangle$ is $\mathscr{H}^1(Z, B_{p^*D})$. By (1), $\mathscr{B}''(V, h_V)$ is of dimension $\dim_{\boldsymbol{R}} \mathscr{H}^1(M, C_D)$ at $[D]$, which is equal to the complex dimension of $\mathscr{E}''(p^*V, p^*h_V)$ at $\langle p^*D \rangle$.

REMARKS. Capria and Salamon [4] constructed interesting families of B_2-connections for some vector bundles over $P^n\boldsymbol{H}$. In a forthcoming paper [12], as an application of Theorem 3.1, we shall clarify the relationship between such families of B_2-connections and the moduli space of Einstein-Hermitian connections on null-correlation bundles over odd dimensional complex projective spaces.

References

[1] M.F. Atiyah: Geometry of Yang-Mills fields, Accademia nazionale dei Lincei scula mormale superiore.

[2] M.F. Atiyah, N.J. Hitchin and I.M. Singer: *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London, Ser. A, **362** (1978), 425–461.

[3] L. Berard Bergery and T. Ochiai: *On some generalizations of the construction of twistor spaces,* in Global Riemannian Geometry (Proc. Symp. Duhram), Ellis Horwood, Chichester, 1982, 52–59.

[4] M.M. Capria and S.M. Salamon: *Yang-Mills fields on quaternionic spaces,* Nonlinearity, **1** (1988), 517–520.

[5] R. Hartshorne: *Stable vector bundles and instantons,* Comm. Math. Phys., **59** (1978), 1–15.

[6] M. Itoh: *The moduli space of Yang-Mills connections over a Kähler sufrace is a complex manifold,* Osaka J. Math., **22** (1985), 845–862.

[7] H. Kim: *Moduli of Hermite-Einstein vector bundles,* to appear in Math. Z.

[8] S. Kobayashi: Differential geometry of complex vector bundles, Iwanami Shoten, Publishers and Princeton University Press, Japan, 1987.

[9] N. Koiso: *Einstein metrics and complex structures,* Invent. Math., **73** (1983), 71–106.

[10] M. Lübke and C. Okonek: *Moduli spaces of simple bundles and Hermitian-Einstein connections,* Preprint.

[11] T. Nitta: *Vector bundles over quaternionic Kähler manifolds,* Tohoku Math. J. **40** (1988), 425–440.

[12] T. Nitta: *Compactification of the moduli spaces of Einstein-Hermitian connections for null-correlation bundles* to appear in Advanced Studies in Pure Mathematics 18, Recent Topics in Complex Geometry, 1989.

[13] T. Nitta and M. Takeuchi: *Contact structures on twistor spaces,* J. Math. Soc. Japan, **39** (1987), 139–162.

[14] S.M. Salamon: *Quaternionic Kähler manifolds,* Invent. Math. **67** (1982), 143–171.

[15] S.M. Salamon: *Quaternionic manifolds,* Symposia Mathematica, **26** (1982), 139–151.

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