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SEMINORMALITY AND F-PURITY IN LOCAL RINGS

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In this paper we consider the properties of seminormality and F-purity in commutative Noetherian local rings, two properties which are closely related, especially in the situation where the ring in question has Krull dimension equal to 1. By making use of this relationship, we obtain a simplified proof of a generalization of a result due to Goto and Watanabe [7], which describes the structure of the completion of a 1-dimensional F-pure ring whose residue field is algbraically closed. Finally, in our main result, we demonstrate that for a 1-dimensional local ring of prime characteristic whose Forbenius endomorphism is finite, the properties of seminormality and F-purity are equivalent if the residue field of the ring is perfect.

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0. Notations and conventions

From this point forth, A will denote a reduced Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k \cong A/\mathfrak{m}$. The classical ring of quotients of A will be denoted by $Q(A)$ and $\bar{A}$ will denote the integral closure of A in $Q(A)$. The term *dimension* will always refer to the Krull dimension of A.

1. Preliminaries

DEFINITION 1.1 (see [6, 1.1]). A ring A is *seminormal* if it satisfies the following equivalent conditions:

(i) if $a \in Q(A)$ and $a^2, a^3 \in A$, then $a \in A$;

(ii) if $a \in Q(A)$ and there exists a positive integer k such that $a^t \in A$ whenever $t \geq k$, then $a \in A$.

In the particular case where A is 1-dimensional, we have a further characterization of seminormality, namely:

(iii) A is seminormal if $\mathfrak{m} = J(\bar{A})$, where $J(\bar{A})$ denotes the Jacobson radical of $\bar{A}$.

DEFINITION 1.2. Suppose that A is a ring of prime characteristic p and let $F: A \to A$ be the Forbenius endomorphism of A. Let A^F denote A when regarded as an A-module via F. Then A is said to be *F-pure* if, for all A-modules E, the map $h_E: E \to A^F \otimes_A E$ defined by $h_E(x) = 1 \otimes_A x$, for all $x \in E$, is injective. We say that F is *finite* when A^F is a finite A-module.

In addition, we shall require the following results which concern the two properties defined above. The first of these appears in [7], and although we do not impose the same restrictions on the ring A that are present in the original statement, the proof that appears therein applies equally well to the more general situation which we consider in this paper.

Proposition 1.3 ([7, (2.2)]). *Let A be an F-pure ring, let $Q = Q(A)$ and let $f_{Q/A}: Q/A \to Q/A$ be defined by the relation*

$$f_{Q/A}(x \bmod A) = x^p \bmod A ,$$

for all $x \in Q(A)$. Then $f_{Q/A}$ is injective.

Concerining the property of seminormality we shall make use of the following results which are stated without proof.

Lemma 1.4 ([11]). *Let k be a field and let L be a reduced Noetherian k-algebra of dimension 0. Then $k + TL[T]$ is seminormal, where T is an indeterminate over L, and $k + TL[T]$ is identified with a subring of $L[T]$ in the natural way.*

Proposition 1.5 ([11, 3]). *Let A be a 1-dimensional ring of prime characteristic p, whose Frobenius endomorphism is finite. Then the following conditions are equivalent:*

(*i*) *A is seminormal;*

(*ii*) *$\hat{A}$ is seminormal;*

(*iii*) *$Gr(A)$ is k-isomorphic to $k + TK[T]$, where $Gr(A)$ denotes the associated graded ring of A with respect to $\mathfrak{m}$ and $K = \bar{A}/J(\bar{A})$;*

(*iv*) *$Gr(A)$ is reduced and seminormal.*

At this point we remark that if A is a 1-dimensional ring of prime characteristic p, whose Frobenius endomorphism is finite, then it follows from [10, Th. 108 and Th. 78] that $\bar{A}$ is a finite A-module. We are therefore justified in replacing the latter condition, which appears in the original statement of 1.5., with the former condition which is more appropriate to the work of this paper.

2. The results

The first result is an immediate consequence of 1.3.

Proposition 2.1. *Let A be a ring of prime characteristic p. If A is F-pure* *then A is seminormal.*

Proof. Suppose that $a\in Q(A)$ and a^2, $a^3\in A$. Then it is easily seen that $a^p\in A$, so that, by 1.3., $a\in A$ and the result follows.

From this point on we focus our attention on the situation where A is a 1-dimensional ring. Concerning this situation we have the folloaing result which sheds light on the relationship between seminormality and the prime ideal structure of a ring.

Theorem 2.2 ([5, 5.1.5., cf. 7, (1.1)]). *Let A be a seminormal ring of dimension* 1, *and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the minimal primes of A. Then for each i, $1\leq i\leq n$,*

(*i*) $\mathfrak{m}=\mathfrak{p}_i\oplus \bigcap_{j\neq i}\mathfrak{p}_j$, *and*

(*ii*) $\mathfrak{m}=\bigoplus_{i=1}^{n}(\bigcap_{j\neq i}\mathfrak{p}_j)$.

Proof. Since A is reduced, the natural map $A\to \bigoplus_1^n A/\mathfrak{p}_i$ is injective, so that A can be regarded as a subring of $\bigoplus_1^n A/\mathfrak{p}_i\subset Q(A)$ under this map. Fix i, $1\leq i\leq n$, let $x\in\mathfrak{m}$, and consider $(0, \cdots, \bar{x}, \cdots, 0)\in \bigoplus_1^n A/\mathfrak{p}_j$. where $\bar{x}\in A/\mathfrak{p}_i$ is the image of x. Now $r(\mathfrak{p}_i\oplus\bigcap_{j\neq i}\mathfrak{p}_j)=\mathfrak{m}$, so that there exists a positive integer s, such that $\mathfrak{m}^t\subseteq\mathfrak{p}_i\oplus\bigcap_{j\neq i}\mathfrak{p}_j$ whenever $t\geq s$. It follows that for each $t\geq s$, there exists $u\in\bigcap_{j\neq i}\mathfrak{p}_j$ such that $x^t-u\in\mathfrak{p}_i$. This implies that $(0, \cdots, \bar{x}, \cdots, 0)\in A$ by 1.1. (ii). There therefore exists an element $v\in\bigcap_{j\neq i}\mathfrak{p}_j$ such that $x-v\in\mathfrak{p}_i$, so that $x\in\mathfrak{p}_i\oplus\bigcap_{j\neq i}\mathfrak{p}_j$, and (i) follows.

To see that (ii) holds, observe that the above argument shows that $\mathfrak{m}\cong\bigoplus_1^n\mathfrak{m}_i$, where $\mathfrak{m}_i$ is the image of $\mathfrak{m}$ in $A/\mathfrak{p}_i$, $1\leq i\leq n$. It follows from (i) that $\mathfrak{m}_i=\mathfrak{m}/\mathfrak{p}_i\cong\bigcap_{i\neq i}\mathfrak{p}_j$, $1\leq i\leq n$, so that $\mathfrak{m}=\bigoplus_{i=1}^n\bigcap_{j\neq i}\mathfrak{p}_j$ as required.

We now make use of 2.1. and 2.2. to prove a structure theorem for a certain class of 1-dimensional F-pure rings, originally proved by Goto and Watanabe under more restrictive hypotheses.

Theorem 2.3 ([7, (1.1)]). *Let A be a* 1*-dimensional ring of prime characteristic p. Suppose that the field $k\cong A/\mathfrak{m}$ is algebraically closed. Then A is F-pure if and only if*

$$\hat{A}\cong k[[X_1, \cdots, X_n]]/(\cdots, X_i X_j, \cdots)_{i\neq j},$$

where $\hat{A}$ denotes the completion of A with respect to $\mathfrak{m}$.

Proof. We begin by noting that A is F-pure if and only if $\hat{A}$ is F-pure by the argument on p. 466 of [4]. Now $\hat{A}$ satisfies the conditions of the statement of the theorem, so that we can assume with no loss of generality that A is complete. It is a straightforward consequence of [4, 1.12] that rings of the form $k[[X_1, \cdots, X_n]]/(\cdots, X_i X_j, \cdots)_{i\neq j}$ are F-pure.

Let us now suppose that A is F-pure and that $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ are the minimal primes of A. As in the proof of 2.2. we can consider A to be a subring of $\bigoplus_1^n A/\mathfrak{p}_i$. Furthermore, by [2, Chap. V, 1.2], $\bar{A}=\bigoplus_1^n \overline{A/\mathfrak{p}_i}$ so that we have

$$A \subseteq \bigoplus_1^n A/\mathfrak{p}_i \subseteq \bigoplus_1^n \overline{A/\mathfrak{p}_i}\,.$$

Since A is complete, it follows that $A/\mathfrak{p}_i$ is complete, $1\leq i\leq n$, so that each $\overline{A/\mathfrak{p}_i}$ is a local ring with maximal ideal $\bar{\mathfrak{m}}_i$, say. By 2.1., A is seminormal so that, by 1.1. (iii),

$$\mathfrak{m} = J(\bar{A}) = \bigoplus_1^n \bar{\mathfrak{m}}_i\,,$$

thus the natural image of $\mathfrak{m}$ in $\overline{A/\mathfrak{p}_i}$ is $\bar{\mathfrak{m}}_i$. Moreover since k is algebraically closed, it follows that $(\overline{A/\mathfrak{p}_i})/\bar{\mathfrak{m}}_i=k$, $1\leq i\leq n$, so that $\overline{A/\mathfrak{p}_i}=A/\mathfrak{p}_i$, and thus each $A/\mathfrak{p}_i$ is integrally closed. It now follows from [1,9.2] that $\bar{\mathfrak{m}}_i$ is a principal ideal, $1\leq i\leq n$, so that by 2.2., there exist elements $x_i\in \bigcap_{j\neq i} \mathfrak{p}_j$, $1\leq i\leq n$, such that

$$\mathfrak{m} = \bigoplus_1^n Ax_i\,.$$

Since A is complete, it is possible to define a surjective k-algebra homomorphism $f\colon k[[X_1, \cdots, X_n]]\to A$, such that $f(X_i)=x_i$, $1\leq i\leq n$. It can easily be verified that $\operatorname{Ker} f=(\cdots, X_iX_j, \cdots)_{i\neq j}$, so that

$$A = k[[X_1, \cdots, X_n]]/(\cdots, X_iX_j, \cdots)_{i\neq j}\,,$$

as required.

Now in [3], Davis considers the situation where A is a 1-dimensional ring such that $\bar{A}$ is a finite A-module and k is algebraically closed. In §3 therein he shows that such a ring is seminormal if and only if $\hat{A}\simeq k[[X_1, \cdots, X_n]]/(\cdots, X_iX_j, \cdots)_{i\neq j}$. In view of this and the remarks which follow 1.5. we are able to deduce the following corollary to 2.3..

Corollary 2.4. *Let A be a 1-dimensional ring of prime characteristic p whose Frobenius endomorphism is finite and suppose that $k\simeq A/\mathfrak{m}$ is algebraically closed. Then A is seminormal if and only if A is F-pure.*

In our main result of this paper, we demonstrate that the equivalence of the two properties will continue to hold if we relax the condition that k be algebraically closed and only insist that it be perfect. We shall require the following auxiliary result.

Lemma 2.5 (cf [8, 4.6]]). *Let $k \subset k'$ be fields of non-zero characteristic p, and suppose that k' is separable over k. Let R be a k-algebra such that $R \otimes_k k'$ is F-pure. Then R is F-pure.*

Proof. This follows in a straightforward way from [8, 4.6] and 1.2.

Theorem 2.6. *Let A be a 1-dimensional ring of prime characteristic p, whose Frobenius endomorphism is finite and whose residue field k is perfect. Then A is F-pure if and only if A is seminormal.*

Proof. If A is F-pure then A is seminormal, by 2.1. Let us now suppose that A is seminormal so that, by 1.5.,

$$Gr(A) \cong_k k + TK[T],$$

where $K = \bar{A}/J(\bar{A})$. Moreover, by 1.5., we can assume that A is complete, so that $k \subset A$, by [10, (28) P]. Now, if $\bar{k}$ denotes the algebraic closure of k, then

$$(k+TK[T]) \otimes_k \bar{k} \cong (k \otimes_k \bar{k}) \oplus (TK[T] \otimes_k \bar{k}) \cong \bar{k} \oplus T(K \otimes_k \bar{k})[T].$$

We claim that, as $\bar{k}$-algebras, $\bar{k} \oplus T(K \otimes_k \bar{k})[T] \cong Gr(A \otimes_k \bar{k})$.

Since $\bar{k}$ is integral and flat over k and since $\bar{k} \subset A \otimes_k \bar{k}$, it easily follows that $\bar{k} \otimes_k \mathfrak{m} = \bar{\mathfrak{m}}$ is the unique maximal ideal of $A \otimes_k \bar{k}$. We have the exact sequence of k-vector spaces

$$O \to \mathfrak{m}^{n+1} \to \mathfrak{m}^n \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to O,$$

which on applying $-\otimes_k \bar{k}$ yields the following exact sequence,

$$O \to \bar{k} \otimes_k \mathfrak{m}^{n+1} \to \bar{k} \otimes_k \mathfrak{m}^n \to \bar{k} \otimes_k \mathfrak{m}^n/\mathfrak{m}^{n-1} \to O.$$

It now follows that, as $\bar{k}$-modules,

$$\bar{k} \otimes_k \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong (\bar{k} \otimes_k \mathfrak{m}^n)/(\bar{k} \otimes_k \mathfrak{m}^{n+1}) \cong \bar{\mathfrak{m}}^n/\bar{\mathfrak{m}}^{n+1}.$$

Therefore

$$\bar{k} \otimes T(K \otimes_k \bar{k})[T] \cong_{\bar{k}} \bar{k} \otimes_k Gr(A) \cong_{\bar{k}} Gr(\bar{k} \otimes_k A). \quad (*)$$

Now K is a finite direct product of fields, each of which is a finite separable extension of k, since $\bar{A}$ is a finite A-module and since k is perfect. It follows from [9, 3.3. (iv)] that the ring $K \otimes_k \bar{k}$ is reduced. In addition, $K \otimes_k \bar{k}$ is Noetherian and zero dimensional, so that by 1.4. and (*), $Gr(A \otimes_k \bar{k})$ is reduced, as is easily seen, and seminormal.

Now $k \subset A$ so that, by a straightforward adaption of the proof of [10, p. 212, Cor. 2], A is a finite $k[[x]]$-module, where x is an indeterminate over k. Hence $A \otimes_k \bar{k}$ is a finite module over $k[[x]] \otimes_k \bar{k}$. We now show that $k[[x]] \otimes_k \bar{k}$ is Noetherian.

Let us first consider the domain $k[[x]]$. It is easily seen that the quotient field of $k[[x]]$ consists of elements of the form $\sum_d^{\infty} k_i x^i$, where $d \in \boldsymbol{Z}$, and it is now a simple matter to verify that k is algebraically closed in $Q(k[[x]])$. This means that k is *maximally algebraic* in $Q(k[[x]])$, in the notation of [13, p. 196] so that by [13, p. 198, Cor. 2], $Q(k[x]]) \otimes_k \bar{k}$ is a domain. This in turn implies that $k[[x]] \otimes_k \bar{k}$ is also a domain, $\bar{k}$ being flat over k. In addition; $\bar{k}$ is integral over k and it follows that $k[[x]] \otimes \bar{k}$ is a 1-dimensional local domain whose maximal ideal is generated by the single element $x \otimes_k 1$, a so that, by [1, p. 84, Ex. 1], $k[[x]] \otimes_k \bar{k}$ is Noetherian. This implies that $A \otimes_k \bar{k}$, a finite $k[[x]] \otimes_k \bar{k}$-module, is itself Noetherian. We now have that $A \otimes_k \bar{k}$ is a 1-dimensional local ring of prime characteristic $\mathfrak{p}$, whose Frobenius endomorphism is easily seen to be finite, so that we can deduce from 1.4. that $A \otimes_k \bar{k}$ is seminormal. Furthermore, since the residue field of $A \otimes_k \bar{k}$ is algebraically closed, it follows from 2.4. that $A \otimes_k \bar{k}$ is F-pure. Now k is perfect, so that $\bar{k}$ is separable over k, and we deduce from 2.5. that A itself is F-pure. This completes the proof.

It is clearly of interest to investigate whether the properties of seminormality and F-purity are equivalent under more general hypothesis than are present in the statement of 2.6. The following example, taken from [12], demonstrates that the result of 2.6. is not necessarily true if we do not impose the condition that the residue field of the ring be perfect.

Example 2.7. [12] Let $A = k[x, y]/(y^2 - x^p - a)$, where k is a non-perfect field of characteristic p, and $a \in k - k^p$, and let $B = Q(A)$. Then A is integrally closed in B, and hence seminormal, but $k(a^{1/p}) \otimes_k A$ is not seminormal. By 2.1., $k(a^{1/p}) \otimes_k A$ cannot be F-pure and an appealing to [8, 4.6], we see that A itself is not F-pure.

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