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# SCATTERING THEORY FOR FIRST ORDER SYSTEMS WITH LONG-RANGE PERTURBATIONS: THE EXISTENCE OF THE MODIFIED WAVE OPERATORS

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## 1. Introduction

The equation of acoustic, electromagnetic and elastic waves can be considered as first order symmetric hyperbolic systems of partial differential equations for $\boldsymbol{C}^m$-valued function $u$ of the form

$$D_t u = \Lambda u\,, \tag{1.1}$$

where

$$\Lambda = E(x)^{-1} \sum_{j=1}^{n} A_j D_j\,.$$

Here $D_t=(1/i)(\partial/\partial t)$ and $D_j=(1/i)(\partial/\partial x_j)$, $A_j$'s are constant $m\times m$ hermitian matrices, and $E(x)$ is a positive definite hermitian matrix. It is measurable and satisfies

$$0<c_1 I\leqq E(x)\leqq c_2 I \qquad \text{for some } c_1 \text{ and } c_2\,.$$

We shall consider the case that there exists a constant matrix $E_0$ and a positive number $\delta$ such that

$$|E(x)-E_0|\leqq C\langle x\rangle^{-\delta} \qquad (\langle x\rangle = (1+|x|^2)^{1/2})\,.$$

Then the equation (1.1) is regarded as the perturbation of the equation

$$D_t u = \Lambda^0 u\,,$$

where

$$\Lambda^0 = E_0^{-1} \sum_{j=1}^{n} A_j D_j\,.$$

When $\delta<1$, the perturbation is said to be short-range, and when $\delta\leqq 1$, it is said to be long-range. In this paper we shall consider a class of first order systems with long-range perturbations.

Now we put $J=E^{-1/2}E_0^{-1/2}$. The operator $W_\pm$ is called the wave operator if the limit

$$W_{\pm}u = \lim_{t\to\pm\infty} e^{it\Lambda} J e^{-it\Lambda^0} P_{ac}(\Lambda^0)u \tag{1.2}$$

exists. From the point of view of the existence and the completeness of wave operators we can say that the general theory of systems with short-range perturbations is at the satisfactory stage (for example [1], [8]). When the perturbation is long-range, as is known for the case of Schrödinger operators, the limit (1.2) does not exist generally, that is, $\Lambda^0$ cannot be regarded as the free operator. Thus our problem is to construct another operator $W_{\pm}^D$ (so-called modified wave operator) which satisfies the intertwining property

$$e^{is\Lambda} W_{\pm}^D = W_{\pm}^D e^{is\Lambda^0}.$$

The fundamental problems of the theory of long-range perturbations are the existence and the completeness of $W_{\pm}^D$. In connection with these problems H. Tamura first proved the limiting absorption principle for uniformly propagative systems with long-range perturbations ([5]). This result has been extended to wide classes ([6], [7], [8] and [9]). For the case of 2-body Schrödinger operators the limiting absorption principle alone enables us to prove the existence and completeness of (modified) wave operators even in the case of long-range perturbations. This is also the case for first order hyperbolic systems when all characteristic roots of the unperturbed operator are simple. However, if we consider the case that characteristic roots are non smooth, for example the Maxwell equation in a crystal optics, we encounter serious difficulties in developing the scattering theory by using the above mentioned stationary results.

In this paper we prove the existence of the modified wave operator for some classes of systems by using the time dependent method. In section 2 we shall state the exact conditions for $\Lambda^0$, which includes the Maxwell equations in biaxial crystals. Their characteristic roots are non smooth, and our theory admits including such equations.

Now we want to define the modified wave operator as the limit

$$W_{\pm}^D u = \lim_{t\to\pm\infty} e^{it\Lambda} J X(t) u \tag{1.3}$$

for some partial isometric operator $X(t)$ instead of $e^{-it\Lambda^0}$. In the construction of $X(t)$ we shall use the solution of the Hamilton-Jacobi equations

$$\frac{\partial W_k}{\partial t} = \lambda_k(\nabla_\xi W_k, \xi), \tag{1.4}$$

where $\lambda_k(x, \xi)$'s are the eigenvalues of the principal symbol of $\Lambda$ (an $m\times m$ matrix).

In [2] Lars Hörmander has proved the existence of the modified wave operators for Schrödinger equations with long-range potentials. Here he solved the Hamilton-Jacobi equation and used the stationary phase method on $\boldsymbol{R}^n$. But in our case, since $\lambda_k(x, \xi)$ is positively homogeneous of degree 1 with respect to $\xi$, its Hessian vanishes everywhere. Hence we cannot apply the Hörmander calculus, and we shall use the stationary phase method on a hypersurface. Then our calculus is rather different from that of [2].

This paper is organized as follows. In section 2 we shall give the exact assumptions for $\Lambda^0$ and some preliminary results. In section 3 we shall define the operator $X(t)$ and show some properties. In section 4 we shall prove the existence of the limit (1.3) by the use of the stationary phase method on a hypersurface.

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## 2. Assumptions and some preliminary results

We are now treating the case that the perturbation is long-range. More precisely we assume

ASSUMPTION (E) $E(x)\in C^\infty(\boldsymbol{R}^n)$ and

$$|\partial_x^\alpha(E(x)-E_0)|\leqq C\langle x\rangle^{-\delta-|\alpha|}$$

for $\delta>0$ and $|\alpha|\geqq 0$.

On $\Lambda^0$ we also require some assumptions. We put

$$\Lambda^0(\xi)=E_0^{-1}\sum_{j=1}^n A_j\xi_j \qquad \text{(the symbol of } \Lambda^0)$$

and

$$\rho_0=\max_{\xi\in\boldsymbol{R}^n}\#\{\text{positive eigenvalues of } \Lambda^0(\xi)\}\,.$$

Then

ASSUMPTION (F) 1) $\Lambda^0$ is strongly propagative, that is, for some $d$

$$\operatorname{rank}\Lambda^0(\xi)=m-d \qquad (\text{for } \xi\neq 0)\,.$$

2) $$\rho_0=(m-d)/2\,.$$

Here $d$ is called the deficit. The condition 2) of (F) is equivalent to that the multiplicities of non-zero eigenvalues are all simple outside a conic unll set. As an example we consider the Maxwell equations in crystals:

$$\nabla\times H-\varepsilon\frac{\partial E}{\partial t}=0\,,\quad \nabla\times E+\mu_0\frac{\partial H}{\partial t}=0\,.$$

Here $\varepsilon=(\varepsilon_{ij})$ is a tensor dielectric constant and $\mu_0$ is a scalor magnetic permeability. Let $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ be eigenvalues of $\varepsilon$. We may assume that

$$\varepsilon_1 \geqq \varepsilon_2 \geqq \varepsilon_3 > 0 .$$

There are three classes which are defined by the condition (1) $\varepsilon_1>\varepsilon_2>\varepsilon_3$, (2) $\varepsilon_1>\varepsilon_2=\varepsilon_3$ or $\varepsilon_1=\varepsilon_2>\varepsilon_3$ and (3) $\varepsilon_1=\varepsilon_2=\varepsilon_3$ (isotropic). The cases of (1) and (2) are covered in our class, and the case of (3) is not covered (refer to C.H. Wilcox [11]).

Similar to the case of $\Lambda^0$ we put

$$\Lambda(x, \xi) = E(x)^{-1} \sum_{j=1}^{n} A_j \xi_j \qquad \text{(the symbol of } \Lambda)$$

and

$$\rho(x) = \max_{\xi \in \boldsymbol{R}^n} \# \{\text{positive eigenvalues of } \Lambda(x, \xi)\} .$$

The eigenvalues of $\Lambda^0(\xi)$ and $\Lambda(x, \xi)$ can be enumerated as follows:

$$\lambda^0_{\rho_0}(\xi) \geqq \cdots \geqq \lambda^0_1(\xi) > \lambda^0_0(\xi) \equiv 0 > \lambda^0_{-1}(\xi) \geqq \cdots \geqq \lambda^0_{-\rho_0}(\xi) \tag{2.1}$$

and

$$\lambda_{\rho(x)}(x, \xi) \geqq \cdots \geqq \lambda_1(x, \xi) > \lambda_0(x, \xi) \equiv 0 > \lambda_{-1}(x, \xi) \geqq \cdots \geqq \lambda_{-\rho(x)}(x, \xi) , \tag{2.2}$$

respectively. If $\lambda^0_j(\xi) \equiv \lambda^0_k(\xi)$ ($\lambda_j(x, \xi) \equiv \lambda_k(x, \xi)$ for fixed $x$), then $j=k$.

Here we note that 2) of (F) guarantees that $\lambda_k(x, \xi)$ is smooth for large $|x|$ when $\lambda^0_k(\xi)$ is smooth. It is not the case if 2) of (F) is not satisfised, and (1.4) does not have classical solutions in general. Hence it seems to be difficult to remove the condition 2) of (F) as long as our method is used.

To study $\Lambda^0$ and $\Lambda$ in one Hilbert space $\mathcal{H}=L^2(\boldsymbol{R}^n; \boldsymbol{C}^m)$ with the usual inner product

$$(u, v) = \int_{\boldsymbol{R}^n} u(x)^* v(x) dx$$

(note that $\Lambda^0$ and $\Lambda$ are not self-adjoint in $\mathcal{H}$), we set

$$\tilde{\Lambda}^0 = \sum_{j=1}^{n} E_0^{-1/2} A_j E_0^{-1/2} D_j$$

and

$$\tilde{\Lambda} = \tilde{E}(x)^{-1/2} \tilde{\Lambda}^0 \tilde{E}(x)^{-1/2} \quad (\tilde{E}(x) = E_0^{-1/2} E(x) E_0^{-1/2}) ,$$

which are self-adjoint operators in $\mathcal{H}$ with domains

$$\mathcal{D}(\tilde{\Lambda}^0) = \mathcal{D}(\tilde{\Lambda}) = \{u \in \mathcal{H};\ \tilde{\Lambda}^0 u \in \mathcal{H}\} .$$

Now we can reduce our problem to studying $\tilde{\Lambda}^0$ and $\tilde{\Lambda}$ in $\mathcal{H}$ (see for example

[9]), and then our purpose is now to construct a partial isomteric operator $\tilde{X}(t)$ in $\mathcal{H}$ and to show the existence of the limits

$$\tilde{W}_{\pm}^{D} = \lim_{t\to\pm\infty} e^{it\tilde{\Lambda}}\tilde{X}(t)\,.$$

Note that the eigenvalues of $\tilde{\Lambda}^0(\xi)$ and $\tilde{\Lambda}(x, \xi)$ (symbols of $\tilde{\Lambda}^0$ and $\tilde{\Lambda}$) coincide with those in (2.1) and (2.2). Clearly $\tilde{E}$ satisfies the Assumption (E) with $E_0=I$, and $\tilde{\Lambda}^0$ satisfies the Assumption (F). Then we can replace $E_0$ with $I$ without loss of generality, and hereafter we shall omit the sign "$\sim$" for simplicity.

Here we give some fundamental concepts which are important to develop the theory of first order symmetric hyperbolic systems. (About these concepts refer to C.H. Wilcox [10] and K. Kikuchi [3]). Let $P_k^0(\xi)$ and $P_k(x, \xi)$ be orthogonal projections on the eigenspaces corresponding to $\lambda_k^0(\xi)$ and $\lambda_k(x, \xi)$, respectively. The properties of $\lambda_k^0(\xi)$ and $P_k^0(\xi)$ are given in [3, (2.5)$\sim$(2.13)]. $\lambda_k(x, \xi)$ and $P_k(x, \xi)$ also satisfy the same properties. Especially $\lambda_k(x, \xi)$'s are positively homogeneous of degree 1 with respect to $\xi$. The slowness surface for $\Lambda^0$ is den defined by

$$S = \{\xi\in \boldsymbol{R}^n;\ \det(I-\Lambda^0(\xi)) = 0\}\ .$$

The condition 1) of the Assumption (F) is equivalent to the fact that $S$ is bounded. we put

$$S_k = \{\xi\in\boldsymbol{R}^n;\ \lambda_k^0(\xi) = 1\}\ , \qquad j = 1, 2, \cdots, \rho_0\,.$$

and then $S=\bigcup_{k=1}^{\rho} S_k$. Let $Z_S^{(1)}$ be a set of all algebraic singularities of $S$. $\lambda_k^0(\xi)$ may not be smooth on $Z_S^{(1)}$. $Z_S^{(2)}$ denotes a set defined in [3, page 579], which contains all points at which the Gaussian curvature vanishes. (For the definition of the Gaussian curvature see [4]). And we put

$$Z_S = Z_S^{(1)} \cup Z_S^{(2)}.$$

In this paper we shall denote for $M\subset\boldsymbol{R}^n$

$$\bar{M} = \{\xi = rs;\ s\in M \quad \text{and} \quad r\in\boldsymbol{R}\}\ .$$

For example $\bar{Z}_S=\{\xi=rs;\ s\in Z_S \text{ and } r\in\boldsymbol{R}\}$. (Note that $\bar{Z}_S$ is a closed null set). The similar concepts are defined for $\Lambda$, for example $S_x$, $Z_{S_x}$ and $\bar{Z}_{S_x}$.

About the eigenvalues and projections we have the following proposition.

**Proposition 2.1.** *Under the Assumptions* (E) *and* (F) *the following facts hold*:

i) *There exists an* $R>0$ *such that*

$$\rho(x) = \rho_0 \quad \textit{when} \quad |x| > R .$$

ii) *Let* $K \subset \boldsymbol{R}^n \backslash \bar{Z}_S$ *be compact. Take* $R$ *large enough. Then* $\lambda_k(x, \xi)$ *and* $P_k(x, \xi)$ *are smooth for* $(x, \xi) \in \{|x| > R\} \times K$ *and satisfies*

$$|\partial_\xi^\alpha \partial_x^\beta (\lambda_k(x, \xi) - \lambda_k^0(\xi))| \leqq C \langle x \rangle^{-\delta - |\beta|}$$

*for any* $\alpha$, $\beta$, $|x| > R$ *and* $\xi \in K$.

iii)
$$|\partial_\xi^\alpha \partial_x^\beta (P_k(x, \xi) - P_k^0(\xi))| \leqq C \langle x \rangle^{-\delta - |\beta|}$$

*for any* $\alpha$, $\beta$, $|x| > R$ *and* $\xi \in K$.

*Here* $C$ *is uniform for* $(x, \xi) \in \{|x| > R\} \times K$.

Proof. Since $\rho_0 = (m-d)/2$, the algebraic equation with respect to $\lambda$

$$\det(\lambda I - \Lambda^0(\xi)) = 0$$

does not have a double root other than $\lambda = 0$ when $\xi \notin \bar{Z}_S^{(1)}$. Then the continuity of the roots gives i) and

$$|\lambda_k(x, \xi) - \lambda_k^0(\xi)| \to 0 \qquad \text{as} \quad |x| \to \infty$$

uniformly for $\xi$ in a compact set $K \subset \boldsymbol{R}^n \backslash \bar{Z}_S^{(1)}$. The rest of the assertions are easily proved by a straightforward calculation. Q.E.D.

In the rest of this section we state a theorem of the stationary phase method on a hypersurface. It will play an important role in the argument of section 4.

Let $\{S_t\}_{t \geqq T}$ be a family of smooth hypersurfaces. For each $t$ a comapct support smooth function $\mu_t(s)$ defined on $S_t$ is given. Suppose that they satisfy

i) For sufficiently small domain $U$ of $\boldsymbol{R}^n$ there exist smooth functions $\{\varphi_t\}$ such that

$$S_t \cap U = \{\varphi_t(\xi) = 0\} .$$

ii) There exists another smooth function $\varphi$ such that

$$|\partial^\alpha \varphi_t - \partial^\alpha \varphi| \leqq C_\alpha t^{-\delta} \qquad \text{for} \quad |\alpha| \geqq 0 .$$

iii) The Gaussian curvature $K_t(s)$ of $S_t$ satisfies

$$0 < c_1 \leqq |K_t(s)| \leqq c_2 \qquad \text{for} \quad s \in \operatorname{supp} \mu_t ,$$

where $c_1$ and $c_2$ are constants independent of $t$.

iv) The derivatives of $\mu_t(s)$ are uniformly bounded with respect to $t$.

We consider here the integral

$$I_t(x) = \int_{S_t} e^{ixs}\mu_t(s)d\!\!{}^-S_t, \quad x\in \boldsymbol{R}^n$$
$$(d\!\!{}^-S_t = (2\pi)^{-(n-1)/2}dS_t)\,.$$

Let $s_t^{\gamma}(\vartheta)$'s be the points where the exterior unit normal of $S_t$ is $\vartheta$, and let $\rho_t(\vartheta)$ be the number of $\{s_t^{\gamma}(\vartheta)\}$. Put

$$\psi_t^{\pm}(s) = \exp\{\pm(\pi i/4)(p_t^+(s)-p_t^-(s))\}\,, \tag{2.3}$$

where $p_t^{\pm}(s)$ denotes the number of positive (negative) principal curvatures of $S_t$ at $s$. Now the following fact holds.

**Theorem 2.2.** *If $\{S_t\}_{t\geq T}$ and $\{\mu_t\}_{t\geq T}$ are given as above, then $I_t(x)$ satisfies the following expansion formula:*

$$\begin{aligned} I_t(x) = &\sum_{\gamma=1}^{\rho_t(\vartheta)} e^{ixs}\psi_t^+(s)\mu_t(s)|K_t(s)|^{-1/2}|_{s=s_t^{\gamma}(\vartheta)}|x|^{-(n-1)/2} \\ &+\sum_{\gamma=1}^{\rho_t(-\vartheta)} e^{ixs}\psi_t^-(s)\mu_t(s)|K_t(s)|^{-1/2}|_{s=s_t^{\gamma}(-\vartheta)}|x|^{-(n-1)/2} \\ &+\sum_{\gamma=1}^{\rho_t(\vartheta)} e^{ixs}C^+(t,s)|K_t(s)|^{-1/2}\psi_t^+(s)|_{s=s_t^{\gamma}(\vartheta)}|x|^{-n/2} \\ &+\sum_{\gamma=1}^{\rho_t(-\vartheta)} e^{ixs}C^-(t,s)|K_t(s)|^{-1/2}\psi_t^-(s)|_{s=s_t^{\gamma}(-\vartheta)}|x|^{-n/2} \\ &+q(t,x)\,, \end{aligned}$$

*where* $\vartheta=x/|x|$,

$$|q(t,x)|\leq C\sum_{|\alpha|\leq n+2}|\partial_\sigma^\alpha\mu_t(s(\sigma))|\,|x|^{-(n+2)/2}$$

*and*

$$|C^{\pm}(t,s)|\leq C\sum_{|\alpha|=0,1}|\partial_\sigma^\alpha\mu_t(s(\sigma))|_{s(\sigma)=s}|$$
$$(\sigma \text{ is a local coordinate})$$

*for some constant C independent of t.*

The proof of Theorem 2.2 can be carried out in the same line of the proofs written in many literatures (see, for example, M. Matsumura [4, §4 and §5]).

## 3. Definition and properties of $X(t)$

We denote the set $\{u\in \boldsymbol{C}^\infty(\boldsymbol{R}^n);\, \hat{u}\in C_0^\infty(\boldsymbol{R}^n\backslash\bar{Z}_S)\}$ by $\mathcal{S}_S$. The unitary group $e^{-it\Lambda^0}$ is given for $u\in\mathcal{S}_S$ by

$$e^{-it\Lambda^0}u = P_0^0(D_x)u+\sum_{|k|=1}^{{}^0d}\int_{R^n} e^{ix\xi-it\lambda_k^0(\xi)}P_k^0(\xi)\hat{u}(\xi)d\!\!{}^-\xi\,. \tag{3.1}$$

We must replace $e^{-it\Lambda^0}$ in the definition of the wave operator by another operator $X(t)$. So we shall seek $X(t)$ in the form

$$X(t)u=\sum_{|k|=1}^{\rho_0}\int_{\Omega_t} e^{ix\xi-iW_k(t,\xi)}P_k^0(\xi)\hat{u}(\xi)d\xi\,, \tag{3.2}$$

where $W_k(t, \xi)$ satisfies the Hamilton-Jacobi equation (1.4) and $\Omega_t$ satisfies $\bigcup_t \Omega_t = \boldsymbol{R}^n\backslash \bar{Z}_S$. The solutions of the Hamilton-Jacobi equations are obtained in a standard way. (Here we state only the case of $t\rightarrow+\infty$).

**Lemma 3.1.** *Under the Assumptions* (E) *and* (F), *let* $\{t_l\}_{l=1}^{\infty}$ *be a given sequence with* $t_1<t_2<\cdots$ *and* $\lim_{l\to\infty} t_l=\infty$. *Then there exist a sequence of conic open sets* $\{\Omega_l\}_{l=0}^{\infty}$ *with* $\Omega_0\subset\Omega_1\subset\cdots$ *and* $\bigcup_{l=0}^{\infty}\Omega_l=\boldsymbol{R}^n\backslash\bar{Z}_S^{(1)}$, *and* $C^\infty$*-functions* $W_k(x, \xi)$ $(|k|=1, \cdots, \rho_0)$ *defined on* $\bigcup_{l=1}^{\infty}[t_l, \infty)\times\Omega_l$ *having the following properties*:

i) $W_k(x, \xi)$ *satisfies*

$$\frac{\partial W_k}{\partial t}=\lambda_k(\nabla_\xi W_k, \xi) \quad on \quad \bigcup_{l=1}^{\infty}[t_l, \infty)\times\Omega_l\,. \tag{3.3}$$

*For any compact set* $K\subset\boldsymbol{R}^n\backslash\bar{Z}_S^{(1)}$

ii) $|\partial_\xi^\alpha W_k(t, \xi)|\leqq C_\alpha t \quad$ *for* $|\alpha|\geqq 1$ *and* $\xi\in K$

iii) $|\partial_\xi^\alpha(t^{-1}W_k(t, \xi)-\lambda_k^0(\xi))|+|\partial_\xi^\alpha(\lambda_k(\nabla W_k, \xi)-\lambda_k^0(\xi))|\leqq C_\alpha t^{-\delta}$ *for* $|\alpha|\geqq 0$ *and* $\xi\in K$

iv) *For* $r>0 \quad W_k(t, r\xi)=rW_k(t, \xi)$

v) $W_k(t, -\xi)=-W_{-k}(t, \xi)$.

The proof of this lemma is similar to that of Lemma 3.8 of [2]. Assertions iv) and v) follow from the fact that $\lambda_k(x, \xi)$'s have the same properties.

From Lemma 3.1 ii) and iii) for $|\alpha|=1$ we have

$$C't\leqq|\nabla W_k(t, \xi)|\leqq Ct\,. \tag{3.4}$$

Now we define $X(t)$ by (3.2) for $u\in\mathcal{S}_S$ with functions $W_k$ constructed in Lemma 3.1 and domains $\Omega_t$ given by

$$\Omega_t=\Omega_l \quad \text{for} \quad t\in[t_l, t_{l+1})\,. \tag{3.5}$$

Let $I_{W_k}(t)$ be operators defined by

$$I_{W_k}(t)u=\int_{\Omega_t} e^{ix\xi-iW_k(t,\xi)}\hat{u}(\xi)d\xi \quad \text{for} \quad u\in\mathcal{S}_S\,,$$

and we put $X_k(t)=I_{W_k}(t)P_k^0(D_x)$. Then $X(t)=\sum_k X_k(t)$. $Y_k(t)$ and $Y(t)$ are other operators given by $Y_k(t)=I_{W_k}(t)P_k(\nabla W_k(D_x), D_x)$ and $Y(t)=\sum_k Y_k(t)$.

Here note that from Proposition 2.1 and Lemma 3.1 the matrices $P_k(\nabla W_k, \xi)$'s are smooth on supp $\hat{u}$ when $t$ is large.

Let $\rho(r)\in C^\infty(\boldsymbol{R}^1)$ be a function satisfying

$$\rho(r)=\begin{cases}1 & \text{if} \quad r\geqq 1\\ 0 & \text{if} \quad r\leqq 1/2,\end{cases}$$

and we put

$$\chi(t, x)=\rho(2C_W|x|/t) \tag{3.6}$$

where $C_W$ is a constant which satisfies

$$C_W^{-1}\leqq|\nabla\lambda_k^0(\xi)|\leqq C_W \quad \text{for} \quad |k|=1, 2, \cdots, \rho_0. \tag{3.7}$$

The existence of such constant easily follows from 1) of Assumption (F).

Now we have

**Lemma 3.2.** i) *Let $u\in\mathcal{S}_S$ be fixed. Take sufficiently large $t$. Then for any $s>0$*

$$\|(1-\chi(t, x))I_{W_k}(t)u\|_{L^2}\leqq C_{s,u}u^{-s}.$$

ii) *$u$ and $t$ are the same as above. Then for any $s>0$*

$$\|\langle x\rangle^{-s}I_{W_k}(t)u\|_{L^2}\leqq C_{s,u}t^{-s}.$$

iii) *$X_k(t)-Y_k(t)\to 0$ strongly as $t\to\infty$.*

Proof i) Lemma 3.1 iii) gives

$$|t^{-1}\nabla W_k(t, \xi)|\geqq|\nabla\lambda_k^0(\xi)|-c_1t^{-\delta},$$

and this implies

$$|\nabla_\xi(x\xi-W_k(t, \xi))|\geqq(C_W^{-1}/4)t$$

when $t$ is large and $(t, x)$ is in the support of $1-\chi(t, x)$. Then the assertion easily follows from the equality

$$e^{ix\xi-iW_k(t,\xi)}=-i\frac{\nabla_\xi(x\xi-W_k)}{|\nabla_\xi(x\xi-W_k)|^2}\nabla_\xi e^{ix\xi-iW_k(t,\xi)}$$

and integration by parts.

ii) The assertion follows from the equality

$$e^{-iW_k(t,\xi)}=-i|\nabla W_k|^{-2}\nabla W_k\cdot\nabla e^{-iW_k(t,\xi)},$$

the inequality (3.4) and integration by parts.

iii) The assertion follows from (3.4) and Proposition 2.1 iii). Q.E.D.

It is easy to see that $X(t)$ is a partial isometry as an operator from

$\mathcal{H}(=L^2(\boldsymbol{R}^n; \boldsymbol{C}^m))$ to $\mathcal{H}$ with the domain $\mathcal{S}_S$. Since $\mathcal{S}_S$ is dense in $\mathcal{H}$, $X(t)$ can be extended to an operator whose domain is the whole space $\mathcal{H}$. We denote this operator also by $X(t)$.

## 4. The existence of the modified wave operators

The purpose of this section is to prove the main theorem:

**Theorem 4.1.** *The modified wave operator*

$$W_{\pm}^D = \operatorname*{s-lim}_{t\to\pm\infty} e^{it\Lambda}X(t) \tag{4.1}$$

*exists, and it is a partially isometric operator with the intertwining property*

$$e^{is\Lambda}W_{\pm}^D u = W_{\pm}^D e^{is\Lambda^0}u \qquad s\in\boldsymbol{R} \quad \textit{and} \quad u\in\mathcal{H}_c(\Lambda^0)\,.$$

Proof If the limit

$$W_{\pm}^D u = \operatorname*{s-lim}_{t\to\pm\infty} e^{it\Lambda}X(t)u$$

exists for a dense set of $u\in\mathcal{H}$, it exists for every $u\in\mathcal{H}$. Hence we may assume $u\in\mathcal{S}_S$. If $t$ is so large that supp $\hat{u}\subset\Omega_t$, we may consider $\Omega_t=\boldsymbol{R}^n$ in (3.2). Then we shall take such $t$ for any fixed $u$.

Here we consider only the case of $W_+^D$. ($W_-^D$ can be treated in the same way).

In the same way as in the proof of intertwining property for the case of the Schrödinger operators (see [2]) we have

$$\lim_{t\to\infty} X(t+s)^*X(t)e^{-it\Lambda^0}u = (I-P_0^0(D_x))u\,,$$

and the intertwining property has been proved. (Note that $P_c(\Lambda^0)=P_{ac}(\Lambda^0)=I-P_0^0(D_x)$).

Now we prove the existence of the limit (4.1). From Lemma 3.2 iii) the existence of $W_+^D u$ is equivalent to the existence of

$$\operatorname*{s-lim}_{t\to+\infty} e^{it\Lambda}Y_k(t)u \qquad (|k|=1,2,\cdots,\rho_0)\,. \tag{4.2}$$

When we have for some $T$

$$\int_T^\infty \left\|\frac{d}{dt}(e^{it\Lambda}Y_k(t)u)\right\|_{L^2}dt<\infty\,,$$

we know that the limit (4.2) exists. Here note the equality

$$i^{-1}e^{-it\Lambda}\frac{d}{dt}(e^{it\Lambda}Y_k(t)u) = \Lambda Y_k(t)u - i\frac{dY_k(t)}{dt}u$$
$$= \int_{R^n} e^{ix\xi - iW_k(t,\xi)}[(\Lambda(x, \xi) - \frac{\partial W_k}{\partial t}(t, \xi))P_k(\nabla W_k, \xi)\hat{u}(\xi)$$
$$+R(t, x, \xi)\hat{u}(\xi)]d\xi ,$$

where

$$|R(t, x, \xi)| \leqq C(\langle x\rangle^{-1-\delta} + t^{-1-\delta}) .$$

From Lemma 3.2 ii) we easily have

$$\|\int_{R^n} e^{ix\xi - iW_k(t,\xi)}R(t, x, \xi)\hat{u}(\xi)d\xi\|_{L^2} \leqq Ct^{-1-\delta} .$$

Let $\chi(t, x)$ be of (3.6). From Lemma 3.2 i) we have

$$\|\int_{R^n} e^{ix\xi - iW_k(t,\xi)}(1-\chi(t, x))$$
$$\cdot(\Lambda(x, \xi) - \frac{\partial W_k}{\partial t}(t, \xi))P_k(\nabla W_k, x)\hat{u}(\xi)d\xi\| \leqq Ct^{-1-\delta} .$$

Hence we have only to consider

$$I(t, x) = \int_{R^n} e^{ix\xi - iW_k(t,\xi)}\chi(t, x) \cdot(\Lambda(x, \xi) - \frac{\partial W_k}{\partial t}(t, \xi))P_k(\nabla W_k, \xi)\hat{u}(\xi)d\xi . \tag{4.3}$$

Our purpose is to prove that the $L^2$-norm of $I(t, x)$ is integrable with respect to $t$.

Here we introduce a new surface which is like the slowness surface:

$$S_k(t) = \{\xi \in \Omega_t;\ t^{-1}W_k(t, \xi) = 1\} \qquad (\Omega_t \text{ is of (3.5)}) .$$

Put $K = \text{supp}\,\hat{u}$ and $\bar{K} = \{\xi = r\xi';\ r \in \boldsymbol{R} \text{ and } \xi' \in K\}$. It follows from (3.4) that $S_k(t) \cap \bar{K}$ is a smooth hypersurface when we take sufficiently large $t$ (depending on $\bar{K}$). We put $\Sigma_t = S_k(t) \cap \bar{K}$.

In (4.3) we make the change of variables $\xi \to (r, s)$ $(r>0, s \in \Sigma_t)$ by $\xi = rs$. From Lemma 3.1 iv) and v) this transformation is surely one to one. We put

$$F_k(t, x, \xi) = \chi(t, x)\,(\Lambda(x, \xi) - \frac{\partial W_k}{\partial t}(t, \xi))\hat{P}_k(\nabla W_k, \xi) . \tag{4.4}$$

In the same way as [10, §4] we have

$$I(t, x) = \int_{-\infty}^{\infty} (\int_{\Sigma_t} e^{i(rxs - W_k(t,rs))}F_k(t, x, rs)\hat{u}(rs)\,|T_t(s)|^{-1}dS(t))\,|r|^{n-1}dr$$
$$= \int_{-\infty}^{\infty} e^{-irt}r\,|r|^{n-1}(\int_{\Sigma_t} e^{irxs}F_k(t, x, s)\hat{u}(rs)\,|T_t(s)|^{-1}dS(t))dr ,$$

where $dS(t)$ is the surface element of $\Sigma_t$ and

$$T_t(s) = t^{-1}\nabla W_k(t, s) \qquad \text{for} \quad s \in S_k(t) . \tag{4.5}$$

Now we put

$$v_k(t, x, y, r) = \int_{\Sigma_t} e^{ixs} F_k(t, y, s)\hat{u}(rs)|T_t(s)|^{-1} dS(t) .$$

Then

$$I(t, x) = \int_{-\infty}^{\infty} e^{-irt} v_k(t, rx, x, r) r |r|^{n-1} dr . \tag{4.6}$$

We shall use the stationary phase method (Theorem 2.2) for the integral on $\Sigma_t$. For any parametrization $s=s(\sigma)$ the gradient of the phase $x\cdot s(\sigma)$ with respect to $\sigma$ satisfies $\vec{\nabla}_\sigma(x\cdot s)=\vec{x}\cdot\vec{\nabla}_\sigma s$. Here note that the column vector of $\vec{\nabla}_\sigma s(\sigma)$ construct the tangent plane of $S_k(t)$ at $s(\sigma)$. Hence the stationary points are the points $s$ where $x$ is normal to $S_k(t)$, that is, $N_t(s)=\pm\vartheta$ $(\vartheta=x/|x|)$. We denote the Gaussian curvature of $S_k(t)$ by $K_t(s)$. Since $K\subset \boldsymbol{R}^n\setminus\bar{Z}_S$, it easily follows that $K_t(s)$ does not vanish for sufficiently large $t$ in $\Sigma_t$. This fact implies that the Gauss map $N_t(s)$ on $\Sigma_t$ is a $C^\infty$-injective map into $S^{n-1}$. We put $\omega_t=N_t(\Sigma_t)$. Then $N_t$ is bijective from $\Sigma_t$ to $\omega_t$. We denote the inverse map of this map by $s_t$.

It is easy to see that the integral of $v_k$ satisfies all assumptions of Theorem 2.2. Hence we have

$$\begin{aligned} v_k(t, x, y, r) &= e^{ix\cdot s} F_k(t, y, s)\hat{u}(rs)|T_t(s)|^{-1}|K_t(s)|^{-1/2} \\ &\quad \cdot \psi_t^+(s)|_{s=s_t(\vartheta)}|x|^{-(n-1)/2} + e^{ix\cdot s}F_k(t, y, s)\hat{u}(rs) \\ &\quad \cdot |T_t(s)|^{-1}|K_t(s)|^{-1/2}\psi_t^-(s)|_{s=s_t(-\vartheta)}|x|^{-(n-1)/2} \\ &\quad + e^{ix\cdot s}C^+(t, y, r, s)|K_t(s)|^{-1/2}\psi_t^+(s)|_{s=s_t(\vartheta)}|x|^{-n/2} \\ &\quad + e^{ix\cdot s}C^-(t, y, r, s)|K_t(s)|^{-1/2}\psi_t^-(s)|_{s=s_t(-\vartheta)}|x|^{-n/2} \\ &\quad + q(t, x, y, r) \\ &=: v_{0,+} + v_{0,-} + v_{1,+} + v_{1,-} + q , \end{aligned} \tag{4.7}$$

where

$$|q(t, x, y, r)| \leqq C(\langle y\rangle^{-\delta} + t^{-\delta})|x|^{-(n+2)/2}$$

and

$$|C^\pm(t, y, r, s)| \leqq C \sum_{|\alpha|=0,1} |\{\partial_\xi^\alpha(F_k\hat{u})|_{\xi=s}\}| \tag{4.8}$$

with constants independent of $t$. It is easy to see that the supports of $C^\pm$ and $q$ with respect to $r$ are compact in $\boldsymbol{R}^1\setminus\{0\}$.

From (4.6) and (4.7)

$$I(t, x) = I_{0,+} + I_{0,-} + I_{1,+} + I_{1,-} + Q ,$$

where

$$I_{j,\pm}(t, x) = \int_{-\infty}^{\infty} e^{-irt} v_{j,\pm}(t, rx, x, r) r |r|^{n-1} dr \qquad (j = 0, 1)$$

and $Q(t, x)$ is defined similarly. $Q$ satisfies the estimate

$$|Q(t, x)| \leqq C|x|^{-(n+2)/2}(\langle x \rangle^{-\delta} + t^{-\delta}). \tag{4.9}$$

Now we put

$$\begin{aligned}\Phi_{\pm}^{0}(\tau, s; x, t) = F_k(t, x, s)|T_t(s)|^{-1}|K_t(s)|^{-1/2} \\ \cdot \int_{-\infty}^{\infty} e^{ir\tau} r |r|^{(n-1)/2} \hat{u}(rs) \psi_t^{\mathrm{sign}(\pm r)}(s) dr\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\Phi_{\pm}^{1}(\tau, s; x, t) = |K_t(s)|^{-1/2} \int_{-\infty}^{\infty} e^{ir\tau} C^{\pm}(t, x, r, rs) \\ \cdot |r|^{n/2-1} \psi_t^{\mathrm{sign}(\pm r)}(s) dr.\end{aligned} \tag{4.11}$$

(Recall that $\psi_t^{\pm}$ is defined in (2.3)). Then $I_{j,\pm}$ are written as

$$I_{j,\pm}(t, x) = \begin{cases} |x|^{-(n-1+j)/1} \Phi_{\pm}^{j}(x \cdot s_t(\pm\vartheta) - t, s_t(\pm\vartheta); x, t) & (\pm\vartheta \in \omega_t) \\ 0 \quad \text{(otherwise).} \end{cases}$$

It easily follows from Lemma 3.1 iv) and (4.5) that $T_t(s) = (s \cdot N_t(s))^{-1} N_t(s)$. This equality implies

$$x \cdot s_t(-\vartheta) - t = -|x|(-\vartheta) \cdot s_t(-\vartheta) - t = -|x|\,|T_t(s_t(-\vartheta))|^{-1} - t.$$

Hence we have

$$|x \cdot s_t(-\vartheta) - t| > C|x| + t. \tag{4.12}$$

On the other hand integration by parts gives

$$\Phi_{-}^{j}(\tau, s; x, t) = \{\cdots\}(-1/i\tau)^2 \int_{-\infty}^{\infty} e^{ir\tau} (\partial/\partial r)^2 \{\cdots\} dr. \tag{4.13}$$

Then from (4.12) and (4.13) $I_{j,-}(t, x)$ $(j=0, 1)$ satisfy the same estimate of (4.9). Now we write

$$I(t, x) = I_{0,+} + I_{1,+} + \tilde{Q}, \tag{4.14}$$

where $\tilde{Q} = I_{0,-} + I_{1,-} + \tilde{Q}$. Since $\tilde{Q}$ satisfies the same estimate of (4.9) and $|x| > (1/4C_W)t$ on the support of $\chi(t, x)$, we have

$$\|\tilde{Q}(t, \cdot)\|_{L^2} \leqq Ct^{-1-\delta}. \tag{4.15}$$

Next we calculate the $L^2$-norm of $I_{j,+}$:

$$\|I_{j,+}(t,\cdot)\|_{L^2}^2=\int_0^\infty\int_{S^{n-1}} I_{j,+}(t,x)^* I_{j,+}(t,x)\,|x|^{n-1}dS^{n-1}d|x|$$
$$=\int_0^\infty\int_{\omega_t}\Phi_+^j(x\cdot s_t(\vartheta)-t,\ s_t(\vartheta);\ x,\ t)^*$$
$$\cdot\Phi_+^j(x\cdot s_t(\vartheta)-t,\ s_t(\vartheta);\ x,\ t)dS_\vartheta^{n-1}d|x|.$$

Now we make a change of variables $s=s_t(\vartheta)$ ($\vartheta\in\omega_t$). It is well-known that

$$dS^{n-1}=|K_t(s)|dS(t).$$

It is easy to see that $\vartheta\cdot s_t(\vartheta)=|T_t(s_t(\vartheta))|^{-1}$ and $x=|x|\vartheta=|x|N_t(s_t(\vartheta))$. Thus

$$\|I_{j,+}(t,\cdot)\|_{L^2}^2=\int_0^\infty\int_{\Sigma_t}\Phi_+^j(|x|\,|T_t(s)|^{-1}-t,\ s;\ |x|N_t(s),\ t)^*$$
$$\cdot\Phi_+^j(|x|\,|T_t(s)|^{-1}-t,\ s;\ |x|N_t(s),\ t)|K_t(s)|dS(t)d|x|.$$

Here we make another change of variable $|x|\to\tau$ by

$$|x|\,|T_t(s)|^{-1}-t=\tau.$$

Then $d|x|=|T_t(s)|d\tau$, and we have

$$\|I_{j,+}(t,\cdot)\|_{L^2}^2=\int_{-t}^\infty\int_{\Sigma_t}\Phi_+^j(\tau,\ s;\ (\tau+t)T_t(s),\ t)^* \tag{4.16}$$
$$\cdot\Phi_+^j(\tau,\ s;\ (\tau+t)T_t(s),\ t)|T_t(s)|\,|K_t(s)|dS(t)d\tau.$$

First we consider the case of $j=0$. Recall the definition (4.10) of $\Phi_+^0$. We begin with $F_k$. From (3.3) and (4.4)

$$F_k(t,x,s)=\chi(t,x)\,(\Lambda(x,s)-\lambda_k(\nabla W_k,s))P_k(\nabla W_k,s)$$
$$=\chi(t,x)\,(\Lambda(x,s)-\Lambda(\nabla W_k,s))P_k(\nabla W_k,s)$$
$$=\chi(t,x)\int_0^1\sum_{\mu=1}^n(x-\nabla W_k)_\mu(\partial_{x_\mu}\Lambda)\,((x-\nabla W_k)\vartheta$$
$$+\nabla W_k,\ s)d\vartheta\cdot P_k(\nabla W_k,s).$$

From (4.5) we have

$$F_k(t,(\tau+t)T_t(s),s)=\chi(t,(\tau+t)T_t(s))$$
$$\cdot\int_0^1\sum_{\mu=1}^n\tau(T_t(s))_\mu(\partial_{x_\mu}\Lambda)\,(\tau T_t(s)\vartheta+tT_t(s),s)d\vartheta\cdot P(tT_t(s),s).$$

This gives

$$|F_k(t,(\tau+t)T_t(s),s)| \tag{4.17}$$
$$\leqq C|\tau|\int_0^1\langle|\tau\vartheta+t|\,|T_t(s)|\rangle^{-1-\delta}d\vartheta\leqq C'|\tau|\langle t\rangle^{-1-\delta}(1+\tau^2/2).$$

Integration by parts gives for any non-negative integer $l$

$$\tag{4.18} \begin{aligned} &\tau^l(1+\tau^2/2)\int_{-\infty}^{\infty} e^{ir\tau}r|r|^{(n-1)/2}\hat{u}(rs)\psi_t^{\mathrm{sign}(\pm r)}(s)dr \\ &= \int_{-\infty}^{\infty} e^{ir\tau}(-D_r)^l(1+D_r^2/2)\,(r|r|^{(n-1)/2}\hat{u}(rs)\psi_t^{\mathrm{sign}(\pm r)}(s))dr\,. \end{aligned}$$

Clearly the right hand side of (4.18) is bounded. Then we easily have for any $s>0$

$$\tag{4.19} |\tau(1+\tau^2/2)\int_{-\infty}^{\infty} e^{ir\tau}r|r|^{(n-1)/2}\hat{u}(rs)\psi_t^{\mathrm{sign}(\pm r)}(s)dr| \leqq C_s|\tau|^{-s}\,.$$

If we estimate the right hand side of (4.10) by using (4.17) and (4.19) with $s=(1-\delta)/2$, we obtain

$$|\Phi_+^0(\tau,\, s;\, (\tau+t)T_t(s),\, t)| \leqq Ct^{-1-\delta}|\tau|^{-(1-\delta)/2}\,.$$

Thus from (4.16) we have

$$\tag{4.20} \|I_{0,+}(t,\,\cdot)\|_{L^2}^2 \leqq C't^{-2(1+\delta/2)}\,.$$

For the case of $j=1$ note the inequality (4.8) and the equality

$$\begin{aligned} \partial_{\xi_\mu}F_k(t,\,x,\,\xi) = \chi(t,\,x)[\{(\partial_{\xi_\mu}\Lambda)\,(x,\,\xi)-(\partial_{\xi_\mu}\Lambda)\,(\nabla W_k,\,\xi) \\ -\sum_{\nu=1}^{n}\partial_{\xi_\mu}\partial_{\xi_\nu}W_k\cdot(\partial_{x_\nu}\Lambda)\,(\nabla W_k,\,\xi)\}P_k(\nabla W_k,\,\xi) \\ +(\Lambda(x,\,\xi)-\Lambda(\nabla W_k,\,\xi))\partial_{\xi_\mu}(P_k(\nabla W_k,\,\xi))]\,, \end{aligned}$$

and in the same way as in the proof of the case of $j=0$ we have

$$\tag{4.21} \|I_{1,+}(t,\,\cdot)\|_{L^2}^2 \leqq C't^{-2(1+\delta/2)}\,.$$

Thus the proof is complete. Q.E.D.

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