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ASYMPTOTIC DISTRIBUTION OF EIGENVALUES FOR SCHRÖDINGER OPERATORS WITH HOMOGENEOUS MAGNETIC FIELDS

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1. Introduction

We consider the Schrödinger operator for one particle subjected to an external potential $-V(x)$, $x=(x_1, x_2, x_3)\in R^3_x$, and a homogeneous magnetic field of magnitude $b>0$ along the x_3 axis. Under a suitable normalization of units, this operator is described in the following form:

$$H = (D_1-bx_2/2)^2+(D_2+bx_1/2)^2+D_3^2-V(x)\,, \quad D_j = -i\partial/\partial x_j\,.$$

If $V(x)\to 0$ as $|x|\to\infty$, then H has essential spectrum beginning at b and discrete spectrum below the bottom b([1]). We denote by $N(\lambda)$, $\lambda>0$, the number of eigenvalues less than $b-\lambda$ of H with repetitoin according to multiplicities. The aim of this paper is to study the asymptotic behavior as $\lambda\to 0$ of $N(\lambda)$ when H has an infinite number of eigenvalues below the bottom b of essential spectrum.

We assume $V(x)$ to satisfy the following

ASSUMPTION $(V)_m$. (i) $V(x)$ is positive and C^1-smooth. (ii) There exists $m>0$ such that

$$C^{-1}\langle x\rangle^{-m}\leqq V(x)\leqq C\langle x\rangle^{-m}\,, \qquad C>1\,,$$

and $|\partial V/\partial x_j|\leqq C_j\langle x\rangle^{-m-1}$, where $\langle x\rangle=(1+|x|^2)^{1/2}$.

We now formulate the main theorem. Under the above assumption, H is essentially self-adjoint in $C_0^\infty(R^3_x)$. We denote by the same notation H its self-adjoint realization in $L^2(R^3_x)$.

Theorem 1. *Assume $(V)_m$ and denote by $N(\lambda)$, $\lambda>0$,, the number of eigenvalues less than $b-\lambda$ of H.*

(i) *If $0<m<2$, then*

$$N(\lambda) = N_0(\lambda;\, V)(1+o(1))\,, \qquad \lambda\to 0\,, \tag{1.1}$$

where

$$N_0(\lambda;\, V) = 2(2\pi)^{-2} b \int (V(x)-\lambda)_+^{1/2} dx \tag{1.2}$$

with $(V(x)-\lambda)_+=\max(0,\, V(x)-\lambda)$ *and the integration with no domain attached is taken over the whole space.*

(ii) *If* $m>2$, *then*

$$N(\lambda) = N_0(\lambda;\, V)(1+o(1))\,, \qquad \lambda \to 0\,, \tag{1.3}$$

where

$$N_0(\lambda;\, V) = (2\pi)^{-1} b\, vol[\{x' = (x_1,\, x_2);\; W(x')>2\lambda^{1/2}\}] \tag{1.4}$$

with $W(x')=\int V(x',\, x_3) dx_3$.

By a simple calculation, we see that $N(\lambda)$ behaves like $O(\lambda^{1/2-3/m})$, $0<m<2$, and like $O(\lambda^{-1/(m-1)})$, $m>2$, as $\lambda\to 0$. It should be noted that $N(\lambda)\to\infty$ as $\lambda\to 0$ even in the case $m>2$. This is one of main differences between the $b=0$ and $b\neq 0$ cases. As is well known, the number of negative eigenvalues is finite in the case $b=0$ for $m>2$.

The class of potentials we here consider is rather restrictive. In section 5 we will extend the above theorem to a wider class of potentials with singularities which includes the Coulomb potential $1/|x|$ as a typical example. This theorem can be also extended to the case of many particle systems. If the bottom of essential spectrum is determined only by two cluster decompositions, then the problem is reduced to the case of one particle systems. We will discuss the details elsewhere.

We conclude this section by stating the recent result related to the main theorem. In [9], the following class of potentials has bene studied:

(i) $\sup_{x'} \int (1+|x_3|)^\rho V(x',\, x_3) dx_3<\infty \qquad \text{for} \quad \rho>2;$

(ii) $W(x')=\Phi(x'/|x'|)|x'|^{-\alpha}+o(|x'|^{-\alpha})$, $\alpha>0$, as $|x'|\to\infty$.

Roughly speaking, $(V)_m$ with $m>3$ implies (i). The argument in [9] seems to rely on the homogeneous property (ii) and uses a special information on the eigenfunctions of the unperturbed operator

$$H_0 = (D_1-bx_2/2)^2+(D_2+bx_1/2)^2+D_3^2\,. \tag{1.5}$$

Thus, our theorem may be considered as a slight extension to a class of slowly decaying potentials without homogeneous property (ii), although the proof here is quite different from that in [9]. In particular, we do not require any information on the eigenfunctions of H_0.

2. Asymptotics for $N(\lambda)$ in the case $0<m<2$

In this section we deal with the case $0<m<2$ and prove the asymptotic formula (1.1). The proof is based on the min-max principle.

Let H_0 be defined by (1.5). Let $Q_R=(0, R)^3$ be the cube domain with side R. We consider H_0 over Q_R under zero Dirichlet boundary conditions and denote by $N_R^D(\lambda)$ the number of eigenvalues less than λ. The following lemma due to Colin de Verdiere [3] plays an essential role in proving (1.1).

Lemma 2.1. *Let the notations be as above. Then there exists $C_0>0$ independent of λ, R and A, $0<A<R/2$, such that*
(i) $N_R^D(\lambda)\geqq(R-A)^3\nu_b(\lambda-C_0A^{-2})$
(ii) $N_R^D(\lambda)\leqq R^3\nu_b(\lambda)$,
where $\nu_b(\lambda)=2(2\pi)^{-2}\sum_{j=0}^{\infty}b(\lambda-(2j+1)b)_+^{1/2}$.

Proof of (1.1). We first give the upper bound for $N(\lambda)$. Let $Q(z; R)$ be the open cube with center z and side R. For any $\delta>0$ small enough, we can construct a disjoint cube covering $\{Q(z_j; R_j)\}_{j=1}^{\infty}$ such that

$$C^{-1}\delta(1+|z_j|)\leqq R_j\leqq C\delta(1+|z_j|)$$

for $C>1$ independent of j and δ ([8]). Let $\{\chi_j\}_{j=1}^{\infty}$ be a partition of unity subject to the above covering with the following properties: (i) $\chi_j\in C_0^{\infty}(R_x^3)$ has support in $Q(z_j; (1+\delta)R_j)$; (ii) $0\leqq\chi_j\leqq 1$ and $\sum_{j=1}^{\infty}\chi_j^2=1$; (iii) $|\partial_x^{\alpha}\chi_j|\leqq C_{\alpha}(\delta R_j)^{-|\alpha|}$ for C_{α} independent of R_j and δ.If we denote by (,) the scalar product in $L^2(R_x^3)$, then we have by partial integration that

$$(Hu, u) = \sum_{j=1}^{\infty}\{(H\chi_j u, \chi_j u)-(|\nabla\chi_j|^2 u, u)\}$$

for $u\in C_0^{\infty}(R_x^3)$. As is easily seen, the number $\#\{k; \operatorname{supp}\chi_k\cap\operatorname{supp}\chi_j\neq\phi\}$ is bounded uniformly in j and δ and also

$$M^{-1}\leqq(1+|z_k|)/(1+|z_j|)\leqq M, \qquad M>1\,,$$

for such a pair (k,j). Thus it follows from property (iii) that

$$(Hu, u)\geqq\sum_{j=1}^{\infty}\{(H\chi_j u, \chi_j u)-C_1\delta^{-2}R_j^{-2}(\chi_j u, \chi_j u)\}. \tag{2.1}$$

We now put

$$V_{+j} = \sup\{V(x); x\in Q(z_j; (1+\delta)R_j\}.$$

Then, by (2.1) and Lemma 2.1, the min-max principle shows that

$$N(\lambda)\leqq(1+\delta)^3\sum_{j=1}^{\infty}R_j^3\nu_b(\mu_{+j}-\lambda) \tag{2.2}$$

with $\mu_{+j}=V_{+j}+C_1\delta^{-2}R_j^{-2}+b$. By assumption $(V)_m$, $0<m<2$, we have $R_j^{-2}=o(V_{+j})$ as $|z_j|\to\infty$ and $V(x)=(1+O(\delta))V_{+j}$ for $x\in Q(z_j; (1+\delta)R_j)$. Hence we see that the term on the right side of (2.2) behv behaves like

$$2(2\pi)^{-2}b\int (V(x)-\lambda)_+^{1/2}\,dx(1+O(\delta))+o(\lambda^{1/2-3/m}) \tag{2.3}$$

as $\lambda\to 0$, where the order relation $O(\delta)$ is uniform in λ. This gives the upper bound for $N(\lambda)$.

The lower bound is obtained in a similar way. We put

$$V_{-j}=\inf\{V(x);\ x\in Q(z_j;\ R_j)\}$$

and use again Lemma 2.1 with $A=\delta R_j$ to obtain

$$N(\lambda)\geqq(1-\delta)^3\sum_{j=1}^{\infty}R_j^3\nu_b(\mu_{-j}-\lambda)$$

with $\mu_{-j}=V_{-j}-C_0\delta^{-2}R_j^{-2}+b$, C_0 being as in Lemma 2.1. The term on the right side obeys the same asymptotic formula as in (2.3). This proves (1.1). □

3. Bounds for $N(\lambda)$ in the case $m>2$

In the present and next sections we deal with the case $m>2$. We first establish a bound for $N(\lambda)$, which is used to prove the asymptotic formula (1.3) in the next section.

Proposition 3.1. *Assume $V(x)$ to satisfy $(V)_m$, $m>2$. Then there exists C_V depending on $V(x)$ such that*

$$N(\lambda)\leqq C_V(\lambda^{-1/(m-1)}+1)$$

for any $\lambda>0$.

Proof. The proof is long and is divided into several steps. Throughout the proof, we fix $b=1$ and assume, without loss of generality, that $V(x)\in C^\infty(R_x^3)$ and

$$|\partial_x^\alpha V(x)|\leqq C_\alpha\langle x\rangle^{-m-|\alpha|} \tag{3.1}$$

for any multi-index α.

3.1. Let $p=(p_1, p_2)$ be the variables dual to $x'=(x_1, x_2)$. We define the linear transformation $R^4\to R^4$ by

$$y_1=p_2+x_1/2\,,\quad q_1=p_1-x_2/2\,,$$
$$y_2=p_1+x_2/2\,,\quad q_2=p_2-x_1/2\,,$$

so that $x_1=y_1-q_2$ and $x_2=y_2-q_1$. Let $S_0=(D_1-x_2/2)^2+(D_2+x_1/2)^2$. The above linear transformation is symplectic and hence we can construct a unitary operator by which S_0 becomes unitarily equivalent to $L_0=-(\partial/\partial y_1)^2+y_1^2$ and also the operator H under consideration becomes unitarily equivalent to

$$L = D_3^2 + L_0 - A_V\,, \tag{3.2}$$

where $A_V = a_V^W(y, D_y, x_3)$ is defined by the Weyl formula

$$A_V f = (2\pi)^{-2} \iint \exp(i(y-y')\cdot q) a_V((y+y')/2, q, x_3) f(y', x_3) dy' dq$$

with the symbol

$$a_V(y, q, x_3) = a_V(y_1, y_2, q_1, q_2, x_3) = V(y_1 - q_2, y_2 - q_1, x_3)\,.$$

3.2. Let $\phi(y_1)$ be the normalized eigenfunction associated with the first eigenvalue 1 of L_0. We define the projection P: $L^2(R^3)$—$L^2(R^3)$ by

$$(Pf)(y, x_3) = \int f(y_1, y_2, x_3)\phi(y_1) dy_1 \phi(y_1)$$

and Q by $Q = Id - P$, Id being the identity operator. Since A_V is non-negative, we have the operator inequality

$$PA_V Q + QA_V P \leqq PA_V P + QA_V Q \tag{3.3}$$

in the form sense and hence $L \geqq L_1 + R_1$, where $L_1 = P(D_3^2 - 2A_V)P + P$ and $R_1 = Q(D_3^2 + L_0 - 2A_V)Q$. We denote by $N(\lambda; L_1)$ and $N(\lambda; R_1)$ the number of eigenvalues less than $1-\lambda$ of L_1 and R_1, respectively.

Lemma 3.2. *$N(\lambda; R_1)$ is bounded uniformly in λ.*

Proof. By definition,

$$QL_0Q \geqq (1/2)QL_0Q + (3/2)Q$$

and hence $N(\lambda; R_1)$ is dominated by the number of eigenvalues less than $-1/2-\lambda$ of the operator $R = D_3^2 + (1/2)L_0 - 2A_V$. As is easily seen, this quantity is bounded uniformly in λ, because R has essential spectrum beginning at $1/2$. Thus the proof is complete. □

3.3. We prove $N(\lambda; L_1)$ to obey the bound as in the proposition. For notational convenience, we write (w, η), (z, ξ) and s for (y_1, q_1), (y_2, q_2) and x_3, respectively. The operator L_1 can be considered as an operator acting on $L^2(R^2_{z,s}) \simeq \text{Range}\, P$. If we write $(Pf)(w, z, s) = g(z, s)\phi(w)$, then we have

$$(PA_V Pf)(w, z, s) = (B_V g)(z, s)\phi(w)\,,$$

where

$$B_V g = (2\pi)^{-1} \iint \exp(i(z-z')\xi) b_V((z+z')/2, \xi, s) g(z', s) dz' d\xi \tag{3.4}$$

with the symbol $b_V(z, \xi, s)$ defined by

$$b_V = (2\pi)^{-1}\iiint \exp(i(w-w')\eta)a_V((w+w')/2, z, \eta, \xi, s)\phi(w)\phi(w')dw'd\eta dw$$

DEFINITION 3.3. We denote by S^d, $-\infty<d<\infty$, the class of all smooth symbols $a(z, \xi, s)$ such that

$$|\partial_z^k\partial_\xi^l a|\leqq C_{kl}(1+|s|+|z|+|\xi|)^{d-k-l}$$

for C_{kl} independent of s.

We define the operator $a^W(z, D_z, s)$ with symbol $a(z, \xi, s)$ by the Weyl formula (3.4) and denote by OPS^d the class of such operators with symbols of class S^d.

Lemma 3.4. *Let* $B_V=b_V^W(z, D_z, s)$ *be defined by* (3.4). *Then* B_V *is of class* OPS^{-m} *and*

$$b_V(z, s, \xi) = V(-\xi, z, s) \qquad (\bmod S^{-m-1}).$$

Proof. We give only a sketch. The proof uses the standard asymptotic expansion method for oscillatory integrals ([5]).

We write $b_V=\int c(w, z, \xi, s)\phi(w)sdw$, where

$$c = (2\pi)^{-1}\iint \exp(-i\eta u)a_V(w+u/2, z, \eta, \xi, s)\phi(w+u)du\,d\eta\,.$$

Since V satisfies (3.1) and since $\phi(w)$ falls off exponentially as $|w|\to\infty$, c is asymptotically expanded as

$$c = \sum_{j=0}^N d_j\partial_\eta^j\partial_w^j(a_V(w, z, \eta, \xi, s)\phi(w))|_{\eta=0}+c_N$$

with some constants $d_j(d_0=1)$, where the remainder c_N satisfies the estimate

$$|\partial_z^k\partial_\xi^l c_N|\leqq C_{Nkl}(1+|s|+|z|+|\xi-w|)^{-m-N-1-k-l}\,.$$

By normalization, $\int|\phi(w)|^2dw=1$ and hence the Taylor expansion formula gives

$$b_V(z, \xi, s) = a_V(0, z, 0, \xi, s)+a(z, \xi, s)$$

with $a\in S^{-m-1}$. This proves the lemma. □

3.4. The proof is complete in this step. Let $\Pi_0=-(\partial/\partial z)^2+z^2$. We can show ([7]) that the fractional power $(\Pi_0+s^2+1)^{d/2}$ is of class OPS^d and hence it follows from Lemma 3.4 that

$$B_V\leqq(\gamma_0/2)(\Pi_0+s^2+1)^{-m/2}$$

for some $\gamma_0>0$. Thus, $N(\lambda; L_1)$ is dominated by the number of eigenvalues less that $-\lambda$ of

$$L_2 = -(\partial/\partial s)^2-\gamma_0(\Pi_0+s^2+1)^{-m/2}.$$

Therefore, we have only to evaluate the number of eigenvalues less than $-\lambda$ of the operator

$$L_{2j} = -(d/ds)^2-\gamma_0(s^2+1+2j+1)^{-m/2}, \qquad j\geqq 0,$$

acting on $L^2(R^1_s)$.

Lemma 3.5 ([10]). *Assume that $v(s)\geqq 0$ satisfies*

$$\int(1+|s|)v(s)ds<\infty$$

and consider the operator $h=-(d/ds)^2-v(s)$. Then:

(i) *The number N of negative eigenvalues of h satisfies*

$$N\leqq 1+\int|s|v(s)ds.$$

(ii) *If $\int|s|v(s)ds<1$, the eigenvalue $E=-\alpha^2$, $\alpha>0$, of the ground state of h satisfies*

$$\alpha\geqq(1/2)\int v(s)ds\left(1-\int|s|v(s)ds\right).$$

The desired bound for $N(\lambda; L_1)$ follows immediately from Lemma 3.5. In fact, if $j\geqq\gamma_1(\lambda^{-1/(m-1)}+1)$ for some $\gamma_1>0$, then L_{2j} has no eigenvalues less than $-\lambda$ and also if $j>\gamma_1(\lambda^{-1/(m-1)}+1)$, then the number of eigenvalues less than $-\lambda$ of L_{2j} is bounded uniformly in λ. Thus the proof is now completed.

4. Asymptotics for $N(\lambda)$ in the case $m>2$

We shall prove the asymptotic formula (1.3) for the case $m>2$.

Proof of (1.3). The proof is done through several steps. Throughout the proof, we again fix $b=1$ and assume $V(x)$ to satisfy (3.1) together with $(V)_m$, $m>2$. It suffices to prove (1.3) for such a class of potentials. In fact, if $V(x)$ satisfies $(V)_m$ only, then $V(x)$ is approximated by a sequence of smooth potentials satisfying (3.1) by the mollifier technique.

4.1. Let $N_0(\lambda; V)$ be defined by (1.4) with $b=1$. We shall prove that

$$\limsup_{\lambda\to 0} N(\lambda)/N_0(\lambda; V)\leqq 1. \tag{4.1}$$

By a similar argument, we can also prove that

$$\liminf_{\lambda\to 0} N(\lambda)/N_0(\lambda;\ V)\geqq 1$$

and hence (1.3) is obtained.

We prove (4.1) only. We keep the same notations as in section 3. We begin by the operator inequality

$$PA_VQ+QA_VP\leqq \delta PA_VP+\delta^{-1}QA_VQ \tag{4.2}$$

for any $\delta>0$ small enough. By (4.2), $L\geqq L_\delta+R_\delta$ for L defined by (3.2), where

$$L_\delta = P(D_s^2-(1+\delta))A_V)P+P\,, \qquad D_s = -i\partial/\partial s\,,$$
$$R_\delta = Q(D_s^2+L_0-(1+\delta^{-1})A_V)Q\,.$$

Let $N(\lambda;\ L_\delta)$ and $N(\lambda;\ R_\delta)$ be the number of eigenvalues less than $1-\lambda$ of L_δ and R_δ, respectively. By the same argument as in the proof of Lemma 3.2, $N(\lambda;\ R_\delta)$ is bounded uniformly in λ. Hence, (4.1) follows, if

$$\lim_{\delta\downarrow 0}\limsup_{\lambda\to 0} N(\lambda;\ L_\delta)/N_0(\lambda;\ V)\leqq 1\,. \tag{4.3}$$

To prove this, we regard L_δ as an operator acting on $L^2(R^2_{z,s})$ and write

$$L_\delta = D_s^2-(1+\delta)B_V+1$$

with $B_V=b_V^W(z,\ D_z,\ s)$ defined by (3.4).

4.2. Let $\|\ \ \|$ denote the operator norm when considered as an operator from $L^2(R_z^1)$ into itself. By assumption $(V)_m$ with $m>2$, we can choose an integer K so large that $\|b_V^W(z, D_z, s)\|\leqq\lambda/2$ for $|s|>(1/2)K\lambda^{-1/2}$. We further take two smooth functions $\chi_j=\chi_j(s;\ K,\ \lambda)$, $1\leqq j\leqq 2$, with the following properties: (i) $0\leqq\chi_j\leqq 1$ and $\sum_{j=1}^2\chi_j^2=1$; (ii) χ_1 has support in $\{s;\ |s|<K\lambda^{-1/2}\}$ and $\chi_1=1$ for $|s|<(1/2)K\lambda^{-1/2}$; (iii)

$$|(\partial/\partial s)\chi_j|\leqq(\beta_0/2)^{1/2}K^{-1}\lambda^{1/2}\leqq\lambda^{1/2}/4$$

for $\beta_0>0$ independent of K and λ. If we denote by $\langle\ ,\ \rangle$ the scalar product in $L^2(R_s^1)$, then

$$\langle D_su,\ D_su\rangle = \textstyle\sum_{j=1}^2\{\langle D_s\chi_ju,\ D_s\chi_ju\rangle-\langle|D_s\chi_j|^2u,\ u\rangle\} \tag{4.4}$$

for $u\in C_0^\infty(R_s^1)$.

Let $B_K=\{(z,\ s);\ |s|<K\lambda^{-1/2}\}$. We denote by $L_{\delta K}^D$ the self-adjoint realization in $L^2(B_K)$ of L_δ under zero Dirichlet boundary conditions and by $N(\lambda;\ L_{\delta K}^D)$ the number of eigenvalues less than $1-\lambda$ of $L_{\delta K}^D$. By the above choice of K and by (4.4), it follows that

$$N(\lambda;\ L_\delta)\leqq N((1-\beta_0K^{-2})\lambda;\ L_{\delta K}^D)\,.$$

As is easily proved,

$$\lim_{\delta\downarrow 0}\limsup_{\lambda\to 0} N_0((1-\delta)\lambda;\ V)/N_0(\lambda;\ V)=1\,. \tag{4.5}$$

Hence, (4.3) follows, if

$$\lim_{\delta\downarrow 0}\lim_{K\uparrow\infty}\limsup_{\lambda\to 0} N(\lambda;\ L^D_{\delta K})/N_0(\lambda;\ V)\leqq 1\,. \tag{4.6}$$

4.3. We now consider $D_s^2=-(d/ds)^2$ over the interval $J_K=(-K\lambda^{-1/2}, K\lambda^{-1/2})$ under zero Dirichlet boundary conditions. This operator has the eigenvalues

$$\lambda_j=\lambda(\pi/2K)^2 j^2,\qquad j\geqq 1\,, \tag{4.7}$$

and the corresponding normalized eigenfunctions

$$\psi_j(s;\ K,\ \lambda)=(-1)^{[j/2]}K^{-1/2}\lambda^{1/4}\sin\,(j((\pi/2K)\lambda^{1/2}s+\pi/2))\,, \tag{4.8}$$

where [ ] denotes the Gauss notation. For later use, we here note that ψ_j behaves like

$$\psi_j=\begin{cases} K^{-1/2}\lambda^{1/4}(1+O(r^2))\,, & \text{for } j \text{ odd}\,,\\ K^{-1/2}\lambda^{1/4}O(r)\,, & \text{for } j \text{ even}\,,\end{cases} \tag{4.9}$$

as $r=(\pi/2K)\lambda^{1/2}s\to 0$.

We set $N=2K^2$ and define the projection P_N: $L^2(B_K)\to L^2(B_K)$ by

$$(P_N f)(z,\ s)=\textstyle\sum_{j=1}^N\int_{J_K}\psi_j(s;\ K,\ \lambda)f(z,\ s)ds\,\psi_j(s;\ K,\ \lambda)$$

and Q_N by $Q_N=Id-P_N$. By an operator inequality similar to (4.2), we obtain $L^D_{\delta K}\geqq A^D_{\delta K}+B^D_{\delta K}$, where

$$\begin{aligned} A^D_{\delta K}&=P_N(D_s^2-(1+2\delta)B_V)P_N+P_N\,,\\ B^D_{\delta K}&=Q_N(D_s^2-(1+\delta+\delta^{-1})B_V)Q_N+Q_N\,. \end{aligned}$$

Let $N(\lambda;\ A^D_{\delta K})$ and $N(\lambda;\ B^D_{\delta K})$ be the number of eigenvalues less than $1-\lambda$ of $A^D_{\delta K}$ and $B^D_{\delta K}$, respectively.

We assert that

$$\lim_{K\uparrow\infty}\limsup_{\lambda\to 0} N(\lambda;\ B^D_{\delta K})/N_0(\lambda;\ V)=0\,.$$

To prove this, we define the operator H_δ acting on $L^2(R^3_x)$ by

$$H_\delta=(D_1-x_2/2)^2+(D_2+x_1/2)^2+2^{-1}D_3^2-(1+\delta+\delta^{-1})V(x)$$

and denote by $N(\lambda;\ H_\delta)$ the number of eigenvalues lessthan $1-\lambda$ of H_δ. By the definition of Q_N,

$$Q_N D_s^2 Q_N \geqq 2^{-1} Q_N D_s^2 Q_N + (K^2/2)\lambda Q_N$$

and hence we obtain by Proposition 3.1 that

$$N(\lambda;\ B_{\delta K}^D) \leqq N((K^2/2+1)\lambda;\ H_\delta) \leqq C_\delta((K^2\lambda)^{-1/(m-1)}+1)\ .$$

This proves the above assertion. Thus, (4.6) follows, if

$$\lim_{\delta \downarrow 0} \lim_{K \uparrow \infty} \limsup_{\lambda \to 0} N(\lambda;\ A_{\delta K}^D)/N_0(\lambda;\ V) \leqq 1\ . \tag{4.10}$$

4.4. To prove (4.10), we regard $A_{\delta K}^D$ as an operator acting on the space $X_N = \Sigma \oplus L^2(R_z^1)$, N summands, and represent $A_{K\delta}^D$ in the matrix form.

Let $\{\lambda_j\}_{j=1}^\infty$ and $\{\psi_j\}_{j=1}^\infty$ be defined by (4.7) and (4.8), respectively. The matrix representation for $P_N D_s^2 P_N$ is given by the diagonal matrix Λ_K with the components λ_j, $1 \leqq j \leqq N$;

$$\Lambda_K = \mathrm{diag}\ \{\lambda_1, \cdots, \lambda_N\}\ , \qquad N = 2K^2\ ,$$

and also the symbol of $P_N B_V P_N$ is represented in the matrix form $\Gamma_K(z, \xi;\ \lambda) = \{\gamma_{jk}(z, \xi;\ \lambda)\}_{1 \leqq j, K \leqq N}$ with

$$\gamma_{jk} = \int_{J_K} \psi_j(s;\ K, \lambda) b_V(z, \xi, s) \psi_k(s;\ K, \lambda) ds\ . \tag{4.11}$$

We now write $\Lambda_K + \lambda Id = \lambda D_K^2$ for $D_K = \mathrm{diag}\ \{\mu_1, \cdots, \mu_N\}$ with $\mu_j = ((\pi/2K)^2 j^2 + 1)^{1/2}$, and define $G_K(\lambda)$: $X_N \to X_N$ by

$$G_K(\lambda) = D_K^{-1} \Gamma_K^W(z, D_z;\ \lambda) D_K^{-1}\ .$$

Let $n(\lambda;\ G_K(\lambda))$ be the number of eigenvalues greater than λ of $G(\lambda)$. Then, $N(\lambda;\ A_{\delta K}^D)$ coincides with $n(\lambda/(1+2\delta);\ G_K(\lambda))$. Hence, (4.10) follows from (4.5), if

$$\lim_{K \uparrow \infty} \limsup_{\lambda \to 0} n(\lambda;\ G_K(\lambda))/N_0(\lambda;\ V) \leqq 1\ . \tag{4.12}$$

4.5. We investigate the behavior as $\lambda \to 0$ of the symbol $\Gamma_K(z, \xi;\ \lambda)$. To do this, we first define the constant matrix Γ_0 of size $N \times N$, $N = 2K^2$, by $\Gamma_0 = (e_0, 0, e_0, 0, \cdots, 0, e_0, 0)$ with the column vector $e_0 = {}^t(1, 0, 1, 0, \cdots, 0, 1, 0)$. As is easily seen, Γ_0 is non-negative and

$$\mathrm{rank}\ \Gamma_0 = 1\ . \tag{4.13}$$

We further define $c_V(z, \xi)$ by $c_V = \int b_V(z, \xi, s) ds$. According to Lemma 3.4, c_V behaves like

$$c_V = W(-\xi, z) + O(\langle z, \xi \rangle)^{-m} \tag{4.14}$$

as $\langle z, \xi \rangle = (1 + z^2 + \xi^2)^{1/2} \to \infty$.

Lemma 4.1. *Let the notations be as above and denote by* $|\ |_0$ *the matrix norm. Then,* $\Gamma_K(z, \xi; \lambda)$ *is represented in the following form:*

$$\Gamma_K = K^{-1}\lambda^{1/2} c_V(z, \xi)\Gamma_0 + R_{1K}(z, \xi; \lambda) + R_{2K}(z, \xi; \lambda),$$

where R_{1K} *and* R_{2K} *satisfy the estimates*

$$|\partial_z^k \partial_\xi^l R_{1K}|_0 \leqq C_{Kkl}\lambda^{1/2+\kappa}\langle z, \xi\rangle^{-(m-1)-k-l},$$
$$|\partial_z^k \partial_\xi^l R_{2K}|_0 \leqq C_{Kkl}\lambda^{1+\kappa}\langle z, \xi\rangle^{-k-l}$$

with $\kappa=(m-2)/(2m)$, $0<\kappa<1/2$.

Proof. Let $J=(-\lambda^{-1/2+\kappa}, \lambda^{-1/2+\kappa})$. We divide the integral (4.11) into two parts; $\gamma_{jk}=\int_J ds+\int_{J_K\setminus J} ds$. Since b_V is of class S^{-m} by Lemma 3.4, the lemma follows from (4.9) at once. □

Let $\Pi_0=-(d/dz)^2+z^2$ again. Then it follows from Lemma 4.1 that for any $\varepsilon>0$ small enough, there exists $\rho_0=\rho_0(\varepsilon, K)$ such that

$$G_K(\lambda) \leqq \lambda^{1/2} F_{\varepsilon K} + \varepsilon\lambda Id$$

for λ, $0<\lambda<\rho_0$, where

$$F_{\varepsilon K} = K^{-1} c_V^W(z, D_z) D_K^{-1}\Gamma_0 D_K^{-1} + \varepsilon(\Pi_0+1)^{-(m-1)/2} Id.$$

If we denote by $n(\lambda; F_{\varepsilon K})$ the number of eigenvalues greater than λ of $F_{\varepsilon K}$ then $n(\lambda; G_K(\lambda)) \leqq n(1-\varepsilon)\lambda^{1/2}; F_{\varepsilon K})$ for λ as above and hence (4.12) follows from (4.5), if

$$\lim_{K\uparrow\infty}\lim_{\varepsilon\downarrow 0}\limsup_{\lambda\to 0} n(\lambda^{1/2}: F_{\varepsilon K})/N_0(\lambda; V) \leqq 1. \tag{4.15}$$

4.6. The proof of (4.1) is completed in this step. We apply the Weyl asymptotic formula for eigenvalues of pseudodifferential operators ([2], [4], etc.) to prove (4.15).

Taking account of relation (4.14), we now define the principal symbol $F_{\varepsilon K}^{(0)}(z, \xi)$ of $F_{\varepsilon K}$ by

$$F_{\varepsilon K}^{(0)} = K^{-1} W(-\xi, z) D_K^{-1}\Gamma_0 D_K^{-1} + \varepsilon\langle z, \xi\rangle^{-(m-1)} Id$$

and denote by $\{\omega_{j\varepsilon K}(z, \xi)\}_{j=1}^N$, $\omega_{1\varepsilon K} \geqq \cdots \geqq \omega_{M\varepsilon K}$, the eigenvalues of $F_{\varepsilon K}^{(0)}(z, \xi)$. Then we obtain by the Weyl asymptotic formula that

$$n(\lambda^{1/2}; F_{\varepsilon K}) = n_0(\lambda; \varepsilon, K)(1+o(1)) \tag{4.16}$$

as $\lambda \to 0$, where

$$n_0(\lambda; \varepsilon, K) = (2\pi)^{-1}\textstyle\sum_{j=1}^N \mathrm{vol}[\{(z, \xi); \omega_{j\varepsilon K}(z, \xi) > \lambda^{1/2}\}].$$

We calculate the eigenvalues $\omega_{j\varepsilon K}$. Since rank $D_K^{-1}\Gamma_0 D_K^{-1}=1$ by (4.13), $\omega_{j\varepsilon K}=\varepsilon\langle z, \xi\rangle^{-(m-1)}$, $2\leqq j\leqq N$, and also the non-zero eigenvalue ν_K of $D_K^{-1}\Gamma_0 D_K^{-1}$ is calculated as

$$\nu_K = \mathrm{tr}(D_K^{-1}\Gamma_0 D_K^{-1}) = \sum_{j=1}^{K^2} \mu_{2j-1}^2 ,$$

so that $K^{-1}\nu_K=1/2+o(1)$ as $K\to\infty$. Thus we have

$$\omega_{1\varepsilon K} = 2^{-1}W(-\xi, z)(1+o(1))+\varepsilon\langle z, \xi\rangle^{-(m-1)} , \qquad K\to\infty ,$$

and hence it follows that

$$\lim_{K\uparrow\infty}\lim_{\varepsilon\downarrow 0}\limsup_{\lambda\to 0} n_0(\lambda;\, \varepsilon, K)/N_0(\lambda;\, V)\leqq 1 .$$

This, together with (4.16), proves (4.15) and hence (4.1). Thus the proof is now complete.

5. Case of singular potentials

As stated in section 1, the asymptotic formula obtained in the regular case is extended to a certain class of potentials with singularities. We assume that:

(A.0) $V(x)$ admits a decomposition $V=V_0(x)+V_1(x)$;

(A.1) $V_0(x)$ satisfies $(V)_m$;

(A.2) $V_1(x)\in L^{3/2}(R_x^3)$ is a real function with compact support.

We do not necessarily assume $V(x)$ to be positive.

Let H_0 be defined by (1.5). Under assumptions (A.0)~(A.2), the multiplication operator V is relatively form-bounded with bound <1 with respect to H_0 and hence $H=H_0-V$ admits a unique self-adjoint realization in $L^2(R_x^3)$.

Theorem 5. *Assume* (A.0)~(A.2). *Then* $N(\lambda)$ *obeys the same asymptotic formula as in Theorem* 1.

For brevity, we fix $b=1$ as before and denote by $N(\lambda;\, V)$ the number of eigenvalues less than $1-\lambda$ of $H=H_0-V$.

Proposition 5.1. *Assume* $V(x)$ *to satisfy* (A.2). *Then*

$$N(\lambda;\, V)\leqq C_{\kappa V}(\lambda^{-\kappa}+1)$$

for any $\kappa>0$ *small enough.*

Proof. Let $\chi_j\in C^\infty(R_x^3)$, $1\leqq j\leqq 2$, be such that: (i) $0\leqq\chi_j\leqq 1$ and $\sum_{j=1}^2\chi_j^2=1$; (ii) χ_1 has compact support and $\chi_1=1$ on the support of $V(x)$. Let $h_0(u, v)$ and $h_V(u, v)$ be sesquilinear form associated with H_0 and $H=H_0-V$, respectively. We again denote by (,) the scalar product in $L^2(R_x^3)$. Then we have by partial integration that

$$h_V(u, u) = \sum_{j=1}^{2} g_V(\chi_j u, \chi_j u) \tag{5.1}$$

for $u \in C_0^\infty(R_x^3)$, where

$$g_V(u, u) = h_V(u, u) - (U_0 u, u)$$

with $U_0(x) = \sum_{j=1}^{2} |\nabla \chi_j|^2 \in C_0^\infty(R_x^3)$.

Since χ_1 has compact support, there exists $U \in L^{3/2}(R_x^3)$ with compact support such that

$$g_V(v, v) - (v, v) \geqq (1/2)(\nabla v, \nabla v) - (Uv, v)$$

for $v = \chi_1 u$. Let $m_1(\lambda)$ be the maximal dimension of subspace in $C_0^\infty(R_x^3)$ such that

$$(1/2)(\nabla v, \nabla v) - (Uv, v) < -\lambda(v, v), \qquad v \in C_0^\infty(R_x^3).$$

Then $m_1(\lambda)$ is dominated by the number of negative eigenvalues of the Schrödinger operator $-(1/2)\Delta - U$ and hence

$$m_1(\lambda) \leqq C \int |U(x)|^{3/2} dx \tag{5.2}$$

by the Rosenbljum-Cwikel-Lieb bound ([6]).

Since χ_2 vanishes on the support of $V(x)$, we can write

$$g_V(v, v) = h_0(v, v) - (U_0 v, v)$$

for $v = \chi_2 u$. Let $m_2(\lambda)$ be the maximal dimension of subspace in $C_0^\infty(R_x^3)$ such that

$$h_0(v, v) - (U_0 v, v) < (1-\lambda)(v, v), \qquad v \in C_0^\infty(R_x^3).$$

Then $m_2(\lambda) = N(\lambda; U_0)$ and hence, by Proposition 3.1,

$$m_2(\lambda) \leqq C_\kappa(\lambda^{-\kappa} + 1) \tag{5.3}$$

for any $\kappa > 0$ small enough. By (5.1), the min-max principle shows that $N(\lambda; V) \leqq m_1(\lambda) + m_2(\lambda)$. Thus the bounds (5.2) and (5.3) prove the proposition. □

Once the above proposition is established, Theorem 5 is verified in the same way as in the proof of Theorem B, [11].

Proof of Theorem 5. Let $N_0(\lambda; V)$ be defined by (1.2), $0 < m < 2$, and (1.4), $m < 2$, with $b = 1$. If $V = V_0 + V_1$ satisfies (A.0)∼(A.2), then

$$\lim_{\lambda \downarrow 0} N_0(\lambda; V)/N_0(\lambda; V_0) = 1. \tag{5.4}$$

We now define the operators $T_j(\lambda)$: $L^2(R_x^3) \to L^2(R_x^3)$, $0 \leqq j \leqq 1$, by

$$T_0(\lambda) = (H_0-1+\lambda)^{-1/2} V_0(H_0-1+\lambda)^{-1/2}$$
$$T_1(\lambda) = (H_0-1+\lambda)^{-1/2} | V_1 | (H_0-1+\lambda)^{-1/2}$$

and $T_\pm(\lambda)$ by $T_\pm(\lambda)=T_0(\lambda)\pm T_1(\lambda)$.

Let $n_0(\lambda; \varepsilon)$, $0<\varepsilon \ll 1$, be the number of eigenvalues greater than $1-\varepsilon$ of $T_0(\lambda)$. Then

$$n_0(\lambda; \varepsilon) = N(\lambda; (1-\varepsilon)^{-1} V_0)$$

and hence by Theorem 1, it follows from (4.5) and (5.4) that

$$\lim_{\varepsilon \downarrow 0} \limsup_{\lambda \to 0} n_0(\lambda; \varepsilon)/N_0(\lambda; V) \leqq 1 . \tag{5.5}$$

Let $n_1(\lambda; \varepsilon)$ be the number of eigenvalues greater than ε of $T_1(\lambda)$. Then

$$n_1(\lambda; \varepsilon) = N(\lambda; \varepsilon^{-1} | V_1 |)$$

and hence by Proposition 5.1,

$$\limsup_{\lambda \to 0} n_1(\lambda; \varepsilon)/N_0(\lambda; V) = 0 . \tag{5.6}$$

For given self-adjoint compact operator T, we denote by $\lambda_j(T)$ the j-th non-negative eigenvalue of T. If T is decomposed as $T=T_0+T_1$, then

$$\lambda_{n+m-1}(T) \leqq \lambda_n(T_0)+\lambda_m(T_1) .$$

This is well known as the Weyl inequality. We apply this to $T_+(\lambda)=T_0(\lambda)+T_1(\lambda)$. Since $N(\lambda; V_0+|V_1|)$ coincides with the number of eigenvalues greater than 1 of $T_+(\lambda)$, we have

$$N(\lambda; V) \leqq N(\lambda; V_0+|V_1|) \leqq n_0(\lambda; 2\varepsilon)+n_1(\lambda; \varepsilon)+1$$

and hence it follows from (5.5) and (5.6) that

$$\limsup_{\lambda \to 0} N(\lambda; V)/N_0(\lambda; V) \leqq 1 .$$

To estimate $N(\lambda; V)$ from below, we write $T_0(\lambda)$ as $T_0(\lambda)=T_-(\lambda)+T_1(\lambda)$ and use the Weyl inequality again to obtain

$$N(\lambda: (1+\varepsilon)^{-1} V_0) \leqq N(\lambda; V_0-|V_1|)+n_1(\lambda; \varepsilon)+1$$

for any $\varepsilon>0$ small enough. Hence, the same argument as above proves that

$$\liminf_{\lambda \to 0} N(\lambda; V)/N_0(\lambda; V) \geqq 1$$

and the proof is complete.

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