Maruyama, K.
Osaka J. Math.
25 (1988), 581-589

e-INVARIANTS ON THE STABLE COHOMOTOPY GROUPS OF LIE GROUPS

Dedicated to Professor Masahiro Sugawara on his 60th birthday

KEN-ICHI MARUYAMA

(Received March 12, 1987)

0. Introduction

The e-invariant was introduced by Adams [2] and Toda [13]. We know that this invariant is very useful and powerful in homotopy theory.

In this paper we will calculate e-invariants of certain elements of stable cohomotopy groups of Lie groups, more precisely, elements which arise from the Hopf construction of representations of Lie groups. We use the definition of the (complex) e-invariant in terms of the Chern character and resultly which is expressed as a tuple of rationals. These will give us informations on orders of above elements. Actually we will observe that our invariants behave well among Lie groups of low rank.

Our method depends on classical Theorems of Adams [1] and the result of a homotopy type of a Thom complex by Held-Sjerve [5]. To compute e-invariants of Lie groups of low rank, we will utilize the determination of the image of Chern character by T. Watanabe [15].

This paper is organized as follows. In Section 1, we will define an e-invariant on the stable cohomotopy group of a Lie group. In Section 2 we will introduce a theorem by which we can obtain our result on the Hopf-construction of a representation. In Section 3, we will show how to compute e-invariants concretely by computing them of few examples. In Section 4, we will consider simple applications of previous sections.

The author wishes to express his gratitude to Professors M. Kamata, T. Watanabe and the referee for their valuable suggestions and criticism.

1. A definition of an e-invariant

Let G be a compact connected Lie group with torsion-free fundamental group. Assume that we are given an element μ of the 0-th reduced stable cohomotopy group $\tilde{\pi}^0(G)$. We shall define an e-invariant of μ in terms of the Chern character. Hodgkin's Theorem [6] states that $K^*(G)$ has no torsion. This implies that μ induces a trivial homomorphism on the K-cohomology and the rational cohomology since $\tilde{\pi}^0(G)$ is finite. Thus we obtain the following commutative diagram in which rows are exact.

$$\begin{array}{ccccccccc}
0 \leftarrow & \tilde{K}(S^0) & \overset{i^*}{\leftarrow} & \tilde{K}(C_\mu) & \overset{j^*}{\longleftarrow} & \tilde{K}^1(G) & \leftarrow 0 \\
& ch \downarrow & & ch \downarrow & & ch \downarrow & \\
0 \leftarrow & \tilde{H}^*(S^0, Q) & \overset{i^*}{\leftarrow} & \tilde{H}^*(C_\mu, Q) & \overset{j^*}{\leftarrow} & H^*(G, Q) & \leftarrow 0
\end{array}$$

Here i and j are natural injection and projection, C_μ is a mapping cone. For the next section we note that homomorphisms i^* and j^* are considered to be defined through Thom isomorphisms for trivial complex vector bundles and $\tilde{H}^*(S^0, Q)$ will be identified as Q, the rational numbers.

Let $\eta_1, \cdots, \eta_m$ and $h_1, \cdots, h_m$ be basis of $\tilde{K}^1(G)$ and $\tilde{H}^{odd}(G, Q)$ respectively and h_i be integral, where $m=2^{\mathrm{rank}\, G-1}$ and $\deg(h_i)=2n_i-1$. Let ξ be a generator of $\tilde{K}(S^0)$ and $\bar{\xi}$ an element of $\tilde{K}(C_\mu)$ such that $i^*\bar{\xi}=\xi$, moreover $\bar{h}$ be an integral generator of $\tilde{H}(C_\mu, Q)$ auch that $i^*\bar{h}=1$. Then the image of $\bar{\xi}$ by the Chern character is

$$ch\, \bar{\xi} = \bar{h}+j^*(\sum_{j=1}^{m} \delta_j h_j)\,,$$

where δ_j's are some rational numbers. If we take another element $\bar{\xi}'$ instead of $\bar{\xi}$, $\bar{\xi}'=\bar{\xi}+j^*(\sum_{l=1}^{m}\gamma_l\eta_l)$ where γ_l's are integers. Since $ch(\eta_i)=\sum_{k=1}^{m}\lambda_{i,k}h_k$ for some $\lambda_{1,k}\in Q$, we obtain a nonsingular matrix $(\lambda_{i,j})$. We call this matrix the fundamental matrix of G and we denote it by $M(G)$. Here we remark that the determinant of the fundamental matrix with some canonical basis is equal to 1 by Atiyah [3]. Anyway we obtain the following equalities.

$$\begin{aligned}
(1.1) \qquad ch\,\bar{\xi}' &= ch\,\bar{\xi}+j^*(\sum_l \gamma_l\, ch\,(\eta_l)) \\
&= \bar{h}+j^*(\sum_j \delta_j h_j)+j^*(\sum_l \gamma_l(\sum_k \lambda_{l,k}h_k)) \\
&= \bar{h}+j^*(\sum_j(\delta_j+\sum_k \gamma_k\lambda_{k,j})h_j)\,.
\end{aligned}$$

In other words, the following formula holds.

$$(1.2) \qquad (\delta_1', \cdots, \delta_m')-(\delta_1, \cdots, \delta_m) = (\gamma_1, \cdots, \gamma_m)M(G)\,.$$

Definition 1.3. We define $e(\mu)$, an e-invariant of μ, by $e(\mu)=(\delta_1, \cdots, \delta_m)M(G)^{-1}$ as an element of the group $Q/Z\oplus\cdots\oplus Q/Z$ (a direct sum of m-copies of Q/Z). Thus we define a homomorphism $e\colon \tilde{\pi}^0(G)\to Q/Z\oplus\cdots\oplus Q/Z$.

REMARK. For the definition in a more broader context, refer [2, 10].

2. Elements obtained by the Hopf construction

Let ρ be a unitary representation of a compact connected Lie group G

with the torsion-free fundamental group and V its representation space. We also denote the action on the unit sphere $S(V)$ by ρ.

$$\rho\colon\ G\times S(V)\to S(V)\,.$$

Applying the Hopf construction to ρ, we have the following element.

$$H(\rho)\colon\ G*S(V)\to \Sigma S(V)\,.$$

$G*S(V)$ is homotopy equivalent to $\Sigma(G\wedge S(V))$ (cf. Toda [12], p. 113). When the dimension of V is sufficiently large, $H(\rho)$ can be considered as an element of $\tilde{\pi}^0(G)$.

We can apply the results of [1] for expressing e-invariant of $H(\rho)$ more concretely. By [5], $C_{H(\rho)}$ is stably homotopy equivalent to the Thom complex of the vector bundle ω over ΣG constructed by ρ as the clutching function and the inclusion $\Sigma S(V)\to C_{H(\rho)}$ can be regarded as the inclusion of the fibre into Thom complex. Then by Theorem 5.1 and Proposition 5.2 [1],

$$\phi_H^{-1}\, ch\, \phi_K(1) = 1+\Sigma\, \alpha_t\, ch_t\, \omega\,,$$

where ch_t is the component of ch in dimension $2t$, α_t is the number defined in [1] by Adams, ϕ_H and ϕ_K are Thom isomorphisms of the bundle ω in K or ordinary cohomology. Thus we can take $\xi=\phi_K(1)$ and resultly we can express $e(H(\rho))$ as follows.

Theorem 2.1. *$e(H(\rho))$, we shortly denote $e(\rho)$, is equal to $(\alpha_{n_1}a_1, \cdots, \alpha_{n_m}a_m)M(G)^{-1}$, where $ch\,\omega=\sum_{i=1}^m a_ib_i$. Here α_k is the number as above and $n_k=(\deg(h_k)+1)/2$.*

From this theorem we can show that $e(\ )$ is natural with respect to the addition of rperesentations.

Proposition 2.2. *Let ρ_1 and ρ_2 be representations of G, then $e(\rho_1+\rho_2)=e(\rho_1)+e(\rho_2)$.*

Proof. Bundles ω_1, ω_2 associated with ρ_1 and ρ_2 (see before Theorem 2.1) are equal to beta constructions $\beta(\rho_1)$, $\beta(\rho_2)$ respectively. By [6], beta construction is aditive and thus we obtain, $ch\,\beta(\rho_1+\rho_2)=ch\,\beta(\rho_1)+ch\,\beta(\rho_2)$. Now the result is clear from Theorem 2.1.

Corollary 2.3. $e(H(\rho_1)\cdot H(\rho_2))=0$.

Proof. From Becker-Schultz [4], $H(\rho_1+\rho_2)=H(\rho_1)+H(\rho_2)-H(\rho_1)\cdot H(\rho_2)$. Our corollary follows easily from Proposition 2.2.

3. Calculations on compact Lie groups of low rank

We shall give e-invariants of representations of Lie groups of low rank [15]. Let G be a compact connected Lie group of rank $G=n$ with the torsion-free fundamental group. As already stated in §1, 2 $H^*(G, Q)=\Lambda_Q(h_1, \cdots, h_n)$, $K^*(G)=\Lambda(\beta(\lambda_1), \cdots, \beta(\lambda_n))$, where $\lambda_j\colon G\to U(m_j)$ is some representation and $\beta(\lambda_j)$ is the β-construction of λ_j (see Hodgkin [6]). The fundamental matrix has the following form.

$$M(G) = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$

Thus we obtain the following by Theorem 2.1.

Proposition 3.1. *Under the above assumptions,* $e(\lambda_j)=((\alpha_{n_1}a_{j,1}, \cdots, \alpha_{n_n}a_{j,n})\times A^{-1}, 0, \cdots, 0))$, *where* $ch\,\beta(\lambda_j)=\sum_{s=1}^{n} a_{j,s}h_s$, α_t *and* n_k *are same as in* §2.

$$R(SU(3)) = Z[\lambda_1, \lambda_2] \quad \text{and} \quad K^*(SU(3)) = E(\beta(\lambda_1), \beta(\lambda_2)) \qquad [7].$$

The Chern character on $\beta(\lambda_1)$ ($i=1, 2$) are obtained in [15] as follows

$$\begin{aligned} ch\,\beta(\lambda_1) &= -x_3+(1/2)x_5\,, \\ ch\,\beta(\lambda_2) &= -x_3+(-1/2)x_5\,. \end{aligned}$$

Thus the fundamental matrix mentioned in Section 1 is the following.

$$\begin{pmatrix} -1 & 1/2 \\ -1 & -1/2 \end{pmatrix}$$

Since $\alpha_2=1/12$, $\alpha_3=0$, we obtain by Theorem 2.1 the following.

$$e(\lambda_1) = e(\lambda_2) = (1/24,\ 1/24)\in Q/Z\oplus Q/Z\,.$$

This implies that $e(\lambda_i)$ ($i=1, 2$) has the order 24.

We can obtain e-invariants similarly on other cases. We shall give the list of some examples. For the structures of representation rings of images of ch refer [7], [15] and [10].

$$R(SU(4)) = Z[\lambda_1, \lambda_2, \lambda_3]. \quad M(SU(4)) = \begin{pmatrix} -1 & 1/2! & -1/3! & \vdots \\ -2 & 0 & 4/3! & \vdots \\ -1 & -1/2! & -1/3! & \vdots \\ \cdots & \cdots & \cdots & \vdots \end{pmatrix}$$

By Proposition 3.1,

	$e(\ )$	Order
λ_1	(19/720, 11/720, 19/720, 0)	$2^4 3^2 5$
λ_2	(11/180, 21/45, 11/180, 0)	$2^2 3^2 5$
λ_3	(19/720, 11/720, 19/720, 0)	$2^4 3^2 5$

.

$$R(SU(5)) = Z[\lambda_1, \lambda_2, \lambda_3, \lambda_4]. \quad M(SU(5)) = \begin{pmatrix} -1 & 1/2! & -1/3! & 1/4! & \vdots \\ -3 & 1/2! & 3/3! & -11/4! & \vdots \\ -3 & -1/2! & 3/3! & 11/4! & \vdots \\ -1 & -1/2! & -1/3! & -1/4! & \vdots \\ \cdots & \cdots & \cdots & \cdots & \vdots \end{pmatrix}$$

	$e(\ )$	Order
λ_1	(3/160, 11/1440, 11/1440, 3/160, 0, ⋯, 0)	$2^5 3^2 5$
λ_2	(11/160, 3/160, 3/160, 11/160, 0, ⋯, 0)	$2^5 5$
λ_3	(11/160, 3/160, 3/160, 11/160, 0, ⋯, 0)	$2^5 5$
λ_4	(3/160, 11/1440, 11/1440, 3/160, 0, ⋯, 0)	$2^2 3^2 5$

$$R(Sp(2)) = Z[\lambda_1, \lambda_2,]. \quad M(Sp(2)) = \begin{pmatrix} 1 & -1/3! \\ 2 & 4/3! \end{pmatrix}$$

	$e(\ )$	Order
λ_1	((19/360, 11/720)	$2^4 3^2 5$
λ_2	(11/90, 11/45)	$2 \cdot 3^2 \cdot 5$

$$R(Sp(3)) = Z[\lambda_1, \lambda_2, \lambda_3]. \quad M(Sp(3)) = \begin{pmatrix} 1 & -1/3! & 1/5! & \vdots \\ 4 & 2/3! & -26/5! & \vdots \\ 6 & 6/3! & 66/5! & \vdots \\ \cdots & \cdots & \cdots & \vdots \end{pmatrix}$$

	$e(\ )$	Order
λ_1	(863/30240, 271/30240, 191/60480, 0)	$2^6 \cdot 3^3 \cdot 5 \cdot 7$
λ_2	(2137/15120, 509/15120, 289/30240, 0)	$2^5 \cdot 3^3 \cdot 5 \cdot 7$
λ_3	(1177/5040, 209/5040, 169/10080, 0)	$2^5 \cdot 3^2 \cdot 5 \cdot 7$

$$R(Sp(4))=Z[\lambda_1,\lambda_2,\lambda_3,\lambda_4].\quad M(Sp(4))=\begin{pmatrix} 1 & -1/3! & 1/5! & -1/7! & \vdots \\ 6 & 0 & -24/5! & 120/7! & \vdots \\ 15 & 9/3! & 15/5! & -1191/7! & \vdots \\ 20 & 16/3! & 80/5! & 2416/7! & \vdots \\ \cdots & \cdots & \cdots & \cdots & \vdots \end{pmatrix}$$

	$e(\ )$	*Order*
λ_1	(33953/1814400, 7297/1814400, 3233/1814400, 2497/3628800, 0, ⋯, 0)	$2^8 3^4 5^2 7$
λ_2	(2063/15120, 397/15120, 143/15120, 97/30240, 0, ⋯, 0)	$2^5\cdot 3^3\cdot 5\cdot 7$
λ_3	(246541/604800, 35309/604800, 12301/604800, 11309/1209600, 0, ⋯, 0)	$2^8 3^3 5^2 7$
λ_4	(66697/113400, 7253/113400, 3817/113400, 2153/226800, 0, ⋯, 0)	$2^4 3^4 5^2 7$

$$R(G_2)=Z[\rho_1,\rho_2].\quad M(G_2)=\begin{pmatrix} 2 & 1/60 \\ 10 & -5/12 \end{pmatrix}$$

	$e(\ )$	*Order*
ρ_1	(53/756, 1/378)	$2^2 3^3 7$
ρ_2	(125/378, 13/756	$2^2 3^3 7$

For the rest of examples we only mention the orders of $e(\ )$.

$R(Spin\ (7))=Z[\lambda_1', \lambda_2', \Delta_7]$.

	Order
$e(\lambda_1')$	$2^2\cdot 3^3\cdot 5\cdot 7$
$e(\lambda_2')$	$2^2\cdot 3^3\cdot 5\cdot 7$
$e(\Delta_7)$	$2^3\cdot 3^3\cdot 5\cdot 7$

$R(Spin\ (8))=Z[\lambda_1, \lambda_2, \Delta_8^+, \Delta_8^-]$.

	Order
$e(\lambda_1)$	$2^3\cdot 3^3\cdot 5\cdot 7$
$e(\lambda_2)$	$2^2 3^2 7$
$e(\Delta_8^+)$	$2^3\cdot 3^3\cdot 5\cdot 7$
$e(\Delta_8^-)$	$2^3\cdot 3^3\cdot 5\cdot 7$

$R(Spin\,(9)) = Z[\lambda_1', \lambda_2', \lambda_3', \Delta_9]$.

	Order
$e(\lambda_1')$	$2^3 3^4 5^2 7$
$e(\lambda_2')$	$2^3 3^4 5^2 7$
$e(\lambda_3')$	$2^3 3^3 5^2 7$
$e(\Delta_9)$	$2^7 3^4 5^2 7$

$R(F_4) = Z[\rho_4, \Lambda^2\rho_4, \Lambda^3\rho_4, \rho_1]$.

	Order
$e(\rho_4)$	$2^{12}3^2 5^3 7^2$
$e(\Lambda^2\rho_4)$	$2^{10}3^2 5^2 7^2 11$
$e(\Lambda^3\rho_3)$	$2^7 3^2 5^2 7^2 13$
$e(\rho_1)$	$2^{11}3^3 5^2 7\cdot 11$

REMARK. $\tilde{\pi}^0(SU(3))$ and $\tilde{\pi}^0(Sp(2))$ can be easily calculated by cell structures; $\tilde{\pi}^0(SU(3)) = Z_8\langle\tilde{\nu}\rangle \oplus Z_3\langle\tilde{\alpha}_1\rangle \oplus Z_2\langle q^*\tilde{\nu}\rangle \oplus Z_2\langle q^*\varepsilon\rangle$, and $\tilde{\pi}^0(Sp(2)) = Z_8\langle\tilde{\nu}\rangle \oplus Z_{16}\langle\tilde{\sigma}\rangle \oplus Z_2\langle q^*\eta\mu\rangle \oplus Z_9\langle\tilde{\alpha}_1\rangle \oplus Z_3\langle q^*\beta_1\rangle \oplus Z_5\langle\tilde{\alpha}_{1,5}\rangle$, for these facts see [14]. By (3.1) and Proposition 2.2, $H(\lambda_1)$ has the order more than 24, thus $H(\lambda_1) = \tilde{\nu} + \tilde{\alpha}_1 + t$, t an element of another summands. Namely $H(\lambda_1)$ is an element of the highest order at 2 and 3 primary components. For $Sp(2)$, also we can see that $H(\lambda_1)$ is an element of the highest order at 2, 3 and 5 primary components. By contrast, on $\tilde{\pi}^0(G_2)$ we can not see whether $H(\rho_1)$ is the highest order using only e-invariants at 2 component because this group contains an element of order 8 [8]. Nevertheless, we claim that $H(\rho_1)$ attains an element of highest order at odd (3, 7) components.

4. An application

We shall show some simple consequences of our computations of previous sections.

Theorem 4.1. $\tilde{\pi}^0(SU(4))_{(odd)} = Z_3 \oplus Z_3 \oplus Z_9 \oplus Z_5 \oplus Z_5$,
$\tilde{\pi}^0(Sp(3))_{(odd)} = Z_{27} \oplus Z_3 \oplus Z_3 \oplus Z_5 \oplus Z_7$.

Proof. Let h^* be a (unreduced) cohomology theory. From the Atiyah-

Hirzebruch spectral sequence associated to the fibration, $F \to E \to S^n$, there exists the following generalized Wang sequence. See Switzer [11], 15.35.

$$\to h^m(E) \to h^m(F) \to h^{m-n+1}(F) \to h^{m+1}(E) \to .$$

Applying this sequence to the cohomotopy theory π^* nad the fibration $SU(3) \to SU(4) \to S^7$,

(4.2) $$\to \pi^{-1}(SU(3)) \to \pi^{-7}(SU(3)) \to \pi^0(SU(4)) \to \pi^0(SU(3)) \to .$$

Since $SU(3)_{(odd)} = S^3_{(odd)} \times S^5_{(odd)}$ (cf [14]) we can see that $\pi^{-1}(SU(3))_{(odd)} = 0$, $\pi^{-7}(SU(3))_{(odd)} = Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_5 \oplus Z_5$ and $\pi^0(SU(3))_{(odd)} = Z_3 \oplus Z_{(odd)}$. Therefore localize (4.2) at odd primes,

$$0 \to Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_5 \oplus Z_5 \to \tilde{\pi}^0(SU(4))_{(odd)} \to Z_3 \to .$$

We have seen in the previous section that there exists an element $H(\rho_1)$ of order more than or equal to 9. Thus we obtain a non-trivial extension of 3 components and this implies the former.

On the other hand $Sp(3)^{11}$, the 11-skeleton of $Sp(3)$, has the form $Sp(2) \cup e^{11}$ and the attaching map $\theta\colon S^{10} \to Sp(2)$ is the generator of $\pi_{10}(Sp(2)) = Z_{120}$ by [9, §2]. Consider $Sp(2)$ as the subcomplex of $Sp(3)$, there is the following cofibration.

(4.3) $$Sp(2) \to Sp(3) \to S^{11} \cup e^{14} \cup e^{18} \cup e^{21} .$$

It is well known that the top attaching maps of $Sp(2)$ and $Sp(3)$ are stably trivial. Thus (4.3) is stably equivalent to the cofibration as follows.

(4.4) $$S^3 \cup e^7 \vee S^{10} \to Sp(3) \to (S^{11} \cup e^{14} \cup e^{18}) \vee S^{21} .$$

We claim that $\tilde{\pi}^0(Sp(3))_{(odd)} = \tilde{\pi}^0(Sp(3)^{14})_{(odd)}$ because $\pi^s_{21}(S^0)$, $\pi^s_{18}(S^0)$ and $\pi^s_{17}(S^0)$ have no odd-torision elements [12]. By (4.4) and the fact $\pi_{13}(Sp(2))_{(odd)} = 0$ [9], $\tilde{\pi}^0(Sp(3)^{14})_{(odd)} = \tilde{\pi}^0(Sp(3)^{11})_{(odd)}$. Now we concern ourself with the 3-primary part. At the prime 3 $Sp(3)^{11}_{(3)} = S^{10} \vee S^3 \cup_{\alpha_1} e^7 \cup e^{11}$ (stably) since the above mentioned attaching map θ factors through the 3-skeleton. Precisely it can be written as $\theta = \alpha_2 i$, where $i\colon S^3 \to S^3 \cup_{\alpha_1} e^7$ is the inclusion map. From this fact and the result on $\tilde{\pi}^0(Sp(2))$ [14] we can easily obtain the following.

(4.5) $$0 \leftarrow Z_9 \leftarrow \tilde{\pi}^0(Y) \leftarrow Z_9 \leftarrow 0 .$$

Here $Y = (S^3 \cup_{\alpha_1} e^7 \cup_{\theta} e^{11})_{(3)}$. Considering the cofibration $S^3_{(3)} \to Y \to (S^7 \vee S^{11})_{(3)}$, we obtain the other short exact sequence as follows.

(4.6) $$0 \leftarrow Z_3 \leftarrow \tilde{\pi}^0(Y) \leftarrow Z_3 \oplus Z_9 \leftarrow 0 .$$

By (4.5), (4.6) and the result that $H(\lambda_1)$ is of order more than or equal to 27, we can conclude that $\tilde{\pi}^0(Y)=Z_3\oplus Z_{27}$. Thus we can show that $\tilde{\pi}(Sp(3))_{(3)}=\tilde{\pi}^0(S^{10})_{(3)}\oplus\tilde{\pi}^0(Y)=Z_3\oplus Z_3\oplus Z_{27}$. For other odd primes there are no non-trivial extensions. This proves the latter.

References

[1] J.F. Adams: *On the group $J(X)$-II*, Topology **3** (1965), 137–171.
[2] J.F. Adams: *On the group $J(X)$-IV*, Topology **5** (1966), 21–71.
[3] M.F. Atiyah: *On the K-theory of compact Lie groups,* Topology **4** (1965), 95–99.
[4] J.C. Becker and R.E. Schultz: *Fixed point indices and left invariant framings,* Springer Lecture Notes in Math. 657, 1–31.
[5] R.P. Held and D.K. Sjerve: *On the homotopy properties of Thom complexes,* Math. Z. **135** (1974), 315–323.
[6] L. Hodgkin: *On the K-theory of Lie groups,* Topology **6** (1967), 1–36.
[7] D. Husemoller: *Fibre bundles*, McGraw-Hill, 1966.
[8] K. Maruyama: *The stable cohomotopy ring of G_2*, Pub. R.I.M.S., Kyoto Univ. **23** (1987), 731–736.
[9] M. Mimura and H. Toda: *Homotopy groups of symplectic groups,* J. Math. Kyoto Univ. **3** (1964), 251–273.
[10] S. Oka: *Homotopy of the exceptional Lie group G_2*, Proc. Edinburgh Math. Soc. **29** (1986), 145–169.
[11] R.M. Switzer: *Algebraic topology-homotopy and homology*, Springer, 1975.
[12] H. Toda: *Composition methods in homotopy groups of spheres*, Ann. Math. Studies 49, Princeton, 1962.
[13] H. Toda: *A survey of homotopy theory*, Sugaku **15** (1963/4), 141–155.
[14] G. Walker: *The stable cohomotopy rings of $SU(3)$ and $Sp(2)$*, Bull. London Math. Soc. **9** (1977), 93–96.
[15] T. Watanabe: *Chern characters on compact Lie groups of low rank,* Osaka J. Math. **22** (1985), 463–488.

Department of Mathematics
Kyushu University 33
Hakozaki, Fukuoka 812
Japan

Current Address:
Department of Mathematics
Faculty of Education
Chiba University
Yayoicho Chiba
Japan