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S^4 DOES NOT HAVE ONE FIXED POINT ACTIONS

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1. Introduction

In this paper we mean smooth actions on manifolds of compact Lie groups simply by actions.

Several authors found one fixed point actions on spheres [9] (or [10]), [12], [13] and [14]. Those spheres have dimensions greater than 5. It is easy to see that the spheres S^n of dimension $n \leqq 2$ do not have one fixed point actions of compact Lie groups. Further it is conjectured among topologists dealing with 3-dimensional manifolds that S^3 has no one fixed point actions of compact Lie groups. The purpose of this paper is to show:

Theorem A. *The 4-dimensional homotopy spheres have no one fixed point actions of compact Lie groups.*

Special cases of this theorem were proved by M. Furuta and W.-Y. Hsiang-E. Straume. Let Σ be an oriented 4-dimensional homotopy sphere.

Theorem (M. Furuta [4]). *Any finite group G can not act on Σ in such a way that* (1) *Σ^G consists of exactly one point and* (2) *each element of G preserves the orientation of Σ.*

Corollary to Theorem 1 of W.-Y. Hsiang-E. Straume [6]. *Any compact connected Lie group can not act on Σ with exactly one fixed point.*

Our proof of Theorem A goes on by showing the following lemmas. For a compact manifold X and for an integer $k \geqq 0$, we denote by X_k the totality of k-dimensional connected components of X. For a set Y, we denote by $|Y|$ the cardinality of Y. Let Ξ be an oriented 4-dimensional homology sphere.

Lemma B. *If a compact Lie group G of dimension $\geqq 1$ acts effectively on Ξ, then Ξ^G is empty or diffeomorphic to S^n with $n \leqq 2$. Especially one has $|\Xi^G_0| = 0$ or 2.*

Lemma C. *If a finite group G acts on Ξ, then one has $|\Xi^G_0| \leqq 2$.*

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For a G-action on Ξ we define $K=K(G, \Xi)$ to be the subgroup of G of elements preserving the orientation of Ξ. If a finite group G acts on Σ^4 with $|\Sigma^G|=1$, then by Furuta's theorem we have $G\neq K$, moreover we will see $|\Sigma_0^K|\geqq 3$ in Section 5. This contradicts Lemma C.

We wish to express our gratitude to M. Furuta for informing us of his result.

2. Preliminary

Let G be a compact Lie group, H a subgroup of G and X a compact G-manifold of dimension n. If X_k^H is non-empty, then take an H-equivariant normal bundle of X_k^H in X. The fibers of it are $(n-k)$-dimensional real H-representations. We call them the *normal representations of* X_k^H *in* X. We remark that if the G-action on X is effective, then the normal representations are faithful.

We frequently use the following well known result.

Theorem (P.A. Smith [1, Theorem 5.1]). *If G is a p-group (p prime) and if it acts on a* mod p *homology sphere X, then X^G is empty or a* mod p *homology sphere.*

The following lemma is well known and easily proved.

Lemma 2.1. *If a compact Lie group G acts on S^n with $n\leqq 2$, then S^G is empty or diffeomorphic to S^m with $m\leqq 2$.*

3. Proof of Lemma B

Let G be a compact Lie group of dimension$\geqq 1$ and Ξ an oriented 4-dimensional homology sphere with G-action. Suppose that the G-action is effective. Let G_0 be the identity component of G.

Proposition 3.1. *If G_0 has an abelian normal subgroup $A\neq\{1\}$, then Ξ^G is empty or diffeomorphic to S^m with $m\leqq 2$.*

Proof. Each element of G_0 preserves the orientation on Ξ. Since the G-action is effective, we have $\dim\Xi^B\leqq 2$ for any subgroup $B\neq 1$ of G_0. Let C be a cyclic subgroup of A of prime order. By Smith's theorem Ξ^C is a sphere of dimension $\leqq 2$. By Lemma 2.1 $\Xi^G=(((\Xi^C)^A)^H)^G$, $H=G_0$, is empty or also a sphere.

Proposition 3.2. *It holds that*

(1) *if $G_0=SO(3)$, then Ξ^G is empty or diffeomorphic to S^m with $m\leqq 1$,*

(2) *if $G_0=SU(2)$, then $|\Xi^G|=0$ or* 2, and

(3) *if* $G_0=SO(4)$, *then* $|\Xi^G|=0$ *or* 2.

Proof. The proof is done under the assumption $\Xi^G\neq\phi$ and the notation $H=G_0$.

(1) Since $SO(3)$ has no irreducible k-dimensional representations for $k=2$ and 4, we have $\dim\Xi^H=1$. Take a dihedral subgroup D of G_0 of order 4. Then we have $\dim\Xi^D=1$. By Smith's theorem Ξ^D is a circle. Thus Ξ^H coincides with Ξ^D. By Lemma 2.1 we have that $\Xi^G=(\Xi^H)^G$ is a sphere of dimension$\leqq 1$.

(2) Since $SU(2)$ has no faithful representations of dimension$\leqq 3$, Ξ^H is a finite set. Furthermore the normal H-representations of Ξ^H in Ξ are unique up to isomorphisms. For any cyclic subgroup C of H of prime order, Ξ^C is a sphere and includes Ξ^H. Observing the normal representations of Ξ^H, we see that Ξ^C consists of exactly two points. For any non-trivial subgroup B of H, we have $1\leqq|\Xi^B|\leqq 2$. Let T be a maximal toral subgroup of H. We have $|\Xi^T|=2$ by Smith's theorem. If $\Xi^T-\Xi^H$ is non-empty, then denote the point by x. There is a subgroup L of H such that (i) L has a normal subgroup Q of order 8 and L/Q has order 3 and (ii) $L\cap T\neq\{1\}$. By Oliver's theorem [11, Proposition 2] we have that $|\Xi^L|=2$, hence $\Xi^L=\Xi^T$. Sinc the smallest subgroup of H which includes T and L is H, we have $H_x=H$. This contradicts the assumption $\{x\}=\Xi^T-\Xi^H$. Thus $\Xi^T=\Xi^H$ and Ξ^G also consists of exactly two points.

(3) The conclusion follows from (2) and the fact that $SO(4)$ has a normal subgroup isomorphic to $SU(2)$.

Proof of Lemma B. Suppose that $|\Xi^G|\neq 0$ nor 2. Then G is a subgroup of $O(4)$ and G_0 is a subgroup of $SO(4)$. By Proposition 3.1 G_0 does not have an abelian normal subgroup except $\{1\}$. Hence G_0 is isomorphic to either one of $SO(3)$, $SU(2)$ and $SO(4)$. This contradicts Proposition 3.2.

4. Proof of Lemma C

Let G be a finite group, Ξ an oriented 4-dimensional homology sphere with G-action and $K=K(G,\Xi)$ the subgroup of G defined in Section 1. Our proof of Lemma C is done under the assumption that the G-action on Ξ is effective and $\Xi^G\neq\phi$.

First we note that G is a subgroup of $O(4)$, K a subgroup of $SO(4)$ and $\dim\Xi^H\leqq 2$ for any non-trivial subgroup H of K.

Proposition 4.1. *Let H be a subgroup of K. Then it holds that*

(1) *if* $\Xi_2^H\neq\phi$, *then H is cyclic, and*

(2) *if* $\Xi_1^H \neq \phi$, *then* $\Xi_2^H = \phi$ *and* H *is dihedral or isomorphic to one of* A_4, S_4 *and* A_5.

Here S_4 stands for the symmetric group on four letters, and A_n, $n=4$ and 5, stand for alternating groups on n letters.

Proof. (1) It follows from the fact that a finite subgroup of $SO(2)$ is cyclic.

(2) A finite subgroup of $SO(3)$ is cyclic, dihedral or isomorphic to one of A_4, S_4 and A_5 (see [5]). Suppose that H is cyclic. Then the normal H-representations have even dimensions. This contradicts $\Xi_1^H \neq \phi$.

Proposition 4.2. *Let* H *be a non-trivial solvable subgroup of* K. *Then* Ξ^H *is (empty or) diffeomorphic to* S^m *with* $m \leqq 2$.

Proof. Take a normal series of subgroups $H(i)$ of H: $\{1\} = H(0) \underset{\neq}{\trianglelefteq} H(1) \underset{\neq}{\trianglelefteq} \cdots \underset{\neq}{\trianglelefteq} H(n) = H$ with $H(i)/H(i-1)$ of prime order. By Smith's theorem and Proposition 4.1, $\Xi^{H(1)}$ is a sphere of dimension $\leqq 2$. Since $\Xi^{H(i)} = (\Xi^{H(i-1)})^{H(i)}$, by induction on i $\Xi^{H(i)}$ are spheres of dimension $\leqq 2$.

Proposition 4.3. *Let* H *be a subgroup of* K *and suppose* H *is isomorphic to* A_5. *Then it holds that*

(1) *if* $\Xi_0^H \neq \phi$, *then* $|\Xi^H| = 1$ *or* 2, *and*

(2) *if* $\Xi^H \neq \phi$ *and* $\Xi_0^H = \phi$, *then* Ξ^H *is diffeomorphic to* S^1.

Proof. (1) Let V be a normal representation of Ξ_0^H in Ξ. Since V is faithful, $V^H = 0$ and $\dim V = 4$, V is an irreducible H-representation. Let C be a cyclic subgroup of H of order 5. Then we have $V^C = 0$, hence $\Xi_0^C \supset \Xi_0^H (\neq \phi)$. By Proposition 4.2, Ξ^C consists of exactly two points. The relation $\Xi^C \supset \Xi^H$ implies that $|\Xi^H| = 1$ or 2.

(2) In the case Ξ^H is a disjoint union of circles. Let D be a dihedral subgroup of H of order 4. Then Ξ^D is a circle by Smith's theorem. Immediately we have $\Xi^H = \Xi^D \cong S^1$.

Proposition 4.4. *Provided* $|\Xi^K| \geqq 3$, *then every Sylow subgroup of* K *is either cyclic or dihedral.*

Proof. Let P be a Sylow subgroup of K. Since P is solvable, Ξ^P is a sphere of dimension 1 or 2 by Proposition 4.2. The conclusion follows from Proposition 4.1.

Now we prove Lemma C. We suppose that $|\Xi_0^G| \geqq 3$, and we will meet with a contradiction.

We note that $K \neq \{1\}$ and $|\Xi^K| \geqq 3$. If K is solvable, then Ξ^K is a sphere, hence $\Xi^G = (\Xi^K)^G$ is also a sphere. We have $|\Xi_0^G| = 0$ or 2. This contradicts the above assumption. Thus K is non-solvable. By Suzuki's theorem [15, p. 671, Theorem B] and Proposition 4.4, there exist subgroups H, L and Z of K such that (1) $[K: H]\leqq 2$, (2) $H=Z\times L$, (3) Z is solvable and (4) L is isomorphic to $PSL(2, q)$. Here q is a prime greater than 4. Since L is non-solvable and $\Xi^L\neq\phi$, L has an irreducible representation of dimension 3 or 4. By Tables 3 and 4 of [8], $PSL(2, q)\cong L$ is nothing but $PSL(2, 5)$. In other words, L is isomorphic to A_5. By Proposition 4.3 it holds that $\Xi^L\cong S^1$ or $|\Xi^L|\leqq 2$. From the assumption that $|\Xi_0^G|\geqq 3$, we have $\Xi^L\cong S^1$. Since $\Xi^H=(\Xi^L)^H$, it is isomorphic to S^0 or S^1, so is Ξ^K. Then Ξ^G is also diffeomorphic to S^m, $m\leqq 1$. This is a contradiction.

5. Proof of Theorem A

By Lemma B, it is sufficient to prove the case in which G is a finite group acting effectively on Σ, an oriented 4-dimensional homotopy sphere. The following arguments go on in this case.

Proposition 5.1. *Provided* $|\Sigma^G|=1$, *then* $|\Sigma^K|$ *is finite and an odd number, where* K *is the subgroup of* G *defined in Section* 1.

Proof. Suppose $|\Sigma^G|=1$. By Proposition 4.2, K is non-solvable. It follows from Proposition 4.1 that $\Sigma^K=\Sigma_0^K \amalg \Sigma_1^K$. It holds that

$$\begin{aligned} 1 = \chi(\Sigma^G) = \chi((\Sigma^K)^G) &= \chi((\Sigma_0^K)^G) + \chi((\Sigma_1^K)^G) \\ &\equiv \chi((\Sigma_0^K)^G) \qquad (\text{mod. } 2) \\ &\equiv \chi(\Sigma_0^K) \qquad (\text{mod. } 2)\,. \end{aligned}$$

Thus $|\Sigma_0^K|$ is an odd number. Especially Σ_0^K is non-empty. If Σ_1^K is non-empty, then K is isomorphic to A_5 by Proposition 4.1. In this case, Proposition 4.3 gives that either Σ_0^K or Σ_1^K is empty. This is a contradiction. Hence we have $\Sigma^K=\Sigma_0^K$.

Now we prove Theorem A. Provided $|\Sigma^G|=1$, then by Furuta's theorem and Proposition 5.1 we have $|\Sigma_0^K|\geqq 3$. This, however, contradicts Lemma C. Thus we get the conclusion of Theorem A.

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