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CASSON-GORDON'S RECTANGLE CONDITION OF HEEGAARD DIAGRAMS AND INCOMPRESSIBLE TORI IN 3-MANIFOLDS

TSUYOSHI KOBAYASHI*

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1. Introduction

Heegaard diagram of a 3-manifold [6] is one of the most fundamental description of the manifold. But it seems that little is known about it. For example, there is no efficient method to decide if the manifold with a given Heegaard diagram is aspherical or not. Recently, Casson-Gordon defined a generalized Heegaard diagram and gave a nice sufficient condition for the Heegaard splitting of being irreducible [3]. In fact, they showed that if a Heegaard diagram of a Heegaard splitting $(V_1, V_2: F)$ satisfies a certain condition, say a *rectangle condition*, and $D_i (\subset V_i)$ $(i=1, 2)$ is an essential disk then $\partial D_1 \cap \partial D_2 \neq \phi$. This result together with the Haken's theorem [5, 7] implies that if the Heegaard diagram satisfies a rectangle condition then the manifold is irreducible. In this paper, firstly, we investigate the effect of a rectangle condition on incompressible tori in the 3-manifold. Actually we prove:

Theorem 1. *Let M be a compact orientable 3-manifold, $(C_1, C_2: F)$ a Heegaard splitting of M and $\mathcal{T}$ a union of mutually disjoint essential tori in M. Suppose that $(C_1, C_2: F)$ satisfies a rectangle condition. Then $\mathcal{T}$ is ambient isotopic to $\mathcal{T}'$ such that $\mathcal{T}' \cap F$ is a union of essential loops on $\mathcal{T}'$.*

As consequences of Theorem 1 we have:

Corollary 1. *Let M be a Haken manifold which is closed or has incompressible toral boundary and $\mathcal{T}$ a union of tori which gives the torus decomposition of M. Suppose that M admits a genus g Heegaard splitting $(C_1, C_2: F)$ which satisfies a rectangle condition. Then $\mathcal{T}$ consists of at most $3g-4$ components. Moreover, if $\mathcal{T}$ consists of $3g-4$ components then $\mathcal{T}$ is ambient isotopic to $\mathcal{T}'$ such that each component of $\mathcal{T}'$ intersects C_1 in a disk.*

For the definition of the torus decomposition in this context, see Section 4

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below.

Corollary 2. *Let M, $\mathcal{T}$, $(C_1, C_2: F)$ be as in Corollary* 1. *Then M is decomposed into at most $3g-3-\beta_1(G)$ components by the torus decomposition, where $\beta_1(G)$ denotes the first Beetti number of the characteristic graph G of M. Moreover, if M is decomposed into $3g-3-\beta_1(G)$ components then $\mathcal{T}$ is ambient isotopic to $\mathcal{T}'$ as in Corollary* 1.

In [8] the author showed that the Haken manifolds with Heegaard splittings of genus g are decomposed into at most $3g-3$ components by the torus decomposition and that, for each g (>1), there are infinitely many Haken manifolds with Heegaard splittings of genus g, which are decomposed into $3g-3$ components by the torus decomposition. Corollary 2 shows that the above estimation can be sharpened if the Heegaard splitting satisfies a rectangle condition. In fact, in [8, Section 8], it is shown that there exist Haken manifolds which do not satisfy the inequality in Corollary 2, so that the tori which give the torus decompositions of them can not be isotoped to positions as in the conclusion of Theorem 1. In Section 7 we show that the estimations in Corollaries 1, 2 are best possible by giving infinitely many examples for each g (>1).

Secondly, in Section 5, we define a strong version of a rectangle condition, say a *strong rectangle condition*, and show that if a Heegaard diagram satisfies a strong rectangle condition then the manifold does not contain an essential torus (Corollary 3). Moreover, we give a sufficient condition for a knot on a Heegaard surface of a 3-manifold of being hyperbolic (Corollary 4).

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2. Preliminaries

Throughout this paper we will work in the piecewise linear category. For the definitions of standard terms in the 3-dimensional topology, we refer to [6, 7]. A *surface* is a connected 2-manifold. A (possibly closed) surface F properly embedded in a 3-manifold M is *essential* if it is incompressible and not parallel to a subsurface of ∂M. A *simple loop* is a connected closed 1-manifold.

Let F be a genus $g(>0)$ closed orientable surface. A genus g *compression body* C is a 3-manifold obtained from $F\times[0, 1]$ by attaching 2-handles along mutually disjoint simple loops on $F\times\{1\}$ and then attaching some 3-handles so that ∂C does not contain 2-spheres (cf. [1]). Then $\partial_0 C$ denotes the component of ∂C corresponding to $F\times\{0\}$. We note that a handlebody V is a compression body with $\partial_0 V=\partial V$. We say that a surface $S(\subset C)$ is *normally embedded* in C if $S\cap\partial_0 C=\partial S$. $(C_1, C_2: F)$ is a *Heegaard splitting* of a 3-manifold M if each C_i is a compression body, $M=C_1\cup C_2$ and $C_1\cap C_2=\partial_0 C_1=\partial_0 C_2=F$. F is called a *Heegaard surface* of M and the genus of F is called the genus of the Heegaard splitting.

Then we have:

Lemma 2.1. *Let $\mathcal{A}$ be a union of mutually disjoint, normally embedded, essential annuli in a genus g compression body C. Suppose that no three components of $\partial\mathcal{A}$ are mutually parallel in $\partial_0 C$. Then $\mathcal{A}$ consists of at most $3g-4$ components.*

Proof. By [4, section 1], $\mathcal{A}$ is boundary compressible in C, i.e. there is a disk Δ in C such that $\Delta\cap\mathcal{A}=\alpha$ is an essential arc in $\mathcal{A}$ and $\Delta\cap\partial_0 C=\beta$ an arc such that $\alpha\cup\beta=\partial\Delta$. Let A be the component of $\mathcal{A}$ which intersects Δ. Then, by performing a surgery on A along Δ, we get an essential disk D properly embedded in C. By moving D by a tiny isotopy, we may suppose that $D\cap\mathcal{A}=\phi$. Since each component of $\mathcal{A}$ is essential, no component of $\partial\mathcal{A}$ is parallel to ∂D in $\partial_0 C$. We note that there are at most $3g-3$ mutually disjoint non-parallel essential simple loops on $\partial_0 C$. Then $\partial\mathcal{A}$ consists of at most $3g-4$ parallel classes. Since no three components of $\partial\mathcal{A}$ are mutually parallel, we see that $\mathcal{A}$ consists of at most $3g-4$ components.

This completes the proof of Lemma 2.1.

Lemma 2.2. *Let F be a closed incompressible surface in a compression body C. Then F is parallel to a component of $\partial C-\partial_0 C$.*

Proof. Let $C=(S\times I)\cup$(2-handles)$\cup$(3-handles) and $l_1, \cdots, l_m$ a system of simple loops on $S\times\{1\}$ along which 2-handles are attached. Then $(\cup l_i)\times[0, 1]\cup$(the cores of the 2-handles) is a system of disks $D_1, \cdots, D_m$ normally embedded in C such that $\cup D_i$ cuts C into handlebodies and (surface)$\times[0, 1]$'s. Since F is incompressible and C is irreducible [4], we may suppose that $F\cap(\cup D_i)=\phi$. Hence F is contained in a component of C cut along $\cup D_i$ which is of the form (surface)$\times[0, 1]$. Hence, by [5], F is parallel to a component of $\partial C-\partial_0 C$.

This completes the proof of Lemma 2.2.

3. Rectangle conditions

In this section we introduce *rectangle conditions* of Heegaard splittings following Casson-Gordon [3] and prove Theorem 1.

Let S be a genus g (>1) closed orientable surface and P_i ($i=1, 2$) a pants (: disk with two holes) embedded in S with $\partial P_i=l_1^i\cup l_2^i\cup l_3^i$. We suppose that ∂P_1 and ∂P_2 intersect transversely. We say that P_1 and P_2 are *tight* if:

(i) there is no 2-gon B in S such that $\partial B=a\cup b$, where a is a subarc of

∂P_1 and b is a subarc of ∂P_2,

(ii) for each pair (l_s^1, l_t^1) with $s \neq t$ and (l_p^2, l_q^2) with $p \neq q$, there is a rectangle R embedded in P_1 and P_2 such that Int $R \cap (\partial P_1 \cup \partial P_2) = \phi$, and the edges of R are subarcs of l_s^1, l_t^1, l_p^2 and l_q^2.

Let $\{l_1, \cdots, l_{3g-3}\}$ ($\{l'_1, \cdots, l'_{3g-3}\}$ resp.) be a system of mutually disjoint simple loops on S such that S-Int$(N(\cup l_i))$ (S-Int$(N(\cup l'_i))$ resp.) consists of $2g-2$ pants $P_1, \cdots, P_{2g-2}$ ($P'_1, \cdots, P'_{2g-2}$ resp.), where $N(.)$ denotes a regular neighborhood. We suppose $\cup l_i$ and $\cup l'_i$ intersect transversely. We say that $\{l_1, \cdots, l_{3g-3}\}$ and $\{l'_1, \cdots, l'_{3-3}\}$ are *tight* if, for each pair (i, j), P_i and P'_j are tight.

Let $(C_1, C_2: F)$ be a Heegaard splitting of a 3-manifold M. We say that $(C_1, C_2: F)$ satisfies a *rectangle condition* if there are tight systems $\{l_1, \cdots, l_{3g-3}\}$ and $\{l'_1, \cdots, l'_{3g-3}\}$ on F such that each l_i (l'_i resp.) is the boundary of a disk or a boundary component of an incompressible, boundary incompressible annulus properly embedded in C_1 (C_2 resp.).

Remark. By using the complete disk system of [4, section 1], we can show that if A is an incompressible, boundary incompressible surface properly embedded in a compression body C then either A is a disk normally embedded in C or A is an annulus such that one component of ∂A is contained in $\partial_0 C$ and the other component is contained in $\partial C - \partial_0 C$.

We say that a Heegaard splitting $(C_1, C_2: F)$ is *strongly irreducible* if there are no essential disks D_1, D_2 normally embedded in C_1, C_2 respectively such that $\partial D_1 \cap \partial D_2 = \phi$. Then Casson-Gordon proved:

Theorem 3.1 [3]. *Suppose that a Heegaard splitting $(C_1, C_2: F)$ satisfies a rectangle condition. Then $(C_1, C_2: F)$ is strongly irreducible.*

Let M, $\mathcal{S}$, $(C_1, C_2: F)$ be as in Theorem 1. The proof of Theorem 1 is done by using the argument of hierarchy for a 2-manifold, isotopy of type $A, \cdots$, which is an idea of Jaco's for proving the Haken's theorem ([7, Chapter II]) and was developed by Ochiai [10]. For the definitions of terminologies, see [7].

By the above remark, we may suppose that each component of $\mathcal{S}_1 = \mathcal{S} \cap C_1$ is a disk and the number of the component is minimal. Then $\mathcal{S}_2 = \mathcal{S} \cap C_2$ is a union of punctured tori. By [7, 8], there is a hierarchy $(\mathcal{S}_2^{(0)}, a_0), \cdots, (\mathcal{S}_2^{(n)}, a_n)$ for $\mathcal{S}_2$ and a sequence of isotopies of type A which realizes the hierarchy in C_2. We may suppose that $a_i \cap a_j = \phi\, (i \neq j)$ and, hence, we can consider $a_0, \cdots, a_n$ are mutually disjoint arcs on $\mathcal{S}_2$. We say that a_i is of *type* 1 if a_i joins distinct components of $\partial \mathcal{S}_2$, a_i is of *type* 2 if a_i joins one component of $\partial \mathcal{S}_2$ and a_i separates the component of $\mathcal{S}_2$ containing a_i and a_i is of *type* 3 if a_i joins one component of $\partial \mathcal{S}_2$ and a_i does not separate the component of $\mathcal{S}_2$ containing a_i [9]. Then we have:

Lemma 3.2. *Let M, $\mathcal{S}$, $\mathcal{S}_i$ $(i=1, 2)$, $(\mathcal{F}_2^{(j)}, a_j)$ $(j=1, \cdots, n)$ be as above.*

Then:

(*i*) *each a_i is not of type* 2,

(*ii*) *for each component of $\partial\mathcal{S}_2$, there is at least one type* 3 *arc which joins the component,*

(*iii*) *if two type* 3 *arcs meet a component of $\partial\mathcal{S}_2$ then $T\cap C_1$ consists of a disk, where T denotes the component of $\mathcal{S}$ containing the arcs and*

(*iv*) *for each component of $\partial\mathcal{S}_2$, there is at most two type* 1 *arcs which meet the component. Moreover, if a_i (a_j resp.) is a type* 1 (*type* 3 *resp.*) *arc such that a_i and a_j meet the same component of $\partial\mathcal{S}_2$ then $j<i$.*

Proof. See [8, section 3].

Proof of Theorem 1. Let M, $\mathcal{S}$, $\mathcal{S}_i$ ($i=1, 2$), $(\mathcal{S}_2^{(j)}, a_j)$ ($j=1, \cdots, n$) be as above. Suppose that $\mathcal{S}_1$ consists of p components. By Lemma 2.2, we see that each component of $\mathcal{S}$ intersects the Heegaard surface F. If $a_0, \cdots, a_{p-1}$ are all of type 3, which meet mutually distinct components of $\partial\mathcal{S}_2$ then, by taking the image of $\mathcal{S}$ after the isotopy of type A at a_{p-1}, we have the conclusion of Theorem 1.

Assume that the above is not true. Then, by Lemma 3.2 (i), we have the following two cases.

Case 1. There exists an i ($<p$) such that a_i is of type 1.

By taking minimal i such that a_i is of type 1, we may suppose that if $j<i$ then a_j is of type 3. Let $\mathcal{S}'$ be the image of $\mathcal{S}$ after the isotopy of type A at a_i, i.e. $\mathcal{S}'\cap C_2=\mathcal{S}_2^{(i+1)}$. Let S_1, S_2 be the components of $\partial\mathcal{S}_2$ which a_i joins. Then, by Lemma 3.2 (ii), (iv), there are two type 3 arcs a_k, a_l ($k, l<i$) such that a_k (a_l resp.) meets S_1 (S_2 resp.). Then $a_k\cup a_l\cup a_i$ cuts the component of $\mathcal{S}$ containing a_i into a disk and an annulus with (possibly empty) holes. This shows that a component of $\mathcal{S}'\cap C_2$ is a disk D. By the minimality of $\mathcal{S}$, we see that D is essential. Since $i<p$, some component of $\mathcal{S}'\cap C_1$ is an essential disk D'. Clearly we have $\partial D\cap\partial D'=\phi$, contradicting Theorem 3.1.

Case 2. There exists a pair i, j ($i<j<p$) such that a_i, a_j are of type 3 and meet the same component of $\partial\mathcal{S}_2$.

Let T be the component of $\mathcal{S}$ which contains a_i, a_j. Then, by Lemma 3.2 (iii), we see that $T\cap C_1$ consists of a disk. Let $\mathcal{S}'$ be the image of the isotopy of type A at a_j. Then we have a contradiction as in Case 1.

This completes the proof of Theorem 1.

4. Proof of Corollaries 1, 2

In this section we prove Corollaries 1, 2 stated in section 1.

Let M be a Haken manifold which is closed or has incompressible toral boundary. Then, by [7], there is a maximal, perfectly embedded Seifert fibered manifold Σ, which is called a characteristic Seifert pair for M. Then $\mathrm{Fr}_M\Sigma$ consists of tori in Int M, where $\mathrm{Fr}_M\Sigma$ denotes the frontier of Σ in M. If a pair of components of the tori are parallel in M then we remove one of them from the system of tori. If a component of the tori is parallel to a component of ∂M then we remove it from the sysetm of tori. By iterating these finitely many times, we get a system of tori $\mathcal{T}$ in M the components of which are mutually non parallel and each component of which is not parallel to a component of ∂M. In this paper we call the decomposition of M by $\mathcal{T}$, the *torus decomposition* of M. Then, we get a graph G, where the edges of G correspond to the components of $\mathcal{T}$ and the vertices of G correspond to the components of $M-\mathcal{T}$. G is called the *characteristic graph* for M.

Let M, $\mathcal{T}$, $(C_1, C_2: F)$ be as in Corollary 1. By Theorem 1, we may suppose that $\mathcal{T}$ intersects F in essential loops on $\mathcal{T}$ and the number of the components of $\mathcal{T}\cap F$ is minimal among all surfaces which are ambient isotopic to $\mathcal{T}$ and intersect F in essential loops. By the minimality, we see that each component of $\mathcal{T}\cap C_i$ $(i=1, 2)$ is an essential annulus normally embedded in C_i. Then we have:

Lemma 4.1. *No three components of $\mathcal{T}\cap F$ are mutually parallel in F.*

Proof. Assume that three components l_1, l_2, l_3 of $\mathcal{T}\cap F$ are mutually parallel in F. We may suppose that $l_1\cup l_2$ ($l_2\cup l_3$ resp.) bounds an annulus A_1 (A_2 resp.) in F such that $A_1\cap A_2=l_2$. Let M_1 (M_2 resp.) be the closure of the component of M-Int $N(\mathcal{T})$ which contains $A_1'=A_1$-Int $N(\mathcal{T})$ ($A_2'=A_2$-Int $N(\mathcal{T})$ resp.). We note that it is possible that $M_1=M_2$. By the minimality of $\mathcal{T}$, we see that A_i' $(i=1, 2)$ is an essential annulus in M_i. Then, by [7], M_i admits a Seifert fibration such that A_i is a union of fibers. Hence a Seifert fibration on M_1 can be extended to M_2 through the component of $N(\mathcal{T})$ which contains l_2. But this contradicts the definition of the torus decomposition.

This completes the proof of Lemma 4.1.

Proof of Corollary 1. By Lemma 4.1, we see that the system of annuli $\mathcal{T}\cap C_i$ $(i=1, 2)$ in C_i satisfies the assumption of Lemma 2.1. Hence we have the conclusion of Corollary 1.

Proof of Corollary 2. By Corollary 1, we see that the characteristic graph G of M contains at most $3g$–4 edges. Hence, by the Euler characteristic formula, we see that G contains at most $3g-3-\beta_1(G)$ vertices so that we have the conclusion of Corollary 2.

5. Strong rectangle condition

In this section we give the definition of the strong rectangle condition and prove an analogy to Theorem 3.1 for essential annuli normally embedded in compression bodies. In fact, we prove:

Theorem 2. *Suppose that a Heegaard splitting* $(C_1, C_2: F)$ *satisfies the strong rectangle condition. Then there are no essential annuli* A_1, A_2 *normally embedded in* C_1, C_2 *respectively such that* $\partial A_1 \cap \partial A_2 = \phi$.

As a consequence of Theorem 2 we have:

Corollary 3. *If a Heegaard splitting satisfies a strong rectangle condition then the manifold does not contain an essential torus.*

First of all, we give the definition of the strong rectangle condition. Let C be a genus $g(>1)$ compression body, and $l_1, \cdots, l_{3g-3}$ a system of mutually disjoint, non-parallel essential simple loops contained in $\partial_0 C$ such that each l_i is a boundary component of a disk or an incompressible, boundary-incompressible annulus. Then $\partial_0 C$-Int $(N(\cup l_i: \partial_0 C))$ consists of $2g$–2 pants $P_1, \cdots, P_{2g-2}$, where $N(\cdot: \partial_0 C)$ denotes a regular neighborhood in $\partial_0 C$. Moreover, when we talk about a strong rectangle condition, we assume that there are two different pants which intersects $N(l_i: \partial_0 C)$ for each i. It is easy to see that this condition is equivalent to:

Each l_i does not separate $\partial_0 C$ into a genus 1 surface and a genus $g-1$ surface.

Let $l \subset \partial_0 C$ be a simple loop which intersects the above $l_1 \cup \cdots \cup l_{3g-3}$ in transverse points. We say that l is *complicated with respect to* $l_1, \cdots, l_{3g-3}$ (or simply *complicated*) if l satisfies:

(i) there is no 2-gon B in $\partial_0 C$ such that $\partial B = a \cup b$, where a is a subarc of l and b is a subarc of $\cup l_i$ and

(ii) for any two boundary components of each pants P_i, there is a subarc a of l joining them in P_i.

Let C, l_i, P_j be as above. Let $R_i (i=1, \cdots, 3g$–$3)$ be the double pants (:disk with three holes) $P_{s_i} \cup N(l_i: \partial_0 C) \cup P_{t_i}$, where $s_i \neq t_i$, $P_{s_i} \cap N(l_i: \partial_0 C) \neq \phi$ and $P_{t_i} \cap N(l_i: \partial_0 C) \neq \phi$. We note that there are six ways of making pair of boundary components of R_i. We say that a simple loop $l(\subset \partial_0 C)$ intersecting $\cup l_i$ in transverse points is *sufficiently complicated with respect to* $l_1, \cdots, l_{3g-3}$ (or simply *sufficiently complicated*) if l satisfies the above condition (i) and:

(iii) for any two boundary components of each R_i, there is a subarc a

of l joining them in R_i.

Then we have:

Lemma 5.1. *If l is sufficiently complicated with respect to $l_1, \cdots, l_{3g-3}$ then l is complicated with respect to $l_1, \cdots, l_{3g-3}$.*

Proof. Let $P_i (i=1, \cdots, 2g-2)$, $R_j (j=1, \cdots, 3g-3)$ be as sbove. Assume that l is not complicated with respect to $l_1, \cdots, l_{3g-3}$. Then there is a pants P_k and a pair of boundary components m_1, m_2 of P_k such that no subarc of l properly embedded in P_k joins m_1 and m_2. Let l_s be the simple loop such that $N(l_s: \partial_0 C) \cap P_k \neq \phi$ and $m_i \cap N(l_s: \partial_0 C) = \phi$ $(i=1, 2)$. Let P_t be the pants such that $P_t \neq P_k$, $P_t \cap N(l_s: \partial_0 C) \neq \phi$, i.e. $R_s = P_t \cup N(l_s: \partial_0 C) \cup P_k$. Since l is sufficiently complicated, there is a subarc a of l properly embedded in R_s such that a joins m_1 and m_2. Hence there is a subarc a' of a which is an essential arc properly embedded in P_t and $\partial a' \subset \partial N(l_s: \partial_0 C)$. Let l_u be a simple loop such that $l_u \neq l_s$ and $N(l_u: \partial_0 C) \cap P_t \neq \phi$. Then a' separates R_u into an annulus and a pants. Hence there is a pair of boundary components of R_u separated by a'. But this contradicts to the fact that l is sufficiently complicated.

This completes the proof of Lemma 5.1.

Now we give the definition of the strong rectangle condition. Let S be a genus $g(>1)$ closed orientable surface and $R_i (i=1, 2)$ a double pants embedded in S with $\partial R_i = l_1^i \cup l_2^i \cup l_3^i \cup l_4^i$. We suppose that ∂R_1 and ∂R_2 intersect transversely. We say that R_1 and R_2 are *tight* if:

(i) there is no 2-gon B in S such that $\partial B = a \cup b$, where a is a subarc of ∂R_1 and b is a subarc of ∂R_2,

(ii) for each pair (l_s^1, l_t^1) with $s \neq t$ and (l_p^2, l_q^2) with $p \neq q$, there is a rectangle R embedded in R_1 and R_2 such that, $\mathrm{Int}\, R \cap (\partial R_1 \cap \partial R_2) = \phi$, and the edges of R are subarcs of l_s^1, l_t^1, l_p^2 and l_q^2.

Let $\{l_1, \cdots, l_{3g-3}\}$ ($\{l'_1, \cdots, l'_{3g-3}\}$ resp.) be a system of mutually disjoint simple loops such that $\cup l_i$ ($\cup l'_i$ resp.) cuts S into $2g-2$ pants. Let $R_1, \cdots, R_{3g-3}$ ($R'_1, \cdots, R'_{3g-3}$ resp.) be a system of double pants obtained from $\{l_1, \cdots, l_{3g-3}\}$ ($\{l'_1, \cdots, l'_{3g-3}\}$ resp.) as above. We say that $\{l_1, \cdots, l_{3g-3}\}$ and $\{l'_1, \cdots, l'_{3g-3}\}$ are *strongly tight* if, for each pair (i, j), R_i and R_j are tight.

Let $(C_1, C_2: F)$ be a Heegaard splitting of a 3-manifold M. We say that $(C_1, C_2: F)$ satisfies a *strong rectangle condition* if there are strongly tight systems $\{l_1, \cdots, l_{3g-3}\}$ and $\{l'_1, \cdots, l'_{3g-3}\}$ on F such that each l_i (l'_i resp.) is the boundary of a disk or a boundary component of an incompressible, boundary incompressible annulus properly embedded in C_1 (C_2 resp.).

Then we have:

Lemma 5.2. *Let $(C_1, C_2: F)$ be a Heegaard splitting of a 3-manifold. If* *two systems of simple loops on F give a strong rectangle condition for* $(C_1, C_2\colon F)$ *then they also give a rectangle condition for* $(C_1, C_2\colon F)$.

The proof is essentially the same as in Lemma 5.1. So we omit it.

Proof of Theorem 2. Let $\{l_1, \cdots, l_{3g-3}\}$, $\{l'_1, \cdots, l'_{3g-3}\}$ be systems of simple loops on F which give a strong rectangle condition for $(C_1, C_2\colon F)$. Let F_i (F'_i resp.) ($i=1, \cdots, 3g-3$) be a disk or an incompressible, boundary incompressible annulus properly embedded in C_1 (C_2 resp.) such that $l_i \subset \partial F_i$ ($l'_i \subset \partial F'_i$ resp.). We may suppose that $F_1, \cdots, F_{3g-3}$ ($F'_1, \cdots, F'_{3g-3}$ resp.) are mutually disjoint. Then C_1-Int $(N(\cup F_i))$ (C_2-Int $(N(\cup F'_i))$ resp.) consists of $2g-2$ components $Q_1, \cdots, Q_{2g-2}$ ($Q'_1, \cdots, Q'_{2g-2}$ resp.). Then let $P_i = Q_i \cap \partial_0 C_1$, $P'_i = Q'_i \cap \partial_0 C_2$. Let $S_i = Q_{m_i} \cup N(l_i\colon C_1) \cup Q_{k_i}$ ($S'_i = Q'_{n_i} \cup N(l'_i\colon C_2) \cup Q'_{p_i}$ resp.), where $m_i \neq k_i$ and $Q_{m_i} \cap N(l_i\colon C_1) \neq \phi$, $Q_{k_i} \cap N(l_i\colon C_1) \neq \phi$ ($n_i \neq p_i$ and $Q'_{n_i} \cap N(l'_i\colon C_2) \neq \phi$, $Q'_{p_i} \cap N(l'_i\colon C_2) \neq \phi$ resp.). Then let $R_i = S_i \cap \partial_0 C_1$ ($R'_i = S'_i \cap \partial_0 C_2$ resp.).

By the general position argument and cut and paste method [6], we may suppose that $F_i \cap A_1$ ($F'_i \cap A_2$ resp.) consists of (possibly empty) arcs properly embedded in F_i (F'_i resp.) and that $\partial A_1 \cap P_i$ ($\partial A_2 \cap P'_i$ resp.) consists of essential arcs in P_i (P'_i resp.).

Suppose that there are components of $(\cup F_i) \cap A_1$ and $(\cup F'_i) \cap A_2$, say α and α', which are inessential arcs in A_1 and A_2. Let β (β' resp.) be the subarc of ∂A_1 (∂A_2 resp.) such that $\partial\beta = \partial\alpha$ ($\partial\beta' = \partial\alpha'$ resp.) and $\alpha \cup \beta$ ($\alpha' \cup \beta'$ resp.) bounds a disk in A_1 (A_2 resp.). We may suppose that $\beta \subset P_1$ and $\beta' \subset P'_1$. Let b_1, b_2 (b'_1, b'_2 resp.) be the boundary components of P_1 (P_2 resp.) such that $(b_1 \cup b_2) \cap \beta = \phi$ ($(b'_1 \cup b'_2) \cap \beta' = \phi$ resp.). Then, by Lemma 5.2, we see that there is a rectangle R in F such that $R \subset P_1$, $R \subset P'_1$ and the edge of R consists of subarcs of b_1, b_2, b'_1 and b'_2. Moreover, a subarc of β' (β resp.) is properly embedded in R and connects b_1 and b_2 (b'_1 and b'_2 resp.), so that $\beta \cap \beta' \neq \phi$. Hence, we may suppose that each component of $(\cup F'_i) \cap A_2$ is an essential arc in A_2.

By [2], we see that there is a train track τ on F such that ∂A_2 is carried by τ and $\tau \cap P'_i$, $\tau \cap N(l_j\colon F)$ look as in Figure 5.1. Since each component of $(\cup F'_i) \cap A_2$ is an essential arc in A_2, we can isotope A_2 so that $\partial A_2 \subset N(\tau\colon F)$ and each component of $Q'_i \cap A_2$ looks like the bottom of a ditch (Figure 5.2).

Let D be a component of $N(F'_i\colon C_2) \cap A_2$. We say that D is of *type a* if the two arcs $D \cap F$ are carried by a path in $\tau \cap N(l'_i\colon F)$, D is of *type b* if the two arcs $D \cap F$ are carried by pairwise different paths in $\tau \cap N(l'_i\colon F)$ (Figure 5.3). Assume that all components of $A_2 \cap (\cup N(F'_i\colon C_2))$ are of type a. Then A_2 is parallel to an annulus in $\partial_0 C$, a contradiction. Hence we have:

Assertion. There exists a component of $A_2 \cap (\cup N(F'_i\colon C_2))$ which is of type b.

We may suppose that $A_2 \cap N(F'_1\colon C_2)$ contains a type b disk D. Let D'

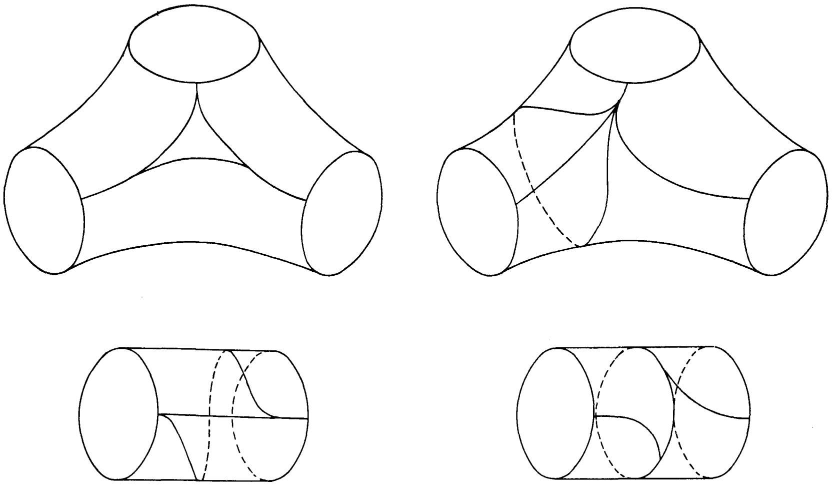


Figure 5.1

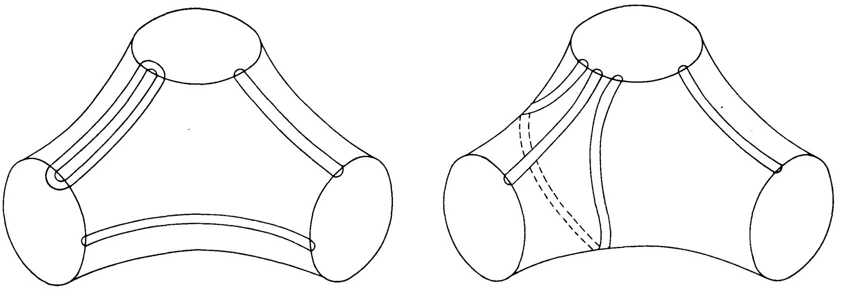
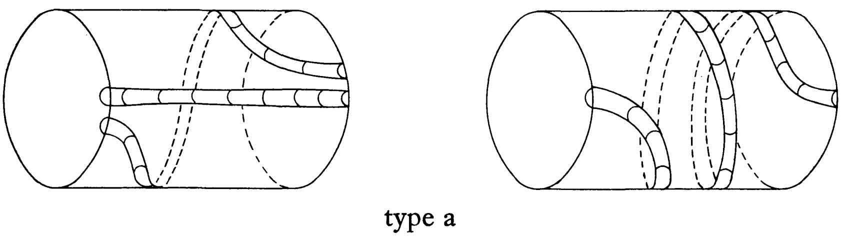
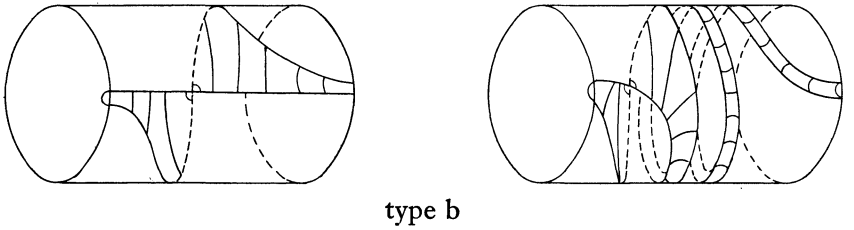


Figure 5.2



type a



type b

Figure 5.3

be the component of $A_2 \cap S_1'$ which contains D. Then $D' \cap F$ consists of two arcs a_1, a_2 properly embedded in R_1'. Since D is of type b, $a_1 \cup a_2 (\subset R_1')$ separates a pair of boundary components l_1, l_2 of R_1' (Figure 5.4). Hence, if a is an arc properly embedded in R_1', which joins l_1 and l_2, then a intersects $a_1 \cup a_2$. Then, by the definition of the strong rectangle condition, we see that a component l of $\partial A_2'$ is sufficiently complicated with respect to $l_1, \cdots, l_{3g-3}$. Now, we have the following two cases.

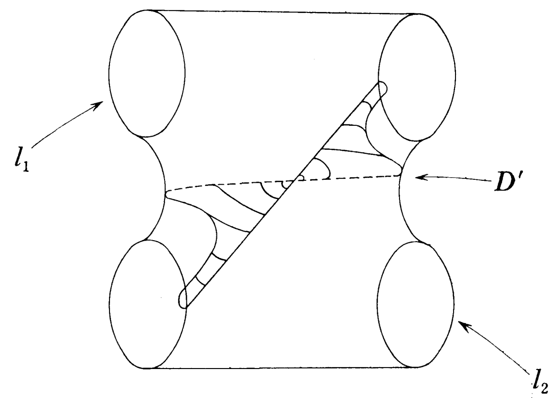


Figure 5.4

Case 1. There is a component a of $A_1 \cap (\cup F_i)$ which is an inessential arc in A_1.

We may suppose that a is innermost, i.e. there is a disk D in A_1 such that $cl(\partial D - \partial A_1) = a$ and $\text{Int}\, D \cap (\cup F_i) = \phi$. Moreover we may suppose that $a \subset F_1$, $D \subset Q_1$. Let $b = cl(\partial D - a)$. Then b is an arc properly embedded in P_1, which meets one boundary component of P_1. By Lemma 5.1, l is complicated with respect to $l_1, \cdots, l_{3g-3}$ so that $l \cap b \neq \phi$. Hence $\partial A_1 \cap \partial A_2 \neq \phi$.

Case 2. Every component of $A_1 \cap (\cup F_i)$ is an essential arc in A_1.

In this case, by the argument as above, there are a double pants R_i and a component E of $A_1 \cap S_i$ such that $E \cap F$ consists of two arcs properly embedded in R_i, which separate a pair of boundary components m_1, m_2 of R_i. Since l is sufficiently complicated, there is a subarc b of l properly embedded in R_i, which connects m_1 and m_2. Hence $b \cap \partial A_1 \neq \phi$, so that $\partial A_1 \cap \partial A_2 \neq \phi$.

This completes the proof of Theorem 2.

Proof of Corollary 3. Let $(C_1, C_2 \colon F)$ be a Heegaard splitting of a 3-manifold M, which satisfies a strong rectangle condition. Assume that M contains an essential torus T. By Lemma 2.2, we see that $T \cap F \neq \phi$. By Theorem 1, Lemma 5.2, we may suppose that each component of $T \cap C_i$ $(i=1, 2)$ is an essential annulus in C_i. Hence there are essential annuli A_1, A_2 in C_1, C_2 respectively such that $\partial A_1 \cap \partial A_2 = \phi$, contradicting Theorem 2.

6. Hyperbolic knots

In this section we give a sufficient condition for a given knot embedded in a Heegaard surface of a 3-manifold is atoroidal by using the concept of sufficiently complicated simple loops defined in section 5.

Theorem 3. *Let K be a simple loop embedded in the Heegaard surface of a Heegaard splitting $(C_1, C_2: F)$ of a 3-manifold M. If K is sufficiently complicated with respect to C_1 and C_2 then M-Int $N(K)$ is irreducible and does not contain an essential torus.*

By using Theorem 3 and a theorem of Thurston [11], we prove:

Corollary 4. *Let K be a simple loop embedded in the Heegaard surface of a Heegaard splitting $(C_1, C_2: F)$ of a 3-manifold M which is closed or has (not necessarily incompressible) toral boundary. If K is sufficiently complicated with respect to C_1 and C_2 then K is a hyperbolic knot, i.e.* Int $(M-K)$ *admits a complete hyperbolic structure of finite volume.*

Lemma 6.1. *Let C be a genus $g(>1)$ compression body and l $(\subset \partial_0 C)$ a simple loop. If l is complicated then $\partial_0 C$-Int $N(l)$ is incompressible in C.*

Proof. Let $l_1, \cdots, l_{3g-3}$ be a system of simple loops on $\partial_0 C$, with respect to which l is complicated, and let $F_1, \cdots, F_{3g-3}$ be a system of mutually disjoint incompressible, boundary incompressible surfaces such that $F_i \cap \partial_0 C = l_i$ $(i=1, \cdots, 3g-3)$.

Assume that $\partial_0 C$-Int $N(l)$ is compressible in C. Let D be a compressing disk. Then we may suppose that D intersects $\cup F_i$ transversely. By cut and paste argument, we may suppose that D intersects $\cup F_i$ in arcs. Moreover we may suppose that there is no 2-gon B in $\partial_0 C$ such that $\partial B = a \cup b$, where a is a subarc of ∂D and b is a subarc of $\cup l_i$. Suppose that $\partial D \cap (\cup l_i) = \phi$. Then ∂D is parallel to some l_i so that ∂D intersects l, a contradiction. Hence $D \cap (\cup F_i) \neq \phi$. Let $\Delta (\subset D)$ be one of the innermost disks, i.e. $\Delta \cap (\cup F_i) = \partial \Delta \cap (\cup F_i) = \alpha$ an arc and $\Delta \cap \partial_0 C = \beta$ an arc such that $\alpha \cup \beta = \partial \Delta$. Let P be the closure of the component of $C - N(\cup F_i)$ such that $\Delta' = \Delta$-Int $(N(\cup F_i))$ is contained in P. Let $\beta' = \beta$-Int $(N(\cup F_i))$. Then β' is an arc properly embedded in the pants $P \cap (\partial_0 C)$ and β' separates two boundary components of $P \cap (\partial_0 C)$. On the other hand, since l is complicated, there is a subarc γ of l properly embedded in $P \cap (\partial_0 C)$ such that γ joins the two boundary components. Hence $\beta' \cap \gamma \neq \phi$ so that $\partial D \cap l \neq \phi$, a contradiction.

This completes the proof of Lemma 6.1.

As a consequence of Lemma 6.1, we have:

Lemma 6.2. *Let K, $(C_1, C_2: F)$, M be as in Theorem* 3. *If K is complicated with respect to C_1 and C_2 then M-*Int $N(K)$ *is irreducible.*

Proof. Assume that M-Int $N(K)$ contains an essential 2-sphere S. Since C_i is irreducible [4], $S \cap (F\text{-Int}\, N(K)) \neq \phi$. We may suppose that the number of the components of $S \cap (F\text{-Int}\, N(K))$ is minimal among all essential 2-spheres in M-Int $N(K)$. Let $D(\subset S)$ be one of the innermost disks, i.e. $\partial D \subset F$, Int $D \cap F = \phi$. Then, by Lemma 6.1, we see that ∂D is contractible in F-Int $N(K)$. Hence, D can be pushed into the other compression body, contradicting the minimality of S.

This completes the proof of Lemma 6.2.

The next lemma is proved implicitly in section 5. So we will just see how the proof proceeds.

Lemma 6.3. *Let C, l be as in Lemma* 6.1. *If l is sufficiently complicated then $(C, \partial_0 C\text{-Int}\, N(l))$ does not contain an essential annulus, i.e. if A is an incompressible annulus properly embedded in $(C, \partial_0 C\text{-Int}\, N(l))$ then A is boundary parallel.*

Outline of proof. Assume that there is an incompressible annulus A properly embedded in $(C, \partial_0 C\text{-Int}\, N(l))$ such that A is not parallel to an annulus in $\partial_0 C$. Let $l_1, \cdots, l_{3g-3}, F_1, \cdots, F_{3g-3}$ be as in the proof of Lemma 6.1. Then, by the proof of Lemma 6.1 and Lemma 5.1, we may suppose that each component of $A \cap (\cup F_i)$ is an essential arc in A. By Assertion of section 5, we see that there is a component S of $C\text{-Int}\,(N(\bigcup_{\mathcal{L}} F_i))$, for a suitable subset $\mathcal{L}$ of $\{1, \cdots, 3g-3\}$, such that $R = S \cap \partial_0 C$ is a double pants and a pair of components of $A \cap R$ separates a pair of boundary components of R, a contradiction.

As a consequence of Lemma 6.3, we have:

Lemma 6.4. *Let C, l be as in Lemma* 6.1. *If l is sufficiently complicated then C is not homeomorphic to the total space of* [0, 1]*-bundle over a surface such that $\partial_0 C$-*Int $N(l)$ *corresponds to the associated* $\{0, 1\}$*-bundle.*

Proof. Assume that C is homeomorphic to the total space of [0, 1]-bundle $C \xrightarrow{p} F$ such that $\partial_0 C$-Int $N(l)$ corresponds to the associated $\{0, 1\}$-bundle. Then F is a genus $g/2$ orientable surface with one hole or a genus g non-orientable surface with one hole. It is easy to see that there is an essential simple loop l in F. Then $p^{-1}(l)$ is an essential annulus properly embedded in $\partial_0 C$-Int $N(l)$, contradicting Lemma 6.3.

This completes the proof of Lemma 6.4.

Proof of Theorem 3. By Lemmas 5.1, 6.2, we see that $M\text{-Int}\,N(K)$ is irreducible. Let T be an incompressible torus in $M\text{-Int}\,N(K)$. Since $F\text{-Int}\,N(K)$ is incompressible in $M\text{-Int}\,N(K)$ (Lemma 6.1), we may suppose that T intersects $F\text{-Int}\,N(K)$ by loops which are essential on T. Moreover we may suppose that the number of the components of $T\cap(F\text{-Int}\,N(K))$ is minimal among all tori which are ambient isotopic to T and intersecting $F\text{-Int}\,N(K)$ by loops which are essential on them. Then, by Lemma 6.3, we see that each component of $T\cap C_i$ $(i=1, 2)$ is an annulus which is parallel to $N(K\colon F)$. Hence T is parallel to $\partial N(K)$.

This completes the proof of Theorem 3.

Proof of Corollary 4. By Theorem 3, we see that $M\text{-Int}\,N(K)$ does not contain an essential torus. Hence, by [11], it is enough to show that M does not admit a Seifert fibration for the proof of Corollary 4. Assume that M admits a Seifert fibration. Then, by Lemma 6.1 and [7, Theorem VI. 34], we see that C_i $(i=1, 2)$ is homeomorphic to the total space of a $[0, 1]$-bundle over a surface, where $F\text{-Int}\,N(l)$ corresponds to the associated $\{0, 1\}$-bundle, contradicting Lemma 6.4.

This completes the proof of Corollary 4.

7. Examples

In this section we will show that for each $g(>1)$ there exist infinitely many closed Haken manifolds which admit genus g Heegaard splittings with rectangle conditions, each of which admits a torus decomposition which satisfies the equality in Corollary 2 (Examples 1, 2, 3). It is clear that such examples show that the estimation in Corollary 1 is best possible. Then we will give examples satisfying the assumptions of Theorems 2, 3 and Corollaries 3, 4 (Examples 4, 5).

Example 1. Let V_1 be a genus 2 handlebody, A_1^1, A_2^1 be a pair of essential annuli properly embedded in V_1 as in Figure 7.1. Let V_2 be a copy of V_1, A_1^2, A_2^2 the annuli in V_2 corresponding to A_1^1, A_2^1 and $h\colon \partial V_1 \to \partial V_2$ the homeomorphism induced from the identification of V_1 and V_2. Then $V_1 \cup_h V_2$ is the connected sum of two $S^2\times S^1$'s. Let $l(\subset \partial V_1)$ be the simple loop in Figure 7.1 and $T\colon \partial V_1 \to \partial V_1$ the right hand Dehn twist along l. For each integer n, set $M_2^{(n)} = V_1 \cup_{h\circ T^n} V_2$. Then $A_1^1\cup A_1^2$, $A_2^1\cup A_2^2$ become tori $T_1^{(n)}$, $T_2^{(n)}$ in $M^{(n)}$. $T_i^{(n)}\cup T_1^{(n)}$ separates $M^{(n)}$ into two components $N_1^{(n)}$, $N_2^{(n)}$, where $N_2^{(n)}$ admits a Seifert fibration with the orbit manifold an annulus with two exceptional fibers of index two such that A_k^l is a union of the fibers and $N_1^{(n)}$ is homeomorphic to the exterior of $(2, 2n)$ torus link such that the core of A_k^l corresponds to a meridian loop [9, section 4]. Hence if $|n|>1$ then $N_1^{(n)}$ admit a Seifert fibration with the orbit manifold an annulus and one exceptional fiber of index $|n|$, whose regular fiber in $\partial N_1^{(n)}$ intersects the core of A_k^l transversely in one point. By [7, Theorem VI. 18], we see that if $|n|>1$ then the Seifert fibration on $N_2^{(n)}$ does not extend to $N_1^{(n)}$. Hence $M^{(n)}=N_1^{(n)}\cup N_2^{(n)}$ is the torus decomposition of $M^{(n)}$ provided $|n|>1$. By the uniqueness of the torus decomposition, we see that $M^{(n)}$ is not homeomorphic to $M^{(m)}$, provided $|n|\neq|m|$.

Let $F^{(n)}=\partial V_1=\partial V_2(\subset M^{(n)})$. Clearly $F^{(n)}$ is a Heegaard surface of $M^{(n)}$. We will show that the Heegaard splitting $(V_1, V_2: F^{(n)})$ satisfies the rectangle condition of if $|n|>1$. Let l_i, l_i' $(i=1, 2, 3)$ be simple loops on $F^{(n)}$ in Figure 7.2. We note that each simple loop bounds a disk in V_1. Recall that T is a right hand Dehn twist along l. Set $l_i^{(n)}=T^n(l_i')$. Then, for the proof of the fact that $(V_1, V_2: F^{(n)})$ satisfies the rectangle condition, it is enough to show that

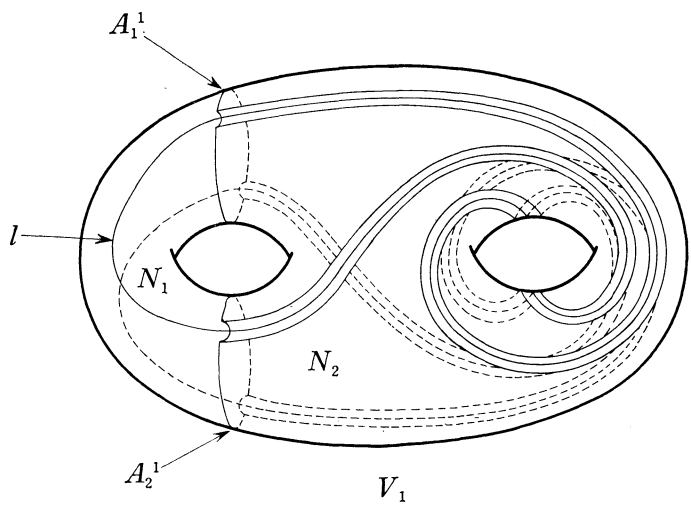
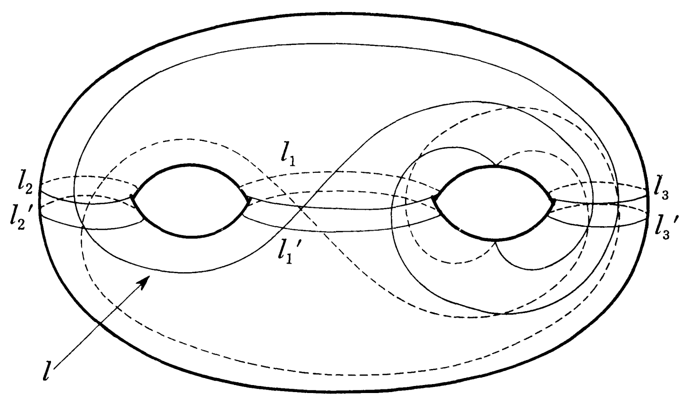


Figure 7.1



Pigure 7.2

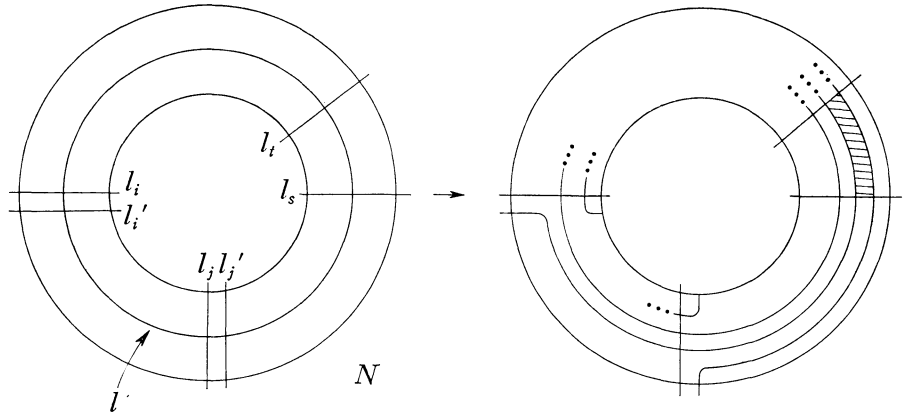


Figure 7.3

$\{l_1, l_2, l_3\}$ and $\{l_1^{(n)}, l_2^{(n)}, l_3^{(n)}\}$ are tight. Let N be a regular neighborhood of l in $F^{(n)}$. We may suppose that $T|_{F^{(n)}-\mathrm{Int}(N)}=id_{F^{(n)}-\mathrm{Int}(N)}$. Then, by seeing the configuration in N, we see that $\{l_1, l_2, l_2\}$ and $\{l_1^{(n)}, l_2^{(n)}, l_3^{(n)}\}$ are tight, provided $|n|>1$ (Figure 7.3).

EXAMPLE 2. Let V_1, A_j^1, l be as in Example 1. Let W be another genus two handlebody, A_3^1, A_4^1 be a pair of essential annuli properly embedded in W as in Figure 7.4, A_5^1 be an annulus embedded in ∂W as in Figure 7.4. Then we get a genus three handlebody V_1^3 from V_1 and W by identifying $N(l\colon \partial V_1)$ and A_5^1. We shall denote the image of A_j^1 $(j=1, \cdots, 5)$ in V_1^3 also by A_j^1. Let m_3 be the image of m in ∂V_1^3, where m is the simple loop on ∂W in Figure 7.4. Let V_2^3 be

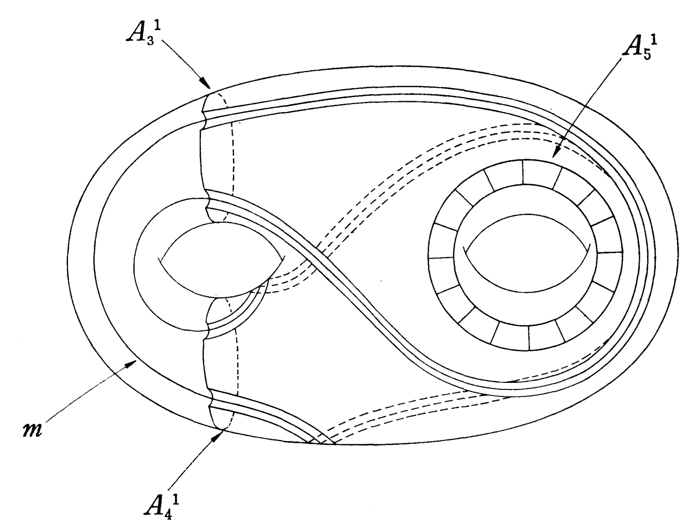


Figure 7.4

a copy of V_1^3, A_j^2 $(j=1, \cdots, 5)$ the annuli in V_2^3 corresponding to A_j^1 and h: $\partial V_1^3 \to \partial V_2^3$ the homeomorphism induced from the identification of V_1^3 and V_2^3. Let T_3: $\partial V_1^3 \to \partial V_1^3$ be the right hand Dehn twist along m_3. For each integer n, set $M_3^{(n)} = V_1^3 \underset{h \circ T_3^n}{\cup} V_2^3$. Then $A_i^1 \cup A_i^2$ $(i=1, \cdots, 5)$ becomes a torus $T_i^{(n)}$ in $M_3^{(n)}$. It is directly seen that $T_1^{(n)} \cup \cdots \cup T_5^{(n)}$ separates $M_3^{(n)}$ into four components $N_1^{(n)}$, $N_2^{(n)}$, $N_3^{(n)}$ and $N_4^{(n)}$, where $N_4^{(n)}$ admits a Seifert fibration with the orbit manifold an annulus and two exceptional fibers of index two, $N_i^{(n)}$ $(i=2, 3)$ admits a Seifert fibration with the orbit manifold a disk with two holes and no exceptional fiber and $N_i^{(n)}$ is homeomorphic to the exterior of $(2, 2n)$ torus link. It is easily seen that this gives a torus decomposition of $M^{(n)}$, provided $|n|>1$ such that the characteristic graph is as in Figure 7.5.

Let $F^{(n)} = \partial V_1^3 = \partial V_2^3 (\subset M^{(n)})$. We can show that the Heegaard splitting $(V_1^3, V_2^3: F^{(n)})$ satisfies the rectangle condition, provided $|n|>1$, by considering the simple loops $l_1, \cdots, l_6$ in Figure 7.6 and the argument in Example 1.

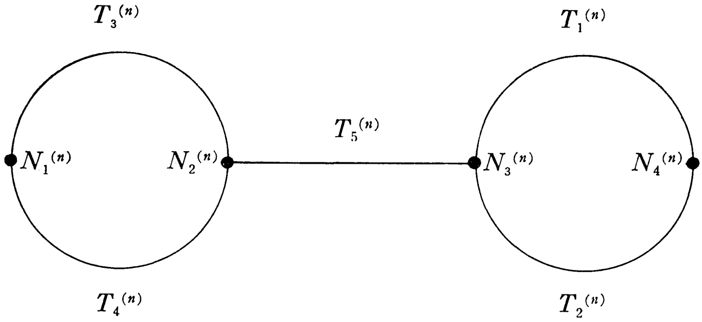


Figure 7.5

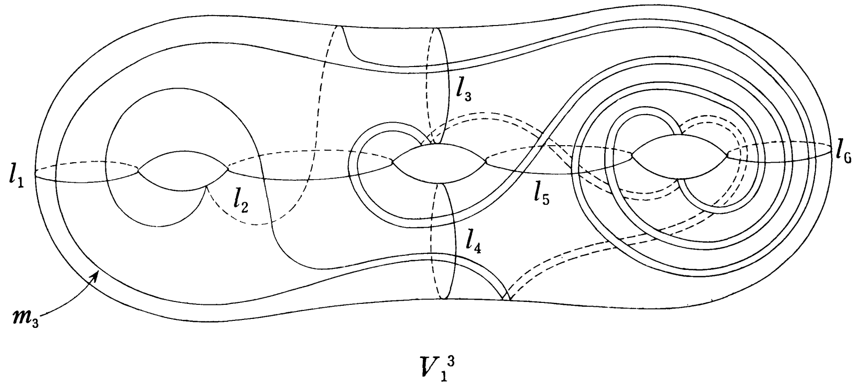


Figure 7.6

EXAMPLE 3 (general construction). We shall construct a family of examples for each $g(>2)$ inductively. The first step of the induction is Example

2. Then suppose that we have:

Induction hypothesis. Let V_1^{g-1} be a genus g-1 handlebody, $\mathcal{A}_1^{g-1}$ a union of $3g$–7, mutually disjoint, essential annuli in V_1^{g-1}, and let V_2^{g-1} be a copy of V_1^{g-1}, and $\mathcal{A}_2^{g-1}$ the union of annuli in V_2^{g-1} corresponding to $\mathcal{A}_1^{g-1}$ and $h: \partial V_1^{g-1} \to \partial V_2^{g-1}$ the homeomorphism induced from the identification of V_1^{g-1} and V_2^{g-1}. Suppose that there is a simple loop m_{g-1} on ∂V_1^{g-1}, which satisfies:

(i) $m_{g-1} \cap \mathcal{A}_1^{g-1} = \phi$,

(ii) let $T_{g-1}: \partial V_1^3 \to \partial V_1^3$ be the right hand Dehn twist along m_{g-1}. Set $M_{g-1}^{(n)} = V_1^{g-1} \underset{h \circ T_{g-1}^n}{\cup} V_2^{g-1}$. Then $M_{g-1}^{(n)}$ is a Haken manifold and $\mathcal{T}^{g-1} = \mathcal{A}_1^{g-1} \cup \mathcal{A}_2^{g-1}$ gives a torus decomposition of $M_{g-1}^{(n)}$ into $2g$–4 components and

(iii) there exists a union of mutually disjoint $3g$–6 disks $\mathcal{D}^{g-1}$ properly embedded in V_1^{g-1} such that $\partial \mathcal{D}^{g-1}$ cuts ∂V_1^{g-1} into $2g$–4 pants $P_1, \cdots, P_{2g-4}$ and, for each pair of boundary components of P_i, there is a subarc of m_{g-1} properly embedded in P_i, which joins the boundary components.

It is easy to see that the above condition (iii) together with the argument in Example 1 shows that the Heegaard splitting $(V_1^{g-1}, V_2^{g-1}: F^{(n)})$ of $M_{g-1}^{(n)}$ satisfies the rectangle condition provided $|n|>1$.

Construction. Let $W, A_3^1, A_4^1, A_5^1, m$ be as in Example 2. Then we get a genus g handlebody from V_1^{g-1} and W by identifying $N(m_{g-1}: \partial V_1^{g-1})$ and A_5^1. We shall denote the image of $A_3^1, A_4^1, A_5^1, \mathcal{A}_1^{g-1}$ in V_1^g also by $A_3^1, A_4^1, A_5^1, \mathcal{A}_1^{g-1}$. Then let $\mathcal{A}_1^g = A_3^1 \cup A_4^1 \cup A_5^1 \cup \mathcal{A}_1^{g-1}$ and $m_g (\subset \partial V_1^g)$ be the image of m. Let V_2^g be a copy of V_1^g and $\mathcal{A}_2^g$ the union of annuli in V_2^g corresponding to $\mathcal{A}_1^g$. Then it is easily checked that $V_i^g, \mathcal{A}_i^g, m_g$ satisfy the above conditions (i), (ii). Moreover, we easily find a union of mutually disjoint $3g$–3 disks $\mathcal{D}^g$, which satisfy the condition (iii). See Figure 7.7.

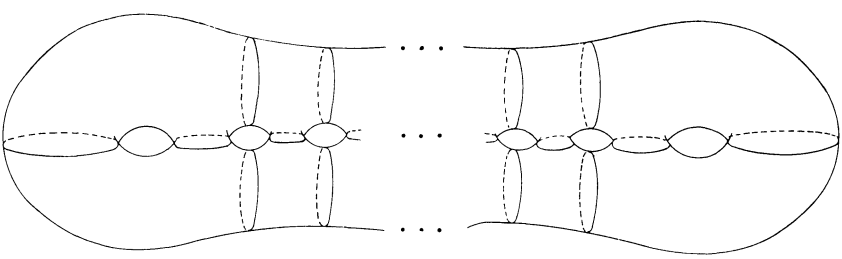


Figure 7.7

EXAMPLE 4. Let V be a genus two handlebody and τ the train track on ∂V as in Figure 7.8. We note that τ is complete, i.e. each component of $\partial V - \tau$ is a 3-gon. Hence τ determines an open set of the projective lamination space of ∂V [2]. Let l be a simple loop which is carried by τ with all weights positive.

Then it is easy to see that l is sufficiently complicated with respect to $\{D_1, D_2, D_3\}$ in Figure 7.8. Let V' be a copy of V and $h: \partial V \to \partial V'$ the homeomorphism induced from the identification. Let $T: \partial V \to \partial V$ be the Dehn twist along l. Then, by seeing the configuration of $T^n(\partial D_1 \cup \partial D_2 \cup \partial D_3)$ and $\partial D_1 \cup \partial D_2 \cup \partial D_3$ in a regular neighborhood of l in ∂V, we see that the Heegaard splitting $(V, V': F)$ of the manifold $V \underset{h \circ T^n}{\cup} V'$ satisfies the strong rectangle condition if $|n|$ is sufficiently large (Figure 7.3). Moreover it is easily verified that if all the weights are greater than two then $(V, V': F)$ satisfies the strong rectangle condition provided $|n| \neq 0$.

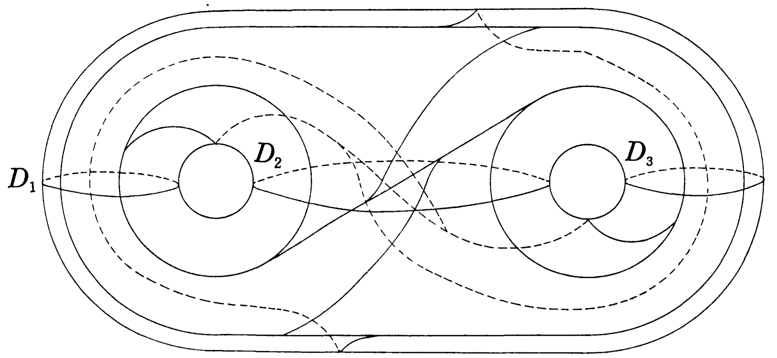


Figure 7.8

EXAMPLE 5. Let $(V, V': F)$ be a genus two Heegaard splitting of the 3-sphere S^3. We draw a picture of F as in Figure 7.9. Let τ be the complete

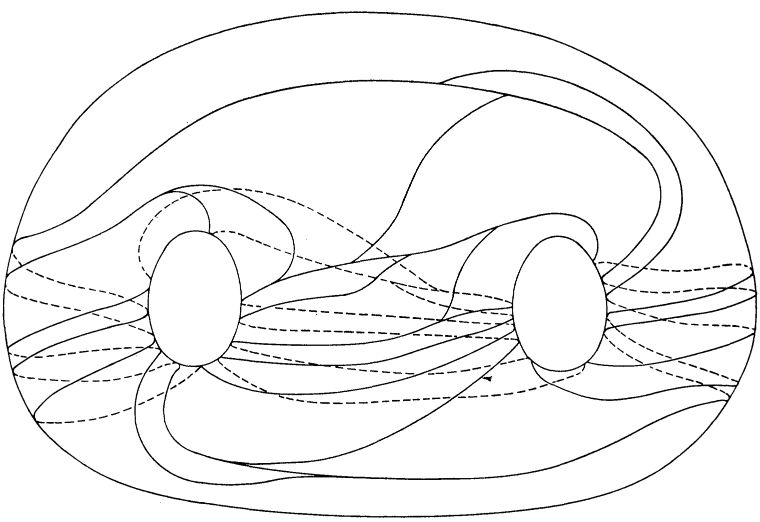


Figure 7.9

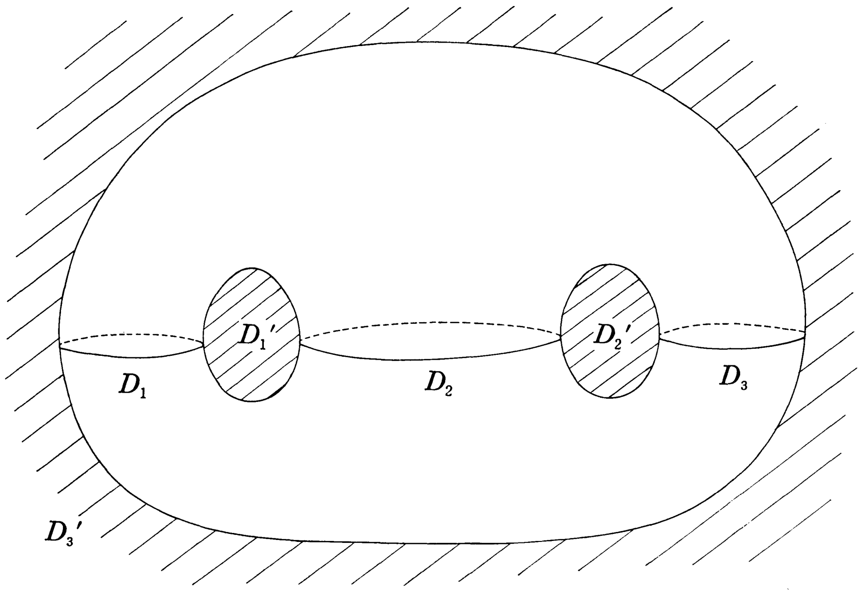


Figure 7.10

train track on F as in Figure 7.9 and $\{D_1, D_2, D_3\}$ ($\{D'_1, D'_2, D'_3\}$ resp.) be a system of disks in V (V' resp.) as in Figure 7.10. We note that Figure 7.9 is obtained from Figure 7.8 by applying Dehn twists twice along ∂D_1, ∂D_2 and ∂D_3 in Figure 7.8. Let l be a simple loop which is carried by τ with all weights positive. Then l is sufficiently complicated with respect to $\{D_1, D_2, D_3\}$ and $\{D'_1, D'_2, D'_3\}$ in Figure 7.10. Hence, by Theorem 4, l is a hyperbolic knot.

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Department of Mathematics
Osaka University,
Toyonaka, Osaka, 560
Japan