Uchida, F.
Osaka J. Math.
25 (1988), 343-352

CERTAIN ASPECTS OF TWISTED LINEAR ACTIONS

Dedicated to Professor Hirosi Toda on his 60th birthday

FUICHI UCHIDA*)

(Received February 18, 1987)

0. Introduction

In the previous paper [2], we have introduced the concept of a twisted linear action which is an analytic action of a non-compact Lie group on a sphere, and we have shown as an example that there have been uncountably many topologically distinct analytic actions of $\boldsymbol{SL}(n, \boldsymbol{R})$ on the $(2n-1)$-sphere.

In this paper, we shall show another aspect of twisted linear actions. In particular, we shall show that there are uncountably many C^1-differentiably distinct but topologically equivalent analytic actions of $\boldsymbol{SL}(n, \boldsymbol{R})$ on a k-sphere for each $k \geqq n \geqq 2$.

1. Twisted linear actions

Throughout this paper, a matrix means only the one with real coefficients.

1.1. Let $\boldsymbol{u}=(u_i)$ and $\boldsymbol{v}=(v_i)$ be column vectors in $\boldsymbol{R}^n$. As usual, we define their inner product by $\boldsymbol{u}\cdot\boldsymbol{v}=\sum_i u_i v_i$ and the length of $\boldsymbol{u}$ by $||\boldsymbol{u}||=\sqrt{\boldsymbol{u}\cdot\boldsymbol{u}}$. Let $M=(m_{ij})$ be a square matrix of degree n. We say that M satisfies the condition $(\boldsymbol{T})$ if the quadratic form

$$\boldsymbol{x}\cdot M\boldsymbol{x} = \sum_{i,j} m_{ij}\, x_i\, x_j$$

is positive definite. It is easy to see that M satisfies $(\boldsymbol{T})$ if and only if

$$(\boldsymbol{T}')\quad \frac{d}{dt}\,||\exp\,(tM)\,\boldsymbol{x}|| > 0 \quad \text{for each} \quad \boldsymbol{x}\in\boldsymbol{R}_0^n = \boldsymbol{R}^n - \{0\},\ t\in\boldsymbol{R}\,.$$

If M satisfies $(\boldsymbol{T}')$, then

$$\lim_{t\to+\infty} ||\exp(tM)\,\boldsymbol{x}|| = +\infty \quad\text{and}\quad \lim_{t\to-\infty} ||\exp(tM)\,\boldsymbol{x}|| = 0$$

for each $\boldsymbol{x}\in\boldsymbol{R}_0^n$, and hence there exists a unique real valued analytic function τ

---

*) Partly supported by the Grants-in-Aid for Scientific as well as Co-operative Research, The Ministry of Education, Science and Culture, Japan.

on $\boldsymbol{R}_0^n$ such that

$$||\exp(\tau(\boldsymbol{x})\, M)\, \boldsymbol{x}|| = 1 \quad \text{for} \quad \boldsymbol{x} \in \boldsymbol{R}_0^n .$$

Therefore, we can define an analytic mapping π^M of $\boldsymbol{R}_0^n$ onto the unit $(n-1)$-sphere S^{n-1} by

$$\pi^M(\boldsymbol{x}) = \exp(\tau(\boldsymbol{x})\, M)\, \boldsymbol{x} \quad \text{for} \quad \boldsymbol{x} \in \boldsymbol{R}_0^n ,$$

if M satisfies the condition $(\boldsymbol{T})$.

1.2. Let G be a Lie group, $\rho: G \to \boldsymbol{GL}(n, \boldsymbol{R})$ a matricial representation, and M a square matrix of degree n satisfying $(\boldsymbol{T})$. We call (ρ, M) a TC-pair of degree n, if $\rho(g)\, M = M \rho(g)$ for each $g \in G$. For a TC-pair (ρ, M) of degree n, we can define an analytic mapping

$$\xi: G \times S^{n-1} \to S^{n-1} \quad \text{by} \quad \xi(g, x) = \pi^M(\rho(g)\, x) ,$$

and we see that ξ is an analytic G-action on S^{n-1}. We call $\xi = \xi^{(\rho, M)}$ a twisted linear action of G on S^{n-1} determined by the TC-pair (ρ, M), and we say that ξ is associated to the matricial representation ρ.

1.3. For a given Lie group G, we introduce certain equivalence relations on TC-pairs. Let (ρ, M) and (σ, N) be TC-pairs of degree n. We say that (ρ, M) is algebraically equivalent to (σ, N) if there exist $A \in \boldsymbol{GL}(n, \boldsymbol{R})$ and a positive real number c satisfying

$$(*) \quad cN = AMA^{-1} \quad \text{and} \quad \sigma(g) = A\rho(g)\, A^{-1} \quad \text{for each} \quad g \in G .$$

We say that (ρ, M) is C^r-equivalent to (σ, N) if there exists a C^r-diffeomorphism f of S^{n-1} onto itself such that the following diagram is commutative:

$$\begin{array}{ccc} G \times S^{n-1} & \xrightarrow{1 \times f} & G \times S^{n-1} \\ \downarrow \xi^{(\rho, M)} & & \downarrow \xi^{(\sigma, N)} \\ S^{n-1} & \xrightarrow{f} & S^{n-1} . \end{array}$$

We call f a G-equivariant C^r-diffeomorphism.

Lemma. *If (ρ, M) is algebraically equivalent to (σ, N), then (ρ, M) is C^∞-equivalent to (σ, N).*

Proof. It has been proved in the previous paper [2], but we give a proof for completeness. Suppose that there exist $A \in \boldsymbol{GL}(n, \boldsymbol{R})$ and a positive real number c satisfying $(*)$. Define analytic mappings h_A and k_A of S^{n-1} into itself by

$$h_A(x) = \pi^N(Ax) \quad \text{and} \quad k_A(y) = \pi^M(A^{-1}y) .$$

Then the composites $h_A k_A$ and $k_A h_A$ are the identity mapping on S^{n-1} by the condition $cN=AMA^{-1}$, and hence h_A is a C^ω-diffeomorphism. Furthermore, the equality

$$h_A(\xi^{(\rho,M)}(g, x)) = \xi^{(\sigma,N)}(g, h_A(x))$$

holds for each $g \in G$ and $x \in S^{n-1}$, by the condition (*). q.e.d.

Theorem ([2], Theorem 3.3). *Let G be a compact Lie group and $\rho: \boldsymbol{G} \to \boldsymbol{GL}(n, \boldsymbol{R})$ a matricial representation. Then any TC-pairs (ρ, M) and (ρ, N) are C^ω-equivalent.*

2. First typical examples

Here we shall study twisted linear actions of $G=\boldsymbol{SL}(n, \boldsymbol{R})$ on the $(nk-1)$-sphere associated to a representation $\rho=\rho_n \otimes I_k$, that is, $\rho(A)=A \otimes I_k$.

2.1. Let A and $B=(b_{ij})$ be square matrices of degrees n and k, respectively. Denote by $A \otimes B$ the Kronecker product written in the form

$$A \otimes B = \begin{pmatrix} b_{11}A & \cdots & b_{1k}A \\ \vdots & & \vdots \\ b_{k1}A & \cdots & b_{kk}A \end{pmatrix} .$$

Let $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_k$ be column vectors in $\boldsymbol{R}^n$. Then the correspondence

$$(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_k) \to \begin{pmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_k \end{pmatrix}$$

defines a linear isomorphism $\iota: M(n, k; \boldsymbol{R}) \to \boldsymbol{R}^{nk}$. Let X and Y be $n \times k$ matrices. As usual, we define their inner product by

$$\langle X, Y \rangle = \operatorname{trace}({}^t XY) ,$$

and the length of X by $\|X\| = \sqrt{\langle X, X \rangle}$. Then ι is an isometry. Furthermore, the equality

$$(A \otimes B)\, \iota(X) = \iota(AX\,{}^tB)$$

holds, where A and B are square matrices of degrees n and k, respectively, and X is an $n \times k$ matrix. In the following, we shall identify $\boldsymbol{R}^{nk}$ with $M(n, k; \boldsymbol{R})$ via the isometry ι.

2.2. We obtain the following lemma directly.

Lemma 2.2. *Let $\bar{M}$ be a square mtarix of degree nk. Then*

$$\bar{M}(A\otimes I_k) = (A\otimes I_k)\bar{M}$$

for each $A\in \boldsymbol{SL}(n, \boldsymbol{R})$, if and only if $\bar{M}=I_n\otimes M$ for some square matrix M of degree k. Furthermore, $I_n\otimes M$ satisfies the condition $(\boldsymbol{T})$ if and only if M satisfies $(\boldsymbol{T})$.

Consequently, $(\rho_n\otimes I_k, I_n\otimes M)$ is a TC-pair for any square matrix M of degree k satisfying $(\boldsymbol{T})$, and any TC-pair $(\rho_n\otimes I_k, \bar{M})$ is written in such a form. Furthermore, TC-pairs $(\rho_n\otimes I_k, I_n\otimes M)$ and $(\rho_n\otimes I_k, I_n\otimes N)$ are algebraically equivalent, if and only if there exist $A\in \boldsymbol{GL}(k, \boldsymbol{R})$ and a positive real number c satisfying $cN=AMA^{-1}$.

2.3. Let M be a square matrix of degree k satisfying $(\boldsymbol{T})$. Denote by ζ^M the twisted linear $\boldsymbol{SL}(n, \boldsymbol{R})$ action on the $(nk-1)$-sphere determined by the TC-pair $(\rho_n\otimes I_k, I_n\otimes M)$. Identifying $\boldsymbol{R}^{nk}$ with $M(n, k; \boldsymbol{R})$ via the isometry ι, we can describe

$$\zeta^M\colon \boldsymbol{SL}(n, \boldsymbol{R})\times S^{nk-1} \to S^{nk-1}$$

as follows. That is, S^{nk-1} can be viewed as the set of all $n\times k$ matrices X with $||X||=1$, and ζ^M is written in the form

$$\zeta^M(A, X) = AX\exp(\theta^t M)$$

for a real number θ which is uniquely determined by the condition

$$||AX\exp(\theta^t M)|| = 1\,.$$

Let $I(M)$ and $O(M)$ denote the isotropy group at

$$\frac{1}{\sqrt{k}}\begin{pmatrix} I_k \\ 0 \end{pmatrix}$$

and the orbit through that point, respectively, with respect to the twisted linear action ζ^M. We obtain the following lemma.

Lemma 2.3. *Suppose $n>k\geqq 2$. Then the isotropy group $I(M)$ is written in the form*

$$I(M) = \left\{ \left(\begin{array}{c|c} \exp(\theta^t M) & * \\ \hline 0 & * \end{array}\right) : \theta\in \boldsymbol{R} \right\}$$

and the orbit $O(M)$ is an open dense subset consisting of all $n\times k$ matrices X with rank $X=k$ and $||X||=1$.

2.4. Suppose that $n>k\geqq 2$ and there exists an $\boldsymbol{SL}(n, \boldsymbol{R})$-equivariant homeomorphism f of S^{nk-1} with a twisted linear action ζ^M onto S^{nk-1} with a twisted linear action ζ^N. Then we obtain $f(O(M))=O(N)$, and hence $I(M)$ and $I(N)$

are conjugate in $\boldsymbol{SL}(n, \boldsymbol{R})$. Finally, we see that there exist $A\in \boldsymbol{GL}(k, \boldsymbol{R})$ and a positive real number c satisfying $cN=AMA^{-1}$, by making use of the fact that M and N satisfy the condition $(\boldsymbol{T})$ and the group $I(M)$ contains a subgroup written in the form

$$\left\{\left(\begin{array}{c|c} I_k & * \\ \hline 0 & I_{n-k} \end{array}\right)\right\}.$$

Summing up the above discussion, we obtain the following result.

Theorem 2.4. *Suppose $n>k\geqq 2$. Then any two of TC-pairs in the form $(\rho_n\otimes I_k, \bar{M})$ are algebraically equivalent if and only if they are C^0-equivalent.*

Consequently, we see that if $n>k\geqq 2$ then there are uncountably many topoloically distinct twisted linear actions of $\boldsymbol{SL}(n, \boldsymbol{R})$ on S^{nk-1} associated to the matricial representation $\rho_n\otimes I_k$. This is a generalization of a result studied in the previous paper [2].

3. Second typical examples

Here we shall stduy twisted linear actions of $G=\boldsymbol{SL}(n, \boldsymbol{R})$ on the $(n+k-1)$-sphere associated to a representation $\rho=\rho_n\oplus I_k$, that is, $\rho(A)=A\oplus I_k$.

3.1. Let A and B be square matrices of degrees n and k, respectively. We denote by $A\oplus B$ the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

of degree $n+k$. We obtain the following lemma.

Lemma 3.1. *Let $n\geqq 2$ and $k\geqq 1$. Let $\bar{M}$ be a square matrix of degree $n+k$. Then*

$$\bar{M}(A\oplus I_k) = (A\oplus I_k)\,\bar{M}$$

for each $A\in \boldsymbol{SL}(n, \boldsymbol{R})$, if and only if $\bar{M}=cI_n\oplus M$ for some square matrix M of degree k and a real number c. Furthermore, $\bar{M}=cI_n\oplus M$ satisfies the condition $(\boldsymbol{T})$, if and only if c is positive and M satisfies $(\boldsymbol{T})$.

3.2. Let M be a square matrix of degree k satisfying $(\boldsymbol{T})$. Denote by χ^M the twisted linear $\boldsymbol{SL}(n, \boldsymbol{R})$ action on the $(n+k-1)$-sphere determined by the TC-pair $(\rho_n\oplus I_k, I_n\oplus M)$. Then χ^M is written in the form

$$\chi^M(A, \boldsymbol{u}\oplus \boldsymbol{v}) = e^{\theta}A\boldsymbol{u}\oplus e^{\theta M}\boldsymbol{v}$$

for a real number θ which is uniquely determined by the condition

$$||e^\theta A\boldsymbol{u}||^2+||e^{\theta M}\boldsymbol{v}||^2 = 1\,,$$

where $\boldsymbol{u}$ is a column vector in $\boldsymbol{R}^n$ and $\boldsymbol{v}$ is a column vector in $\boldsymbol{R}^k$ satisfying $||\boldsymbol{u}||^2+||\boldsymbol{v}||^2=1$.

3.3. Let us define closed subgroups $L(n)$ and $N(n)$ of $\boldsymbol{SL}(n, \boldsymbol{R})$ by the forms

$$L(n) = \left\{ \left(\begin{array}{c|c} 1 & * \cdots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array} \right) \right\},\ N(n) = \left\{ \left(\begin{array}{c|c} \lambda & * \cdots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array} \right) : \lambda>0 \right\}.$$

Denote by $F(M)$ the fixed point set of $L(n)$ with respect to the twisted linear action χ^M. Then we obtain the following lemma.

Lemma 3.3. *With respect to the twisted linear action* χ^M,

$$F(M) = \{a\boldsymbol{e}_1\oplus\boldsymbol{v}\colon a^2+||\boldsymbol{v}||^2 = 1\}\,,$$

where $\boldsymbol{e}_1={}^t(1, 0, \cdots, 0)\in\boldsymbol{R}^n$. *The isotropy group at* $\boldsymbol{0}\oplus\boldsymbol{v}$ *coincides with* $\boldsymbol{SL}(n, \boldsymbol{R})$, *the one at* $\pm\boldsymbol{e}_1\oplus\boldsymbol{0}$ *coincides with* $N(n)$, *and if* $a||\boldsymbol{v}||\neq0$, *then the one at* $a\boldsymbol{e}_1\oplus\boldsymbol{v}$ *coincides with* $L(n)$.

3.4. Notice that the normalizer $N(L(n))$ of $L(n)$ acts on $F(M)$ naturally via χ^M, the identity component of $N(L(n))$ coincides with $N(n)$, and the factor group $N(L(n)/L(n)$ is naturally isomorphic to the multiplicative group $\boldsymbol{R}^\times$ consisting of non-zero real numbers.

Let us investigate the induced $N(L(n))/L(n)$ action on $F(M)$ via χ^M. Leaving fixed any point $a\boldsymbol{e}_1\oplus\boldsymbol{v}$ of $F(M)$ satisfying $a||\boldsymbol{v}||\neq0$, we have a real valued analytic function $\theta=\theta(\alpha)$ on $\boldsymbol{R}^\times$ determined by

$$\chi^M\left(\left(\begin{array}{c|c} \alpha & * \cdots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array} \right),\ a\boldsymbol{e}_1\oplus\boldsymbol{v} \right) = e^\theta\alpha\, a\boldsymbol{e}_1\oplus e^{\theta M}\boldsymbol{v}$$

and $(e^\theta\alpha a)^2+||e^{\theta M}\boldsymbol{v}||^2=1$. Then $\theta(-\alpha)=\theta(\alpha)$ and

$$\frac{d\theta}{d\alpha}<0<\frac{d}{d\alpha}\,(e^\theta\alpha)$$

for $\alpha>0$. Furthermore, we obtain

$$\lim_{\alpha\to+\infty}\theta(\alpha) = -\infty, \quad \lim_{\alpha\to+\infty} e^\theta\alpha = |a|^{-1}, \quad \lim_{\alpha\to+\infty}||e^{\theta M}\boldsymbol{v}|| = 0\,,$$

and

$$\lim_{\alpha\to 0+} e^{\theta}\alpha = 0,\ \lim_{\alpha\to 0+} e^{\theta M}v = \pi^M(v)\,.$$

3.5. Here we shall show the following result.

Theorem 3.5. *Let M, N be any square matrices of degree k satisfying the condition ($\boldsymbol{T}$). Then there exists an $\boldsymbol{SL}(n, \boldsymbol{R})$-equivariant homeomorphism f of S^{n+k-1} with a twisted linear action χ^M onto S^{n+k-1} with a twisted linear action χ^N.*

Proof. By the above investigation, we can construct uniquely an $N(L(n))/L(n)$-equivariant homeomorphism f_0 of $F(M)$ onto $F(N)$ satisfying the following conditions

$$f_0(ae_1\oplus v) = ae_1\oplus v \quad \text{for} \quad |a|=1 \quad \text{or} \quad 1/\sqrt{2}\,,$$

and

$$f_0(0\oplus\pi^M(v)) = 0\oplus\pi^N(v) \quad \text{for} \quad \|v\| = 1/\sqrt{2}\,.$$

Next we consider the following diagram

$$\begin{array}{ccc} \boldsymbol{SO}(n)\times F(M) & \xrightarrow{\psi_1} & S^{n+k-1} \\ \big\downarrow 1\times f_0 & & \big\downarrow f \\ \boldsymbol{SO}(n)\times F(N) & \xrightarrow{\psi_2} & S^{n+k-1}\,, \end{array}$$

where

$$\psi_1(K, x) = \chi^M(K, x) = (K\oplus I_k)\,x\,,$$
$$\psi_2(K, x) = \chi^N(K, x) = (K\oplus I_k)\,x\,.$$

By the construction of f_0, we see that $\psi_1(K, x)=\psi_1(K', x')$ if and only if $\psi_2(K, f_0(x))=\psi_2(K', f_0(x'))$, and hence we obtain a unique bijection f of S^{n+k-1} onto itself satisfying

$$f\circ\psi_1 = \psi_2\circ(1\times f_0)\,.$$

Then f is a homeomorphism, because ψ_1 and ψ_2 are closed continuous mappings. Finally, we show that f is $\boldsymbol{SL}(n, \boldsymbol{R})$-equivariant. Let $A\in\boldsymbol{SL}(n, \boldsymbol{R})$, $K\in\boldsymbol{SO}(n)$ and $x\in F(M)$. Then, there are $B\in\boldsymbol{SO}(n)$ and $U\in N(n)$ such that $AK=BU$, and hence

$$\begin{aligned} f(\chi^M(A, \psi_1(K, x))) &= f(\chi^M(AK, x)) = f(\chi^M(BU, x)) \\ &= f(\psi_1(B, \chi^M(U, x))) = \psi_2(B, f_0(\chi^M(U, x))) \\ &= \psi_2(B, \chi^N(U, f_0(x))) = \chi^N(BU, f_0(x)) \\ &= \chi^N(AK, f_0(x)) = \chi^N(A, \psi_2(K, f_0(x))) \\ &= \chi^N(A, f(\psi_1(K, x)))\,. \end{aligned}$$

Consequently, we see that f is an $\boldsymbol{SL}(n, \boldsymbol{R})$-equivariant homeomorphism of S^{n+k-1} with the action χ^M onto S^{n+k-1} with the action χ^N. q.e.d.

3.6. Next we shall show the following result.

Theorem 3.6. *Let M, N be square matrices of degree k satisfying the condition $(\boldsymbol{T})$. If there exists an $\boldsymbol{SL}(n, \boldsymbol{R})$-equivariant C^1-diffeomorphism f of S^{n+k-1} with a twisted linear action χ^M onto S^{n+k-1} with a twisted linear action χ^N, then*

$$N = PMP^{-1}$$

for some $P \in \boldsymbol{GL}(k, \boldsymbol{R})$.

Proof. By the existence of such an equivariant C^1-diffeomorphism f, we obtain an $N(L(n))/L(n)$-equivariant C^1-diffeomorphism f_0: $F(M) \to F(N)$. Considering points whose isotropy groups coincide with $N(n)/L(n)$, we can assume

$$f_0(\boldsymbol{e}_1 \oplus \boldsymbol{0}) = \boldsymbol{e}_1 \oplus \boldsymbol{0} .$$

Then we obtain an isomorphism

$$df_0\colon T_{e_1 \oplus 0} F(M) \to T_{e_1 \oplus 0} F(N)$$

of tangential representation spaces of the isotropy group $N(n)/L(n)$.

Here we consider the representation space $T_{e_1 \oplus 0} F(M)$. Denote by $F(M)_+$ an open subset of $F(M)$ consisting of $a\boldsymbol{e}_1 \oplus \boldsymbol{v}$ with $a > 0$, and define

$$\psi^M\colon F(M)_+ \to \boldsymbol{R}^k \quad \text{by} \quad \psi^M(a\boldsymbol{e}_1 \oplus \boldsymbol{v}) = \exp(-(\log a)\, M)\, \boldsymbol{v} .$$

Then ψ^M is a C^ω-diffeomorphism satisfying $\psi^M(e_1 \oplus \boldsymbol{0}) = \boldsymbol{0}$. Considering the C^ω-diffeomorphism ψ^M, we see that the tangential representation of the isotropy group $N(n)/L(n) \cong \boldsymbol{R}$ on $T_{e_1 \oplus 0} F(M)$ is equivalent to a representation

$$\sigma^M\colon \boldsymbol{R} \to \boldsymbol{GL}(k, \boldsymbol{R}) \quad \text{defined by} \quad \sigma^M(\lambda) = \exp(-\lambda M) .$$

The existence of the isomorphism df_0 of tangential representation spaces assures that the representations σ^M and σ^N are equivalent, and hence the equality $N = PMP^{-1}$ holds for some $P \in \boldsymbol{GL}(k, \boldsymbol{R})$. q.e.d.

Notice that the twisted linear actions χ^M are new concrete examples for analytic $\boldsymbol{SL}(n, \boldsymbol{R})$-actions on a sphere investigated in [1].

4. Concluding remark

With respect to the first typical examples, we obtain a classification theorem only for the case $n > k \geq 2$ in §2. It seems to be difficult to obtain a similar result for the remaining case $k \geq n \geq 2$ in general. Here we consider the case $n = k = 3$.

The following matrices satisfy the condition ($\boldsymbol{T}$).

$$\text{(Type 1)} \quad M_1(a,b)=\begin{pmatrix}1&0&0\\0&a&0\\0&0&b\end{pmatrix};\ 1\leqq a\leqq b$$

$$\text{(Type 2)} \quad M_2(a,b)=\begin{pmatrix}1&a&0\\-a&1&0\\0&0&b\end{pmatrix};\ a>0,\ b>0$$

$$\text{(Type 3)} \quad M(a)=\begin{pmatrix}1&1&0\\0&1&0\\0&0&a\end{pmatrix};\ a>0$$

$$\text{(Type 4)} \quad M_0=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}.$$

Furthermore, if a matrix M of degree 3 satisfy the condition ($\boldsymbol{T}$), then M is similar to only one of the above matrices up to positive scalar multiplication. Here we say that M is similar to N up to positive scalar multiplication if there exist a non-singular matrix A and a positive real number c such that $AMA^{-1}=cN$.

Denote by $S(M)$ the 8-sphere with the twisted linear $\boldsymbol{SL}(3, \boldsymbol{R})$ action ζ^M (see §2.3), where M is a square matrix of degree 3 satisfying the condition ($\boldsymbol{T}$). We obtain the following result.

Theorem. (0) *If $S(M)$ and $S(M')$ are equivariantly C^1-diffeomorphic, then M is similar to M' up to positive scalar multiplication.*

(1) *If $S(M)$ and $S(M')$ are equivariantly homeomorphic, then M and M' have the same type in the above sense.*

(2) *If $S(M_1(a, b))$ and $S(M_1(a', b'))$ are equivariantly homeomorphic, then $(a', b')=(a, b)$ or $(a', b')=(a^{-1}b, b)$.*

(3) *If $S(M_2(a, b))$ and $S(M_2(a', b'))$ are equivariantly homeomorphic, then $a=a'$.*

(4) *If $S(M(a))$ and $S(M(a'))$ are equivariantly homeomorphic, then $a=a'$ or $aa'=1$.*

Proof. We give only an outline of the proof. The fixed point set $S(M)^{L(3)}$ of the restricted $L(3)$-action is a 2-sphere and the fixed point set $S(M)^{N(3)}$ of the restricted $N(3)$-action is a disjoint union of low dimensional spheres, where $L(3)$ and $N(3)$ are closed subgroups of $\boldsymbol{SL}(3, \boldsymbol{R})$ defined in §3.3.

If we consider homeomorphism classes of $S(M)^{N(3)}$, we can distinguish a matrix of (Type i) from that of (Type j) except for the case $(i, j)=(2, 4)$. Furthermore, we can prove (0) by considering a tangential representation of $N(3)/L(3)$ on the tangent space of the 2-sphere $S(M)^{L(3)}$ at isolated fixed points of the restricted $N(3)$-action.

Denote by $H(P)$ a closed subgroup of $\boldsymbol{SL}(3, \boldsymbol{R})$ consisting of all matrices in the form

$$\left(\begin{array}{c|c} e^{\theta P} & * \\ \hline 0 & * \end{array}\right), \quad \theta \in \boldsymbol{R}$$

where P is a square matrix of degree 2. We can prove the remaining part of the theorem by considering homeomorphism classes of the fixed point sets $S(M)^{H(P)}$ of the restricted $H(P)$-action. For $P=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that $S(M_0)^{H(P)}$ is a 1-sphere but $S(M_2(a, b))^{H(P)}$ is a 0-sphere, and hence we can distinguish M_0 from any matrix of (Type 2). By $P=\begin{pmatrix} 1 & c \\ -c & 1 \end{pmatrix}$ for $c>0$, we can prove (3). By $P=\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$ for $c>0$, we can prove (2) and (4). q.e.d.

References

[1] F. Uchida: *Real analytic* SL(n,R) *actions on spheres,* Tohoku Math. J. **33** (1981), 145–175.

[2] F. Uchida: *On a method to construct analytic actions of non-compact Lie groups on a sphere,* Tohoku Math. J. **39** (1987), 61–69.

Department of Mathematics
Faculty of Science
Yamagata University
Yamagata 990
Japan