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FIXED-POINT FREE SU(n)-ACTIONS ON ACYCLIC MANIFOLDS

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0. Introduction

Group actions without fixed points are called fixed-point free actions. The existence of fixed-point free actions on acyclic manifolds has been studied by R. Oliver in [9] and [10]. In [10], he showed that, for any compact Lie group G, if the identity component G_0 is non-abelian, or G/G_0 is non-solvable, then there is a fixed-point free smooth G-action on a disk (and hence on a compact $\boldsymbol{Z}_p$-acyclic smooth manifold). So, for such a G, it is natural to ask the following question: What is the minimum dimension of disks (or compact $\boldsymbol{Z}_p$-acyclic smooth manifolds) with fixed-point free smooth G-actions? On the question, R. Oliver proved in [11] that when $G=\mathrm{SO}(3)$, the minimum dimension of such disks is 8. But, for any compact connected simple Lie group of rank≥ 2, the minimum dimension of such disks (or compact $\boldsymbol{Z}_p$-acyclic smooth manifolds) has not been determined.

The object of this paper is to investigate the minimum dimension of compact $\boldsymbol{Z}_p$-acyclic (p; prime) smooth manifolds with fixed-point free smooth $\mathrm{SU}(p+1)$-actions. Our main results for the object are Theorems A and B below. The notation $X_m(p+1, p)$ in those theorems denotes a compact $\boldsymbol{Z}_p$-acyclic smooth manifold with a fixed-point free smooth $\mathrm{SU}(p+1)$-action, which is constructed in Example 2.3 (Section 2). From the construction, the $\mathrm{SU}(p+1)$-action on $X_m(p+1, p)$ satisfies all assumptions in Theorem A. So, from Theoerm A, we see that $X_m(p+1, p)$ ($p\neq 3$) has the minimum dimension in all compact $\boldsymbol{Z}_p$-acyclic smooth manifolds which have fixed-point free smooth $\mathrm{SU}(p+1)$-actions satisfying the assumptions in Theorem A. In particular, Theorem B shows that, without any assumption on orbits, $X_m(3, 2)$ has the minimum dimension in all compact $\boldsymbol{Z}_2$-acyclic smooth manifolds with fixed-point free smooth $\mathrm{SU}(3)$-actions. In Theorem A, the group $\tilde{S}_p$ denotes the inverse image of the subgroup $S_p= S_p\times 1\subset S_{p+1}$ (symmetric group of $p+1$ letters) by the natural projection from $N(\mathrm{S}T_{p+1})$ onto $N(\mathrm{S}T_{p+1})/\mathrm{S}T_{p+1}\cong S_{p+1}$, where $N(\mathrm{S}T_{p+1})$ is the normalizer of the maximal torus $\mathrm{S}T_{p+1}$ of $\mathrm{SU}(p+1)$ specified in 1.1. The group $\tilde{S}_{p+1}$ coincides with $N(\mathrm{S}T_{p+1})$ itself. For the other notations, see the list at the end of Introduction.

Theorem A. *Let p $(p\neq 3)$ be a prime number and let X be a $\boldsymbol{Q}$-acyclic smooth manifold with a fixed-point free smooth* SU$(p+1)$*-action. If* $X^{S(U(p)\times U(1))}\neq\Phi$, $X^{\tilde{S}_{p+1}}\neq\Phi$ *and* $X^{\tilde{S}_p}$ *is connected, then dim* $X\geq\frac{1}{4}(p+1)^2(p+4)p-1=$ *dim* $X_m(p+1,p)$ *holds.*

Theorem B. *Let X be a compact $\boldsymbol{Z}_2$-acyclic smooth manifold with a fixed-point free smooth* SU(3)*-action. Then dim* $X\geq 26=$ *dim* $X_m(3,2)$ *holds.*

In Section 1, we recall weight systems of G-manifolds and state some results about induced representations, which are useful to prove our main results. In 2.1 (Section 2), under the condition $p<n<2p$ (p; prime), we give a method to construct compact $\boldsymbol{Z}_p$-acyclic smooth manifolds with fixed-point free smooth SU(n)-actions. By the method, for fixed n and p, we can construct many different compact $\boldsymbol{Z}_p$-acyclic SU(n)-manifolds without fixed points. We will write all of them by the same notation $X(n,p)$. And in 2.3 (Section 2), we give a typical and useful example of $X(n,p)$ denoted by $X_m(n,p)$. When $n=p+1$, the manifold $X_m(n,p)$ just coincides with $X_m(p+1,p)$ in Theorems A and B. In Section 3, we prove that if $n=p+1$ $(p\neq 3)$, then $X_m(n,p)$ has the minimum dimension in $X(n,p)$'s (see Theorem C). And then we prove Theorem A. Lastly, in Section 4 we give the proof of Theorem B.

From now on, manifolds and actions mean smooth manifolds and smooth actions. And notations that are not defined in the later sections can be found in the list below:

(1) Let G be a compact group.

$RRep(G)$=the set of real representations of G,

$CRep(G)$=the set of complex representations of G,

$Irr(G)$=the set of irreducible complex representations of G,

$\hat{G}$=the set of equivalence classes of all irreducible complex representations of G.

(2) Let H be a closed subgroup of G above. For $\varphi\in RRep(G)$ (or $CRep(G)$) and $\psi\in RRep(H)$ (or $CRep(H)$),

$Ind_H^G\psi$=the induced represenation of G by ψ,

$Res_H\varphi$=the restriction of φ to H,

$d_F(\varphi)$=the degree of φ over $F=\boldsymbol{R}$ or $\boldsymbol{C}$,

$dim_F V$=the dimension of a vector space V over F,

W_φ=the representation space of φ,

$\#\Omega(\varphi)$=the number of weights in $\Omega(\varphi)$ (see 1.1),

$k\varphi$=the direct sum of k copies of φ,

φ_c=the complexification of $\varphi\in RRep(G)$,

φ^*=the conjugate representation of $\varphi\in CRep(G)$,

$S^a\varphi$=the a-th symmetric power of φ,
$\Lambda^a\varphi$=the a-th exterior power of φ,
θ=1-dimensional trivial representation over $\boldsymbol{R}$ or $\boldsymbol{C}$.

(3) Let G, G_1 and G_2 be compact groups. For $\varphi_1, \varphi_2 \in RRep(G)$ (or $CRep(G)$) and $\psi_i \in RRep(G_i)$ (or $CRep(G_i)$) ($i=1, 2$),
$\varphi_1\varphi_2$=the tensor product representation of G by φ_1 and φ_2,
$\psi_1 \otimes \psi_2$=the tensor product representation of $G_1 \times G_2$ by ψ_1 and ψ_2,
$\varphi_1 - \varphi_2$=the representation φ such that $\varphi \oplus \varphi_2$ is equivalent to φ_1,
$\varphi_1 \simeq \varphi_2$ means that φ_1 and φ_2 are equivalent,
$G_1 \cong G_2$ means that groups G_1 and G_2 are isomorphic.

(4) For subgroups K_i of U(n_i) ($i=1, 2$),
$\mathrm{S}(K_1 \times K_2) = \{X \times Y \in K_1 \times K_2 \mid \det X \det Y = 1\}$.

(5) Finally, for a G-space X,
$X^H = \{x \in X \mid h \cdot x = x \text{ for all } h \in H\}$,
G_x=the isotropy subgroup of x,
dim X=the dimension of a manifold X.

When $X^G = \Phi$, as mentioned in the first paragraph, the G-action is called *fixed-point free*.

1. Preliminaries

1.1. Weight systems of G-manifolds. Weight systems of G-manifolds play an important role in this paper. So we first recall weight systems of representations and then state the definition of (geometric) weight systems of G-manifolds introduced by W.Y. Hsiang (see [6]).

Let G be a compact connected Lie group, T a maximal torus of G and $W(G)=N(T)/T$ the Weyl group. Denote by $L(T)$ the Lie algebra of T. Then the restriction $Res_T\varphi$ of any $\varphi \in CRep(G)$ is decomposed into $k_1\varphi_1 \oplus k_2\varphi_2 \oplus \cdots \oplus k_m\varphi_m$, where each φ_i is an irreducible representation of degree 1 and $k_i\varphi_i$ denotes the direct sum of k_i copies of φ_i. Via the exponential map of $L(T)$ onto T, each φ_i can be identified with an element in the dual space $L(T)^*$ of $L(T)$. The set $k_1\{\varphi_1\} \cup \cdots \cup k_m\{\varphi_m\}$ whose elements are regarded as elements in $L(T)^*$ is called the *weight system* of φ, and is denoted by $\Omega(\varphi)$. Here $k_i\{\varphi_i\}$ denotes the union of k_i copies of $\{\varphi_i\}$. And each element in $\Omega(\varphi)$ is called a *weight* of φ. If $\varphi \in RRep(G)$, then by $\Omega(\varphi)$ we also denote the weight system of φ_c. In both cases, $\Omega(\varphi)$ is invariant under the action of $W(G)$ induced from the natural action of $W(G)$ on $L(T)^*$ (see 5.18 in [1]). And φ is equivalent to another representation φ' if and only if $\Omega(\varphi)=\Omega(\varphi')$. When G is disconnected, we can define the weight system of a representation φ of G by $\Omega(Res_{G_0}\varphi)$. Here G_0 is the identity component of G.

Let G be also a compact connected Lie group and X a $\boldsymbol{Q}$-acyclic manifold with a G-action Ψ. For $x \in X$ and $g \in G$, the differential of $\Psi(g, \ )$ at x induces a real representation Ψ_x of G_x on the tangent space at x. The Ψ_x is called the *tangential representation* at x. Since X is $\boldsymbol{Q}$-acyclic, by Smith theory (see Chapter III in [3]) the submanifold X^T is also $\boldsymbol{Q}$-acyclic, and consequently, connected. Thus the weight system of the tangential representation Ψ_{x_0} at $x_0 \in X^T$ does not depend on the choice of points. This weight system $\Omega(\Psi_{x_0})$ which is an invariant of the given action Ψ is defined to be the (*geometric*) *weight system* of a $\boldsymbol{Q}$-acyclic G-manifold X and will be denoted by $\Omega(X)$. Clearly X^T is $W(G)$-invariant, and hence $\Omega(X)$ is invariant under the naturally induced $W(G)$-action. The $W(G)$-action on $\Omega(X)$ is equal to the $W(G)$-action on $\Omega(\Psi_{x_0})$ which is mentioned before.

Especially put $G=\mathrm{U}(n)$. The Lie algebra is the set of skew Hermitian matrices of degree n, and a Cartan subalgebra $\mathfrak{T}$ is chosen as the subset of diagonal matrices (see 2.31, 4.16 in [1]). Let T_n be the maximal torus of $\mathrm{U}(n)$ with $L(T_n)=\mathfrak{T}$, and $\{x_i\}$ the coordinate with respect to the canonical basis for $L(T_n)$. Then every weight of representations, and hence every element in $\Omega(X)$, is expressed as a linear form λ: $(x_1, x_2, \cdots, x_n) \to a_1x_1+a_2x_2+\cdots+a_nx_n$ where $a_i \in \boldsymbol{Z}$. We write the λ by $\sum_{i=1}^{n} a_ix_i$. The Weyl group $W(G) \cong S_n$ (symmetric group of n letters) acts on $\Omega(X)$ by permutations on x_i's. When $G=\mathrm{SU}(n)$, we choose a maximal torus $\mathrm{S}T_n$ consisting of matrices in T_n with the determinant$=1$. Then $L(\mathrm{S}T_n)$ is a subspace of $L(T_n)$ with the trace$=0$, and every weight takes the form λ above with the relation $\sum_{i=1}^{n} x_i=0$. Throughout this paper, the notation $\sum_{i=1}^{n} a_ix_i$ denotes a weight in the above meaning and the *natural* $W(\mathrm{SU}(n))$ ($\cong W(\mathrm{U}(n)) \cong S_n$)-action on a set of weights means the action induced by permutations on x_i's. We often regard a weight as an element in $Irr(T)$.

1.2. Induced representations. Let G be a group and H its subgroup. When G is a finite group, we find in [12] various important properties about induced representations $Ind_H^G\varphi$. Most of those are naturally extended to the case that G is a compact group and H is a closed subgroup of finite index. We state here two of those which are needed in the later sections. The proofs are the same as those of the corresponding propositions in [12]. Note that $Ind_H^G\varphi$ is defined by the same method as in the case of a finite group G.

Proposition 1.2.1 (cf. Proposition 22 in [12]). *Let G be a compact group, and let K and H be its closed subgroups such that the index $(G; H)$ is finite. Put $H_s=sHs^{-1}\cap K$ for a representative s of the double cosets $K\backslash G/H$. Then, for each $\varphi \in CRep(H)$, $Res_K Ind_H^G\varphi$ is equivalent to $\bigoplus_{s\in K\backslash G/H} Ind_{H_s}^K\varphi^s$, where $\varphi^s \in CRep(H_s)$ is defined by $\varphi^s(x)=\varphi(s^{-1}xs)$ for all $x \in H_s$.*

Proposition 1.2.2 (cf. Proposition 23 in [12]). *Let G, H and φ be the same as in Proposition* 1.2.1. *Then $Ind_H^G\varphi$ is irreducible if and only if the following a)* *and b) hold: a) φ is irreducible, b) for each $s\in G-H$, $Res_{H_s}\varphi$ and φ^s have no irreducible direct summand in common, where $H_s=sHs^{-1}\cap H$ and $\varphi^s\in CRep(H_s)$ is defined by $\varphi^s(x)=\varphi(s^{-1}xs)$ for each $x\in H_s$.*

1.3. Representations of group extensions. For any compact group K, we write by $\hat{\gamma}$ the equivalence class of $\gamma\in Irr(K)$.

Let G be a compact group and H its closed normal subgroup, that is, consider an extension $1\to H\to G\to G/H\to 1$. Then G acts naturally on $\hat{H}$ by $(g\cdot\gamma)(h)=\gamma(g^{-1}hg)$ for $\hat{\gamma}\in\hat{H}$, $g\in G$ and $h\in H$. Dneote by G_γ the isotropy subgroup of $\hat{\gamma}$. G_γ is a closed subgroup of G containing H. If the index $(G;H)$ is finite, then $\hat{G}$ is described in terms of representations of H as follows.

Proposition 1.3. *Let G be a compact group and H a closed normal subgroup such that $(G;H)$ is finite. Let $\hat{\gamma}$ be any element of $\hat{H}$ and let $\check{G}_\gamma=\{\hat{\nu}\in\hat{G}_\gamma\mid Res_H\nu$ is equivalent to a multiple of $\gamma\}$. Then $\hat{G}$ is a disjoint union*

$$\hat{G}=\bigcup_{\hat{\gamma}\in\hat{H}/G}\{\widehat{(Ind^G_{G_\gamma}\nu)}\mid\hat{\nu}\in\check{G}_\gamma\}\ .$$

Moreover if γ extends to $\gamma'\in Irr(G_\gamma)$, then ν is equivalent to $\sigma'\gamma'$, where σ' is the natural lift of $\sigma\in Irr(G_\gamma/H)$ to $Irr(G_\gamma)$.

Proof. Take $\hat{\rho}\in\hat{G}$. Then we may put $W_{Res_H\rho}=\bigoplus_{i,j}W_{\gamma_{i,j}}=\bigoplus_{1\leq i\leq k}W_i$, where $\gamma_{i,j}\in Irr(H)$, $W_i=\bigoplus_j W_{\gamma_{i,j}}$ and $\gamma_{i,j}\simeq\gamma_{s,t}$ if and only if $i=s$. This decomposition $\bigoplus_{i\leq i\leq k}W_i$ is unique, that is, it does not depend on the choice of bases (see Theorem 8 and 4.3 (e) in [12]). In general, for an H-invariant subspace V of W_ρ, $\rho(g)V=\{\rho(g)v\mid v\in V\}$ becomes a gHg^{-1}-invariant subspace. Thus, for each $g\in G$, there holds $W_{Res_H\rho}=W_{Res_{gHg^{-1}}\rho}=\rho(g)W_{Res_H\rho}=\rho(g)\,(\bigoplus_{1\leq i\leq k}W_i)$ as H-spaces because of $gHg^{-1}=H$. This shows that $\rho(g)$ permutes W_i's. Since ρ is irreducible, the permutation by G is transitive. Furthermore the subgroup $\{g\in G\mid\rho(g)W_1=W_1\}$ equals $G_{\gamma_{1,1}}$, and W_1 becomes an irreducible $G_{\gamma_{1,1}}$-space. Therefore if we write this irreducible representation of $G_{\gamma_{1,1}}$ by ν, then $\hat{\nu}\in\check{G}_{\gamma_{1,1}}$ and ρ is expressed as $Ind^G_{G_{\gamma_{1,1}}}\nu$. Clearly the equivalence class of $Ind^G_{G_{\gamma_{1,1}}}\nu$ depends only on the class of $\hat{\gamma}_{1,1}$ in $\hat{H}/G$. Consequently we have $\hat{G}\subseteqq\bigcup_{\hat{\gamma}\in\hat{H}/G}\{\widehat{(Ind^G_{G_\gamma}\nu)}\mid\hat{\nu}\in\check{G}_\gamma\}$.

Conversely take $\hat{\nu}\in\check{G}_\gamma$. Suppose that $G_\gamma\neq G$ and, for some $s\in G-G_\gamma$, $Res_{(G_\gamma)_s}\nu$ and ν^s have an irreducible direct summand in common (see Proposition 1.2.2 for the definitions of $(G_\gamma)_s$ and ν^s). Then $Res_H\nu$ and $Res_H\nu^s$ also have a common direct summand, because $(G_\gamma)_s$ contains H. Thus we have $\gamma\simeq\gamma^s$. This contradicts $s\notin G_\gamma$, and hence from Proposition 1.2.2, $Ind^G_{G_\gamma}\nu$ is irreducible. Consequently we have $\bigcup_{\hat{\gamma}\in\hat{H}/G}\{\widehat{(Ind^G_{G_\gamma}\nu)}\mid\hat{\nu}\in G_\gamma\}\subseteqq\hat{G}$.

Next we show the second statement. Let $A\otimes B$ denote the tensor product of matrices A and B, and let I_k denote the identity matrix of degree k. Take $\check{\nu}\in\check{G}_\gamma$. We may assume $Res_H\nu=m\gamma$. Thus, for each $h\in H$ and $g\in G_\gamma$, we have $\nu(ghg^{-1})=I_m\otimes\gamma(ghg^{-1})=(I_m\otimes\gamma'(g))(I_m\otimes\gamma(h))(I_m\otimes\gamma'(g)^{-1})$ and $\nu(ghg^{-1})=\nu(g)\nu(h)\nu(g)^{-1}=\nu(g)(I_m\otimes\gamma(h))\nu(g)^{-1}$. This shows that $\nu(g)^{-1}(I_m\otimes\gamma'(g))$ and $I_m\otimes\gamma(h)$ commute. And hence there is an automorphism $F(g)$ of $W_{m\theta}$ such that $\nu(g)^{-1}(I_m\otimes\gamma'(g))=F(g)\otimes I_{d_C(\gamma)}$ holds. The F satisfies $F(hg)=F(g)$ for each $h\in H$. Therefore by setting $\sigma(\bar{g})=F(\bar{g})^{-1}$ for $\bar{g}\in G_\gamma/H$, we have $\sigma\in Irr(G_\gamma/H)$ and $\nu=\sigma'\gamma'$. The irreducibility of σ follows from that of ν. q.e.d.

REMARK 1.3. Proposition 1.3 holds for locally compact groups with some conditions. See Theorem 8.1 in [8] or Chapter III, Theorem 2 in [7] for the details.

2. SU(n)-manifolds $X(n, p)$ and $X_m(n, p)$

Throughout this section, we fix an integer $n\geq 3$ and a prime number p with $p<n<2p$. The purpose of this section is to construct certain compact $\boldsymbol{Z}_p$-acyclic SU(n)-manifolds without fixed points which will be denoted by $X(n, p)$ and $X_m(n, p)$. In 2.1 we state the method to construct $X(n, p)$. In 2.2 we prove that $X(n, p)$ is $\boldsymbol{Z}_p$-acyclic. And in 2.3, we define $X_m(n, p)$ as a typical example of $X(n, p)$.

For each $k>0$, let π_k(resp. $s\pi_k$) denote the natural projection from $N(T_k)$ (resp. $N(\mathrm{S}T_k)=\mathrm{S}N(T_k)$) to $N(T_k)/T_k\cong S_k$ (resp. $N(\mathrm{S}T_k)/\mathrm{S}T_k\cong \mathrm{S}_k$). Here T_k (resp. $\mathrm{S}T_k$) is the maximal torus of U(k) (resp. SU(k)) specified in 1.1 and $N(T_k)$ (resp. $N(\mathrm{S}T_k)$) is the normalizer of T_k (resp. $\mathrm{S}T_k$) in U(k) (resp. SU(k)). Especially, we denote $\mathrm{S}T_n$ and $s\pi_n$ by T and π respectively. And, for a subgroup H of S_n, we write $\pi^{-1}(H)$ by $\tilde{H}$. Furthermore, for a subgroup K of SU(n), we write $Res_K Ad_{\mathrm{SU}(n)}-Ad_K$ simply by ι_K, where Ad_L is the adjoint representation of a group L. For a compact group G and a point x in a G-manifold X, $Res_{G_x}Ad_G-Ad_{G_x}$ means the tangential representation at x restricted on the orbit G/G_x.

2.1. First we consider $\varphi_1\in RRep(\tilde{S}_n)$ and $\varphi_2\in RRep(\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p)))$ satisfying Condition A below.

CONDITION A.

i) There exist points in $W_{\varphi_i}(i=1, 2)$ whose isotropy subgroups coincide with $\widetilde{S_p\times S_{n-p}}$.

ii) $Res_{\widetilde{S_p\times S_{n-p}}}(\varphi_1\oplus\iota_{\tilde{S}_n})\simeq Res_{\widetilde{S_p\times S_{n-p}}}(\varphi_2\oplus\iota_{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p))})$.

Given such a pair (φ_1, φ_2), a fixed-point free SU(n)-manifold can be constructed as follows: To simplify the notations, we put $H=\widetilde{S_p\times S_{n-p}}$, $K_1=\tilde{S}_n$, $K_2=\mathrm{S}(\mathrm{U}(p)\times U(n-p))$ and $G=\mathrm{SU}(n)$. Take two disk bundles

$$X_i = G \times_{K_i} {}_{\varphi_i}D^{d_{\mathbf{R}}(\varphi_i)} \quad (i = 1, 2)\,,$$

where K_i acts on $D^{d_{\mathbf{R}}(\varphi_i)}$ via φ_i. Then, from i) in Condition A, two submanifolds

$$Y_i = G \times_{K_i} (K_i \times_H {}_{\psi_i}D^{d_{\mathbf{R}}(\psi_i)}) = G \times_H {}_{\psi_i}D^{d_{\mathbf{R}}(\psi_i)} \quad (i = 1, 2)$$

are equivariantly embedded into $\partial X_i (i=1, 2)$ respectively. Here ψ_i is the real representation of H defined by

$$\psi_i \oplus (Res_H Ad_{K_i} - Ad_H) = Res_H \varphi_i - \theta$$

and H acts on $D^{d_{\mathbf{R}}(\psi_i)}$ via ψ_i. The above θ is the 1-dimensional trivial representation over $\mathbf{R}$. Moreover, from ii) in Condition A, the above ψ_1 and ψ_2 are really equivalent, and hence Y_1 and Y_2 are equivariantly diffeomorphic. Thus, by identifying ∂X_1 with ∂X_2 along $Y_1 = Y_2$, we obtain an SU(n)-manifold with boundary. Here SU(n) acts on X_1 and X_2 by the left translations. This resulting SU(n)-manifold will be denoted by $X(n, p)$. Clearly the SU(n)-action on $X(n, p)$ is fixed-point free.

Taking another pair (φ_1, φ_2), another SU(n)-manifold is constructed by the above method. Thus $X(n, p)$ depends on the choice of (φ_1, φ_2). But we will write all of such manifolds by the same notation $X(n, p)$.

REMARK 2.1.1. In Section 11 of [13], all connected subgroups of SU(k) ($k>1$) with the maximal rank are determined up to an automorphism of SU(k). From the result, it follows that a proper subgroup of SU(k) (resp. U(k)) containing $N(ST_k)$ (resp. $N(T_k)$) must be $N(ST_k)$ (resp. $N(T_k)$) itself.

REMARK 2.1.2. In Condition A, i) for φ_1 can be always replaced by the condition $dim_{\mathbf{R}} W_{\varphi_1}^{\widetilde{S_p \times S_{n-p}}} > dim_{\mathbf{R}} W_{\varphi_1}^{\tilde{S}_n}$. Because a proper subgroup of S_n containing $S_p \times S_{n-p}$ is only $S_p \times S_{n-p}$. Now suppose $n=p+1$. Then, from Remark 2.1.1, it is seen that a proper subgroup of S(U(p)×U(1)) containing $\widetilde{S_p \times S_1} = \tilde{S}_p$ coincides with $\tilde{S}_p$ itself. Thus, if $n=p+1$, then i) for φ_2 is also replaced by the condition $dim_{\mathbf{R}} W_{\varphi_2}^{\tilde{S}_p} > dim_{\mathbf{R}} W_{\varphi_2}^{S(U(p) \times U(1))}$.

REMARK 2.1.3. The weight system of any representation of $\tilde{S}_n = N(T)$ is clearly invariant under the natural W(SU(n)) ($\cong S_n$)-action (see 1.1). Thus, from ii) in Condition A, the following is automatically derived: iii) $\Omega(\varphi_1 \oplus \iota_{\tilde{S}_n}) = \Omega(\varphi_2 \oplus \iota_{S(U(p) \times U(n-p))})$ is invariant under the natural W(SU(n))-action. This condition iii) is a necessary condition in order that the above $X(n, p)$ becomes a $\mathbf{Q}$-acyclic SU(n)-manifold, that is, that $\Omega(X(n, p))$ is W(SU(n))-invariant (see 1.1).

2.2. Let q_1 and q_2 be the natural projections from SU(n)/$\widetilde{S_p \times S_{n-p}}$ to SU(n)/$\tilde{S}_n$ and to SU(n)/S(U(p)×U($n-p$)) respectively. Then, from the following Proposition 2.2 and Mayer-Vietoris exact sequence for $(X(n, p), X_1, X_2)$ in 2.1,

$X(n, p)$ becomes a $\boldsymbol{Z}_p$-acyclic manifold with boundary.

Proposition 2.2. *If* $*>0$, *then via* $q_1^*+q_2^*$, $H^*(\mathrm{SU}(n)/\tilde{S}_n;\ \boldsymbol{Z}_p)\oplus H^*(\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p));\ \boldsymbol{Z}_p)$ *is isomorphic to* $H^*(\mathrm{SU}(n)/\widetilde{S_p\times S_{n-p}};\ \boldsymbol{Z}_p)$.

Before proving Proposition 2.2, we state a basic theorem about cohomologies of groups.

Theorem 2.2 (see Chapter III, Proposition 10.4 in [4]). *Let G be a group, M a G-module and H a subgroup of G such that the index $(G; H)$ is finite and invertible in M. Then, via the restriction map res_H^G, $H^*(G, M)$ is isomorphic to the set of G-invariant elements of $H^*(H, M)$.*

The restriction map res_H^G is a homomorphism of $H^*(G, M)$ to $H^*(H, M)$ given by regarding M as an H-module. And an element z of $H^*(H, M)$ is called a *G-invariant element* if $res^H_{H\cap gHg^{-1}}\, z = res^{gHg^{-1}}_{H\cap gHg^{-1}}\, g^*z$ holds for all $g\in G$. Here g^* is the inverse of the map of $H^*(gHg^{-1}, M)$ to $H^*(H, M)$ which is induced from the map sending h to ghg^{-1} for $h\in H$. See Chapter III, Section 8 in [4] for the precise definition.

Proof of Proposition 2.2. Let $S_n\to E_{S_n}\to B_{S_n}$ be the universal bundle for S_n. Consider the following commutative diagram:

$$
\begin{array}{lcccccc}
 & & & \overset{\pi_A}{\nearrow} & \mathrm{SU}(n)/\tilde{S}_n & & \\
 & & & & \uparrow & & \\
\text{(A)} & \mathrm{SU}(n)/T \to E_A = E_{S_n}\underset{S_n}{\times}\mathrm{SU}(n)/T & & & \xrightarrow{\quad p_A \quad} & & B_{S_n} \\
 & \| \qquad\quad \uparrow \tilde{q}_1 & & & \downarrow q_1 & & \uparrow \bar{q}_1 \\
 & & & \overset{\pi_B}{\nearrow} & \mathrm{SU}(n)/\widetilde{S_p\times S_{n-p}} & & \\
\text{(B)} & \mathrm{SU}(n)/T \to E_B = E_{S_n}\underset{S_p\times S_{n-p}}{\times}\mathrm{SU}(n)/T & & & \xrightarrow{\quad p_B \quad} & & B_{S_p\times S_{n-p}}.
\end{array}
$$

In the diagram, the maps $q_1, \bar{q}_1, \tilde{q}_1, p_A, p_B, \pi_A$ and π_B are all projections and $\tilde{q}_1$ is the bundle map between the fibre bundles (A)$=(\mathrm{SU}(n)/T\to E_A\to B_{S_n})$ and (B)$=(\mathrm{SU}(n)/T\to E_B\to B_{S_p\times S_{n-p}})$. Clearly π_A and π_B yield the isomorphisms π_A^* and π_B^* on the cohomologies. Thus, through π_A^* and π_B^*, we may identify q_1^* with $\tilde{q}_1^*$.

It is known that the $E_2^{s,t}$-term of Serre's spectral sequence for the fibre bundle (A) with the coefficient group $\boldsymbol{Z}_p$, $H^s(B_{S_n};\ \mathcal{H}^t(\mathrm{SU}(n)/T;\ \boldsymbol{Z}_p))$, is isomorphic to the cohomology of the group S_n, $H^s(S_n, H^t(\mathrm{SU}(n)/T;\ \boldsymbol{Z}_p))$. Here $\mathcal{H}^t(\mathrm{SU}(n)/T;\ \boldsymbol{Z}_p)$ is the Serre's local coefficient system of $\{H^t(p_A^{-1}(x);\ \boldsymbol{Z}_p)\,|\,x\in B_{S_n}\}$, and the coefficient $H^t(\mathrm{SU}(n)/T;\ \boldsymbol{Z}_p)$ of $H^s(S_n, H^t(\mathrm{SU}(n)/T;\ \boldsymbol{Z}_p))$ is regarded as an S_n-module via the characteristic homomorphism of $\mathcal{H}^t(\mathrm{SU}(n)/T;\ \boldsymbol{Z}_p)$. That is, S_n ($\cong W(\mathrm{SU}(n))$) acts on $H^*(\mathrm{SU}(n)/T;\ \boldsymbol{Z}_p)=\boldsymbol{Z}_p[y_1, y_2, \cdots, y_n]/\{s_1, s_2, \cdots, s_n\}$ (see Section 20 in [2]) by permutations on y_i's, where the degree of each y_i is two and s_j is the elementary j-th symmetric polynomial of $y_1, y_2, \cdots, y_n$. Similarly the $E_2^{s,t}$-term for (B), $H^s(B_{S_p\times S_{n-p}}; \mathcal{H}^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p))$, is isomorphic to $H^s(S_p\times S_{n-p}, H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p))$. It is clear that the homomorphism $(\tilde{q}_1, \bar{q}_1)^*$ on $E_2^{s,t}$-terms induced by the bundle map $\tilde{q}_1$ corresponds to the restriction map $res^{S_n}_{S_p\times S_{n-p}}$ of $H^s(S_n, H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p))$ to $H^s(S_p\times S_{n-p}, H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p))$.

Let $S_{n,p}$ be a p-Sylow subgroup of S_n and $N(S_{n,p})$ the normalizer of $S_{n,p}$ in S_n. Then, from the assumption $p<n<2p$, we may fix $S_{n,p}$ as the cyclic subgroup $Z_p\times 1$ of $S_p\times 1$. Thus we have $N(S_{n,p})\subset S_p\times S_{n-p}$ and $S_{n,p}\cap gS_{n,p}g^{-1}=\{1\}$ for any $g\notin N(S_{n,p})$,. This shows that, for $s>0$, an element in $H^s(S_{n,p}, H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p))$ is S_n-invariant if and only if it is $S_p\times S_{n-p}$-invariant. Hence, for $s>0$, Theorem 2.2 gives the isomorphism $(res^{S_p\times S_{n-p}}_{S_{n,p}})^{-1}\, res^{S_n}_{S_{n,p}}=res^{S_n}_{S_p\times S_{n-p}}$:

$$H^s(S_n, H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p)) \underset{\rightarrow}{\cong} H^s(S_p\times S_{n-p}, H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p)) .$$

And consequently $E_2^{s,t}$-terms for (A) and (B) are isomorphic via $(\tilde{q}_1, \bar{q}_1)^*$ if $s>0$. On the other hand, for $E_2^{0,t}$-terms $(t>0)$, the followings hold:

$$E_2^{0,t} \text{ for (A)} = H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p)^{S_n} = 0\,,\quad E_2^{0,t} \text{ for (B)} = H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p)^{S_p\times S_{n-p}} .$$

Now consider the projections q: $\mathrm{SU}(n)/T\to \mathrm{SU}(n)/\widetilde{S_p\times S_{n-p}}$ and $q_3=q_2\circ q$: $\mathrm{SU}(n)/T\to\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p))$. Then q^* equals the composite of maps: $H^t(\mathrm{SU}(n)/\widetilde{S_p\times S_{n-p}}; \boldsymbol{Z}_p) \underset{\text{surjection}}{\rightarrow} E_\infty^{0,t} \text{ for (B)} \underset{\text{injection}}{\subset} E_2^{0,t}$ for (B). Furthermore, by applying Theorems 14.2 and 20.3 in [2] to the fibre bundle $\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p))/T \to\mathrm{SU}(n)/T\overset{q_3}{\to}\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p))$, we see that q_3^*: $H^t(\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p)); \boldsymbol{Z}_p)\to H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p)$ is injective and the image of q_3^* coincides with $H^t(\mathrm{SU}(n)/T; \boldsymbol{Z}_p)^{W(\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p)))}=E_2^{0,t}$-term for (B). Therefore q_2^* is injective and q^* is surjective to $E_2^{0,t}$ for (B). Consequently the differential $d_r^{0,t}$ for (B) is a zero map for all r.

From these observations, we get, for $*>0$,

$$H^*(\mathrm{SU}(n)/\widetilde{S_p\times S_{n-p}}; \boldsymbol{Z}_p) \overset{q_1^*+q_2^*(q_3^*)^{-1}}{\underset{\leftarrow}{\cong}} H^*(\mathrm{SU}(n)/\tilde{S}_n; \boldsymbol{Z}_p)\oplus E_2^{0,*} \text{ for (B)}$$
$$\overset{\text{identity}\oplus q_3^*}{\underset{\leftarrow}{\cong}} H^*(\mathrm{SU}(n)/\tilde{S}_n; \boldsymbol{Z}_p)\oplus H^*(\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p)); \boldsymbol{Z}_p)\,. \quad \text{q.e.d.}$$

REMARK 2.2. In general, ecah $X(n,p)$ becomes $\boldsymbol{Z}_q$-acyclic (q; prime) if q does not divide $|S_n|=n!$. The proof is the same as that of Proposition 2.2.

2.3. Here we give a typical example of $X(n,p)$, that is, of representations φ_1 and φ_2 satisfying Condition A.

EXAMPLE **2.3.** Let u_k $(k\geq 2)$ be the standard irreducible complex representation of $\mathrm{U}(k)$ of degree k. Then $Res_{\mathrm{SU}(k)}(S^2u_kS^2u_k^*-u_ku_k^*)$ is irreducible and has a real form. In fact it is equivalent to a self conjugate direct summand of $(S^2Ad_{\mathrm{SU}(k)})_c$. Put $\eta_c=Res_{\mathrm{SU}(n)}(S^2u_nS^2u_n^*-u_nu_n^*)$. Lemmas 2.3.1 and 2.3.2 below show that $Res_{\tilde{\mathfrak{S}}_n}\eta$ and $Res_{S(U(p)\times U(n-p))}\eta$ have direct summands equivalent to $\iota_{\tilde{\mathfrak{S}}_n}\oplus\theta$ and $\iota_{S(U(p)\times U(n-p))}\oplus\theta$ respectively. Thus we can define φ_1 and φ_2 by $\varphi_1= Res_{\tilde{\mathfrak{S}}_n}\eta-\iota_{\tilde{\mathfrak{S}}_n}-\theta$ and $\varphi_2=Res_{S(U(p)\times U(n-p))}\eta-\iota_{S(U(p)\times U(n-p))}-\theta$. From the definition, φ_1 and φ_2 clearly satisfy (ii) in Condition A. Before and after Lemma 2.3.2, it is shown that they satisfy (i). We denote by $X_m(n, p)$, the $X(n, p)$ constructed by the above φ_1 and φ_2. And we write these φ_1 and φ_2 by $\varphi_{1,m}$ and $\varphi_{2,m}$ respectively. Obviously, $dim\ X_m(n, p)$ is equal to $d_C(\eta_c)-1=\frac{1}{4}n^2(n+3)(n-1)-1$.

Lemma 2.3.1. *$Res_{U(p)\times U(n-p)}(S^2u_n\ S^2u_n^*-u_nu_n^*)$ is equivalent to $(S^2u_p\ S^2u_p^* -u_pu_p^*)\otimes\theta\oplus\theta\otimes(S^2u_{n-p}\ S^2u_{n-p}^*-u_{n-p}\ u_{n-p}^*)\oplus S^2u_p\otimes S^2u_{n-p}^*\oplus S^2u_p^*\otimes S^2u_{n-p}\oplus\{(S^2u_p)\ u_p^*-u_p\}\otimes u_{n-p}^*\oplus u_p^*\otimes\{(S^2u_{n-p})\ u_{n-p}^*-u_{n-p}\}\oplus\{(S^2u_p^*)\ u_p-u_p^*\}\otimes u_{n-p}\oplus u_p\otimes\{(S^2u_{n-p}^*)\ u_{n-p}-u_{n-p}^*\}\oplus(u_pu_p^*-\theta)\otimes(u_{n-p}\ u_{n-p}^*-\theta)\oplus(u_pu_p^*-\theta)\otimes\theta\oplus\theta\otimes(u_{n-p}u_{n-p}^* -\theta)\oplus u_p\otimes u_{n-p}^*\oplus u_p^*\otimes u_{n-p}\oplus\theta$ if $n>p+1$, and equivalent to $(S^2u_p\ S^2u_p^*-u_p\ u_p^*)\otimes \theta\oplus(S^2u_p)\otimes(u_1^*)^2\oplus(S^2u_p^*)\otimes(u_1)^2\oplus\{(S^2u_p)\ u_p^*-u_p\}\otimes u_1^*\oplus\{(S^2\ u_p^*)\ u_p-u_p^*\}\otimes u_1\oplus (u_pu_p^*-\theta)\otimes\theta\oplus u_p\otimes u_1^*\oplus u_p^*\otimes u_1\oplus\theta$ if $n=p+1$. Moreover each direct summand is irreducible and $Res_{S(U(p)\times U(n-p))}(u_p\otimes u_{n-p}^*\oplus u_p^*\otimes u_{n-p})$ is equivalent to $(\iota_{S(U(p)\times U(n-p))})_c$.*

The proof of Lemma 2.3.1 is a routine work. So we omit it.

Take $\rho_k\in RRep(N(T_k))$ $(k\geq 2)$ as follows: The representation space W_{ρ_k} is a vector space over $\boldsymbol{R}$ with a basis $\{s_{ij}=s_{ji}\,|\,1\leq i<j\leq k\}$ on which $N(T_k)$ acts by $\sigma(s_{ij})=s_{\pi_k(\sigma)(i)\pi_k(\sigma)(j)}$ for $\sigma\in N(T_k)$. It is clear that $dim_{\boldsymbol{R}}\ W_{\rho_k}^{N(T_k)}=1$ and $dim_{\boldsymbol{R}}\ W_{\rho_k-\theta}^{N(T_{k_1})\times N(T_{k_2})}>0$ hold for any positive integers k_1, k_2 with $k_1+k_2=k$. Especially denote $Res_{N(ST_n)}\ \rho_n$ by ρ. Then by taking $(p, n-p)$ as (k_1, k_2), we have $dim_{\boldsymbol{R}}\ W_{\rho-\theta}^{\widetilde{S_p\times S_{n-p}}}>0$. This is equivalent to that there exists a point in $W_{\rho-\theta}$ whose isotropy subgroup is just $\widetilde{S_p\times S_{n-p}}$(see Remark 2.1.2). Thus, from Lemma 2.3.2, $\varphi_{1,m}$ satisfies i) in Condition A.

Lemma 2.3.2. *For each $k\geq 2$, $Res_{N(T_k)}(S^2u_k\ S^2u_k^*-u_ku_k^*)$ has a direct summand equivalent to $(\rho_k)_c\oplus(Res_{N(T_k)}\ Ad_{U(k)}-Ad_{N(T_k)})_c$. Especially, $Res_{\tilde{\mathfrak{S}}_n}(S^2u_n\ S^2u_n^* -u_nu_n^*)=Res_{\tilde{\mathfrak{S}}_n}\ \eta_c$ has a direct summand equivalent to $\rho_c\oplus(\iota_{\tilde{\mathfrak{S}}_n})_c$.*

Proof. It is sufficient to prove the first statement. Let $\{e_i\,|\,1\leq i\leq k\}$ (resp. $\{e_i^*\,|\,1\leq i\leq k\}$) be the standard basis for W_{u_k} (resp. $W_{u_k^*}$), and put $S_{ij}=\frac{1}{2}(e_i\otimes e_j +e_j\otimes e_i)$ and $S_{ij}^*=\frac{1}{2}(e_i^*\otimes e_j^*+e_j^*\otimes e_i^*)$. Then $\{S_{ij}\otimes S_{st}^*\,|\,1\leq i\leq j\leq k,\ 1\leq s\leq t \leq k\}$ becomes a canonical basis for $W_{S^2u_kS^2u_k^*}$. Now consider three subspaces W_i $(i=1, 2, 3)$ of $W_{S^2u_kS^2u_k^*}$ whose bases are $B_1=\{\sum_{i=1}^{k}S_{ij}\otimes S_{it}^*\,|\,1\leq j,\ t\leq k\}$, $B_2=\{S_{ij} \otimes S_{ij}^*\,|\,1\leq i<j\leq k\}$ and $B_3=\{S_{ij}\otimes S_{ii}^*,\ S_{ij}\otimes S_{jj}^*\,|\,1\leq i<j\leq k\}$ respectively. It is clear that W_2 and W_3 are in the complement of W_1. Moreover we see that W_1 is equivalent to $W_{u_ku_k^*}$ as U(k)-spaces, and that, as $N(T_k)$-spaces, W_2 and W_3 are

equivalent to $W_2'=W_{(\rho_k)_c}$ and $W_3'=W_{(Res_{N(T_k)}Ad_{U(k)}-Ad_{N(T_k)})_c}$ respectively. Thus, up to equivalence, W_2' and W_3' can be regarded as $N(T_k)$-invariant subspaces of $W_{S^2u_kS^2u_k^*-u_ku_k^*}$. q.e.d.

Put $\xi_1=Res_{S(U(p)\times U(n-p))}(S^2u_p\, S^2u_p^*-u_pu_p^*)\otimes\theta$ and $\xi_2=Res_{S(U(p)\times U(n-p))}\,\theta\otimes(S^2u_{n-p}\, S^2u_{n-p}^*-u_{n-p}u_{n-p}^*)$. Then from Lemma 2.3.2 and Remark 2.1.1, we see that there exist points in W_{ξ_1} and W_{ξ_2} whose isotropy subgroups coincide with $S(N(T_p)\times U(n-p))$ and $S(U(p)\times N(T_{n-p}))$, respectively. Thus, in $W_{\xi_1\oplus\xi_2}$ there exists a point with the isotropy subgroup $S(N(T_p)\times U(n-p))\cap S(U(p)\times N(T_{n-p}))=\widetilde{S_p\times S_{n-p}}$. On the other hand, from Lemma 2.3.1, $(\varphi_{2,m})_c$ has a direct summand equivalent to $\xi_1\oplus\xi_2$ if $n>p+1$, and to ξ_1 if $n=p+1$. Therefore $\varphi_{2,m}$ satisfies i) in Condition A.

3. The minimum dimension of $X(n, p)$'s and the proof of Theorem A

Let $X(n,p)$ and $X_m(n,p)$ be $\boldsymbol{Z}_p$-acyclic SU(n)-manifolds constructed in Section 2. The purpose of this section is to prove Theorem A in Introduction and Theorems C and D below. The notations are the same as in the previous sections.

Theorem C. *Suppose $n=p+1(p\neq 3)$. Then, for any $X(n,p)$, we have $\dim X(n,p)\geq \dim X_m(n,p)$.*

Theorem D. *Let p $(p\neq 3)$ be a prime number and let X be a $\boldsymbol{Q}$-acyclic manifold with a fixed-point free SU($p+1$)-action. If $\dim X^{\tilde{S}_p}>\dim X^{S(U(p)\times U(1))}\geq 0$ and $X^{\tilde{S}_p}$ is connected, then $\dim X\geq\frac{1}{4}(p+1)^2(p+4)p-1=\dim X_m(p+1,p)$ holds.*

Let $\varphi_2, \varphi_{2,m}\in RRep(S(U(p)\times U(1)))$ be the representations constructing $X(p+1,p)$ and $X_m(p+1,p)$. Then Theorem C is equivalent to $d_{\boldsymbol{R}}(\varphi_2)\geq d_{\boldsymbol{R}}(\varphi_{2,m})$ for any φ_2. Recall here that every φ_2 satisfies: (1) $\dim_{\boldsymbol{R}}W_{\varphi_2}^{\tilde{S}_p}>\dim_{\boldsymbol{R}}W_{\varphi_2}^{S(U(p)\times U(1))}$, (2) $\Omega(\varphi_2\oplus\iota_{S(U(p)\times U(1))})$ is invariant under the natural $W(SU(p+1))$-action (see Remarks 2.1.2 and 2.1.3). So, in 3.1–3.5, we first investigate which representations φ of $S(U(p)\times U(1))$ satisfy the above (1), (2) and $d_{\boldsymbol{R}}(\varphi)\leq d_{\boldsymbol{R}}(\varphi_{2,m})$. The goal is to prove Proposition 3.5, that is, that if $\varphi\in RRep(S(U(p)\times U(1)))$ satisfies the above (1), (2) and $d_{\boldsymbol{R}}(\varphi)\leq d_{\boldsymbol{R}}(\varphi_{2,m})$, then φ must be equivalent to $\varphi_{2,m}$. From this we obtain Theorem C immediately. The proof of Theorem D follows that of Theorem C, and Theorem A is given as a corollary of Theorem D. The complete proofs of Theorems A, C and D are given in the last subsection 3.6.

Throughout this section, we often identify U(p) with the subgroup S(U(p)× U(1)) of SU($p+1$). Then the maximal tori T_p and ST_{p+1} specified in 1.1 are identified, and the normalizer $N(T_p)$ of T_p is regarded as the subgroup $\widetilde{S_p\times S_1}=$

$\tilde{S}_p$ of $\tilde{S}_{p+1}$ in Section 2. Furthermore, any weight $\sum_{i=1}^{p+1} b_i x_i$ ($\sum_{i=1}^{p+1} x_i=0$) of $\varphi\in CRep$ ($S(U(p)\times U(1))$) corresponds to a weight $\sum_{i=1}^{p}(b_i-b_{p+1})x_i$ of φ regarded as a representation of $U(p)$. Here $\{x_i \mid 1\leq i\leq p+1, \sum_{i=1}^{p+1} x_i=0\}$ is the coordinate with respect to the canonical basis for $L(ST_{p+1})$ (see 1.1). To simplify the notations, we denote the identified ST_{p+1} and T_p by T, and omit to describe the relation $\sum_{i=1}^{p+1} x_i=0$ on each weight $\sum_{i=1}^{p+1} b_i x_i$ if there is no confusion. And, unless otherwise specified, p denotes a prime number. The other notations are the same as in the previous sections.

3.1. Here we give a dimension formula for $\varphi\in Irr(U(p))$. Denote by h_ξ the highest weight of a complex representation ξ of a semisimple Lie group G.

Take $SU(p)$ as G. Then each weight of an arbitrary representation of G is uniquely expressed as a linear combination of $x_1, x_2, \cdots, x_{p-1}$ from the relation $\sum_{i=1}^{p} x_i=0$. Hence we can fix on a set of weights the lexicographical ordering with respect to the basis $x_1>x_2>\cdots>x_{p-1}$. In the rest of this section, h_ξ for $\xi\in CRep$ ($SU(p)$) means the highest weight under this ordering. It is known that h_ξ for $\xi\in Irr(SU(p))$ can be expressed as $\sum_{j=1}^{p-1} a_j(\sum_{i=1}^{j} x_i)$ by non-negative integers $\{a_j\}$ (see Theorem 0.9 in [5]).

For $\varphi\in Irr(U(p))$, suppose $h_{Res_{SU(p)}\varphi}=\sum_{j=1}^{p-1} a_j(\sum_{i=1}^{j} x_i)=\sum_{i=1}^{p-1}(\sum_{j=i}^{p-1} a_j)x_i$. Then by using Weyl's dimension formula (see Theorem 0.24 in [5]), we get

$$d_C(\varphi)=d_C(Rres_{SU(p)}\,\varphi)=\prod_{j=1}^{p-1}\left[\frac{1}{(p-j)!}\prod_{k=j}^{p-1}\{(\sum_{i=j}^{k} a_i)+(k-j+1)\}\right]. \quad (3.1)$$

Moreover, for the φ, there exists a unique integer a_p such that φ is equivalent to an irreducible direct summand of $(\Lambda u_p)^{a_1}(\Lambda^2 u_p)^{a_2}\cdots(\Lambda^p u_p)^{a_p}$. That is, up to equivalence, φ is determined by $h_{Res_{SU(p)}\varphi}=\sum_{i=1}^{p-1}(\sum_{j=i}^{p-1} a_j)x_i$ and the integer a_p. So, in order to distinguish φ, we often describe $h_{Res_{SU(p)}\varphi}$ by the form $\sum_{i=1}^{p}(\sum_{j=i}^{p} a_j)x_i$ $(=\sum_{i=1}^{p-1}(\sum_{j=i}^{p-1} a_j)x_i+a_p\sum_{i=1}^{p} x_i)$.

3.2. In this subsection, we give two necessary conditions (3.2.1) and (3.2.2) in order that $\varphi\in Irr(U(p))\setminus\{\theta\}$ satisfies $dim_C W_\varphi^{\tilde{S}_p}>0$. The conditions play an essential role in proving Lemma 3.3.

Lemma 3.2. *Suppose that $\varphi\in Irr(U(p))\setminus\{\theta\}$ satisfies $dim_C W_\varphi^{\tilde{S}_p}>0$. Then $\Omega(\varphi)$ contains $\{2(x_i-x_j), (x_i-x_j) \mid 1\leq i<j\leq p\}$.*

Proof. Let $K=U(2)\times\overbrace{U(1)\times\cdots\times U(1)}^{p-2}$ $(\subset U(p))$ and let $N_K(T)$ be the normalizer of T in K. Furthermore, take an element $r\in\tilde{S}_p$ corresponding to a cyclic permutation of p letters $1, 2, \cdots, p$, and put $K_i=r^iK(r^i)^{-1}$. If we suppose $W_\varphi^K=W_\varphi^{N_K(T)}$, then $W_\varphi^{U(p)}=\bigcap_{i=1}^{p} W_\varphi^{K_i}=\bigcap_{i=1}^{p} W_\varphi^{N_{K_i}(T)}=W_\varphi^{\tilde{S}_p}$ holds. This contradicts the assumption $dim_C W_\varphi^{\tilde{S}_p}>0$. Thus we have $W_\varphi^K=(W_\varphi^{T^\alpha})^{N(T^\alpha)/T^\alpha}=(W_\varphi^{T^\alpha})^{SO(3)}\subsetneqq(W_\varphi^{T^\alpha})^{N_K(T)/T^\alpha}=(W_\varphi^{T^\alpha})^{O(2)}$. Here T^α is the corank one subtorus of T whose Lie algebra $L(T^\alpha)$ is perpendicular to $\alpha=x_1-x_2$. That is, $L(T^\alpha)$ consists of diagonal matrices diag $(ix_1, ix_2, \cdots, ix_p)$ with $x_1=x_2$. This shows that the restricted representation φ_α of SO(3) on $W_\varphi^{T^\alpha}$ has at least one irreducible direct summand of degree ≥ 5. Therefore 2α and α are contained in $\Omega(\varphi_\alpha)$, and consequently $\Omega(\varphi)$ $(\supset\Omega(\varphi_\alpha))$ contains $\{2(x_i-x_j), (x_i-x_j)\,|\,1\leq i<j\leq p\}$. q.e.d.

For φ in Lemma 3.2, put $h_{Res_{SU(p)}\varphi}=\sum_{i=1}^{p}(\sum_{j=i}^{p} a_j)\, x_i$ by non-negative integers $a_j(1\leq j\leq p-1)$ and an integer a_p. Then $h_{Res_{SU(p)}\varphi}$ is higher than or equal to $2(x_1-x_p)$, and hence $\{a_j\}$ satisfies

$$4\leq \sum_{j=1}^{p-1} a_j\,. \tag{3.2.1}$$

Moreover, from the assumption $dim_C W_\varphi^T\geq dim_C W_\varphi^{\tilde{S}_p}>0$, $Res_T(\Lambda u_p)^{a_1}(\Lambda^2 u_p)^{a_2}\cdots(\Lambda^p u_p)^{a_p}$ has a trivial summand. Thus $\{a_j\}$ has the relation

$$\sum_{k=1}^{p} k a_k = 0\,. \tag{3.2.2}$$

3.3. Put $l=d_R(\varphi_{2,m})=dim\ X_m(p+1,p)-dim\ SU(p+1)/S(U(p)\times U(1))=\frac{1}{4}p^2(p+3)^2-(p+1)$.

Lemma 3.3. *Suppose that $\varphi\in Irr(U(p))\setminus\{\theta\}$ satisfies $dim_C W_\varphi^{\tilde{S}_p}>0$. Moreover suppose that $d_C(\varphi)\leq l$ if φ has a real form, and $2d_C(\varphi)\leq l$ if φ has no real form. Then φ is equivalent to one of the following σ_j's and the conjugate σ_j^*'s; each σ_j is the direct summand of ψ_j with $h_{Res_{SU(p)}\psi_j}=h_{Res_{SU(p)}\sigma_j}$ and $d_C(\sigma_j)=d_j$.*

σ_1: $\psi_1=(u_2)^{2m}(\Lambda^2u_2)^{-m}$ $(2\leq m\leq 10)$ $\quad d_1=2m+1 \quad p=2$

σ_2: $\psi_2=(u_3)^4(\Lambda^2u_3)(\Lambda^3u_3)^{-2} \quad d_2=35 \quad p=3$

σ_3: $\psi_3=(u_3)^6(\Lambda^3u_3)^{-2} \quad d_3=28 \quad p=3$

σ_4: $\psi_4=(u_3)^3(\Lambda^2u_3)^3(\Lambda^3u_3)^{-3} \quad d_4=64 \quad p=3$

σ_5: $\psi_5=(u_p)^2(\Lambda^{p-1}u_p)^2(\Lambda^pu_p)^{-2} \quad d_5=\frac{1}{4}p^2(p+3)(p-1) \quad p\geq 3$

REMARK 3.3.1. Denote by $R(U(p))$ the complex representation ring of $U(p)$ and by t the homomorphism of $R(U(p))$ which sends a complex representation to the conjugate representation. Then $H=ker(1-t)/Im(1+t)$ becomes an algebra over $\boldsymbol{Z}_2$ with generators $\Lambda^i u_p\, \Lambda^{p-i}\, u_p(\Lambda^p u_p)^{-1}$ $(2\leq 2i\leq p)$ (see 7.3 in [1]). Thus neither σ_2 nor σ_3 has any real form.

Lemma 3.3 is given by calculations of degrees which are straight but long. So we give only the outline here.

Outline of the proof. Put $h_{Res_{SU(p)}\varphi}=\sum_{i=1}^{p}(\sum_{j=i}^{p} a_j)\, x_i$. Then $\{a_j\}$ must satisfy (3.2.1) and (3.2.2). By applying (3.1) to φ under these conditions on $\{a_j\}$, we see that if $d_C(\varphi)\leq l$, then $\{a_j \mid 1\leq j\leq p-1\}$ is one of the followings: (1) $p=3$, $a_1=4$, $a_2=1$, (2) $p=3$, $a_1=1$, $a_2=4$, (3) $p=3$, $a_1=a_2=3$, (4) $p=3$, $a_1=6$, $a_2=0$, (5) $p=3$, $a_1=0$, $a_2=6$, (6) $p=3$, $a_1=9$, $a_2=0$, (7) $p=3$, $a_1=0$, $a_2=9$, (8) $p=5$, $a_1=3$, $a_2=1$, $a_3=a_4=0$, (9) $p=5$, $a_1=a_2=0$, $a_3=1$, $a_4=3$, (10) $p=5$, $a_1=5$, $a_2=a_3=a_4=0$, (11) $p=5$, $a_1=a_2=a_3=0$, $a_4=5$, (12) $p\geq 3$, $a_1=a_{p-1}=2$, the other $a_j=0$, (13) $p=2$, $a_1=2m$, $2\leq m\leq 10$. Here only the φ corresponding to (3), (12) or (13) has a real form (see Remark 3.3.1). And the φ corresponding to (1), (4), (6), (8) or (10) is conjugate to that corresponding to (2), (5), (7), (9) or (11), respectively. So from the assumption $2d_C(\varphi)\leq l$, (6), (7), (8) and (9) are excluded. Moreover we see that $\varphi=S^5u_5(\Lambda^5u_5)^{-1}$ corresponding to (10) does not have $2(x_i-x_j)$ as its weight. Thus, from Lemma 3.2, (10) and (11) are also excluded. Consequently the required result is obtained. q.e.d.

Proposition 3.3. *Suppose that* $\varphi\in RRep(\mathrm{S}(\mathrm{U}(p)\times \mathrm{U}(1)))$ *satisfies the conditions*: (i) $d_{\boldsymbol{R}}(\varphi)\leq l$, (ii) $dim_{\boldsymbol{R}} W_{\varphi}^{\tilde{S}_p} > dim_{\boldsymbol{R}} W_{\varphi}^{\mathrm{S}(\mathrm{U}(p)\times \mathrm{U}(1))}$, (iii) $\Omega(\varphi\oplus\iota_{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))})$ *is invariant under the natural* $W(\mathrm{SU}(p+1))$*-action. Then* φ_c *has a direct summand equivalent to* $Res_{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}(S^2u_p\, S^2u_p^*-u_pu_p^*)\otimes\theta$ *where* $S^2u_p\, S^2u_p^*-u_pu_p^*\in Irr$ $(\mathrm{U}(p))$ and $\theta\in Irr(U(1))$.

Proof. Identify $RRep(\mathrm{S}(\mathrm{U}(p)\times \mathrm{U}(1)))$ with $RRep(\mathrm{U}(p))$. Then from (i) and (ii), φ_c has an irreducible direct summand satisfying all assumptions in Lemma 3.3. On the other hand, $S^2u_p\, S^2u_p^*-u_pu_p^*$ is equivalent to σ_5 in Lemma 3.3 if $p\geq 3$, and to σ_1 for $m=2$ if $p=2$. So we have only to show that, up to equivalence, any othei σ_i in Lemma 3.3 can not appear in φ_c. We give the proof only for the case of $p=2$. The other cases are proved by the same method.

Suppose that σ_1 for some $m(2\leq m\leq 10)$ appears in φ_c. Since the σ_1 coincides with $S^{2m}u_2(\Lambda^2u_2)^{-m}$, the assumption (iii) derives

$$\Omega(\varphi_c-\sigma_1)\supseteqq\{\pm k(x_1-x_3),\ \pm k(x_2-x_3)\,|\,2\leq k\leq m\}\ .$$

This shows $4(m-1)\leq\#\Omega(\varphi_c-\sigma_1)\leq d_C(\varphi_c-\sigma_1)\leq l-(2m+1)=21-2m$, that is, $2\leq m\leq 4$. (Note that if we suppose that σ_2, σ_3 or σ_4 appears in φ_c, then the contradiction is derived at this step. For $\sigma_i(i=2,3)$, we consider $\Omega(\varphi_c-\sigma_i-\sigma_i^*)$ because they have no real form.) Suppose $m\geq 3$. In general, if an irreducible complex representation of SU(2) has $2kx_1$ (resp. $(2k+1)\,x_1$) as its weight, then $\{0, \pm 2hx_1 \mid 1 \leq h \leq k\}$ (resp. $\{\pm(2k+1)x_1 \mid 0 \leq h \leq k\}$) are also contained in the weight system. This shows $\Omega(Res_{SU(2)}(\varphi_c - \sigma_1)) \supseteqq \{\pm kx_1, \pm kx_1 \mid 2 \leq k \leq m\} \cup \{0, 0, \pm x_1, \pm x_1\}$. Thus from (iii) we get $\#\Omega(\varphi_c - \sigma_1) > 21 - 2m$. Hence we conclude $m=2$. q.e.d.

3.4. From now on, for a set S of weights, let mS denote the union of m copies of S and $W(SU(p+1))S$ the orbit containing S under the natural $W(SU(p+1))$-action.

Suppose that $\varphi \in RRep(S(U(p) \times U(1)))$ satisfies all assumptions in Proposition 3.3. Then we may put $\psi = \varphi_c - Res_{S(U(p) \times U(1))}(S^2 u_p S^2 u_p^* - u_p u_p^*) \otimes \theta$. In this subsection, we investigate the weight system of ψ.

First we can easily calculate

$$\begin{aligned}
&\Omega((\iota_{S(U(p)\times U(1))})_c \oplus Res_{S(U(p)\times U(1))}(S^2 u_p S^2 u_p^* - u_p u_p^*) \otimes \theta) \\
&\quad = {}_pC_2\{0\} \cup \{\pm(x_i - x_{p+1}) \mid 1 \leq i \leq p\} \\
&\quad \cup (p-1)\{\pm(x_i - x_j) \mid 1 \leq i < j \leq p\} \cup \{\pm 2(x_i - x_j) \mid \\
&\quad 1 \leq i < j \leq p\} \cup \{\pm(2x_i - x_j - x_k) \mid 1 \leq j < k \leq p,\ 1 \leq i \leq p, \\
&\quad i, j, k;\ \text{each other different}\} \cup \{(x_i + x_j) - (x_h + x_k) \\
&\quad \mid 1 \leq i < j \leq p,\ 1 \leq h < k \leq p,\ i, j, h, k;\ \text{each other different}\}.
\end{aligned}$$

Here the last two parts of the union are excluded if $p=2$, and the last one is excluded if $p=3$. Thus, from (iii) in Proposition 3.3, we see that $\Omega(\psi)$ contains

$$(p-2)\{\pm(x_i - x_{p+1}) \mid 1 \leq i \leq p\} \cup \{\pm 2(x_i - x_{p+1}) \mid 1 \leq i \leq p\}. \tag{3.4.1}$$

If $p \geq 3$, $\Omega(\psi)$ also contains

$$\begin{aligned}
&\{\pm(2x_{p+1} - x_j - x_k) \mid 1 \leq j < k \leq p\} \cup \{\pm(2x_i - x_j - x_{p+1}) \mid \\
&\quad i \neq j,\ 1 \leq i, j \leq p\}.
\end{aligned} \tag{3.4.2}$$

And if $p \geq 5$, the following is added more in $\Omega(\psi)$.

$$\begin{aligned}
&\{\pm(x_i + x_{p+1} - x_h - x_k) \mid 1 \leq h < k \leq p,\ 1 \leq i \leq p, \\
&\quad i, h, k\text{: each other different}\}.
\end{aligned} \tag{3.4.3}$$

The above (3.4.1), (3.4.2), (3.4.3) and (i) in Proposition 3.3 imply that if another weight ω is in $\Omega(\psi)$, then it satisfies the following inequalities:

$$\begin{aligned}
\#W(SU(p+1))\{\pm\omega\} &\leq l - d_C(S^2 u_p S^2 u_p^* - u_p u_p^*) - \#\{(3.4.1), (3.4.2), (3.4.3)\} \\
&= \begin{cases} p^2 + 2p - 1 & \text{if } p \geq 5, \\ p^3 - 2p^2 + 4p - 1 = 20 & \text{if } p = 3, \\ p^3 + p^2 + p - 1 = 13 & \text{if } p = 2. \end{cases}
\end{aligned} \tag{3.4.4}$$

Lemma 3.4. *Let* $\omega=\sum_{i=1}^{p+1} b_i x_i$ *be a non-zero weight which satisfies the inequalities* (3.4.4). *If* $p\geq 3$, *then* ω *is in* $W(\mathrm{SU}(p+1))\{\pm\omega_o\}$, *where* ω_o *is one of followings*: $a(x_1-x_2)$, ax_1, $a(x_1+x_2)$, $a(\sum_{i=1}^{\frac{1}{2}(p+1)} x_i)$ $(p=5)$.

Proof. We may put $\omega_0=\sum_{i=1}^{p+1} c_i x_i$ so that $(c_1, c_2, \cdots, c_{p+1})=(\overbrace{d_1, \cdots, d_1}^{j_1}, \overbrace{d_2, \cdots,}^{j_2} \cdots \overbrace{d_k, \cdots, d_k}^{j_k}, \overbrace{0, \cdots, 0}^{j_{k+1}})$, $j_1\leq j_2\leq\cdots\leq j_k$ and d_j's are each other different non-zero integers. Furthermore, from the relation $\sum_{i=1}^{p+1} x_i=0$, we may assume $j_k\leq i_{k+1}$. Suppose $k\geq 3$. Then $\# W(\mathrm{SU}(p+1))\{\pm\omega_o\}\geq {}_{p+1}C_{j_1}\times{}_{p+1-j_1}C_{j_2}\times{}_{p+1-j_1-j_2}C_{j_3}$ $\geq(p+1)(p+1-j_1)(p+1-j_1-j_2)\geq(p+1)\left(p+1-\frac{p+1}{4}\right)\left(p+1-\frac{p+1}{2}\right)$ holds. This contradicts (3.4.4). Similarly we can show that if $k=1$ or 2, then ω_o must be one of four types in the lemma. The details are omitted. q.e.d.

Proposition 3.4. *Suppose that* $\varphi\in RRep(\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1)))$ *satisfies all assumptions in Proposition* 3.3 *and* $p\geq 5$. *Then we have* $\Omega'(\psi)=\Omega(\psi)-$*zero weights* $=\{(3.4.1), (3.4.2), (3.4.3)\}\cup W(\mathrm{SU}(p+1))\{x_1-x_2\}$ *where* $\psi=\varphi_c-Res_{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}$ $(S^2u_p\, S^2u_p^*-u_pu_p^*)\otimes\theta$.

Proof. We first note $\Omega'(\psi)\neq\{(3.4.1), (3.4.2), (3.4.3)\}$. If the equality holds, then $h_{Res_{\mathrm{SU}(p)}\psi}$ is $\omega_1=2x_1-(x_p+x_{p+1})\equiv 2x_1+\sum_{i=1}^{p-1} x_i$, and hence ψ has an irreducible direct summand equivalent to $Res_{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}\{(S^2u_p)u_p^*-u_p\}\otimes u_1^*$. Thus $W(\mathrm{SU}(p+1))\{x_1-x_2\}$ must be added to $\{(3.4.1), (3.4.2), (3.4.3)\}$ because of $\Omega(\{(S^2u_p)u_p^*-u_p\}\otimes u_1^*)\supseteqq(p-1)\{x_i-x_{p+1}\,|\,1\leq i\leq p\}$. Therefore, to get the required result, it is sufficient to prove that, except x_1-x_2, any ω_o in Lemma 3.4 is not contained in $\Omega'(\psi)-\{(3.4.1), (3.4.2), (3.4.3)\}$.

From (3.4.4) we may put

$$\Omega'(\psi)=\{(3.4.1), (3.4.2), (3.4.3)\}\cup W(\mathrm{SU}(p+1))\{\pm\omega_o\} \tag{3.4.5}$$

$$\text{or}\quad =\{(3.4.1), (3.4.2), (3.4.3)\}\cup W(\mathrm{SU}(p+1))\{\pm a_1x_1, \pm a_2x_1, \cdots, \pm a_kx_1\} \quad \left(k\leq\frac{1}{2}(p-1)\right) \tag{3.4.6}$$

$$\text{or}\quad =\{(3.4.1), (3.4.2), (3.4.3)\}\cup W(\mathrm{SU}(p+1))\ \{a(x_1+x_2+x_3), \pm bx_1\}\,. \tag{3.4.7}$$

The case of (3.4.7) may occur only for $p=5$. Moreover, the degree of ψ clearly satisfies

$$d_C(\psi)\leq l-d_C(S^2u_p\,S^2u_p^*-u_pu_p^*)=p^3+3p^2-p-1\,. \tag{3.4.8}$$

(1) *Case of* (3.4.5). First suppose $\omega_o=a(x_1+x_2)\,(a\leq 2)$, $ax_1(a\leq 3)$ or $a(x_1+x_2+x_3)$ $(a\leq 2,\ p=5)$. Then the above ω_1 becomes $h_{Res_{SU(p)}\psi}$. Thus $W(\mathrm{SU}(p+1))\{x_1-x_2\}$ is contained in $\Omega(\psi)$. This contradicts (3.4.5). Next suppose $\omega_o=a(x_1+x_2)\,(a\geq 3)$, $ax_1(a\geq 4)$, $a(x_1-x_2)\,(a\geq 2)$ or $a(x_1+x_2+x_3)\,(a\geq 3,\ p=5)$. We replace $\omega_o=a(x_1-x_2)$ by $a(x_1-x_p)$. Then, in any case, ω_o becomes $h_{Res_{SU(p)}\psi}$, and hence $Res_{SU(p)}\psi$ has an irreducible direct summand ρ with $h_\rho=\omega_o$. The ρ has a real form only when $\omega_o=a(x_1-x_p)$. Thus by applying (3.1) to ρ, we see $d_C(\psi)\geq d_C(\rho)>p^3+3p^2-p-1$ if $\omega_o=a(x_1-x_p)$, and $d_C(\psi)\geq d_C(\rho\oplus\rho^*)>p^3+3p^2-p-1$ if $\omega_o=a(x_1+x_2)$ or $a(x_1+x_2+x_3)$. These contradict (3.4.8). And if $\omega_o=ax_1$, then $(a-1)\,x_1+x_2\in\Omega(S^au_p)$ is contained in $\Omega(Res_{SU(p)}\psi)$ because of $\rho\simeq S^au_p$. This contradicts (3.4.5).

(2) *Case of* (3.4.6). If $a_i\leq 3$ for all i, then ω_1 is $h_{Res_{SU(p)}\psi}$. If $a_i\geq 4$ for some i, then so is $\omega_o=a_ix_1$. Thus this case does not occur by the same reason as in (1).

(3) *Case of* (3.4.7). If $a\geq 3$ or $b\geq 4$, then $\omega_o=a(x_1+x_2+x_3)$ or bx_1 becomes $h_{Res_{SU(p)}\psi}$. And if $a\leq 2$ and $b\leq 3$, then so does ω_1. Thus, by the same reason as in (1), neither do this case occur.

From these results, only x_1-x_2 remains as ω_o in (3.4.5) and hence the proposition is proved. q.e.d.

3.5. $\varphi\in RRep(\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1)))$ $(p\neq 3)$ satisfying all assumptions in Proposition 3.3 is really equivalent to $\varphi_{2,m}$ in Example 2.3. That is, we have the following Proposition 3.5. As in the previous subsections, we put $\psi=\varphi_c-Res_{S(U(p)\times U(1))}(S^2u_p\,S^2u_p^*-u_pu_p^*)\otimes\theta$.

Proposition 3.5. *Suppose that $\varphi\in RRep(\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1)))$ satisfies all assumptions in Proposition* 3.3 *and $p\neq 3$. Then $(\varphi\oplus\iota_{S(U(p)\times U(1))})_c$ is equivalent to $Res_{S(U(p)\times U(1))}(S^2u_{p+1}\,S^2u_{p+1}^*-u_{p+1}\,u_{p+1}^*)-\theta$.*

Proof. *Case of $p\geq 5$.* From Proposition 3.4, we first see that $2x_1-x_p-x_{p+1}$ in (3.4.2) becomes $h_{Res_{SU(p)}\psi}$. Hence ψ has $\psi_1\simeq Res_{S(U(p)\times U(1))}\{(S^2u_p)\,u_p^*-u_p\}\otimes u_1^*$ and ψ_1^* as its direct summands. Note that ψ_1 does not have a real form. Next investigate $h_{Res_{SU(p)}(\psi-\psi_1-\psi_1^*)}$. It is x_1-x_p and hence $\psi-\psi_1-\psi_1^*$ has an irreducible direct summand $\psi_2\simeq Res_{S(U(p)\times U(1))}(u_pu_p^*-\theta)\otimes\theta$. Similarly we see $\psi-\psi_1-\psi_1^*-\psi_2=Res_{S(U(p)\times U(1))}\{(S^2u_p)\otimes(u_1^*)^2\oplus(S^2u_p^*)\otimes u_1^2\}$. This shows $(\varphi\oplus\iota_{S(U(p)\times U(1))})_c\simeq Res_{S(U(p)\times U(1))}(S^2u_{p+1}\,S^2u_{p+1}^*-u_{p+1}\,u_{p+1}^*)-\theta$ (see Lemma 2.3.1).

Case of $p=2$. From (3.4.1), $\Omega'(\psi)$ contains $2(x_1-x_3)$. Thus ψ has an irreducible direct summand ψ_1 with $2(x_1-x_3)\in\Omega'(\psi_1)$. Regard ψ as a representation of U(2). Then we see $\psi_1\simeq S^{6-2c}u_2(\Lambda^2u_2)^c$ with $6-2c>0$ (that is, $c\leq 2$), and ψ_1^* also becomes a summand of ψ. Because $\widehat{\mathrm{U}(2)}$ is $\{S^au_2(\Lambda^2u_2)^b\,|\,a\geq 0,\ b\in\boldsymbol{Z}\}$

in which only the type of $S^{-2b}u_2(\Lambda^2 u_2)^b$ has a real form. Note that if $6-2c=0$, then $2(x_1-x_3)$ is not in $\Omega(S^{6-2c}u_2(\Lambda^2 u_2)^c)$. Moreover, from the assumption $14-4c=2d_C(\psi_1)\leq d_C(\psi)\leq l-d_C(S^2u_2\, S^2u_2^*-u_2\, u_2^*)=17$, we get $0\leq c\leq 2$. Put $c=0$ or 1. Then $\sharp W(\mathrm{SU}(3))\{\Omega'(\psi_1\oplus\psi_1^*)-(3.4.1)\}>13$ is derived. This contradicts (3.4.4). Therefore we conclude $c=2$. Next calculate $W(\mathrm{SU}(3))\{\Omega'(\psi_1\oplus\psi_1^*)-(3.4.1)\}$. Then we have $3x_1\in\Omega'(\psi-\psi_1-\psi_1^*)$. Hence by the same way as above, $\psi-\psi_1-\psi_1^*$ has $\psi_2\simeq S^3u_2$ and ψ_2^* as its direct summands. Repeating this argument, we can show that ψ regarded as a representation of U(2) is equivalent to $(S^2u_2)(\Lambda^2u_2)^2\oplus(S^2u_2^*)(\Lambda^2u_2)^{-2}\oplus S^3u_2\oplus S^3u_2^*\oplus(S^2u_2)(\Lambda^2u_2)^{-1}$. Thus, from Lemma 2.3.1, we get the required result. q.e.d.

3.6. Here we give the complete proofs of Theorems C, D and A in this order.

Proof of Theorem C. Suppose $dim\ X(p+1,p)\leq dim\ X_m(p+1,p)$. Then $\varphi_2\in RRep(\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1)))$ constructing the $X(p+1,p)$ satisfies all assumptions in Proposition 3.3 (see Remarks 2.1.2 and 2.1.3). Thus, from Proposition 3.5, we have $\varphi_2\simeq\varphi_{2,m}$, and consequently $dim\ X(p+1,p)=dim\ X_m(p+1,p)$. This completes the proof. q.e.d.

We remark here that Theorem C is obtained only from the conditions on φ_2, that is, (1) $dim_{\boldsymbol{R}}W_{\varphi_2}^{\tilde{S}_p}>dim_{\boldsymbol{R}}W_{\varphi_2}^{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}$ and (2) $\Omega(\varphi_2\oplus\iota_{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))})$ is $W(\mathrm{SU}(p+1))$-invariant.

Proof of Theorem D. Since $X^{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}\neq\Phi$ and $X^{\mathrm{SU}(p+1)}=\Phi$, there exists a point x with $\mathrm{SU}(p+1)_x=\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))$. Let s_x denote the slice representation at x. Then $dim_{\boldsymbol{R}}W_{s_x}^{\tilde{S}_p}>dim_{\boldsymbol{R}}W_{s_x}^{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}$ holds. Otherwise we have $dim\ X^{\tilde{S}_p}=dim\ X^{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}$ because $X^{\tilde{S}_p}$ is connected. Since X is a $\boldsymbol{Q}$-acyclic $\mathrm{SU}(p+1)$-manifold, $\Omega(X)=\Omega(s_x\oplus\iota_{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))})$ is clearly $W(\mathrm{SU}(p+1))$-invariant (see 1.1). Thus s_x satisfies the above (1) and (2). And hence, to get the theorem we have only to replace φ_2 in the proof of Theorem C by s_x. q.e.d.

Proof of Theorem A. Suppose $dim\ X^{\tilde{S}_p}=dim\ X^{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}$. Then we have $X^{\tilde{S}_p}=X^{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}$, and hence $X^{\mathrm{SU}(p+1)}=X^{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(1))}\cap X^{\tilde{S}_{p+1}}\neq\Phi$. This is a contradiction. Thus the required result follows from Theorem D. q.e.d.

REMARK 3.6.1. From the definition, any $X(p+1,p)$ clearly satisfies all assumptions in Theorem D and in Theorem A.

REMARK 3.6.2. When $p=3$, there exists an $X(4,3)$ whose dimension is less than that of $X_m(4,3)$. First put $H_1=H_2=\langle(34)\rangle$, $H_3=\langle(23)\rangle$, $H_4=\langle(12),(34)\rangle$ and $H_5=1\times S_3$, where $(mn)\in S_4$ denotes the transposition between m and n. Next take five irreducible representations, $\psi_1=u_1\otimes u_1^*\otimes\theta\in Irr(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(2))$, $\psi_2=(u_1)^2\otimes(u_1^*)^2\otimes\theta\in Irr(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(2))$, $\psi_3=(u_1)^2\otimes(\Lambda^2u_2^*)\otimes\theta\in Irr(\mathrm{U}(1)\times\mathrm{U}(2)\times\mathrm{U}(1))$, $\psi_4=(\Lambda^2u_2)\otimes(\Lambda^2u_2^*)\in Irr(\mathrm{U}(2)\times\mathrm{U}(2))$, $\psi_5=(u_1)^4\otimes\theta\in Irr$

(U(1)×U(3)), and define $\gamma'_i \in Irr(\tilde{H}_i)$ by $Res_{\tilde{H}_i}\psi_i$ $(1 \leq i \leq 5)$. Furthermore, let $\sigma_i \in Irr(H_i)$ $(i=1, 3)$ be non-trivial irreducible representations of H_i of degree 1 and $\lambda \in Irr(S_4)$ the standard permutation representation of S_4 of degree 3. Denote by σ'_i and λ' the lifts of σ_i and λ to $\tilde{H}_i$ and $\tilde{S}_4$ respectively. Then, as φ_1 and φ_2 constructing the above $X(4, 3)$, we can take the followings: $(\varphi_1)_c = (\bigoplus_{i=1,3} Ind_{\tilde{H}_i}^{\tilde{S}_4} \gamma'_i\, \sigma'_i) \oplus (\bigoplus_{i=2,4,5} Ind_{\tilde{H}_i}^{\tilde{S}_4} \gamma'_i) \oplus Ind_{\tilde{H}_3}^{\tilde{S}_4}(\gamma'_3)^* \sigma'_3 \oplus Ind_{\tilde{H}_5}^{\tilde{S}_4}(\gamma'_5)^* \oplus \lambda'$, $(\varphi_2)_c = Res_{S(U(3)\times U(1))}\{(S^2 u_3 S^2 u_3^* - u_3 u_3^*) \otimes \theta \oplus (S^4 u_3) \otimes \theta \oplus (S^4 u_3^*) \otimes \theta \oplus (S^2 u_3) \otimes (u_1^*)^2 \oplus (S^2 u_3^*) \otimes u_1^2 \oplus \theta \otimes (u_1)^4 \oplus \theta \otimes (u_1^*)^4\}$. These φ_1 and φ_2 really satisfy Condition A in 2.1. The proof is omitted here. Anyway $dim\ X(4, 3) = d_{\boldsymbol{R}}(\iota_{\tilde{S}_4} \oplus \varphi_1) = d_{\boldsymbol{R}}(\iota_{S(U(3)\times U(1))} \oplus \varphi_2) = 77$ is less than $dim\ X_m(4, 3) = 83$. And the weight system of $\psi = (\varphi_2)_c - Res_{S(U(3)\times U(1))} (S^2 u_3\, S^2 u_3^* - u_3 u_3^*) \otimes \theta$ is given by $\{(3.4.1), (3.4.2)\} \cup W(SU(4))\{\pm 4x_1, x_1 + x_2 - x_3 - x_4\}$ (cf. Proposition 3.4).

4. Proof of Theorem B

The purpose of this section is to prove Theorem B in Introduction. The notations are the same as in the previous sections.

First we classify fixed-point free SU(3)-actions on compact $\boldsymbol{Z}_2$-acyclic manifolds X into two types: (I) $X^{S(U(2)\times U(1))} \neq \Phi$, (II) $X^{S(U(2)\times U(1))} = \Phi$. Theorem B is given by showing

(i) every X of type (I) satisfies all assumptions in Theorem A, and hence $dim\ X \geq 26 = dim\ X_m(3,2)$,

(ii) there is no X of type (II) with $dim\ X < 26 = dim\ X_m(3,2)$.

It is easy to show (i). The most part of this section is offered to show (ii). Roughly speaking, under the assumption that X is an SU(3)-manifold of type (II) with $dim\ X < 26$, in 4.3 and 4.4 we prepare some conditions on the weight system $\Omega(X)$ and on the tangential representations at certain points in X. And then we prove that any SU(3)-manifold of type (II) can not really satisfy all of the conditions in 4.3 and 4.4. As seen in the proof of Proposition 4.3, every SU(3)-manifold X of type (II) contains a compact $\boldsymbol{Z}_2$-acyclic submanifold X^{T^α} with a fixed-point free SO(3)-action such that $\Omega(X) \supset \Omega(X^{T^\alpha})$. In order to determine $\Omega(X)$, it is useful to investigate $\Omega(X^{T^\alpha})$. So, in two subsections 4.1 and 4.2 preceding 4.3, we first state a known result for actions of finite groups on $\boldsymbol{Z}_p$-acyclic manifolds in [9], and study the weight systems of compact $\boldsymbol{Z}_2$-acyclic manifolds with fixed-point free SO(3)-actions. The final proof of Theorem B is given in the last subsection 4.5.

4.1. For prime numbers p and q, $\mathcal{G}_p^q$ denotes the family of finite groups G with series of normal subgroups as follows: $H \lhd K \lhd G$, where H is a p-group, K/H a cyclic group and G/K a q-group. Write $\cup_q \mathcal{G}_p^q$ by $\mathcal{G}_p$. Then by calculating Euler characteristics of fixed point sets, we have the following. For the details, see Lemma 1 and Proposition 1 in [9].

Proposition 4.1. *If a finite group G can act on a compact $\boldsymbol{Z}_p$-acyclic manifold without fixed points, then G is not in $\mathcal{G}_p$.*

4.2. Throughout this subsection, let Y be a compact $\boldsymbol{Z}_2$-acyclic manifold with an SO(3)-action Ψ, and specify the maximal torus T' of SO(3) by SO(2)$=\left\{\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3)\right\}$. Furthermore $\{t\}$ denotes the coordinate with respect to the canonical basis for $L(T')$. Then $mt \in L(T')$ (see 1.1) corresponds to the representation of SO(2) defined by $(\mathrm{A}, v) \to \mathrm{A}^m v$ for $\mathrm{A} \in \mathrm{SO}(2)$, $v \in \boldsymbol{R}^2$, and $\Omega(Y)$ is written in the form $a_0\{0\} \cup a_1\{\pm t\} \cup a_2\{\pm 2t\} \cup \cdots$.

The object of this subsection is to describe $\Omega(Y)$ when Ψ is fixed-point free. To do this, we can apply the procedure in Section 1, [11] where the weight system for a fixed-point free SO(3)-action on a disk has been described by comparing the tangential representations at fixed points by O(2) and an octahedral subgroup. Consequently we obtain Proposition 4.2.2 below which follows from Lemmas 4.2.1–4.2.3 and Proposition 4.2.1 stated before it. The details of the proofs are omitted here. Because the proofs are the same as those of the corresponding lemmas and propositions in [11], except that, in our case, the fixed point set Y^P by a p group P is not always connected if $p \neq 2$.

For any $g \in \mathrm{SO}(3)$ and any $y \in Y$ fixed by g, $\chi_y(g)$ denotes the character of g for the tangential representation at y. And by $\Omega_g(Y)$, we denote the character of $g \in \mathrm{SO}(2)$ for the representation of SO(2) giving $\Omega(Y)$. Furtheremore Z_n, D_n and O denote a cyclic subgroup of order n, a dihedral subgroup of order $2n$ and an octahedral subgroup of SO(3) respectively. Both D_n and O are in $\mathcal{G}_2$. From Proposition 4.1 and the fact that Y^P is connected for a 2 group P (see Chapter III in [3]), we get Lemmas 4.2.1, 4.2.2 and consequently Proposition 4.2.1.

Lemma 4.2.1 (cf. Lemma 1 in [11]). *$Y^{\mathrm{O}(2)}$ and Y^{O} are both non-empty.*

Lemma 4.2.2 (cf. Lemma 2 in [11]). *Assume that $g_1, g_2 \in \mathrm{SO}(3)$ are conjugate elements of 2 power order and $y_1, y_2 \in Y$ are points fixed by g_1 and g_2 respectively. Then $\chi_{y_1}(g_1) = \chi_{y_2}(g_2)$ holds.*

Propositton 4.2.1 (cf. Proposition 1 in [11]). *Suppose that Ψ is fixed-point free. Then the followings hold:*

(1) $Y^{O} \neq \Phi$; *for any $y \in Y^{O}$ and any $g \in O$ of 2 power order, there exists $g' \in \mathrm{SO}(2)$ conjugate to g such that $\Omega_{g'}(Y) = \chi_y(g)$,*

(2) $Y^{\mathrm{O}(2)} \neq \Phi$; *$\Omega(Y)$ is the weight system of an actual representation of* O(2),

(3) *$\Omega(Y)$ contains t,*
(4) *dim $Y^{D_4}>$dim Y^{O},*
(5) *dim $Y^{Z_4}>$dim $Y^{Z_{4s}}$ for all $s\geq 2$.*

(1) and (2) follow from Lemmas 4.2.1 and 4.2.2. (3) is based on that Y has an orbit of type SO(3)/O(2) and $t\in\Omega(\iota_{O(2)})$. And (4) is given from the fact that $Y^{O}\subsetneqq Y^{D_4}\supsetneqq Y^{O(2)}$, $Y^{O}\cap Y^{O(2)}=Y^{SO(3)}$ and Y^{D_4} is connected. Similarly we obtain (5). Note that (1) and (2) do not depend on whether Ψ is fixed-point free or not.

It is well known that $O(\cong S_4)$ has just five irreducible real representations (see Section 5 in [12]). For those, we will use the same notations W_1, W_1', W_2', W_3 and W_3' as in [11]. The W_i (resp. W_i') denotes the orientable (resp. unorientable) representation of degree i. Especially W_1 is the trivial representation and W_2' is the only non-trivial representation such that D_4 has positive dimensional fixed point set on the representation space. Lemma 4.2.3 below follows from (1) in Proposition 4.2.1.

Lemma 4.2.3 (cf. Lemma 3 in [11]). *Put $\Omega(Y)=a_0\{0\}\cup a_1\{\pm t\}\cup a_2\{\pm 2t\}\cup\cdots$, and for all $s\geq 1$, set $k_s=a_s+a_{2s}+a_{3s}+\cdots$. Let Ψ_y denote the tangential representation at $y\in Y^{O}$. Then Ψ_y is equivalent to $bW_1\oplus(-a_0+k_1-k_2-2k_4+b)W_1'\oplus(a_0-k_1+2k_2+k_4-b)W_2'\oplus(k_1-2k_2+k_4)W_3\oplus(k_2-k_4)W_3'$, where b is a non-negative integer.*

Proposition 4.2.2 (cf. Proposition 2 in [11]). *Suppose that Ψ is fixed-point free. And let $\{a_i\}$ and $\{k_s\}$ be the non-negative integers defined in Lemma* 4.2.3. *Then $\Omega(Y)$ satisfies the followings*: (i) $a_1>0$, (ii) $k_2>k_4$, (iii) $0\leq k_1-2k_2\leq a_0$ *and* (iv) $k_4>k_{4s}$ *for all $s\geq 2$.*

(i) and (iv) immediately follow from (3) and (5) in Proposition 4.2.1 respectively, and (iii) from (2) by the same reason as in [11]. If (1) in Proposition 4.2.1 holds, then all coefficients of Ψ_y in Lemma 4.2.3 are non-negative. Moreover if (4) holds, then the coefficient for W_2' of Ψ_y is positive. From these we have (ii).

4.3. Return to SU(3)-actions. In the rest of this section, by T we denote the maximal torus ST_3 of SU(3) specified in 1.1, and by T^{α} the corank one subtorus whose Lie algebra is perpendicular to $\alpha=x_1-x_2$ (see the proof of Lemma 3.2). As in Section 2, for a subgroup H of $N(T)/T\cong S_3$, $\tilde{H}$ means $\pi^{-1}(H)$, where π is the projection from $N(T)$ to S_3. Furthermore let π_{α} be the projection from $N(T^{\alpha})=S(U(2)\times U(1))$ to $N(T^{\alpha})/T^{\alpha}\cong SO(3)$ which is given by $Ad_{S(U(2)\times U(1))}-\theta$, and for a subgroup K of SO(3), we write $\pi_{\alpha}^{-1}(K)$ by K_{α}. Groups D_n and O are the same as in 4.2.

In this subsection, from now we suppose that X is a compact Z_2-acyclic SU(3)-*manifold of type* (II) (*that is,* $X^{S(U(2)\times U(1))}=\Phi$) *with dim $X<26$.*

Proposition 4.3. *The non-zero weight system* $\Omega'(X)$ *is*

$$\{\pm(x_i-x_j),\ \pm 4(x_i-x_j),\ \pm m(x_i-x_j),\ \pm n(x_i-x_j)\,|\,1\leq i<j\leq 3,\ \sum_{i=1}^{3} x_i=0\}$$

for $m\equiv 2$ *(mod 4),* $n\equiv 1$ *(mod 2).*

Proof. Since $X^{S(U(2)\times U(1))}=(X^{T^\alpha})^{N(T^\alpha)/T^\alpha}=(X^{T^\alpha})^{SO(3)}=\Phi$ and by Smith theory X^{T^α} is $\boldsymbol{Z}_2$-acyclic, Proposition 4.2.2 can be applied to the restricted SO(3)-action on X^{T^α}. The α corresponds to t in Lemma 4.2.3 and we may put $\Omega(X^{T^\alpha})=a_0\{0\}\cup a_1\{\pm\alpha\}\cup a_2\{\pm 2\alpha\}\cup\cdots$. Let $k_s=a_s+a_{2s}+\cdots(s\geq 1)$. First *dim* $X<26$ implies $k_1\leq 4$. Furthermore, from Proposition 4.2.2, we get $a_1>0$, $k_1=4$, $k_2=2$, $k_4=a_4=1$ and $k_{4s}=0$ for $s\geq 2$. This shows $\Omega'(X^{T^\alpha})=\{\pm\alpha,\ \pm 4\alpha,\ \pm m\alpha,\ \pm n\alpha\}$ where $m\equiv 2$ (mod 4) and $n\equiv 1$ (mod 2). Hence $\Omega'(X)$ is given as in the proposition, because *dim* $X<26$ and $\Omega(X)$ is invariant under the natural W(SU(3))-action (see 1.1). q.e.d.

By Proposition 4.3, *we may assume dim* $X=24$ *or* 25. So put $d=\mathit{dim}\,X-24$ and $\Lambda=\{\alpha,\ 4\alpha,\ m\alpha,\ n\alpha\}$, where $\alpha=x_1-x_2$ and m, n are as in Proposition 4.3.

Since S_3, $O\in\mathcal{G}_2$, $X^{\tilde{S}_3}$ and $X^{O_\alpha}=(X^{T^\alpha})^O$ are both non-empty (see Proposition 4.1). Thus there are points x and y in X such that the isotropy subgroups $SU(3)_x=\tilde{S}_3$ (see Remark 2.1.1) and $SU(3)_y\supseteqq O_\alpha$ respectively. Denoting by φ_x and φ_y the tangential representation at x and the restriction of that at y to O_α respectively, they satisfy the conditions:

$$\begin{aligned}&\text{(a)}\quad \Omega(\varphi_x)=\Omega(X)=d\{0\}\cup W(SU(3))\,\Lambda,\ (d=0\text{ or }1)\,,\\ &\text{(b)}\quad Res_{(D_4)_\alpha}\varphi_x\simeq Res_{(D_4)_\alpha}\varphi_y\,.\end{aligned}\tag{4.3}$$

The condition (a) follows from Proposition 4.3 and (b) from the facts that $X^{\tilde{S}_3}\subseteqq X^{(D_4)_\alpha}\supseteqq X^{O_\alpha}$ and D_4 is a 2-group, and consequently $X^{(D_4)_\alpha}$ is connected.

4.4. In this subsection, we investigate the above φ_x and φ_y more precisely.

First we fix some subgroups of SO(3). Set $SO(2)=\left\{\begin{pmatrix}a&b&0\\c&d&0\\0&0&1\end{pmatrix}\in SO(3)\right\}$ and $O(2)=\left\{\begin{pmatrix}a&b&0\\c&d&0\\0&0&\pm1\end{pmatrix}\in SO(3)\right\}$. For any $n\geq 2$, Z_n is the cyclic subgroup of SO(2) of order n and D_n is the subgroup generated by Z_n and $\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix}$. And fix O as the subgroup of all 3-square matrices with one non-zero entry in every row and column.

Lemma 4.4.1. $(\varphi_x)_c$ *is equivalent to* $\bigoplus_{\gamma\in\Lambda}(Ind_T^{\tilde{S}_3}\gamma)\oplus d\zeta'$*, where* $\zeta'\in CRep(\tilde{S}_3)$

is the lift of $\zeta \in CRep(S_3)$ with $d_C(\zeta)=1$. Moreover each direct summand is irreducible.

Proof. Consider the extension $1 \to T \to \tilde{S}_3 \to S_3 \to 1$. Clearly S_3 acts on $W(\mathrm{SU}(3))\,\Lambda$ freely. Thus, by Proposition 1.3 and (a) in (4.3), we have the quired result. q.e.d.

Let S be a set of representatives of double cosets $(D_4)_\alpha \backslash \tilde{S}_3/T$, that is, $S=\{1, \begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}\}$. Then for each $t \in S$, $tTt^{-1} \cap (D_4)_\alpha = (Z_4)_\alpha$ holds. Thus, from Lemma 4.4.1 and Proposition 1.2.1, we have

$$\begin{aligned} Res_{(D_4)_\alpha}\{(\varphi_x)_c - d\zeta'\} &\simeq \bigoplus_{\gamma\in\Lambda} Res_{(D_4)_\alpha}\, Ind_T^{\tilde{S}_3}\gamma \\ &\simeq \bigoplus_{\gamma\in\Lambda}\big(\bigoplus_{t\in S} Ind_{(Z_4)_\alpha}^{(D_4)_\alpha}\gamma^t\big)\,. \end{aligned} \tag{4.4}$$

Here $\gamma^t \in Irr((Z_4)_\alpha)$ is defined as in Proposition 1.2.1.

Lemma 4.4.2. *If $\gamma=a(x_1-x_2)$, $a \not\equiv 0$ (mod 4) and $t \neq 1$, then $Ind_{(Z_4)_\alpha}^{(D_4)_\alpha}\gamma^t$ is irreducible.*

Proof. Put $g=\mathrm{diag}(e^{-\frac{\pi}{4}i}, e^{\frac{\pi}{4}i}, 1) \in (Z_4)_\alpha$ and take any $s \in (D_4)_\alpha - (Z_4)_\alpha$. Then by calculating $(\gamma^t)^s(g)$ and $\gamma^t(g)$, we see $(\gamma^t)^s \not\simeq \gamma^t$, where $(\gamma^t)^s$ is defined as in Proposition 1.2.2. Thus from Proposition 1.2.2, we obtain the lemma. q.e.d.

Consider the central extension $1 \to T^\alpha \to O_\alpha \to O \to 1$. This is a restriction of $1 \to T^\alpha \to N(T^\alpha) = \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1)) \to \mathrm{SO}(3) \to 1$. Thus the product of elements (g_1, h_1) and (g_2, h_2) in $O_\alpha = O \times T^\alpha$ (as sets) is given by $(g_1, h_1)\circ(g_2, h_2) = (g_1 g_2, \chi(g_1, g_2) h_1 h_2)$, where $\chi(g_1, g_2)$ is 1 or $\mathrm{diag}(-1, -1, 1)$. This shows that if $\gamma \in Irr(T^\alpha)$ is the form ax_1 and $a \equiv 0$ (mod 2), then γ extends to $\gamma' \in Irr(O_\alpha)$. In fact, γ' can be defined by $\gamma'(g, h) = \gamma(h)$. Therefore, from Proposition 1.3, the lemma below follows.

Lemma 4.4.3. $\hat{O}_\alpha = \bigcup_{\hat{\gamma}\in\hat{T}^\alpha/O_\alpha} \{\hat{\nu} \mid \hat{\nu} \in (\check{O}_\alpha)_\gamma = \check{O}_\alpha\}$. *Moreover, if $\gamma = ax_1$ and $a \equiv 0$ (mod 2), then ν is equivalent to $\sigma'\gamma'$, where σ' and γ' are the lifts of $\sigma \in Irr(O)$ and γ to O_α respectively.*

4.5. Now we can prove Theorem B.

Proof of Theorem B. Let X be a compact $\boldsymbol{Z}_2$-acyclic manifold with a fixed-point free SU(3)-action. As mentioned before, we classify X into two types: (I) $X^{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))} \neq \Phi$, (II) $X^{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))} = \Phi$. First suppose that X is of type (I). Since $S_3 \in \mathcal{G}_2$ and $S_2 \cong \boldsymbol{Z}_2$, $X^{\tilde{S}_3} \neq \Phi$ and $X^{\tilde{S}_2}$ is connected. Thus X satisfies all assumptions in Theorem A, and hence we have $dim\, X \geq dim\, X_m(3, 2) = 26$. Next suppose that X is of type (II) and $dim\ X<26$. Then $\Omega'(X)$ is given as in Proposition 4.3, and we can take the representations $\varphi_x \in RRep(\tilde{S}_3)$ and $\varphi_y \in RRep(O_\alpha)$ satisfying (4.3). From (4.4) and Lemma 4.4.2, $Res_{(D_4)_\alpha}(\varphi_x)_c$ has an irreducible direct summand equivalent to $Ind_{(Z_4)_\alpha}^{(D_4)_\alpha} m(x_1-x_2)^t$ $(t \neq 1)$ of degree 2. And from (b) in (4.3), we have $Res_{T^\alpha}\varphi_y \cong Res_{T^\alpha}\varphi_x$. Hence, from (b) in (4.3) and Lemma 4.4.3, $Res_{(D_4)_\alpha}(\varphi_y)_c$ has a direct summand which is equivalent to $Res_{(D_4)_\alpha}[\sigma' Res_{T^\alpha}\{m(x_1-x_2)^t\}']$ $(\sigma \in Irr(O))$ and is decomposed into an irreducible direct summand of degree 2 and others. This implies that σ is of degree 3. Because the restriction of any other irreducible representation of O to D_4 is a sum of representations of degree 1 (see Section 5 in [12]). Therefore, at least two weights γ with $Res_{T^\alpha}\gamma = \pm 3mx_1$ are added more to $\Omega'(X)$. This is a contradiction. Thus we have $dim\ X \geq 26$. q.e.d.

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