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THE IMBEDDING PROBLEM OF 3-MANIFOLDS INTO 4-MANIFOLDS

Dedicated to Professor Hiroshi Toda on his 60th birthday

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We consider mainly the case $n=3$ of the following general *Imbedding Problem* in the topological category:

Under what relations between an n-manifold M and an $(n+1)$*-manifold W, both closed, connected and oriented, does there exist an imbedding from M to W?*

Since the problem is trivial for $n\leq 2$, the case $n=3$ is the first appearing non-trivial case. In general, for any n, there are two kinds of imbeddings from M to W. An imbedding f from M to W is said to be of *type* I or II, according to whether $W-fM$ is connected or not. If such an imbedding f exists, then we say that *M is type* I *or* II *imbedded in W*. If f is of type II, then $W-fM$ is seen to have exactly two components, since the boundary map $\partial\colon H_1(W, W-fM; Z_2)\to\tilde{H}_0(W-fM; Z_2)$ is onto and there is a duality isomorphism $H_1(W, W-fM; Z_2)\cong H^n(fM; Z_2)$ $(\cong Z_2)$ (cf. Spanier [Sp; p. 342]). It is possible to characterize the type of an imbedding $f\colon M\to W$ in terms of homology. In fact, f is of type II or I according to whether the homomorphism $f_*\colon H_n(M; Z_2)\to H_n(W; Z_2)$ is trivial or not. This is proved by examining the following commutative diagram:

$$\begin{array}{ccccc} H^n(W; Z_2) & \xrightarrow{i^*} & H^n(fM; Z_2) & & \\ \cong\uparrow & & \uparrow\cong & & \\ H_1(W; Z_2) & \xrightarrow{j_*} & H_1(W, W-fM; Z_2) & \xrightarrow{\partial} & \tilde{H}_0(W-fM; Z_2) \to 0\,, \end{array}$$

where the vertical maps are the duality isomorphisms (cf. [Sp]). For example, if $\beta_1(W; Z)=0$, then we see from the Poincaré duality and the universal coefficient theorem that any imbedding from M to W is of type II. A typical example of a type I imbedding is $M\overset{\cong}{\to}1\times M\subset S^1\times M=W$. Let $n=3$. First we show that there is an estimate of $\beta_2(W; Z)$ by $\beta_1(M; Z)$ or by certain integral invariants of an infinite cyclic covering of M, provided that M is topologically type

II imbedded in W. By this estimate, we find *infinitely many M which are smoothly type* I *imbedded in some smooth 4-manifolds having the Q-homology of $S^1\times S^3$, but not topologically type* II *imbeddable in any W with $\beta_2(W; Z)<r$, for each $r>0$* (See Theorem 2.5). This suggsets that the treatment of type I imbeddings is more difficult than that of type II imbeddings, because if M is type II imbedded in W, then M is also type I imbedded in some W' with $\beta_2(W'; Z)=\beta_2(W; Z)$ [For example, take $W'=W\sharp S^1\times S^3$]. We can avoid this difficulty by considering punctured imbeddings instead of type I imbeddings. We denote by M^o a compact punctured manifold of M. Then our main result is that *there is an estimate of $\beta_2(W; Z_2)$ by $\beta_1(M; Z)$ or by certain integral invariants of an infinite cyclic covering of the double DM^o, provided that M^o is topologically imbedded in W.* This estimate enables us to find *infinitely many M such that M^o are not topologically imbeddable in any W with $\beta_2(W; Z_2)<r$, for each $r>0$* (See Theorem 3.2). This research was initially planned in the piecewise-linear category (cf. [K, 1], [K, 2]), but after Freedman's work [F], it became a standard fact that there is a great difference between the piecewise-linear and topological imbeddabilities. In fact, Freedman showed that all homology 3-spheres are imbedded in S^4 by locally flat topological imbeddings, but, as it is well-known, not by piecewise-linear imbeddings. This is the reason why we are converted to the topological category.

In §1 we describe briefly the signature theorem for an infinite cyclic covering of a compact oriented $4m$-manifold with boundary, given in [K, 4]. From this, we derive an estimate of the $4m$-manifold by integral invariants of an infinite cyclic covering of the boundary. Several properties on an infinite cyclic covering of a closed $(4m-1)$-manifold are also given here. In §2 we discuss the estimate of a type II imbedding and its consequence, and in §3, the estimate of a punctured imbedding and its consequence. In §4 we remark that similar results hold in the case $n=4m-1$ $(m>1)$.

1. The signature theorem for an infinite cyclic covering

Consider a pair $(B, \dot{\gamma})$ where B is a compact oriented $(4m-1)$-manifold and $\dot{\gamma}\in H^1(B; Z)$. Using the infinite cyclic covering space $\tilde{B}$ of B associated with $\dot{\gamma}$, we have defined in [K, 3] integral invariants, $\sigma_a^{\dot{\gamma}}(B)$, $a\in[-1, 1]$, of the proper oriented homotopy equivalence class of $(B, \dot{\gamma})$. The invariant $\sigma_a^{\dot{\gamma}}(B)$ is called the *local signature* of $(B, \dot{\gamma})$ *at* a and vanishes except a finite number of a. The sum $\sum_{a\in[-1,1]}\sigma_a^{\dot{\gamma}}(B)$ is called the *signature* of $(B, \dot{\gamma})$ and denoted by $\sigma^{\dot{\gamma}}(B)$. Next, consider a pair (X, γ) where X is a compact oriented $4m$-manifold and $\gamma\in H^1(X; Z)$. Using the infinite cyclic covering space $\tilde{X}$ of X associated with γ, we have also defined in [K, 4] two kinds of integral invariants, $\tau_{a-0}^{\gamma}(X)$ for $a\in(-1, 1]$ and $\tau_{a+0}^{\gamma}(X)$ for $a\in[-1, 1)$, of the proper oriented homotopy equivalence class of (X, γ). The following theorem, which we call the *signature theorem*, was proved in [K, 4]:

Theorem 1.1. *Assume that* $(B, \dot{\gamma})$ *is the boundary of* (X, γ) *with a compact oriented* $4m$*-manifold* X *and* $\gamma \in H^1(X; Z)$. *Then*

$$\tau^{\gamma}_{a-0}(X) - \text{sign } X = \sum_{x \in [a,1]} \sigma^{\dot{\gamma}}_x(B) \quad \text{and} \quad \tau^{\gamma}_{a+0}(X) - \text{sign } X = \sum_{x \in (a,1]} \sigma^{\dot{\gamma}}_x(B)\,.$$

Note that $\sigma^{\dot{\gamma}}_{-1}(B)$ does not appear in the above identities. To simplify the notations, we denote $\tau^{\gamma}_{a+0}(X)$ by $\tau^{\gamma}_a(X)$ and the sum $\sum_{x \in (a,1]} \sigma^{\dot{\gamma}}_x(B)$ by $\tau^{\dot{\gamma}}_a(B)$. Let $\tau^{\gamma}_1(X) = \lim_{a \to 1-0} \tau^{\gamma}_a(X)$ and $\tau^{\dot{\gamma}}_1(B) = \lim_{a \to 1-0} \tau^{\dot{\gamma}}_a(B)$ $(=\sigma^{\dot{\gamma}}_1(B))$. Then the signature theorem implies the identity

$$\tau^{\gamma}_a(X) - \text{sign } X = \tau^{\dot{\gamma}}_a(B)$$

for all $a \in [-1, 1]$. Note that $\sigma^{\dot{\gamma}}_{-1}(B) + \tau^{\dot{\gamma}}_{-1}(B) = \sigma^{\dot{\gamma}}(B)$. Let (Y, A) be a pair such that Y is a compact manifold and A is a compact submanifold. Let $(\tilde{Y}, \tilde{A})$ be the infinite cyclic covering space pair of (Y, A) associated with an element $\gamma \in H^1(Y; Z)$. Let $\langle t \rangle$ be the covering transformation group with a specified generator t. Let $\Lambda = Z\langle t \rangle$ and $\Gamma = Q\langle t \rangle$. Since $H_*(\tilde{Y}, \tilde{A}; Z)$ is a finitely generated Λ-module and Λ is Noetherian, we see that the kernel of $t-1: H_*(\tilde{Y}, \tilde{A}; Z) \to H_*(\tilde{Y}, \tilde{A}; Z)$ is a finitely generated abelian group. We denote this rank by $\kappa^{\gamma}_*(Y, A; Z)$. It also equals the Q-dimension of the kernel of $t-1: H_*(\tilde{Y}, \tilde{A}; Q) \to H_*(\tilde{Y}, \tilde{A}; Q)$. The following is easily obtained (cf. [K, 1; Lemma 1.1]):

Lemma 1.2. *For any integer* $d \neq 0$, $\kappa^{d\gamma}_*(Y, A) = \kappa^{\gamma}_*(Y, A)$.

Let $TH_*(\tilde{Y}, \tilde{A}; Q)$ be the Γ-torsion part of $H_*(\tilde{Y}, \tilde{A}; Q)$, which is a finitely generated Γ-module, and $BH_*(\tilde{Y}, \tilde{A}; Q) = H_*(\tilde{Y}, \tilde{A}; Q)/TH_*(\tilde{Y}, \tilde{A}; Q)$, which is Γ-free. We denote this rank by $\beta^{\gamma}_*(Y, A; Q)$. We use the signature theorem to prove the following:

Lemma 1.3. *Let* B *be a closed oriented* $(4m-1)$*-manifold and* $\dot{\gamma} \in H^1(B; Z)$ *and* d *be a non-zero integer.*

(1) *For a real number* θ *such that* $\cos d\theta \neq \pm 1$ *and* $\sigma^{\dot{\gamma}}_{\cos d\theta}(B) = 0$, *we have* $\tau^{d\dot{\gamma}}_{\cos\theta}(B) = \tau^{\dot{\gamma}}_{\cos d\theta}(B)$ *and* $\sigma^{d\dot{\gamma}}_{\cos\theta}(B) = 0$,

(2) $\sigma^{d\dot{\gamma}}(B) = \sigma^{\dot{\gamma}}(B)$ *(if* d *is odd) or* 0 *(if* d *is even).*

The following is direct from Lemma 1.3:

Corollary 1.4. (1) $\tau^{d\dot{\gamma}}_1(B) = \tau^{\dot{\gamma}}_1(B)$,

(2) $\tau^{d\dot{\gamma}}_{-1}(B) = \tau^{\dot{\gamma}}_{-1}(B)$ *(if* d *is odd) or* $\tau^{\dot{\gamma}}_1(B)$ *(if* d *is even),*

(3) $\sigma^{d\dot{\gamma}}_{-1}(B) = \sigma^{\dot{\gamma}}_{-1}(B)$ *(if* d *is odd) or* $-\sigma^{\dot{\gamma}}_1(B)$ *(if* d *is even),*

(4) *If* $\cos d\theta \neq \pm 1$, *then* $\sigma^{d\dot{\gamma}}_{\cos\theta}(B) = \text{sign}(\sin\theta \sin|d|\theta)\, \sigma^{\dot{\gamma}}_{\cos d\theta}(B)$,

(5) *If* $\cos d\theta=\pm 1$ *but* $\cos\theta\neq\pm 1$, *then* $\sigma^{d\dot{\gamma}}_{\cos\theta}(B)=0$.

1.5. Proof of Lemma 1.3. First, assume that $(B, \dot{\gamma})$ is the boundary of a pair (X, γ). Let $\tilde{X}$ and $\tilde{X}^{(d)}$ be the infinite cyclic covering spaces of X associated with γ and $d\gamma$, respectively. Let $A(t)$ be a t-Hermitian matrix, which is the Γ-intersection matrix associated with a Γ-basis $e_1, e_2, \cdots, e_r$ of $BH_{2m}(\tilde{X}; Q)$. By [K, 1; Lemma 1.1], we can consider $e_1, e_2, \cdots, e_r$ as a Γ-basis for $BH_{2m}(\tilde{X}^{(d)}; Q)$, associated with which the Γ-intersection matrix is $A(t^d)$. Since $\sigma^{\dot{\gamma}}_{\cos d\theta}(B)=0$, it follows from the signature theorem that

$$\begin{aligned}\tau^{\dot{\gamma}}_{\cos d\theta}(B) &= \tau^{\gamma}_{\cos d\theta\pm 0}(X)-\operatorname{sign} X\\ &=\lim_{d\nu\to d\theta\pm 0}\operatorname{sign} A(e^{id\nu})-\operatorname{sign} X\\ &=\lim_{\nu\to\theta\pm 0}\operatorname{sign} A((e^{i\nu})^d)-\operatorname{sign} X\\ &=\tau^{d\gamma}_{\cos\theta\pm 0}(X)-\operatorname{sign} X\\ &=\tau^{d\dot{\gamma}}_{\cos\theta}(B)\quad\text{and}\quad \sigma^{d\dot{\gamma}}_{\cos\theta}(B)=0\,,\end{aligned}$$

showing (1). For (2) note that $\sigma^{\dot{\gamma}}(B)$ is the α-invariant of the double covering space of B associated with the Z_2-reduction $\dot{\gamma}(2)\in H^1(B; Z_2)$ of $\dot{\gamma}$ (See [K, 4; Lemma 4.3]). Since it is similar for $\sigma^{d\dot{\gamma}}(B)$, we see that $\sigma^{d\dot{\gamma}}(B)=\sigma^{\dot{\gamma}}(B)$ (if d is odd) or 0 (if d is even), showing (2). If $(B, \dot{\gamma})$ is not a boundary, then some multiple $N(B, \dot{\gamma})$ $(N>0)$ is a boundary (cf. [K, 4; Remark 1.6]) and we obtain the identities (1), (2) on $N(B, \dot{\gamma})$ in place of $(B, \dot{\gamma})$. Dividing them by N, we obtain the desired (1), (2). This completes the proof.

For an abelian group H, let tH be the torsion part and $bH=H/tH$. Let X be a compact oriented $4m$-manifold with boundary B. Let $\hat{\beta}_*(X; Z)$ be the rank of the cokernel of the natural homomorphism $H_*(B; Z)\to H_*(X; Z)$. Note that any intersection matrix on $bH_{2m}(X; Z)$ has the rank $\hat{\beta}_{2m}(X; Z)$, by Poincaré duality.

Theorem 1.6. *Assume that for some non-zero integer d,* $(B, d\dot{\gamma})$ *is the boundary of a pair* (X, γ) *with a compact oriented 4m-manifold X and* $\gamma\in H^1(X; Z)$. *Then for all a,*

$$|\tau^{\dot{\gamma}}_a(B)|-\kappa^{\dot{\gamma}}_{2m-1}(B)\leq\hat{\beta}_{2m}(X; Z)+|\operatorname{sign} X|\,.$$

Proof. By Lemma 1.2, $\kappa^{\dot{\gamma}}_{2m-1}(B)=\kappa^{d\dot{\gamma}}_{2m-1}(B)$. By Lemma 1.3(1),

$$\max_{a\in[-1,1]}\tau^{\dot{\gamma}}_a(B)=\max_{a\in[-1,1]}\tau^{d\dot{\gamma}}_a(B)\quad\text{and}\quad\min_{a\in[-1,1]}\tau^{\dot{\gamma}}_a(B)=\min_{a\in[-1,1]}\tau^{d\dot{\gamma}}_a(B)\,.$$

Thus, we may assume $d=1$. Let $(\tilde{X}, \tilde{B})$ be the infinite cyclic covering space pair of (X, B) associated with γ. Let $\hat{\beta}^{\gamma}_*(X; Q)$ be the Γ-rank of the cokernel of the natural homomorphism $H_*(\tilde{B}; Q)\to H_*(\tilde{X}; Q)$. By the exact sequence of

$(\tilde{X}, \tilde{B})$, we have

$$\hat{\beta}^{\gamma}_{2m}(X; Q) = \Sigma^{2m}_{q=0}(-1)^q \beta^{\gamma}_q(X, B; Q) + \Sigma^{2m-1}_{q=0}(-1)^q \beta^{\dot{\gamma}}_q(B; Q) + \Sigma^{2m-1}_{q=0}(-1)^{q+1} \beta^{\gamma}_q(X; Q).$$

From the Wang exact sequence

$$\rightarrow H_q(\tilde{X}, \tilde{B}; Q) \xrightarrow{t-1} H_q(\tilde{X}, \tilde{B}; Q) \xrightarrow{p_*} H_q(X, B; Q) \rightarrow H_{q-1}(\tilde{X}, \tilde{B}; Q) \xrightarrow{t-1} H_{q-1}(\tilde{X}, \tilde{B}; Q) \rightarrow,$$

we see that $\beta_q(X, B; Z) = \beta^{\gamma}_q(X, B; Q) + \kappa^{\gamma}_q(X, B) + \kappa^{\gamma}_{q-1}(X, B)$. Similarly, $\beta_q(B; Z) = \beta^{\dot{\gamma}}_q(B; Q) + \kappa^{\dot{\gamma}}_q(B) + \kappa^{\dot{\gamma}}_{q-1}(B)$ and $\beta_q(X; Z) = \beta^{\gamma}_q(X; Q) + \kappa^{\gamma}_q(X) + \kappa^{\gamma}_{q-1}(X)$. Note that $\hat{\beta}_{2m}(X; Z) = \Sigma^{2m}_{q=0}(-1)^q \beta_q(X, B; Z) + \Sigma^{2m-1}_{q=0}(-1)^q \beta_q(B; Z) + \Sigma^{2m-1}_{q=0}(-1)^{q+1} \beta_q(X; Z)$. Then we have

$$\hat{\beta}^{\gamma}_{2m}(X; Q) = \hat{\beta}_{2m}(X; Z) - \kappa^{\gamma}_{2m}(X, B) + \kappa^{\dot{\gamma}}_{2m-1}(B) - \kappa^{\gamma}_{2m-1}(X) \leq \hat{\beta}_{2m}(X; Z) + \kappa^{\dot{\gamma}}_{2m-1}(B).$$

The inequality $|\tau^{\gamma}_a(X)| \leq \hat{\beta}^{\gamma}_{2m}(X; Q)$ is directly obtained from the definition of $\tau^{\gamma}_a(X)$ (cf. [K, 4]). Therefore, by the signature theorem,

$$\begin{aligned} |\tau^{\dot{\gamma}}_a(B)| &\leq |\tau^{\gamma}_a(X)| + |\text{sign } X| \\ &\leq \hat{\beta}^{\gamma}_{2m}(X; Q) + |\text{sign } X| \\ &\leq \hat{\beta}_{2m}(X; Z) + \kappa^{\dot{\gamma}}_{2m-1}(B) + |\text{sign } X|. \end{aligned}$$

This completes the proof.

Corollary 1.7. *Under the assumption of Theorem* 1.6,

$$|\tau^{\dot{\gamma}}_a(B)| \leq \beta_{2m}(X; Z) + |\text{sign } X|$$

for all a.

Proof. By the proof of Theorem 1.6, $|\tau^{\dot{\gamma}}_a(B)| \leq |\tau^{\gamma}_a(X)| + |\text{sign } X|$ and $|\tau^{\gamma}_a(X)| \leq \hat{\beta}^{\gamma}_{2m}(X; Q) \leq \beta^{\gamma}_{2m}(X; Q) \leq \beta^{\gamma}_{2m}(X; Q) + \kappa^{\gamma}_{2m}(X) + \kappa^{\gamma}_{2m-1}(X) = \beta_{2m}(X; Z)$, completing the proof.

REMARK 1.8. In Theorem 1.6 and Corollary 1.7, if we replace $\tau^{\dot{\gamma}}_a(B)$ with $\sigma^{\dot{\gamma}}(B)$, then the resulting inequalities do not hold in general. Some counterexample was given in [K, 4; 4.5].

2. Type II imbeddings

Theorem 2.1. *Assume that M is topologically type II imbedded in W. Then* $\beta_1(M; Z) \leq \beta_2(W; Z)/2$ *or there is an indivisible element* $\dot{\gamma} \in H^1(M; Z)$ *such* *that for all a*

$$|\tau_a^{\dot{\gamma}}(M)|-\kappa_1^{\dot{\gamma}}(M)\leq\beta_2(W;Z)+|\text{sign } W|\leq 2\beta_2(W;Z).$$

Proof. Assume that $\beta_1(M;Z)>\beta_2(W;Z)/2$. Regard $f: M\subset W$. Since it is of type II and $H_1(W, W-M;Z)\cong H^3(M;Z)\cong Z$, the boundary map $\partial: H_1(W, W-M;Z)\to\tilde{H}_0(W-M;Z)$ is an isomorphism, so that the natural homomorphism $H_1(W-M;Z)\to H_1(W;Z)$ is onto. Using Quinn's handle straightening lemma [Q], we can kill $H_1(W;Q)$ without changing $\beta_2(W;Z)$ by a surgery on $W-M$. We assume $\beta_1(W;Z)=0$. By Ancel/Cannon [A/C], the imbedding $f_P=f\times\text{id}: M_P=M\times CP^2\subset W\times CP^2=W_P$ is homotopic to a bi-collared imbedding $f_P': M_P\to W_P$, which is also of type II. Let $f_P' M_P=M_P'$. M_P' splits W_P into two compact connected submanifolds E', E''. To see that $\beta_1(W_P-M_P';Z)\neq 0$, suppose that $H_1(W_P-M_P';Q)=0$. Then $H_1(E';Q)=H_1(E'';Q)=0$ and $\beta_2(E', M_P';Z)\geq\beta_1(M_P';Z)$ and $\beta_2(E'', M_P';Z)\geq\beta_1(M_P';Z)$. Hence $\beta_2(W, M;Z)=\beta_2(W_P, M_P';Z)=\beta_2(E', M_P';Z)+\beta_2(E'', M_P';Z)\geq 2\beta_1(M_P';Z)=2\beta_1(M;Z)$. Since $H_3(W;Q)=H_1(W;Q)=0$, we see from the exact sequence of (W, M) that $\beta_2(W, M;Z)=\beta_1(M;Z)+\beta_2(W;Z)-\beta_2(M;Z)=\beta_2(W;Z)$, so that $\beta_2(W;Z)\geq 2\beta_1(M;Z)$, contradicting our assumption. Therefore, $\beta_1(W_P-M_P';Z)=\beta_1(E';Z)+\beta_1(E'';Z)\neq 0$. Say $\beta_1(E';Z)\neq 0$. Let $\gamma\in H^1(E';Z)$ be any non-zero element. Since the natural map $H_1(M_P';Q)\to H_1(E';Q)$ is onto, $\dot{\gamma}_P'=\gamma|M_P'\in H^1(M_P';Z)$ is not zero. Write $\dot{\gamma}_P'=d\dot{\gamma}_P$ for an integer $d\neq 0$ and an indivisible element $\dot{\gamma}_P$. By Theorem 1.6,

$$|\tau_a^{\dot{\gamma}_P}(M_P')|-\kappa_3^{\dot{\gamma}_P}(M_P')\leq\hat{\beta}_4(E';Z)+|\text{sign } E'|.$$

Let $\dot{\gamma}\in H^1(M;Z)$ correspond to $\dot{\gamma}_P$. Directly, $\kappa_3^{\dot{\gamma}_P}(M_P')=\kappa_1^{\dot{\gamma}}(M)$. By [K, 3], $\tau_a^{\dot{\gamma}_P}(M_P')=\tau_a^{\dot{\gamma}}(M)$. Let $H'\subset H_4(E';Q)$ and $H''\subset H_4(E'';Q)$ be Q-subspaces of dimensions $\hat{\beta}_4(E';Z)$ and $\hat{\beta}_4(E'';Z)$ on which Q-intersection matrices are non-singular, respectively.

Lemma 2.2. *The composite* $H'\oplus H''\subset H_4(E';Q)\oplus H_4(E'';Q)\xrightarrow{i'_*+i''_*} H_4(W_P;Q)\xrightarrow{\text{projection}} H_2(W;Q)\otimes H_2(CP^2;Z)$ *is injective, where* i'_* *and* i''_* *are natural maps.*

Assuming this lemma, we have $\hat{\beta}_4(E';Z)+\hat{\beta}_4(E'';Z)\leq\beta_2(W;Z)$. By the Novikov addition theorem, sign $E'+$sign $E''=$sign $W_P=$sign W. Since $|\text{sign } E''|\leq\hat{\beta}_4(E'';Z)$, it follows that

$$|\tau_a^{\dot{\gamma}}(M)|-\kappa_1^{\dot{\gamma}}(M)\leq\hat{\beta}_4(E';Z)+|\text{sign } E'|\leq\beta_2(W;Z)-\hat{\beta}_4(E'';Z)+|\text{sign } W|+|\text{sign } E''|\leq\beta_2(W;Z)+|\text{sign } W|.$$

This completes the proof except the proof of Lemma 2.2.

2.3. Proof of Lemma 2.2. Using the intersection pairing Int_{W_P} on $H_4(W_P; Q)$, we see that $i'_*+i''_*\,|H'\oplus H''$ is injective, whose image we denote by H. Let $x\in H$ be non-zero and write $x=x_0+x_2+x_4$ with $x_i\in H_i(W; Q)\otimes H_{4-i}(CP^2; Z)$. If $x_4\neq 0$, then there is an element $x'_0\in H_0(W; Q)\otimes H_4(CP^2; Z)$ with $\mathrm{Int}_{W_P}(x_4, x'_0)\neq 0$. Then $\mathrm{Int}_{W_P}(x, x'_0)=\mathrm{Int}_{W_P}(x_4, x'_0)\neq 0$. But, x'_0 is represented by a cycle in M'_P and hence $\mathrm{Int}_{W_P}(H, x'_0)=0$, which is a contradiction. Thus, $x_4=0$ and $x=x_0+x_2$. Note that there is an element $x'=x'_0+x'_2$ in H with $\mathrm{Int}_{W_P}(x, x')\neq 0$. Then $\mathrm{Int}_{W_P}(x, x')=\mathrm{Int}_{W_P}(x_2, x'_2)\neq 0$, and $x_2\neq 0$. This completes the proof of Lemma 2.2.

Since any imbedding from M to W with $\beta_1(W; Z)=0$ is of type II, the following is direct from Theorem 2.1:

Corollary 2.4. *If M is topologically imbedded in any W with $H_*(W; Q)\cong H_*(S^4; Q)$ and $\beta_1(M; Z)\neq 0$, then there is an indivisible element $\dot\gamma\in H^1(M; Z)$ such that $|\tau^{\dot\gamma}_a(M)|\leq\kappa^{\dot\gamma}_1(M)$ for all a.*

This answers in part Problem 3.20 of Kirby's Problem List [Ki] (cf. [G/L]). Note that there are many M which are smoothly imbedded in S^4 and have $|\tau^{\dot\gamma}_a(M)|=\kappa^{\dot\gamma}_1(M)\neq 0$ for an indivisible $\dot\gamma$ and all a. For example, let M be the torus bundle over S^1 with monodromy matrix $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\dot\gamma$, the element represented by the bundle projection. Directly, we see that M is smoothly imbedded in S^4 and $|\tau^{\dot\gamma}_a(M)|=\kappa^{\dot\gamma}_1(M)=1$ for all a.

Theorem 2.5. *For any positive integers r, r', there are infinitely many M having all of the following properties* (0)–(4):

(0) *$H_*(M; Z)\cong H_*(\sharp_r S^1\times S^2; Z)$ and $\kappa^{\dot\gamma}_1(M)=0$ and $|\tau^{\dot\gamma}_{-1}(M)|\geq r'$ for all indivisible elements $\dot\gamma\in H^1(M; Z)$,*

(1) *M is smoothly type II imbedded in $\sharp_r S^2\times S^2$,*

(2) *M is smoothly type I imbedded in a smooth 4-manifold W^* with $tH_q(W^*; Z)\cong Z_2$(if $q=1,2$) or 0(if $q\neq 1, 2$) and $bH_*(W^*; Z)\cong H_*(S^1\times S^3; Z)$,*

(3) *M^o is smoothly imbedded in a smooth 4-manifold W^{**} with $tH_q(W^{**}; Z)\cong Z_2$ (if $q=1, 2$) or 0(if $q\neq 1, 2$) and $bH_*(W^{**}; Z)\cong H_*(S^4; Z)$,*

(4) *M is not topologically type II imbeddable in any W with $\beta_2(W; Z)<2r$ and $\beta_2(W; Z)+|\mathrm{sign}\, W|<r'$.*

REMARK 2.6. We can conclude from Theorem 2.5 that Theorem 2.1 can not apply to type I imbeddings and if $\beta_1(M; Z)\leq\beta_2(W; Z)/2$, then $|\tau^{\dot\gamma}_a(M)|-\kappa^{\dot\gamma}_1(M)$, $a\in[-1, 1]$, do not, in general, restrict $\beta_2(W; Z)$ in Theorem 2.1. Cooper [C] has obtained a result corresponding to (4) in the piecewise-linear category.

2.7. Proof of Theorem 2.5. Let k be any invertible knot in S^3 with $|\sigma(k)|\geq r'$, where $\sigma(k)$ denotes the signature of the knot k. Let $M(k)$ be the 0-surgery manifold of k. Note that $\sigma^{\dot{\gamma}^*}(M(k))=\sigma(k)$ and $\sigma_{-1}^{\dot{\gamma}^*}(M(k))=\kappa_1^{\dot{\gamma}^*}(M(k))=0$ for any generator $\dot{\gamma}^*\in H^1M(k); Z)\cong Z$. By Lemma 1.3, $\tau_{-1}^{d\dot{\gamma}^*}(M(k))=\tau_{-1}^{\dot{\gamma}^*}(M(k))=\sigma(k)$ (if d is odd) or 0 (if d is even), for $\sigma_{-1}^{d\dot{\gamma}^*}(M(k))=\sigma_{-1}^{\dot{\gamma}^*}(M(k))=0$. Let M be the r-fold connected sum of $M(k)$. Then $H_*(M; Z)\cong H_*(\#_r S^1\times S^2; Z)$ and $\kappa_1^{\dot{\gamma}}(M)=0$ and $|\tau_{-1}^{\dot{\gamma}}(M)|\geq s|\sigma(k)|\geq r'$ for any indivisible element $\dot{\gamma}\in H^1(M; Z)$, where s is the number of the summands $M(k)$ of M such that $\dot{\gamma}|M(k)$ is an odd multiple of $\dot{\gamma}^*$. This shows (0). For (1) note that there is a piecewise-linearly imbedded 2-sphere $S^2(k)$ in $S^2\times S^2$ which is homotopic to $S^2\times q$ and has just one non-locally flat point represented by the knot k (See Suzuki [Su]). Since $S^2(k)$ has the self-intersection number 0, we see that the boundary of a (smooth) regular neighborhood of $S^2(k)$ in $S^2\times S^2$ is diffeomorphic to $M(k)$, so that M is smoothly imbedded in $\#_r S^2\times S^2$, showing (1). For (2) we use that k is invertible. From this, we have an orientation-preserving diffeomorphism h of $M(k)$ with $h_*=-1$ on $H_1(M(k); Z)$. Let W be the mapping torus of h. Then $tH_q(W; Z)\cong Z_2$ (if $q=1, 2$) or 0 (if $q\neq 1, 2$) and $bH_*(W; Z)\cong H_*(S^1\times S^3; Z)$. We may consider that h sends a 3-disk D^3 in $M(k)$ to itself by the identity. Let W^{**} be a closed 4-manifold obtained from W by replacing $S^1\times D^3\subset W$ by $D^2\times\partial D^3$. We have $tH_q(W^{**}; Z)\cong Z_2$(if $q=1, 2$) or 0(if $q\neq 1, 2$) and $bH_*(W^{**}; Z)\cong H_*(S^4; Z)$. $M(k)^o=M(k)$-Int D^3 is smoothly imbedded in W^{**} and the connected sum $M(k)\# T_g$, T_g the solid torus of genus g, is smoothly imbedded in $M(k)^o$ and hence in W^{**}. A boundary-disk sum, M^o, of r copies of $M(k)^o$ is smoothly imbedded in $(M(k)\# T_g)\times[0, 1]$ with $g=r-1$. Thus, using a collar of $M(k)\# T_g$ in W^{**}, we see that M^o is smoothly imbedded in W^{**}. Let W^* be a closed 4-manifold obtained from W^{**} by replacing a tubular neighborhood $T(\partial M^o)=S^2\times D^2$ of ∂M^o in W^{**} by $D^3\times\partial D^2$, where the framing of $T(\partial M^o)=S^2\times D^2$ is chosen so that some $S^2\times p$ ($p\in\partial D^2$) is a boundary-parallel 2-sphere in M^o. We see that M is smoothly type I imbedded in W^* and $tH_q(W^*; Z)\cong Z_2$(if $q=1, 2$) or 0 (if $q\neq 1, 2$) and $bH_*(W^*; Z)\cong H_*(S^1\times S^3; Z)$, showing (2) and (3). For (4) suppose that M is topologically type II imbedded in W with $\beta_2(W; Z)<2r$ and $\beta_2(W; Z)+|\text{sign } W|<r'$. Since $\beta_1(M; Z)=r>\beta_2(W; Z)/2$, we have, by (0) and Theorem 2.1, an indivisible element $\dot{\gamma}\in H^1(M; Z)$ such that

$$r'\leq|\tau_{-1}^{\dot{\gamma}}(M)|-\kappa_1^{\dot{\gamma}}(M)\leq\beta_2(W; Z)+|\text{sign } W|<r',$$

which is a contradiction. This completes the proof.

3. Punctured imbeddings

Let α be the standard reflection on the double DM^o of M^o.

DEFINITION. An element $\dot{\gamma}\in H^1(DM^o; Z)$ is *Z_2-asymmetric* if the Z_2-reduction $\dot{\gamma}(2)\in H^1(DM^o; Z_2)$ of $\dot{\gamma}$ has $\alpha^*(\dot{\gamma}(2))\neq\dot{\gamma}(2)$.

Theorem 3.1. *Assume that M^o is topologically imbedded in W. Then $\beta_1(M; Z)\leq\beta_2(W; Z_2)/2$ or there is a Z_2-asymmetric indivisible element $\dot{\gamma}\in H^1(DM^o; Z)$ such that for all a*

$$|\tau_a^{\dot{\gamma}}(DM^o)|-\kappa_1^{\dot{\gamma}}(DM^o)\leq\beta_2(W; Z)+|\mathrm{sign}\, W|\leq 2\beta_2(W; Z)\,.$$

Proof. Assume that $\beta_1(M; Z)>\beta_2(W; Z_2)/2$. Regard $f\colon M^o\subset W$. Since $H_1(W, W-M^o; Z)\cong H^3(M^o; Z)=0$, the natural homomorphism $H_1(W-M^o; Z)\to H_1(W; Z)$ is onto. By [Q], we can kill $H_1(W; Q)$ without changing $\beta_2(W; Z)$ by a surgery on $W-M^o$. We assume $\beta_1(W; Z)=0$. Choose mutually disjoint $S^1\times D_i^3$, $i=1, 2, \cdots, s$, in $W-M^o$ (by using [Q]) so that the cores $S^1\times 0_i$, $i=1, 2, \cdots, s$, represent a basis for $H_1(W; Z_2)$. Let $F=W-\cup_{i=1}^s S^1\times D_i^3$. By [$A/C$] and a boundary collar technique, the imbedding $f_P=f\times\mathrm{id}\colon M_P^o=M^o\times CP^2\subset F\times CP^2=F_P$ is homotopic to a bi-collared imbedding $f_P'\colon M_P^o\to F_P$. Let $N=M^o\times CP^2\times[0, 1]$ be a collar of $f_P'M_P^o$ in F_P. Construct $W^*=F\cup_{i=1}^s D^2\times S_i^2$ identifying $S^1\times\partial D_i^3$ with $\partial D^2\times S_i^2$ for all i. Then $\beta_2(W^*; Z)=\beta_2(W; Z_2)$ and $\beta_1(W^*; Z_2)=0$. Let $W_P=W\times CP^2$, $W_P^*=W^*\times CP^2$, $E=W_P$-Int N and $E^*=W_P^*$-Int N. Note that there is an epimorphism $\mu\colon H_1(E; Z)\to H_1(E^*; Z)$. We show that $\beta_1(E^*; Z)\neq 0$. Suppose $H_1(E^*; Q)=0$. By Poincaré duality, $H_7(E^*, \partial E^*; Q)=0$. But, $H_7(E^*, \partial E^*; Q)\cong H_7(W_P^*, N; Q)\cong H_7(W_P^*, M_P^o; Q)\cong H_3(W^*, M^o; Q)\otimes H_4(CP^2; Z)$. Thus, $H_3(W^*, M^o; Q)=0$. Since $\partial\colon H_2(E^*, \partial E^*; Q)\to H_1(\partial E^*; Q)$ is onto and $H_2(E^*, \partial E^*; Q)\cong H_2(W_P^*, N; Q)\cong H_2(W_P^*, M_P^o; Q)\cong H_2(W^*, M^o; Q)\otimes H_0(CP^2; Z)$ and $\partial E^*\cong DM^o\times CP^2$, we see that $\beta_2(W^*, M^o; Z)\geq\beta_1(DM^o; Z)=2\beta_1(M; Z)$. Using $H_3(W^*, M^o; Q)=0$, we obtain from the exact sequence of (W^*, M^o) that $\beta_2(W; Z_2)=\beta_2(W^*; Z)=\beta_2(W^*, M^o; Z)\geq 2\beta_1(M; Z)$, contradicting our assumption. Therefore, $\beta_1(E^*; Z)\neq 0$. Take any indivisible element $\gamma^*\in H^1(E^*; Z)$. Then $\gamma^*(2)\in H^1(E^*; Z_2)$ is not zero. Note that $\partial E^*=\partial E=\partial N=DM_P^o(=DM^o\times CP^2)$. By the Mayer/Vietoris sequence, the natural homomorphism $H^1(N; Z_2)\oplus H^1(E^*; Z_2)\to H^1(DM_P^o; Z_2)$ is injective, for $H^1(W_P^*; Z_2)=0$. Thus, $\gamma^*(2)|DM_P^o\in H^1(DM_P^o; Z_2)$ is non-zero and there are an odd integer d and an indivisible element $\dot{\gamma}_P\in H^1(DM_P^o; Z)$ such that $d\dot{\gamma}_P=\gamma^*|DM_P^o$. Let $\dot{\gamma}\in H^1(DM^o; Z)$ be an indivisible element corresponding to $\dot{\gamma}_P$. We show that $\dot{\gamma}$ is Z_2-asymmetric. If $\alpha^*(\dot{\gamma}(2))=\dot{\gamma}(2)$, then $\dot{\gamma}_P(2)=d\dot{\gamma}_P(2)$ lies in the image of the natural homomorphism $H^1(N; Z_2)\to H^1(DM_P^o; Z_2)$, so that the natural homomorphism $H^1(N; Z_2)\oplus H^1(E^*; Z_2)\to H^1(DM_P^o; Z_2)$ is not injective, a contradiction. Hence $\dot{\gamma}$ is Z_2-asymmetric. Let $\gamma=\mu^{\sharp}(\gamma^*)\in H^1(E; Z)$. Then $\gamma|DM_P^o=d\dot{\gamma}_P$. By Theorem 1.6, $|\tau_a^{\dot{\gamma}_P}(DM_P^o)|-\kappa_3^{\dot{\gamma}_P}(DM_P^o)\leq\hat{\beta}_4(E; Z)+|\mathrm{sign}\, E|$ for all a. By [K, 3], $\tau_a^{\dot{\gamma}_P}(DM_P^o)=\tau_a^{\dot{\gamma}}(DM^o)$. Directly, $\kappa_3^{\dot{\gamma}_P}(DM_P^o)=\kappa_1^{\dot{\gamma}}(DM^o)$. By Lemma 2.2, $\hat{\beta}_4(E; Z)\leq\beta_2(W; Z)$. By the Novikov addition theorem, sign $E=$sign $W_P=$sign W, for sign $N=0$. It follows that $|\tau_a^{\dot\gamma}(DM^o)|-\kappa_1^{\dot\gamma}(DM^o)\leq\beta_2(W;Z)+|\text{sign } W|$ for all a. This completes the proof.

Theorem 3.2. *For any positive integers r, r', there are infinitely many M having all of the following properties* (0)–(4):

(0) $H_*(M;Z)\cong H_*(\#_r S^1\times S^2;Z)$ *and* $\kappa_1^{\dot\gamma}(DM^o)=0$ *and* $|\tau_{-1}^{\dot\gamma}(DM^o)|\geq r'$ *for all Z_2-asymmetric indivisible elements* $\dot\gamma\in H^1(DM^o;Z)$,

(1) *M is smoothly type II imbedded in* $\#_r S^2\times S^2$,

(2) *M is smoothly type I imbedded in a smooth 4-manifold W^* with* $tH_q(W^*;Z)\cong\oplus_r Z_2$ (*if* $q=1, 2$) *or* 0 (*if* $q\neq1, 2$) *and* $bH_*(W^*;Z)\cong H_*(S^1\times S^3;Z)$,

(3) *M^o is smoothly imbedded in a smooth 4-manifold W^{**} with* $tH_q(W^{**};Z)\cong\oplus_r Z_2$(*if* $q=1, 2$) *or* 0 (*if* $q\neq1, 2$) *and* $bH_*(W^{**};Z)\cong H_*(S^4;Z)$,

(4) *M^o is not topologically imbeddable in any W with* $\beta_2(W;Z_2)<2r$ *and* $\beta_2(W;Z)+|\text{sign } W|<r'$.

Remark 3.3. We can conclude from Theorem 3.2 that $|\tau_a^{\dot\gamma}(DM^o)|-\kappa_1^{\dot\gamma}(DM^o)$, $a\in[-1, 1)$, do not restrict $\beta_2(W;Z)$ if $\beta_1(M;Z)\leq\beta_2(W;Z_2)/2$ and this inequality can not be replaced by $\beta_1(M;Z)\leq\beta_2(W;Z)/2$, in Theorem 3.1.

3.4. Proof of Theorem 3.2. Let k_i, $i=1, 2, \cdots, r$, be invertible knots in S^3 such that $|\sigma(k_j)|\geq r'+\Sigma_{i=1}^{j-1}|\sigma(k_i)|$, $j=1, 2, \cdots, r$. Let $M_i=M(k_i)$ and $M=\#_{i=1}^r M_i$. Then $H_*(M;Z)\cong H_*(\#_r S^1\times S^2;Z)$. Let $\dot\gamma\in H^1(DM^o;Z)$ be any Z_2-asymmetric indivisible element. Directly, $\kappa_1^{\dot\gamma}(DM^o)=0$. Write $DM^o=(M_1\#\bar M_1)\#(M_2\#\bar M_2)\#\cdots\#(M_r\#\bar M_r)$, where $\bar M_i^o=\alpha M_i^o$. Let $M_{i(j)}\#\bar M_{i(j)}$, $j=1, 2, \cdots, s$, be all of the summands of DM^o such that $\dot\gamma|M_{i(j)}\#\bar M_{i(j)}$ is still Z_2-asymmetric, where $1\leq i(1)<i(2)<\cdots<i(s)\leq r$. Since $\tau_{-1}^{d\dot\gamma^*}(M_i)=-\tau_{-1}^{d\dot\gamma^*}(\bar M_i)=\sigma(k_i)$ (if d is odd) or 0 (if d is even) for a generator $\dot\gamma^*\in H^1(M_i;Z)$ (cf. Lemma 1.3), it follows that $\tau_{-1}^{\dot\gamma}(DM^o)=\varepsilon_1\sigma(k_{i(1)})+\varepsilon_2\sigma(k_{i(2)})+\cdots+\varepsilon_s\sigma(k_{i(s)})$, $\varepsilon_j=\pm1$, and $|\tau_{-1}(DM^o)|\geq|\sigma(k_{i(s)})|-\Sigma_{j=1}^{s-1}|\sigma(k_{i(j)})|\geq r'$, showing (0). Since M_i is smoothly imbedded in $S^2\times S^2$ (cf. 2.7), M is smoothly imbedded in $\#_r S^2\times S^2$ (by a type II imbedding), showing (1). Since k_i are invertible, there is an orientation-preserving diffeomorphism h of M with $h_*=-1$ on $H_1(M;Z)$. Let W^* be the mapping torus of h. We have $tH_q(W^*;Z)\cong\oplus_r Z_2$ (if $q=1, 2$) or 0 (if $q\neq1, 2$) and $bH_*(W^*;Z)\cong H_*(S^1\times S^3;Z)$, showing (2). For (3) we can kill $bH_1(W^*;Z)$ by a surgery on W^*-M^o to obtain a desired W^{**}. For (4) suppose that there is an imbedding from M^o to W with $\beta_2(W;Z_2)<2r$ and $\beta_2(W;Z)+|\text{sign } W|<r'$. Since $\beta_1(M;Z)=r>\beta_2(W;Z_2)/2$, we see from (0) and Theorem 3.1 that there is a Z_2-asymmetric indivisible element $\dot\gamma\in H^1(DM^o;Z)$ such that $r'\leq|\tau_{-1}^{\dot\gamma}(DM^o)|-\kappa_1^{\dot\gamma}(DM^o)\leq\beta_2(W;Z)+|\text{sign } W|<r'$, which is a contradiction. This completes the proof of Theorem 3.2.

4. Higher dimensional analogues

We consider the case $n=4m-1$ $(m>1)$ only. The argument of this case is simpler than the case $n=3$, because any topological imbedding from a compact oriented n-manifold to W is homotopic to a bi-collared imbedding by [A/C].

Theorem 4.1. *Assume that M is topologically type II imbedded in W. Then $\beta_1(M;Z)\leq\beta_2(W;Z)$ or there is an indivisible element $\dot{\gamma}\in H^1(M;Z)$ such that for all a*

$$|\tau_a^{\dot{\gamma}}(M)|-\kappa_{2m-1}^{\dot{\gamma}}(M)\leq\beta_{2m}(W;Z)+|\text{sign } W|\leq 2\beta_{2m}(W;Z)\,.$$

Proof. The proof is analogous to that of Theorem 2.1. We give the outline only. Regard $M\subset W$. We can assume that $H_1(W;Q)=0$ by a surgery on $W-M$ and M splits W into two manifolds E', E''. Assume $\beta_1(M;Z)>\beta_2(W;Z)$. By the Mayer/Vietoris sequence, we have $\beta_1(E';Z)+\beta_1(E'';Z)>0$. Say $\beta_1(E';Z)>0$. Let $\gamma\in H^1(E';Z)$ be any non-zero element. Then $\gamma|M\in H^1(M;Z)$ is non-zero, since the natural map $H_1(M;Q)\to H_1(E';Q)$ is onto. The rest of the proof follows from Theorem 1.6, the inequalities $\hat{\beta}_{2m}(E';Z)+\hat{\beta}_{2m}(E'';Z)\leq\beta_{2m}(W;Z)$, $|\text{sign } E''|\leq\hat{\beta}_{2m}(E'';Z)$ and the Novikov addition theorem. This completes the proof.

Theorem 4.2. *For any positive integers r, r', there are infinitely many smooth M having all of the following properties* (0)–(4):

(0) *$H_*(M;Z)\cong H_*(\#_r S^1\times S^{n-1};Z)$ and $\kappa_{2m-1}^{\dot{\gamma}}(M)=0$ and $|\tau_{-1}^{\dot{\gamma}}(M)|\geq r'$ for all indivisible elements $\dot{\gamma}\in H^1(M;Z)$,*

(1) *M is smoothly type II imbedded in a smooth $(n+1)$-manifold homotopy equivalent to $\#_r S^2\times S^{n-1}$,*

(2) *M is smoothly type I imbedded in a smooth $(n+1)$-manifold W^* with $tH_q(W^*;Z)\cong Z_2$ (if $q=1, n-1$) or 0 (if $q\neq 1, n-1$) and $bH_*(W^*;Z)\cong H_*(S^1\times S^n;Z)$,*

(3) *M^o is smoothly imbedded in a smooth $(n+1)$-manifold W^{**} with $tH_q(W^{**};Z)\cong Z_2$(if $q=1, n-1$) or 0 (if $q\neq 1, n-1$) and $bH_*(W^{**};Z)\cong H_*(S^{n+1};Z)$,*

(4) *M is not topologically type II imbeddable in any W with $\beta_2(W;Z)<r$ and $\beta_{2m}(W;Z)+|\text{sign } W|<r'$.*

Proof. We take any invertible smooth $(n-2)$-knot K in S^n with $|\sigma(K)|\geq r'$ (cf. Levine [L]). Construct $F=D^{n+1}\cup D^{n-1}\times D^2$ identifying a tubular neighborhood $T(K)=S^{n-2}\times D^2$ of K in $S^n=\partial D^{n+1}$ with $\partial D^{n-1}\times D^2$. Let $M(K)=\partial F$. Then $H_*(M(K);Z)\cong H_*(S^1\times S^{n-1};Z)$ and the double DF is homotopy equivalent to $S^2\times S^{n-1}$. It is now an easy exercise (cf. 2.7) that the r-fold connected sum, M, of $M(K)$ has (0)–(4) by using Theorem 4.1 for (4). This completes the proof.

In the following, we can take M^o to be any compact connected oriented n-manifold such that ∂M^o is non-empty connected and $\beta_1(\partial M^o; Z)=0$:

Theorem 4.3. *Assume that M^o is topologically imbedded in W. Then $\beta_1(M^o; Z)\leq \dim_{Z_2} H^2(W; Z)\otimes Z_2$ or there is a Z_2-asymmetric indivisible element $\dot{\gamma}\in H^1(DM^o; Z)$ such that for all a*

$$|\tau_a^{\dot{\gamma}}(DM^o)|-\kappa_{2m-1}^{\dot{\gamma}}(DM^o)\leq\beta_{2m}(W; Z)+|\text{sign } W|\leq 2\beta_{2m}(W; Z)\,.$$

Proof. Regard $M^o\subset W$. We can assume without changing $\beta_{2m}(W; Z)$ and $\dim_{Z_2} H^2(W; Z)\otimes Z_2$ that $\beta_1(W; Z_2)=0$ by a surgery on $W-M^o$ (then $\beta_2(W; Z)=\dim_{Z_2} H^2(W; Z)\otimes Z_2$) and M^o has a collar $N\cong M^o\times[0, 1]$ in W. Let $E=M-\text{Int } N$. Then $\partial E=DM^o$. Assume that $\beta_1(M^o; Z)>\beta_2(W; Z)$. By the Mayer/Vietoris sequence, the natural homomorphism $H_1(DM^o; Q)\rightarrow H_1(N; Q)\oplus H_1(E; Q)$ is onto. Since $\beta_1(DM^o; Z)=2\beta_1(M^o; Z)$ and $\beta_1(N; Z)=\beta_1(M^o; Z)$ and the kernel of this epimorphism is the image of $\partial: H_2(W; Q)\rightarrow H_1(DM^o; Q)$, we see that $\beta_1(E; Z)>0$. Let $\gamma\in H^1(E; Z)$ be any indivisible element. Since the natural homomorphism $H^1(N; Z_2)\oplus H^1(E; Z_2)\rightarrow H^1(DM^o; Z_2)$ is injective, we see that $\gamma|DM^o\in H^1(DM^o; Z)$ is Z_2-asymmetric. The desired inequality now follows from Theorem 1.6, the inequality $\hat{\beta}_{2m}(E; Z)\leq\beta_{2m}(W; Z)$ and the Novikov addition theorem, completing the proof.

Theorem 4.4. *For any positive integers r, r', there are infinitely many smooth M having all of the following properties* (0)–(4):

(0) *$H_*(M; Z)\cong H_*(\#_r S^1\times S^{n-1}; Z)$ and $\kappa_{2m-1}^{\dot{\gamma}}(DM^o)=0$ and $|\tau_{-1}^{\dot{\gamma}}(DM^o)|\geq r'$ for all Z_2-asymmetric indivisible elements $\dot{\gamma}\in H^1(DM^o; Z)$,*

(1) *M is smoothly type II imbedded in a smooth $(n+1)$-manifold homotopy equivalent to $\#_r S^2\times S^{n-1}$,*

(2) *M is smoothly type I imbedded in a smooth $(n+1)$-manifold W^* with $tH_q(W^*; Z)\cong\oplus_r Z_2$ (if $q=1, n-1$) or 0 (if $q\neq 1, n-1$) and $bH_*(W^*; Z)\cong H_*(S^1\times S^n; Z)$,*

(3) *M^o is smoothly imbedded in a smooth $(n+1)$-manifold W^{**} with $tH_q(W^{**};Z)\cong\oplus_r Z_2$(if $q=1, n-1$) or 0 (if $q\neq 1, n-1$) and $bH_*(W^{**}; Z)\cong H_*(S^{n+1}; Z)$,*

(4) *M^o is not topologically imbeddable in any W with $\dim_{Z_2} H^2(W; Z)\otimes Z_2<r$ and $\beta_{2m}(W; Z)+|\text{sign } W|<r'$.*

Proof. Take any invertible smooth $(n-2)$-knots K_i, $i=1, 2, \cdots, r$, in S^n such that $|\sigma(K_j)|\geq r'+\Sigma_{i=1}^{j-1}|\sigma(K_i)|$, $j=1, 2, \cdots, r$. The connected sum $M=M(K_1)\#M(K_2)\#\cdots\#M(K_r)$ is proved to have (0)–(4), by using Theorem 4.3 for (4) (cf. 3.4). This completes the proof.

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