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ON THE SCHUR INDICES OF CERTAIN IRREDUCIBLE CHARACTERS OF REDUCTIVE GROURS OVER FINITE FIELDS

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Introduction. Let $\boldsymbol{F}_q$ be a finite field with q elements, of characteristic p. Let G be a connected, reductive linear algebraic group defined over $\boldsymbol{F}_q$, with Frobenius endomorphism F, and let G^F denote the group of F-fixed points of G. In [13], we investigated, under the assumption that the centre Z of G is connected, the rationality-properties of the characters λ^{G^F} of G^F induced by certain linear characters λ of a Sylow p-subgroup of G^F and, using the results obtained there, proved some propositions concerning the Schur indices of the semisimple or regular irreducible characters of G^F. In this paper, we shall treat the general case, that is, the case that Z is not necessarily connected. The main results are stated and proved in § 2. In particular, we get the following (see Corollary 1 to Proposition 1, § 2):

Theorem. *Any irreducible Deligne-Lusztig character* $\pm R_T^\theta$ *of* G^F ([4]) *has the Schur index at most two over the field* $\boldsymbol{Q}$ *of rational numbers.*

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1. Some lemmas. Let G and F be as above. Let B be an F-stable Borel subgroup of G with the unipotent radical U and T an F-stable maximal torus of B. For a root α of G (with respect to T), let U_α denote the root subgroup of G associated with α. Let $U.$ be the subgroup of U generated by the non-simple positive root subgroups U_α (the ordering on the roots is the one determined by B). Then $U/U.$ is commutative and can be regarded as the direct product $\prod_{\alpha\in\Delta} U_\alpha$, where Δ is the set of simple roots. As $FU.=U.$, F acts on $U/U.=\prod_{\alpha\in\Delta} U_\alpha$ and this action is the one induced by the maps F: $U_\alpha\to FU_\alpha$, $\alpha\in\Delta$. Let ρ be the permutation on the roots α given by $FU_\alpha=U_{\rho\alpha}$ and let I

be the set of orbits of ρ on Δ. For $i\in I$, put $U_i=\prod_{\alpha\in i} U_\alpha$. Then $U/U.=\prod_{i\in I} U_i$ and, as each U_i is F-stable, we have $U^F/U.^F=\prod_{i\in I} U_i^F$. For each $i\in I$, put $q_i=q^{|i|}$ and take one simple root γ_i in i. Then, for each i, there is an isomorphism ϕ_i of U_i^F with the additive group of $\boldsymbol{F}_{q_i}$ such that $\phi_i(tut^{-1})=\gamma_i(t)\phi_i(u)$ for $u\in U_i^F$ and $t\in T^F$ (cf. Proof of 11.8 of Steinberg [17] and Carter [3], pp. 76–77). Thus the family $\phi=(\phi_i)_{i\in I}$ defines an isomorphism

$$\phi\colon U^F/U.^F=\prod_{i\in I} U_i^F \xrightarrow{\sim} \prod_{i\in I} \boldsymbol{F}_{q_i} \tag{1}$$

so that, for $u=\prod_{i\in I} u_i$ with $u_i\in U_i^F$ for $i\in I$ and $t\in T^F$, we have

$$\phi(tut^{-1})=\prod_{i\in I} \lambda_i(t)\phi_i(u_i)\,. \tag{2}$$

Now let Λ be the set of characters λ of U^F such that $\lambda\,|\,U.=1$ and Λ_0 the set of characters λ in Λ such that $\lambda\,|\,U_i^F\neq 1$ for all $i\in I$. Then we have

Lemma 1. *Let $\lambda\in\Lambda_0$. Then λ^{G^F} is multiplicity-free (Gel'fand-Graev, Yokonuma, Steinberg) and any irreducible Deligne-Lusztig character $\pm R_T^\theta$ of G^F occurs in λ^{G^F} (Deligne-Lusztig).*

By embedding G in the connected, reductive group $G_1=(G\times T)/\{(z, z^{-1})\,|\,z\in Z\}$ (Z is the centre of G) with connected centre and the same derived group ([4], 5.18) and (as to the second assertion) using properties of Green functions (cf. [3], 7.2.8 and 7.7), we are reduced to the case that Z is connected. In this case the lemma is proved in [4], Theorem 10.7 (or in [3], 8.1.3 and 8.4.5).

Our purpose is to study the rationality of the characters $\lambda^{G^F}, \lambda\in\Lambda$. Suppose $p=2$. Then, by (1), $U^F/U.^F$ is an elementary abelian 2-group, so that, for any $\lambda\in\Lambda$, λ, hence λ^{G^F} is realiazable in $\boldsymbol{Q}$. Therefore, from now on, we shall assume that $p\neq 2$.

Lemma 2. *Let ν be a primitive element of $\boldsymbol{F}_p$ (i.e. $\boldsymbol{F}_p^\times=\langle\nu\rangle$). Then there exists an element t in T^F such that $t^{p-1}=1$ (possibly $t^{(p-1)/2}=1$) and $\alpha(t)=\nu^2$ for all simple roots α.*

It suffices to prove the lemma for the derived group G' of G, hence for the simply-connected covering of G'. If G is a simply-connected semisimple group, then we have $G=G_1\times\cdots\times G_m$, where, for $1\leqq i\leqq m$, G_i is an F-stable simply-connected semisimple closed subgroup of G whose simple components are permuted by F cyclically, and the truth of the lemma for each G_i will imply that for G. If $G=G_1\times FG_1\times\cdots\times F^{n-1}G_1$, where G_1 is an F^n-stable simply-connected simple closed subgroup of G for some $n\geqq 1$, then T and B, hence the set of simple roots has the corresponding decomposition, and it is easy to see that the truth of the lemma for G_1 with Frobenius map F^n implies that for

G (cf. [17], 11.2 (b)). Thus we are reduced to the case that G is a simply-connected simple group.

Suppose therefore that G is such a group. Let $X(T)=\mathrm{Hom}(T, \boldsymbol{G}_m)$ and $Y(T)=\mathrm{Hom}(\boldsymbol{G}_m, T)$, and let $\langle\ ,\ \rangle$: $X(T)\times Y(T)\to\boldsymbol{Z}$ be the natural pairing given by $\langle\chi, \chi^\vee\rangle=$degree of $\chi\circ\chi^\vee$ for $\chi\in X(T)$ and $\chi^\vee\in Y(T)$. Let $\alpha_1, \cdots, \alpha_l$ be the simple roots (as to the numbering of the simple roots, we follow that of Bourbaki [2]) and let $\alpha_1^\vee, \cdots, \alpha_l^\vee$ be the corresponding simple coroots. Then, as G is simply-connected, we have $Y(T)=\langle\alpha_1^\vee, \cdots, \alpha_l^\vee\rangle_Z$, so that the mapping h: $(x_1, \cdots, x_l)\to\prod_{i=1}^{l}\alpha_i^\vee(x_i)$ defines an isomorphism of $(\boldsymbol{G}_m)^l$ with T. Then, for $1\leqq i\leqq l$, we have

$$\alpha_i(h(x_1, \cdots, x_l)) = \prod_{j=1}^{l} x_j^{\langle\alpha_i,\alpha_j^\vee\rangle},$$

where $(\langle\alpha_i, \alpha_j^\vee\rangle)_{1\leqq i,j\leqq l}$ is the Cartan matrix of G. We define an action of F on $Y(T)$ by $F(\chi^\vee)=F\circ\chi^\vee$ for $\chi^\vee\in Y(T)$. Then we have

$$F(\alpha_i^\vee) = q(\rho\alpha_i)^\vee$$

for $1\leqq i\leqq l$ (see [15], 11.4.7). It readily follows that, for $s\in T$, $s=h(x_1, \cdots, x_l)$, we have $Fs=s$ if and only if $x_j=x_i^q$ if $\rho\alpha_i=\alpha_j$. Thus the proof of the lemma has been reduced to solving the following problem:

Find an element $t=h(x_1, \cdots, x_l)$ with $x_i\in\boldsymbol{F}_p^\times$ for $1\leqq i\leqq l$ such taht $\prod_{j=1}^{l} x_j^{\langle\alpha_i,\alpha_j^\vee\rangle}=\nu^2$ for $1\leqq i\leqq l$ and that $x_j=x_i^q$ (hence $x_j=x_i$) if $\rho\alpha_i=\alpha_j$.

When G is adjoint, by the proof of Theorem 1 of [13], there is an element s in T^F of order $p-1$ such that $\alpha(s)=\nu$ for all simple roots α. Hence it suffices to take $t=s^2$. Suppose therefore that G is not adjoint. Then, as $p\neq 2$, G is any one of the following types (Steinberg [17], 11.6; also see [3], 1.19): A_l $(l\geqq 1)$, B_l $(l\geqq 2)$, C_l $(l\geqq 2)$, D_l $(l\geqq 3)$, E_6, E_7, ${}^2A_l$ $(l\geqq 1)$, ${}^2D_l$ $(l\geqq 3)$, ${}^3D_4$, ${}^2E_6$. In each case, an element t of T^F having the property of the lemma (i.e. an solution t of the problem above) can be given as follows (the Cartan matrices are listed up in the appendices of [2]):

Type	t		
A_l ${}^2A_l$	$h(x_1, \cdots, x_l)$	$x_i=\nu^{i(l-i+1)}$	$(1\leqq i\leqq l)$
B_l	$h(x_1, \cdots, x_{l-1}, \nu^{l(l+1)/2})$	$x_i=\nu^{i(2l-i+1)}$	$(1\leqq i\leqq l-1)$
C_l	$h(x_1, \cdots, x_l)$	$x_i=\nu^{i(2l-i)}$	$(1\leqq i\leqq l)$
D_l ${}^2D_l$	$h(x_1, \cdots, x_{l-2}, \nu^{l(l-1)/2}, \nu^{(l-1)/2})$	$x_i=\nu^{i(2l-i-1)}$	$(1\leqq i\leqq l-2)$
E_6 ${}^2E_6$	$h(\nu^{16}, \nu^{22}, \nu^{30}, \nu^{42}, \nu^{30}, \nu^{16})$		
E_7	$h(\nu^{34}, \nu^{49}, \nu^{66}, \nu^{96}, \nu^{75}, \nu^{52}, \nu^{27})$		
${}^3D_4$	$h(\nu^6, \nu^{10}, \nu^6, \nu^6)$		

This completes the proof of Lemma 2.

Lemma 3. *Assume that q is an even power of p. Then there exists an element t in T^F such that $t^{2(p-1)}=1$ (possibly $t^{p-1}=1$) and $\alpha(t)=v$ for all simple roots α.*

As in the proof of Lemma 2, we can be reduced to the case that G is a simply-connected simple group. When G is adjoint Lemma 3 is proved in the proof of Theorem 1 of [13]. When G is not adjoint t can be given by replacing each v in the above table with an element $\varepsilon\in \boldsymbol{F}_q$ such that $\varepsilon^2=v$. (We note that, when G is a simply-connected simple group, an element $s=h(x_1, \cdots, x_l)$ of T has the property of Lemma 3 if and only if the x_i satisfy: (i) $x_i^{2(p-1)}=1$ for $1\leqq i\leqq l$, (ii) $\prod_{j=1}^{l} x_j^{\langle\alpha_i,\alpha_j^\vee\rangle}=v$ for $1\leqq i\leqq l$, and (iii) $x_j=x_i^q$ if $\rho\alpha_i=\alpha_j$.)

In the following, for an integer m and a prime number r, $\mathrm{ord}_r m$ denotes the exponent of the r-part of m.

Lemma 4. *Assume that G is a (non-adjoint) simply-connected simple group of any one of the following types: A_l with $2|l$ or $\mathrm{ord}_2(l+1)>\mathrm{ord}_2(p-1)$; ${}^2A_l$ with $2|l$; B_l with $4|l(l+1)$; D_l with either (a) $4|l(l-1)$ or (b) $\mathrm{ord}_2(l-1)=1$ and $p\equiv-1 \pmod 4$; ${}^2D_l$ with $4|l(l-1)$; ${}^3D_4$; E_6; ${}^2E_6$. Then there exists an element $t\in T^F$ such that $t^{p-1}=1$ and $\alpha(t)=v$ for all simple roots α.*

In fact, for an element $s=h(x_1, \cdots, x_l)$ of T, s satisfies the property of Lemma 4 if and only if the x_i satisfy: (i) $x_i\in F_p^\times$, (ii) $\prod x_j^{\langle\alpha_i,\alpha_j^\vee\rangle}=v$ for $1\leqq i\leqq l$, and (iii) $x_j=x_i^q$ (hence $x_j=x_i$) if $\rho\alpha_i=\alpha_j$. By solving these equations, we find that an element t having the property of the lemma can be given as follows:

Type	t		
A_l ${}^2A_l$ $2\|l$	$h(x_1, \cdots, x_l)$	$x_i=v^{i(l-i+1)/2}$	$(1\leqq i\leqq l)$
A_l $\mathrm{ord}_2(l+1)>\mathrm{ord}_2(p-1)$	$h(x_1, \cdots, x_l)$	$x_1=v^{(el+p-1)/2e}$	$\left(e=\left(\frac{l+1}{2}, p-1\right)\right)$
		$x_i=v^{-i(i-1)/2}x_1^i$	$(2\leqq i\leqq l)$
B_l $4\|l(l+1)$	$h(x_1, \cdots, x_{l-1}, v^{l(l+1)/4})$	$x_i=v^{i(2l-i+1)/2}$	$(1\leqq i\leqq l-1)$
D_l ${}^2D_l$ $4\|l(l-1)$	$h(x_1, \cdots, x_{l-2}, v^{l(l-1)/4}, v^{l(l-1)/4})$	$x_i=v^{i(2l-i-1)/2}$	$(1\leqq i\leqq l-2)$
D_l $\mathrm{ord}_2(l-1)=1$ $p\equiv-1 \pmod 4$	$h(x_1, \cdots, x_{l-2}, v^{(l^2-l+p-1)/4}, v^{(l^2-l+3p-3)/4})$	$x_i=v^{i(2l+p-i-2)/2}$	$(1\leqq i\leqq l-2)$
${}^3D_4$	$h(v^3, v^5, v^3, v^3)$		
E_6 ${}^2E_6$	$h(v^8, v^{11}, v^{15}, v^{21}, v^{15}, v^8)$		

REMARK. If (at least) G is split over $\boldsymbol{F}_q$, then Lemmas 2, 4 above are implicit in Lehrer's work [12] where he showed a method to calculate the image $a(T^F)$ of T^F under the morphism a: $T\to(\boldsymbol{G}_m)^l$ given by $a(s)=\prod_{i=1}^{l}\alpha_i(s)$ when G

is a simply-connected simple group (he has carried out the calculation when G is a classical group). For our purpose, it is essential to know the order of t (cf. § 2 below).

2. The main results. We recall that $p\neq 2$. Let ζ_p be a primitive p-th root of unity in the field $\boldsymbol{C}$ of complex numbers. Let $\hat{\boldsymbol{F}}_q=\mathrm{Hom}(\boldsymbol{F}_q, \boldsymbol{C}^\times)$ (we consider $\boldsymbol{F}_q$ as an additive group) and fix $\chi\in\hat{\boldsymbol{F}}_q$, $\chi\neq 1$. For $a\in\boldsymbol{F}_q$, define $\chi_a\in\hat{\boldsymbol{F}}_q$ by $\chi_a(x)=\chi(ax)$ for $x\in\boldsymbol{F}_q$. Then we have $\hat{\boldsymbol{F}}_q=\{\chi_a\,|\,a\in\boldsymbol{F}_q\}$ and $\{\chi^\tau\,|\,\tau\in\mathrm{Gal}(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q})\}=\{\chi_a\,|\,a\in\boldsymbol{F}_p^\times\}$.

In the following, if χ is a character of a finite group and L is a field of characteristic zero, $L(\chi)$ is the field generated over L by the values of χ. If χ is irreducible, then $m_L(\chi)$ denotes the Schur index of χ with respect to L. If L is an algebraic number field and v is a place of L, then L_v is the completion of L at v. Now let k be the quadratic subfield $\boldsymbol{Q}(\sqrt{\varepsilon p})$, $\varepsilon=(-1)^{(p-1)/2}$, of $\boldsymbol{Q}(\zeta_p)$.

Proposition 1. *Let G, F be as in Introduction. Let $\lambda\in\Lambda$, $\lambda\neq 1$. Then we have the following:*

(i) *λ^{G^F} takes all its values in k; if $p\equiv -1$ (mod 4), λ^{G^F} is realizable in k; if $p\equiv 1$ (mod 4), then, for any finite place v of k, λ^{G^F} is realizable in k_v.*

(ii) *Assume that q is an even power of p. Then λ^{G^F} takes all its values in $\boldsymbol{Q}$ and, for any prime number $r\neq p$, λ^{G^F} is realizable in $\boldsymbol{Q}_r$.*

(iii) *If G is an adjoint semisimple group or any one of the groups described in Lemma 4, then λ^{G^F} is realizable in $\boldsymbol{Q}_r$.*

Proof of (i). Let t be an element of T^F having the property of Lemma 2. Then $z=t^{(p-1)/2}$ lies in the centre Z^F of G^F since $\alpha(z)=1$ for all simple roots α. Put $c=|\langle z\rangle|$ ($c=1$ or 2). Let $M=\langle t\rangle U^F$. Then M acts on Λ by $\lambda^m(u)=\lambda(mum^{-1})$ ($\lambda\in\Lambda$, $m\in M$, $u\in U^F$). Let $\lambda\in\Lambda$, $\lambda\neq 1$. Then, by (1), λ can be expressed as $\lambda=(\lambda_i)_{i\in I}$ with $\lambda_i\in\hat{F}_{q_i}$ for $i\in I$. And, by (2), we have

$$\lambda^t=((\lambda_i)_{\gamma_i(t)})_{i\in I}=((\lambda_i)_{\nu^2})_{i\in I}=(\lambda_i^{\sigma^2})_{i\in I}=\lambda^{\sigma^2},$$

where σ is a suitable generator of $\mathrm{Gal}(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q})$. Thus, on U^F, we have

$$\lambda^M=c\sum_{j=1}^{(p-1)/2}\lambda^{t^j}=c\sum_{j=1}^{(p-1)/2}\lambda^{\sigma^{2j}},$$

hence $\boldsymbol{Q}(\lambda^M)=\boldsymbol{Q}(\zeta_p)^{\langle\sigma^2\rangle}=k$. Therefore the values of $\lambda^{G^F}=(\lambda^M)^{G^F}$ lie in k.

Suppose $t^{(p-1)/2}=1$. Then λ^M is irreducible. By Gow's argument [7], p. 104, we have $m_k(\lambda^M)=1$: $\lambda^M|\langle t\rangle=$the character of the regular representation of $\langle t\rangle$, hence $\langle\lambda^M, 1_{\langle t\rangle}\rangle_{\langle t\rangle}=1$; hence, by Schur's theorem (see e.g. Feit [5], 11.4), $m_k(\lambda_M)=1$. Thus λ^M, hence $\lambda^{G^F}=(\lambda^M)^{G^F}$ is realizable in k.

Assume that $t^{(p-1)/2}\neq 1$. Then λ^M is reducible and is equal to the sum $\mu_0+\mu_1$ where, for $i=0$, 1, μ_i is the irreducible character of M induced by the linear character of $\langle z\rangle U^F$ given by $z^j u \to (-1)^{ji}\lambda(u)$ $(j=0, 1)$. We have $\boldsymbol{Q}(\mu_0)=\boldsymbol{Q}(\mu_1)=k$. For $i=0, 1$, the simple direct summand A_i of the group algebra $k[M]$ of M over k corresponding to μ_i is isomorphic over k to the cyclic algebra $((k(\zeta_p)/k, \sigma^2, (-1)^i)$ over k (cf. Proof of Proposition 3.5 of Yamada [18]). A_0 clearly splits over k, hence $m_k(\mu_0)=1$ and μ_0 is realizable in k. If $p\equiv -1 \pmod 4$, then -1 is a norm in $k(\zeta_p)/k$, hence A_1 splits over k. Thus, in this case, μ_1, hence $\lambda^M=\mu_0+\mu_1$ is realizable in k. Suppose $p\equiv 1 \pmod 4$. Then A_1 has non-zero invariants only at two real places of k (see Janusz [10], Proposition 3). Thus, for any finite place v of k, μ_1, hence $\lambda^M=\mu_0+\mu_1$ is realizable in k_v.

Proof of (ii). Let t be an element of T^F having the property of Lemma 3, and put $M=\langle t\rangle U^F$. Then, as $\lambda^t=\lambda^\sigma$ $(\lambda\neq 1)$, on U^F, we have

$$\lambda^M = c\sum_{j=1}^{p-1}\lambda^{t^j} = c\sum_{j=1}^{p-1}\lambda^{\sigma^j} \qquad (c=|\langle t^{p-1}|\rangle).$$

Thus $\boldsymbol{Q}(\lambda^M)=\boldsymbol{Q}(\zeta_p)^{\langle\sigma\rangle}=\boldsymbol{Q}$.

If $t^{p-1}=1$, then λ^M is irreducible and Gow's argument shows that $m_{\boldsymbol{Q}}(\lambda^M)=1$, hence λ^{G^F} is realizable in $\boldsymbol{Q}$. Suppose $t^{p-1}\neq 1$. Then λ^M is reducible and is equal to the sum $\mu_0+\mu_1$, where, for $i=0, 1$, μ_i is the irreducible character of M induced by the linear character of $\langle t^{p-1}\rangle U^F$ given by $(t^{p-1})^j u \to u(-1)^{ji}\lambda(u)$. We have $\boldsymbol{Q}(\mu_0)=\boldsymbol{Q}(\mu_1)=\boldsymbol{Q}$. For $i=0, 1$, the simple direct summand A_i of $\boldsymbol{Q}[M]$ corresponding to μ_i is isomorphic over $\boldsymbol{Q}$ to $(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q}, \sigma, (-1)^i)$. A_0 splits, hence μ_0 is realizable in $\boldsymbol{Q}$. A_1 has the invariants $\frac{1}{2} \bmod 1$ at ∞, p and $0 \bmod 1$ at any other place of $\boldsymbol{Q}$. Thus, for any prime number $r\neq p$, μ_1, hence $\lambda^M=\mu_0+\mu_1$ is realizable in $\boldsymbol{Q}_r$.

Proof of (iii). When G is adjoint the assertion is contained in Theorem 1 of [13]. Assume that G is not adjoint. Let t be an element of T^F having the property of Lemma 4 and put $M=\langle t\rangle U^F$. Then λ^M is irreducible and $\boldsymbol{Q}(\lambda^M)=\boldsymbol{Q}$. And, by Gow's argument, we have $m_{\boldsymbol{Q}}(\lambda^M)=1$. Thus λ^M, hence $\lambda^{G^F}=(\lambda^M)^{G^F}$ is realizable in $\boldsymbol{Q}$.

We note that, for $G=SL_n$, Sp_{2n}, Proposition 1 is proved by Gow [7], [8].

Corollary 1. *Let G, F be as in Proposition 1. Recall that $p\neq 2$. Let be χ an irreducible character of G^F such that $\langle\chi, \lambda^{G^F}\rangle_{G^F}=1$ for some $\lambda\in\Lambda$ (any irreducible component of λ^{G^F} for $\lambda\in\Lambda_0$ has this property (see Lemma 1)). Then we have $m_{\boldsymbol{Q}}(\chi)\leqq 2$. Thus, in particular, we have $m_{\boldsymbol{R}}(\chi)\leqq 2$ for any irreducible Deligne-Lusztig character $\chi=\pm R_{T'}^{\theta}$ of G^F.* If $\lambda=1$, then λ^{G^F} is realizable in $\boldsymbol{Q}$, hence we have $m_{\boldsymbol{Q}}(\chi)=1$. Assume that $\lambda\neq 1$. Let r be any prime number and v a place of k lying above r. Then, by Proposition 1, we have $m_{k_v}(\chi)=1$, hence $m_{\boldsymbol{Q}_r}(\chi)\leqq 2$ as $[k_v(\chi)\colon \boldsymbol{Q}_r(\chi)]\leqq 2$. We also have $m_{\boldsymbol{R}}(\chi)\leqq 2$. Thus, $m_{\boldsymbol{Q}}(\chi)$, being the least common multiple of the $m_{Q_w}(\chi)$ with w running over all places of $\boldsymbol{Q}$, is at most two. The last assertion follows from this fact and Lemma 1.

Corollary 2. *Assume that q is an even power of p. Let χ be an irreducible character of G^F such that $\langle \chi, \lambda^G \rangle_{G^F}=1$ for some $\lambda \in \Lambda$. Then, for any prime number $r \neq p$, we have $m_{Q_r}(\chi)=1$.*

This follows at once from Proposition 1, (ii).

Corollary 3. *Assume that G is an adjoint semisimple group or any one of the groups described in Lemma 4. Let χ be an irreducible character of G^F such that $\langle \chi, \lambda^{G^F} \rangle_{G^F}=1$ for some $\lambda \in \Lambda$. Then we have $m_Q(\chi)=1$.*

This follows from Proposition 1, (iii).

Corollary 4. *Let G, F be as in Proposition* 1. *Assume that p is a good prime for G* ([16], I, 4.1). *Let χ be an irreducible character of G^F and let u be a regular unipotent element in G^F. Then $\chi(u)$ is an algebraic integer in k, and if $p \nmid \chi(1)$, we have $m_Q(\chi) \leqq 2$.*

We first note that, as p is good for G, $U.^F$ is equal to the derived group of U^F, hence Λ is the set of linear characters of U^F (Howlett [9], Lehrer [11]), and that, if $u \in U^F$, then $\mu(u)=0$ for any non-linear irreducible character μ of U^F (Lehrer [11]).

Let $\mathcal{O}_k$ be the ring of integers in k. We show that $\chi(u)$ belongs to $\mathcal{O}_k$. We may assume that $u \in U^F$ as u is conjugate to an element of U^F. Let t be an element of T^F having the property of Lemma 2, and let $\Lambda_1, \cdots, \Lambda_r$ be the orbits of $\langle t \rangle$ on Λ. Thus, as $\chi^t=\chi$, if we put $a_\lambda=\langle \chi, \lambda \rangle_{U^F}$ for $\lambda \in \Lambda$, a_λ is constant on each Λ_i. Hence we have

$$\chi(u) = \sum_{\lambda \in \Lambda} a_\lambda \lambda(u) = \sum_{i=1}^{r} a_i \Big(\sum_{\lambda \in \Lambda_i} \lambda(u) \Big) ,$$

where $a_i=a_\lambda$ on Λ_i. Each $\sum_{\lambda \in \Lambda_i} \lambda(u)$ is stable under the action of $\langle t \rangle$, hence under the action of $\langle \sigma^2 \rangle$. Thus $\chi(u) \in \mathcal{O}_k$.

To prove the second assertion, we embed G in G_1 as in the proof of Lemma 1. Assume that $p \nmid \chi(1)$ and take an irreducible character χ_1 of G_1^F such that $\langle \chi, \chi_1 | G^F \rangle_{G^F} \neq 0$. Then, by the Clifford theory, we have $\chi_1 | G^F = e(\chi^{(1)}+\chi^{(2)}+\cdots+\chi^{(s)})$, where e is a positive integer dividing $(G_1^F : G^F)$ and $\chi^{(1)}, \chi^{(2)}, \cdots, \chi^{(s)}$ are the G_1^F-conjugates of $\chi=\chi^{(1)} (s \,|\, (G_1^F : G^F))$. Let r be any prime number and v a place of k lying above r. Put $m_v=m_{k_v}(\chi^{(1)})=\cdots=m_{k_v}(\chi^{(s)})$. For $1 \leqq i \leqq s$ and for $\lambda \in \Lambda$, put $a_\lambda^{(i)}=\langle \chi^{(i)}, \lambda \rangle_{U^F}$. Then, by Proposition 1, (i), m_v divides the $a_\lambda^{(i)}$, $1 \leqq i \leqq s$, $\lambda \in \Lambda$. As $p \nmid (G_1^F : G^F)$, $p \nmid \chi_1(1)$, so that, by a theorem of Green-Lehrer-Lusztig (see [3], 8.3.6), we have $\chi_1(u)=\pm 1$. Therefore we have the expression

$$\pm 1/m_v = \chi_1(u)/m_v = \{e\cdot \sum_{i=1}^{s} \chi^{(i)}(u)\}/m_v = e\cdot\sum_{i=1}^{s}\sum_{\lambda\in\Lambda}(a_\lambda^{(i)}/m_v)\cdot\lambda(u),$$

where the right-hand side is an algebraic integer and the left-hand side is a rational number. Hence $m_v=1$, and $m_{Q_r}(\chi)\leqq 2$. As r is an arbitrary prime number, we hence have $m_Q(\chi)\leqq 2$. This completes the proof of Corollary 4.

Corollary 5. *Assume that q is an even power of p and that p is good for G. Let u be a regular unipotent element in G^F. Then, for any irreducible character χ of G^F, $\chi(u)$ is a rational integer, and if $p\not|\chi(u)$, we have $m_{Q_r}(\chi)=1$ for any prime number $r\neq p$.*

The proof is similar to the proof of Corollary 4 (we use Proposition 1, (ii)).

Corollary 6. *Let G be an adjoint semisimple group or any one of the groups described in Lemma 4. Assume that p is good for G. Let u be a regular unipotent element in G^F and let χ be an irreducible charactre of G^F. Then $\chi(u)$ is a rational integer and if $p\not|\chi(u)$, we have $m_Q(\chi)=1$.*

REMARK. Lehrer [12] has calculated the values of the cuspidal irreducible characters of G^F at the regular unipotent elements of G^F when G is a semisimple group. As to the upper bound of the indices of the characters of related finite groups, we reffer to Gow [8] for classical finite groups and Benard [1] and Feit [6] for the sporadic simple groups.

Let G be a connected, reductive algebraic group over an algebraically closed field K of characteristic $p>0$ and F a surjective endomorphism of G such that G^F is finite. Then Lemma 2 still holds for such G^F, so that the statements in Proposition 1, (i) and in Corollary 1 (except for the comment for Lemma 1) hold for G^F. Assume that K is an algebraic closure of $\boldsymbol{F}_p$ and that some power of F is the Frobenius endomorphism relative to a rational structure on G over a finite subfield of K. Then Lemma 1 holds for G^F (cf. Carter [3], 8.1.3 and 8.4.5), so that all the statements in Corollary 1, hence the theorem in Introduction holds for G^F. If p is good for G, then the theorem of Green-Lehrer-Lusztig holds for G^F (if Z is connected: see [3], 8.3.6), so that Corollary 4 holds for G^F.

3. Example. We calculate all the local indices of the cuspidal irreducible Deligne-Lusztig characters $\pm R_{T'}^{\theta}$ of $SL_n(\boldsymbol{F}_q)$ when q is an even power of p $(\neq 2)$.

Let G be SL_n and F the endomorphism $(g_{ij})\to(g_{ij}^q)$ (q may be any power of any prime p). Let T' be a minisotropic maximal torus of G and let $W=N_G(T')^F/T'^F$ (T' is unique up to G^F-con conjugate). Then, taking an element γ of order $(q^n-1)/(q-1)$ in $\boldsymbol{F}_{q^n}^{\times}$, we have $T'^F=\langle t_0\rangle$, where t_0 is G-conjugate to

$\mathrm{diag}(\gamma, \gamma^q, \cdots, \gamma^{q^{n-1}})$, and $W=\langle w_0\rangle \simeq \boldsymbol{Z}/n\boldsymbol{Z}$, where w_0 is defined by $t_0^{w_0}=\dot{w}_0 t_0 \dot{w}_0^{-1} = t_0^q$ ($\dot{w}_0 \in N_G(T')^F$ represents w_0). (All these statements can be easily checked by using [16], II, 1.3, 1.10 and 1.14.) W acts on $\hat{T}'^F=\mathrm{Hom}(T'^F, \boldsymbol{C}^\times)$ by $\theta^w(s)=\theta(s^w)$ for $w\in W$, $\theta\in\hat{T}'^F$ and $s\in T'^F$. If θ is in general position, i.e., no non-identity element of W fixes θ, then $(-1)^{n-1}R_{T'}^\theta$ is a cuspidal irreducible character of $G^F=SL_n(\boldsymbol{F}_q)$ ([4], 7.4, 8.3).

Let $\theta\in\hat{T}'^F$. Then, by [4], 4.2, for $g\in G^F$, if $g=su=us$ (s semisimple, u unipotent) is its Jordan decomposition, we have

$$(3)\qquad R_T^\theta(g)=\frac{1}{|Z_G(s)^F|}\sum_{\substack{h\in G^F\\ h^{-1}sh\in T'}} Q_{hT'h^{-1},Z_G(s)}(u)\cdot\theta(h^{-1}sh),$$

where the $Q_{hT'h^{-1},Z_G(s)}$ are Green functions of $Z_G(s)$ (which is connected since G is simply-connected). It follows that, if s is not conjugate in G^F to any element of T'^F, we have $R_{T'}^\theta(g)=0$, and if $s\in T^{F\prime}$, we have

$$(4)\qquad R_{T'}^\theta(g)=Q_{T',Z_G(s)}(u)\frac{1}{|W(s)|}\sum_{w\in W}\theta^w(s),$$

where $W(s)=\{w\in W \mid s^w=s\}$ (we note that the minisotropic maximal tori of $Z_G(s)$ form a single $Z_G(s)^F$-conjugacy class (cf. [16], II, 1.3, 1.10 and 1.14) and that any two elements of T' that are conjugate in G^F are conjugate under the action of W). Thus, as the Green functions take integeral values, by putting $\theta(t_0)=\zeta$, we get from (4):

$$(5)\qquad Q(R_{T'}^\theta)=Q(\sum_{w\in W}\theta^w)=Q(\zeta+\zeta^q+\cdots+\zeta^{q^{n-1}}).$$

Lemma 5. *Assume that θ is in general position. Let $q=p^m$. We further assume that n is even. Then we have*

$$\mathrm{ord}_2[\boldsymbol{Q}_p(R_{T'}^\theta):\boldsymbol{Q}_p]=\mathrm{ord}_2 m.$$

Let ϕ be the automorphism of $\boldsymbol{Q}_p(\zeta)$ defined by $\zeta^\phi=\zeta^q$. Then ϕ has order n (by assumption) and we have $\boldsymbol{Q}_p(\zeta)^{\langle\phi\rangle}=\boldsymbol{Q}_p(R_{T'}^\theta)$ (cf. (5)). Let $f=[\boldsymbol{Q}_p(\zeta):\boldsymbol{Q}_p]$ and $e=|\langle\zeta\rangle|$. Then f is equal to the least integer $h\geqq 1$ subject for the condition: $p^h\equiv 1 \pmod e$ (see Serre [14], p. 85). As $\phi^n=1$ and $\phi^i\neq 1$ for $1\leqq i\leqq n-1$, we find that $f\mid mn$ but $f\nmid mi$ for $1\leqq i\leqq n-1$ [in fact, if $f\mid mi$, then $p^f-1\mid p^{mi}-1$, hence $e\mid p^{mi}-1$, hence $\phi^i=1$]. This shows that $\mathrm{ord}_r f=\mathrm{ord}_r m+\mathrm{ord}_r n$ for any prime divisor r of n. Thus, in particular, we have $\mathrm{ord}_2 f=\mathrm{ord}_2 m+\mathrm{ord}_2 n$. As $[\boldsymbol{Q}_p(\zeta):\boldsymbol{Q}_p(R_{T'}^\theta)]=[\boldsymbol{Q}_p(\zeta):\boldsymbol{Q}_p(\zeta)^{\langle\phi\rangle}]=n$, we hence have $\mathrm{ord}_2[\boldsymbol{Q}_p(R_{T'}^\theta):\boldsymbol{Q}_p]=\mathrm{ord}_2 m$, as desired.

REMARK. Professor K. Iimura showed to the author (by an elementary proof) that $n=f/(m,f)$ and $[\boldsymbol{Q}_p(\zeta)^{\langle\phi\rangle}:\boldsymbol{Q}_p]=(m,f)$.

Proposition 2. *Let χ be any cuspidal irreducible Deligne-Lusztig character $(-1)^{n-1}R_{T'}^{\theta}$ of $G^F=SL_n(\boldsymbol{F}_q)$, where we assume that q is an even power of $p\neq 2$. Then, if n is odd or $ord_2 n\geqq 2$, we have $m_{\boldsymbol{Q}}(\chi)=1$. Assume that $ord_2 n=1$. Then we have $m_{\boldsymbol{Q}_r}(\chi)=1$ for any prime number r and $m_{\boldsymbol{Q}}(\chi)=m_{\boldsymbol{R}}(\chi)\leqq 2$. And we have $m_{\boldsymbol{R}}(\chi)=2$ if an only if χ is real and $\chi(-1_n)=-\chi(1_n)$ (i,.e. $\theta(-1_n)=-1$).*

REMARK. Let χ be as above. Assume that n is even and let $n=2m$. Fixing a generator θ_0 of $\hat{T}'^F$, put $\theta=\theta_0^i$. Then the following can be shown:

(i) χ is real if and only if $\dfrac{q^m-1}{q-1}\,|\,i$.

(ii) Assume that $\mathrm{ord}_2 n=1$ and let $i=\dfrac{q^m-1}{q-1}i'$ with $i'\in\boldsymbol{Z}$ (hence χ is real).

Then $\theta(-1_n)=1$ if and only if i' is even, and the latter condition is equivalent to the condition that $\theta\,|\,Z^F=1$.

Proof of Proposition 2. Let $\lambda\in\Lambda_0$. Then, by Lemma 1, we have $\langle\chi, \lambda^{G^F}\rangle_{G^F}=1$. Thus, if n is odd or $\mathrm{ord}_2 n>\mathrm{ord}_2(p-1)$, by Proposition 1, (iii), we have $m_{\boldsymbol{Q}}(\chi)=1$. Assume that $1\leqq\mathrm{ord}_2 n\leqq\mathrm{ord}_2(p-1)$. Let t be an element of T^F having the property of Lemma 3. Then, under our assumption, we have $t^{p-1}=-1_n$ (cf. Proof of Lemma 4 and Proof of Lemma 3.3 (a) of Gow [8]). Let us use the notation of the proof of Proposition 1, (ii). Then $\lambda^M=\mu_0+\mu_1$. As $\mu_i(-1_n)=(-1)^i\mu_i(1_n)$ for $i=0$, 1, by Schur's lemma, we have $\langle\chi, \mu_0\rangle_M=1$ if $\chi(-1_n)=\chi(1_n)$, and $\langle\chi, \mu_1\rangle_M=1$ if $\chi(-1_n)=-\chi(1_n)$. As μ_0 is realizable in $\boldsymbol{Q}$, we have $m_{\boldsymbol{Q}}(\chi)=1$ in the first case. Assume that $\chi(-1_n)=-\chi(1_n)$. If r is any prime number $\neq p$, then μ_1 is realizable in $\boldsymbol{Q}_r$, hence we have $m_{\boldsymbol{Q}_r}(\chi)=1$. As q is an even power of p, by Lemma 5, we have $2\,|\,[\boldsymbol{Q}_p(\chi)\colon \boldsymbol{Q}_p]$. Hence $A_1\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p(\chi)$ splits (see [14], Chap. XIII, § 3, Prop. 7), hence μ_1 is realizable in $\boldsymbol{Q}_p(\chi)$. Hence we have $m_{\boldsymbol{Q}_p}(\chi)=m_{\boldsymbol{Q}_p(\chi)}(\chi)=1$. Thus we have $m_{\boldsymbol{Q}}(\chi)=m_{\boldsymbol{R}}(\chi)$. If χ is real, we must have $m_{\boldsymbol{R}}(\chi)=2$ since otherwise χ will be realizable in $\boldsymbol{R}$, so that, by Schur's theorem, we have $(2=)m_{\boldsymbol{R}}(\chi_1)\,|\,\langle\chi, \mu_1\rangle_M=1$, a contradiction. If $\mathrm{ord}_2 n\geqq 2$, then χ cannot be real since G^F contains a central element z of order 4 such that $z^2=-1_n$ and $\chi(z)=\pm\sqrt{-1}\chi(1_n)$ ([7], p. 107). Finally, we note that, by [4], 1.22, we have $\chi(-1_n)=-\chi(1_n)$ if and only if $\theta(-1_n)=-1$. This completes the proof of Proposition 2.

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