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Osaka J. Math.
25 (1988), 135–147

REAL 2-BLOCKS OF CHARACTERS OF FINITE GROUPS

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(Received September 5, 1986)

Let G be a finite group and let p be a prime divisor of $|G|$. Let B be a p-block of irreducible complex characters of G. It is straight forward to show that the complex conjugates of the irreducible characters in B form another p-block of G, $\bar{B}$, say. B is said to be a real p-block if $B=\bar{B}$. Brauer's concept of the defect groups of a block is well known and has been extensively investigated. In this paper, we will show that if B is a real 2-block of G, there is a related concept of extended defect groups of B, which can be explained in the following manner. It is easily proved that a non-principal real 2-block has at least one non-identity real defect class, K, say. The Sylow 2-subgroups of the extended centralizers of the elements of K comprise a complete conjugacy class of 2-groups, which we call the extended defect groups of K. We then show that for a non-principal real 2-block, we obtain the same conjugacy class of 2-groups for any choice of real defect class and associated extended defect groups. This unique conjugacy class will be referred to as the class of extended defect groups of the real 2-block. The principal 2-block is also real but is somewhat anomalous in that it is the only real 2-block of maximal defect and its unique real defect class is the identity class.

We obtain various general results on the number of real 2-blocks and, in particular, show that if 2^a is the 2-part of $|G|$, the number of real blocks of defect $a-1$ equals the number of real 2-regular classes of defect $a-1$. We also show that, if B is a real 2-block of defect zero and w is an involution that generates an extended defect group of B, then there exists a Sylow 2-subgroup Q of G with $Q\cap Q^w=1$. This generalizes part of a well-known theorem of Green on defect groups and Sylow intersections.

We mention here one example of how the conjugacy theorem for extended defect groups yields group-theoretic information. Suppose that G is a simple group of Lie type in characteristic 2. The block of G corresponding to the Steinberg character of G is a real block of defect zero. Let w be an involution that generates an extended defect group of the block. Using properties of the Steinberg character, we can prove that any real element of odd order in G must be inverted by a conjugate of w, and consequently can be written as a product of two involutions conjugate to w. Our Sylow intersection theorem for real 2-

blocks shows that w is derived from the long element w_0 of the Weyl group of G, which transforms the set of positive roots into the set of negative roots.

The subject of real p-blocks was introduced by Brauer in [1] and he obtained a number of theorems on real p-blocks valid for all primes p. However, it seems to us that the prime 2 yields the best theory of real blocks. Various results on real 2-blocks were obtained by the author in an earlier paper, [4], but the present approach to the subject is independent of this previous work.

1. Introductory material

We assume some familiarity with block theory and refer to Chapter 15 of [7] for background material. Let G be a finite group and let ε be a primitive $|G|$-th root of unity in the complex numbers. Let R be the ring of algebraic integers in the field $Q(\varepsilon)$. Let P be a maximal ideal of R that contains the rational prime p, let A be the ring of P-local integers in $Q(\varepsilon)$ and let πA be the unique maximal ideal of A, where π is some uniformizing element. The finite field $F=A/\pi A$ is a splitting field of characteristic p for G and all its subgroups.

Given a conjugacy class K of G, we let K^{-1} denote the conjugacy class whose members are the inverses of the elements in K. Let $Z=Z(F[G])$ denote the center of the group ring $F[G]$ and let $\hat{K}$ denote the class sum given by $\hat{K}=\sum_K x$, the sum extending over all elements x of K. We can define an involutory automorphism * on Z by setting

$$(\hat{K})^* = (\hat{K}^{-1})$$

and extending to all of Z by linearity.

The p-blocks of G correspond in a one-to-one manner with the primitive idempotents of Z and also with the homomorphisms (central characters) $Z\rightarrow F$. If the p-block B corresponds to the primitive central idempotent e and central character λ, we will express this correspondence in the form $B\leftrightarrow e\leftrightarrow\lambda$. We recall that an irreducible complex character χ of G determines a central character $\bar{\omega}$ by

$$\bar{\omega}(\hat{K}) = |K|\chi(g)/\chi(1) \pmod{\pi A},$$

where g is a representative of the class K.

Now if e is a primitive idempotent of Z, the idempotent e^* must also be primitive. Similarly, if λ is a central character, the function λ^* defined by

$$\lambda^*(\hat{K}) = \lambda(\hat{K}^*)$$

is also a central character. Clearly, if λ is associated with e, so that $\lambda(e)=1$, λ^* is associated with e^*. We are now in a position to characterize real blocks by showing that they correspond to idempotents fixed by *.

Lemma 1.1. *Let $B \leftrightarrow e \leftrightarrow \lambda$ be a p-block of G. Then B is a real p-block if and only if $e^*=e$.*

Proof. Suppose that B is a real p-block, and let χ be an irreducible complex character of G in B. Then we have $\lambda=\bar{\omega}$, where $\bar{\omega}$ is defined as before. However $\bar{\chi}$ is also in B and so $\bar{\chi}$ defines the same central character on Z as χ does. Since $\bar{\chi}(g)=\chi(g^{-1})$, we see that

$$\bar{\omega}(\hat{K}) = \bar{\omega}(\hat{K}^{-1}),$$

which means that $\lambda=\lambda^*$ and thus that $e=e^*$. The converse part is proved in a similar manner. This completes the proof.

Suppose now that $B \leftrightarrow e \leftrightarrow \lambda$ is a p-block of G and write

$$e = \sum a_i \hat{K}_i,$$

where the a_i are in F. Recall that a defect class for B is a class K_j such that $a_j \neq 0$ and $\lambda(\hat{K}_j) \neq 0$.

Lemma 1.2. *Let $B \leftrightarrow e \leftrightarrow \lambda$ be a real non-principal 2-block of G. Then B has a real non-identity defect class.*

Proof. If we write $e = \sum a_i \hat{K}_i$, we have

$$\sum a_i \hat{K}_i = \sum a_i \hat{K}_i^*,$$

since $e=e^*$ by Lemma 1.1. Thus the class sums $\hat{K}_i$ and $\hat{K}_i^*$ occur in e with equal coefficients. Now we break the sum for e into two parts,

$$e = \sum a_j \hat{K}_j + \sum a_k(\hat{K}_k + \hat{K}_k^*),$$

where the first sum runs over real classes and the second over non-real classes. We have

$$\lambda(e) = 1 = \sum a_j \lambda(\hat{K}_j) + \sum a_k(\lambda(\hat{K}_k) + \lambda(\hat{K}_k^*)).$$

However, as B is real and F has characteristic 2,

$$a_k(\lambda(\hat{K}_k) + \lambda(\hat{K}_k^*)) = 2a_k \lambda(\hat{K}_k) = 0.$$

Thus there must be an index j such that K_j is a real class, $a_j \neq 0$ and $\lambda(\hat{K}_j) \neq 0$. Therefore, we have found a real defect class for B.

We claim now that K_j cannot be the identity class K_1. For suppose that the identity class is a defect class for B. It follows that B has maximal defect. Now it is easily seen that the only real 2-regular class of maximal defect is the identity class and therefore K_1 is the only real defect class for B. Thus our reasoning above shows that

$$1 = a_1 \lambda(\hat{K}_1).$$

Let μ be the central character of Z corresponding to the principal block. We clearly have $\mu = \mu^*$ and $\mu(\hat{K}_1) = 1$. However $\mu(e) = 0$, as B is non-principal. Repeating our argument above and using the fact that K_1 is the only real defect class for B, we obtain the contradiction

$$0 = \mu(e) = a_1 \mu(\hat{K}_1) \neq 0.$$

Thus we have shown that K_j is not the identity class and our lemma is proved.

Corollary 1.3. *The only real 2-block of G of maximal defect is the principal block.*

The corollary was proved by Brauer in [1] using a different approach.

2. Conjugacy of extended defect groups

Let K be a class of G and let $\delta(K)$ denote the collection of defect groups of K for the prime 2. If K is a real class, the Sylow 2-subgroups of the extended centralizers $C^*(x)$ of the elements x of K form a conjugacy class of 2-groups, which we call the extended defect groups of K. We let $\delta^*(K)$ denote the collection of extended defect groups. Unless K consists of involutions or the identity, an element of $\delta^*(K)$ contains an element of $\delta(K)$ as a subgroup of index 2.

We have seen in the first section that a real non-principal 2-block B has a real non-identity defect class L. As is well known, such a class consists of 2-regular elements. If K is another real defect class for B, we intend to show that $\delta^*(L) = \delta^*(K)$. Knowing this, the subgroups in $\delta^*(L)$ can be called unambiguously the extended defect groups of B. Before proving this, we will prove a more general result that extends a well-known fact in defect group theory.

Theorem 2.1. *Let $B \leftrightarrow e \leftrightarrow \lambda$ be a real non-principal 2-block of G. Let L be a real defect class for B and let U be an element of $\delta^*(L)$. Let K be any real non-identity class of G. If $\lambda(\hat{K}) \neq 0$, U is contained in the extended centralizer of an element z of K. Moreover, if K does not consist of involutions, U does not centralize z.*

Proof. Since λ is a homomorphism and L is a defect class for B, we have

$$\lambda(\hat{K}\hat{L}) = \lambda(\hat{K}) \lambda(\hat{L}) \neq 0.$$

Write

$$\hat{K}\hat{L} = \sum m_i \hat{K}_i,$$

where the K_i are classes and the m_i are in $GF(2)$. Since K, L are real classes, we have $\hat{K}^*=\hat{K}$, $\hat{L}^*=\hat{L}$ and thus $(\hat{K}\hat{L})^*=\hat{K}\hat{L}$, since $*$ is an automorphism. It follows that

$$\sum m_i \hat{K}_i = \sum m_i \hat{K}_i^* .$$

We have then

$$0 \neq \sum m_i \lambda(\hat{K}_i) .$$

The arguments of Section 1 can be applied immediately to deduce that there must be at least one class sum $\hat{K}_j$ occurring in $\hat{K}\hat{L}$ with $m_j \neq 0$, $\hat{K}_j=\hat{K}_j^*$ and $\lambda(\hat{K}_j)\neq 0$. Write J for this class K_j and observe that J is real.

Now let g be a fixed element of J and put

$$W = \{(x, y): x \in K, y \in L, xy = g\} .$$

It is well known that as $m_j=1$ in $GF(2)$, $|W|$ is odd. Let S be a Sylow 2-subgroup of $C(g)$ and T a Sylow 2-subgroup of $C^*(g)$ containing S. Then, unless $g^2=1$, S has index 2 in T. We observe that as $\lambda(\hat{J})\neq 0$, S contains an element of $\delta(L)$ by Theorem 15.31 of [7]. Now S acts by conjugation on W, with an element u of S sending the pair (x, y) to the pair (x^u, y^u). Define

$$W_0 = \{(x, y) \in W : x^u = x,\ y^u = y \qquad \text{for all} \quad u \in S\} .$$

We have $|W_0| \equiv |W| \pmod 2$, as S is a 2-group, and thus $|W_0|$ is odd. Now if $(x, y) \in W_0$, S is contained in Sylow 2-subgroups of $C(x)$, $C(y)$. However, as $y \in L$, a Sylow 2-subgroup of y is in $\delta(L)$. Since we saw that S contains an element of $\delta(L)$, it follows that S is in $\delta(L)$. (The argument so far is part of the usual proof of the conjugacy of defect groups of a block.)

Suppose, if possible, that $g^2=1$. Then for (x, y) in W_0, we have

$$xy = g = g^{-1} = y^{-1}x^{-1} = (y^{-1}x^{-1}y)\,y^{-1} .$$

Since K, L are real classes and $y^{-1}x^{-1}y$, y^{-1} clearly commute with S, the pair $(y^{-1}x^{-1}y, y^{-1})$ is in W_0. The association of (x, y) with $(y^{-1}x^{-1}y, y^{-1})$ is involutory and thus, as W_0 has odd order, there must exist (x, y) in W_0 with

$$x = y^{-1}x^{-1}y , \quad y = y^{-1} .$$

This implies that $y^2=1$ and hence $y=1$, since y is in a defect class and must have odd order. However, as B is assumed to be non-principal, Corollary 1.3 shows that B cannot have maximal defect and thus the identity class is not a defect class. Therefore, we see that $g^2\neq 1$ and S has index 2 in T.

Take t in $T-S$ and any (x, y) in W_0. Then we have

$$xy = g = (g^t)^{-1} = (y^t)^{-1}(x^t)^{-1}y^t\,(y^t)^{-1} = vw ,$$

where $v=(y^t)^{-1}(x^t)^{-1}y^t$, $w=(y^t)^{-1}$. Since t normalizes S, we check that $(v, w)\in W_0$. Thus with the pair (x, y), we associate the pair (v, w). However, this association is involutory, as t^2 is in S and hence centralizes x and y. Again, there must be a pair (a, b) in W_0 with

$$b=(b^t)^{-1}, \quad a=(b^t)^{-1}(a^t)^{-1}b^t .$$

Thus t is $C^*(b)$ and therefore T must be in $\delta^*(L)$, since it has the appropriate order (recall that S is in $\delta(L)$). We also have

$$a=bt^{-1}a^{-1}tb^{-1}$$

and thus a is inverted by bt^{-1}. Since b has odd order, we can write $b=b^{-2r}$ for some integer r. Set $d=b^r$. Then we have

$$d^{-1}t^{-1}d=d^{-1}t^{-1}dtt^{-1}=d^{-2}t^{-1}=bt^{-1} .$$

Since d commutes with S, we see from the calculation above that $d^{-1}Td$ is contained in $C^*(a)$. Thus our theorem holds with $U=d^{-1}Td$ and $z=a$. The final sentence of the theorem is obvious, since a is inverted by bt^{-1}, which is in U. This completes the proof.

Corollary 2.2. *Let $B\leftrightarrow e\leftrightarrow\lambda$ be a real non-principal 2-block of G and let K, L be real defect classes for B. Then we have $\delta^*(L)=\delta^*(K)$.*

Proof. Let U, V be elements of $\delta^*(L)$, $\delta^*(K)$, respectively. Since $\lambda(\hat{L})\neq 0\neq\lambda(\hat{K})$, the previous theorem shows that U is contained in an element of $\delta^*(K)$ and V is contained in an element of $\delta^*(L)$. The corollary is now clear.

In view of the result just proved, we let $\delta^*(B)$ denote the conjugacy class of subgroups $\delta^*(L)$, where L is any real defect class for B. Similarly, $\delta(B)$ will denote the class of defect groups of B.

Recall that the Frobenius-Schur indicator $\varepsilon(\chi)$ of an irreducible character χ is given by

$$\varepsilon(\chi)=1/|G|\sum\chi(g^2) .$$

In [4], we proved that a real 2-block always contains a real-valued irreducible character χ of height 0 with $\varepsilon(\chi)=1$. Making use of the work above, another result of [4] on the existence in a real 2-block of real-valued characters χ with $\varepsilon(\chi)=-1$ can now be stated more precisely.

Theorem 2.3. *Let B be a real non-principal 2-block of G. Let U be an element of $\delta^*(B)$ and V the subgroup of index 2 in U that is in $\delta(B)$. Let V' denote the derived subgroup of V. Then B contains a real-valued irreducible character χ of height zero with $\varepsilon(\chi)=-1$ if and only if U/V' is a non-split extension of V/V'.*

3. Estimates of the number of real 2-blocks

There are a number of general results that bound the number of p-blocks having a given defect group in terms of the number of p-regular classes having the same defect group. In this section, we will show that most of these results have analogues for real 2-blocks.

Lemma 3.1. *The number of real non-principal 2-blocks of G that have defect group V and extended defect group U (where $|U: V|=2$) does not exceed the number of real non-identity 2-regular classes that have defect group V and extended defect group U.*

Proof. Let $e_1, \cdots, e_t$ be all the primitive central idempotents of Z corresponding to the real 2-blocks with defect group V and extended defect group U, and let $\lambda_1, \cdots, \lambda_t$ be the corresponding central character. We have

$$\lambda_i(e_j) = \delta_{ij}\,, \quad 1 \leq i, j \leq t\,,$$

where δ_{ij} is Kronecker's delta. Write

$$e_j = \sum a_{js} \hat{K}_s$$

where the sum runs over all 2-regular classes K_s. Then we obtain

$$\delta_{ij} = \sum a_{js} \lambda_i(\hat{K}_s)\,, \quad 1 \leq i, j \leq t\,.$$

We have seen in Section 1 that if K_s is not a real class, its contribution to the sum above is cancelled by the contribution of K_s^*. Thus in the sums above, we need only consider real 2-regular classes.

Take now a real 2-regular class K_s for which $a_{js} \lambda_i(\hat{K}_s) \neq 0$ for some i, j. The work of Section 2 shows that $V \in \delta(K_s)$ and $U \in \delta^*(K_s)$. It follows that if $K_1, \cdots, K_r$ are the real 2-regular classes with $V \in \delta(K_i)$, $U \in \delta^*(K_i)$, $1 \leq i \leq r$, our sums above can be replaced by

$$\delta_{ij} = \sum_{s=1}^{r} a_{js} \lambda_i(\hat{K}_s)\,, \quad 1 \leq i, j \leq t\,.$$

Thus the $t \times r$ matrix whose (i, s)-entry is $\lambda_i(\hat{K}_s)$ has rank t and therefore $t \leq r$, which is what we wanted to prove.

A well-known theorem of Brauer and Nesbitt states that for an arbitrary prime p, the number of p-blocks of G that have maximal defect equals the number of p-regular classes of maximal defect. We show next that there is an analogous statement for real 2-blocks whose defect is one less than maximal.

Theorem 3.2. *Let G be a finite group of even order and let U be a Sylow 2-subgroup of G. Let V be a subgroup of index 2 in U. Then the number of real 2-blocks of G that have defect group V equals the number of real 2-regular classes* *of G that have defect group V.*

Proof. In a previous paper, [5], we showed that the number r of 2-blocks of G that have defect group V equals the number of 2-regular classes K_i, $1 \leq i \leq r$, for which $V \in \delta(K_i)$. Let $\lambda_1, \cdots, \lambda_r$ be the central characters corresponding to these r blocks. The rank estimation technique used to prove Lemma 3.1 shows that the $r \times r$ matrix E whose (i, j)-entry is $\lambda_i(\hat{K}_j)$ is invertible.

We define now an action of a cyclic group H generated by an element g of order 2 on the class sums by

$$\hat{K}_i^g = \hat{K}_i^*$$

and an action on the central characters by

$$\lambda_i^g = \lambda_i^* .$$

We see that H permutes the rows and columns of the matrix E, and we have

$$\lambda_i^g(\hat{K}_j^g) = \lambda_i(\hat{K}_j), \quad 1 \leq i, j \leq r .$$

Since H is cyclic and E is invertible, a suitably modified version of the usual characteristic zero Brauer permutation lemma implies that the number of λ_i fixed by H equals the number of K_i fixed by H. But since λ_i is fixed by H if and only if λ_i arises from a real block, whereas $\hat{K}_i$ is fixed by H if and only if K_i is real, we conclude that our theorem must be true.

Although we are concentrating on real 2-blocks in this paper, the method of proof of Theorem 3.2 yields information on real p-blocks of maximal defect for odd p, as the next result shows.

Theorem 3.3. *Let p be an odd prime. The number of real p-blocks of G having maximal defect equals the number of real p-regular classes of maximal defect in G.*

Proof. Suppose that $K_1, \cdots, K_r$ are the p-regular classes of G of maximal defect. Let $\lambda_1, \cdots, \lambda_r$ be the central characters of the r p-blocks of maximal defect of G. The proof of Lemma 3.1 shows that the $r \times r$ matrix $E = (\lambda_i(\hat{K}_j))$ is invertible. Following the proof of Theorem 3.2, we define an action of a cyclic group H of order 2 on the rows and columns of E. We can deduce our result by applying Brauer's permutation lemma (again in a suitably modified form to deal with the fact that we are not working over a field of characteristic zero).

Theorem 3.3 reduces to the well-known theorem of Frobenius and Schur on the number of real-valued irreducible characters of G if we choose p not to divide $|G|$.

4. Extended defect groups and Sylow intersections

A well-known theorem of Green, [6], states that if Q is a defect group of a p-block of G, there is an element g in G that centralizes Q and a Sylow p-subgroup P of G such that $P\cap P^g=Q$. In particular, if G has a p-block of defect zero, two Sylow p-subgroups of G intersect trivially. In this section, we intend to give a refined version of this latter fact for real 2-blocks of defect zero. We use a modification of Scott's proof of Green's theorem, as described in Goldschmidt's book, [3].

Let G act as a permutation group on the finite set Ω and let $D_1, \cdots, D_r$ denote the orbits of G acting on $\Omega\times\Omega$. For each i, and g in G, define

$$S_i(g) = \{\alpha\in\Omega\colon (\alpha, \alpha^g)\in D_i\}$$
$$\theta_i(g) = |S_i(g)|.$$

The functions θ_i are called orbital characters. It is easy to check that they are class functions on G. For each orbit D_i, define the orbit D_{i*} by

$$D_{i*} = \{(\beta, \alpha)\colon (\alpha, \beta)\in D_i\} .$$

The orbit D_i is said to be self-paired if $D_i=D_{i*}$.

Lemma 4.1. $\theta_{i*}(g) = \theta_i(g^{-1})$

Proof. This follows since $\alpha\in S_{i*}(g)$ if and only if $\alpha\in S_i(g^{-1})$.
Using the fact that θ_i is a class function, we have the following lemma.

Lemma 4.2. *If g is a real element of G, $\theta_{i*}(g)=\theta_i(g)$.*

The following result is proved in [3], 12.9.

Lemma 4.3. *Let G act on Ω and let θ be the permutation character of this action. Let θ_i, $1\le i\le r$, be the orbital characters. Then for each irreducible character χ of G, there exist algebraic integers α_i, $1\le i\le r$, such that*

$$(\chi, \theta)_G\chi = \chi(1)\sum \alpha_i\theta_i/|D_i|.$$

If we follow the proof of this result in [3], it is not hard to obtain the next lemma.

Lemma 4.4. *Assume the notation of Lemma 4.3. If χ is real-valued, we have $\alpha_i=\alpha_{i*}$.*

We can now proceed to the proof of our main result on Sylow intersections and defect groups. We follow the argument of [3], 12.12.

Theorem 4.5. *Let χ be a real-valued irreducible character of G of defect zero modulo 2. Let g be any element of a real defect class for the block of G as-*

sociated with χ and let t be an involution of G that inverts g. Then there exists a Sylow 2-subgroup P of G that contains t such that $P \cap P^g = 1$.

Proof. Let Q be a Sylow 2-subgroup of G and let Ω be the set of right cosets of Q in G. The permutation character θ of G acting on Ω equals 1_Q^G. Since χ vanishes on non-identity elements of Q, we have

$$(\chi, \theta) = (\chi, 1_Q^G) = (\chi_Q, 1_Q) \not\equiv 0 \qquad (\mathrm{mod}\ 2)\,.$$

Set $|D_i| = |G\colon Q| n_i$, where n_i is a power of 2 dividing $|Q|$.

Let g be any real element of G with $\chi(g) \not\equiv 0\ (\mathrm{mod}\ \pi A)$. By Lemma 4.3, we have

$$(\chi, \theta)\chi(g) = |G\colon Q|^{-1} \sum \chi(1)\alpha_i \theta_i(g)/n_i\,.$$

Since n_i is a power of 2 dividing $|Q|$ and $\chi(1)$ is divisible by the 2-part of $|G|$, all the terms in the sum above are 2-local integers in A. We also know that the sum is non-zero modulo πA. As χ is real-valued, we have $\alpha_i = \alpha_{i^*}$ by Lemma 4.4, and as g is real, $\theta_i(g) = \theta_{i^*}(g)$. Moreover, $n_i = n_{i^*}$ for all i. Thus, we can see that there must exist an index i such that

$$i = i^*\,, \quad n_i = |Q|\,, \quad \theta_i(g) \ \text{ is odd.}$$

This means that the orbit $D = D_i$ is self-paired and $|S_i(g)|$ is odd.

We claim now that the cyclic group generated by t acts on $S = S_i(g)$. For let $\beta \in S$. We have $(\beta, \beta g) \in D$, from which it follows that $(\beta t, \beta g t) \in D$. As $gt = tg^{-1}$, we see that $(\beta t, \beta t g^{-1}) \in D$, and as D is self-paired, we have $(\beta t g^{-1}, \beta t) \in D$ and thus $(\beta t, \beta t g) \in D$. This means that $\beta t \in S$ and proves our claim.

Since S is acted on by a group of order 2 and $|S|$ is odd, there must be some β in S with $\beta t = \beta$. Let β correspond to the coset Qx. Then we have $t \in Q^x$ and $(Qx, Qxg) \in D$. If we set $h = xgx^{-1}$, we have $(Q, Qh) \in D$ and since $|D| = |G|$, it follows that

$$Q \cap hQh^{-1} = 1\,.$$

Thus $Q^x \cap Q^{h^{-1}x} = 1$ and if we put $P = Q^x$, we have

$$P \cap P^g = 1 \quad \text{and} \quad t \in P\,.$$

This proves our theorem.

Corollary 4.6. *Let B be a real 2-block of G of defect zero and let t be an involution that generates an extended defect group of B. Then there exists a Sylow 2-subgroup Q of G that contains t and involution u conjugate to t such that $Q \cap Q^u = 1$.*

Proof. Let x be an element of a real defect class for B that is inverted by

t. By the previous theorem, there is a Sylow 2-subgroup Q containing t with $Q \cap Q^x = 1$. But $Q^x = Q^{tx}$ and it is well known that $u = tx$ is an involution conjugate to t. This proves the corollary.

5. The Steinberg character of a group of Lie type in even characteristic

Let $\bar{k}$ be the algebraic closure of a finite field k of characteristic 2 and let $\bar{G}$ be a connected simple algebraic group defined over k. Let G be the fixed-point subgroup of a Frobenius morphism of $\bar{G}$. Then G is said to be a finite group of Lie type. G has a unique irreducible character χ of 2-defect zero, called the Steinberg character. In particular, χ defines a real 2-block $B \leftrightarrow e \leftrightarrow \lambda$ of G. Moreover, if g is any 2-regular element of G, $|\chi(g)| = $2-part of $|C(g)|$. It follows that if K is the conjugacy class containing g,

$$|K|\chi(g)/\chi(1) \equiv 1 \pmod 2$$

and thus $\lambda(\hat{K}) = 1$. We obtain an immediate corollary of Theorem 2.2.

Theorem 5.1. *Let G be a group of Lie type in characteristic* 2 *(as defined above). There exists a unique conjugacy class C of involutions in G such that every real* 2*-regular element of G is inverted by an element of C. Equivalently, each real* 2*-regular element is a product of two elements of C.*

To describe the class C in more detail, we recall some information about root systems. Let Φ be the set of roots associated with G and let W be the Weyl group of G. Let Π be a fundamental system of roots. If Φ^+ denotes the set of positive roots with respect to Π and Φ^- the set of negative roots, there exists a unique involution w_0 in W with $w_0(\Phi^+) = \Phi^-$. Sometimes w_0 is called the long element of the Weyl group, since it is the unique element of greatest possible minimal length when expressed as a product of fundamental reflections. In general, w_0 depends on the choice of Π, although it is determined up to conjugacy. For certain types of G, w_0 acts as $-I$ on Φ and is thus unique for all choices of Π. This occurs when G is of type B_n, D_{2n}, G_2, F_4, E_7 or E_8. In such cases, it can be shown that all 2-regular (semisimple) elements of G are real.

We will show now that the conjugacy class C of Theorem 5.1 is related to w_0 in a straightforward manner. Let U be a Sylow 2-subgroup of G, generated by root subgroups corresponding to roots in Φ^+, and let $B = UH$ be the normalizer of U, with $H \cap U = 1$. Let N be the normalizer of H in G. Then we have

$$N/H \cong W.$$

Since H has odd order, there exists an involution of N that represents w_0 in N/H

and for brevity, we will also let w_0 denote such an involution. With this notation, we have the following result.

Theorem 5.2. *The conjugacy class C described in Theorem* 5.1 *is the class containing the involution* w_0.

Proof. We know that there is an involution t in C with $U\cap U^t=1$. Now by the Bruhat decomposition, we have $G=BNB$. Thus we can write $t=bnc$, where b, $c\in B$ and $n\in N$. Since b normalizes U, we have

$$U\cap U^s=1\,,\quad s=b^{-1}tb\,.$$

We put

$$s=ncb=nd\,,$$

where $d=cb\in B$. Again we must have

$$U\cap U^n=1\,.$$

However, it is known that in this case nN equals the element w_0 of $N/H\cong W$ (see, for example, Section 8.4 of [2]). Thus we can write $n=w_0h$, for some h in H. Put $d=uh'$, with u in U, h' in H. Then we have

$$s=w_0huh'=w_0vg\,,$$

where $v\in U$, $g\in H$. As s is an involution,

$$1=w_0vgw_0vg$$

and since w_0 is an involution, we have

$$w_0v^{-1}w_0^{-1}=w_0^{-1}gw_0vg$$

The element on the right is in B, that on the left is in V, the Sylow 2-subgroup of G generated by root subgroups corresponding to roots in Φ^-. Since $B\cap V=1$ by Lemma 7.12 of [2], we must have $v=1$ and thus

$$w_0^{-1}gw_0=g^{-1}.$$

Now g has odd order and so $g=g^{2r}$ for some integer r. Then

$$s=w_0g=w_0g^{2r}=g^{-r}w_0g^r\,.$$

It follows that s is conjugate to w_0 and hence C contains w_0, as required.

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