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Osaka J. Math.
25 (1988), 123-134

LEFT SERIAL RINGS OVER WHICH EVERY RIGHT MODULE WITH HOMOGENEOUS TOP IS A DIRECT SUM OF HOLLOW MODULES

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(Received March 20, 1986)
(Revised May 15, 1987)

Let R be a left and right artinian ring with identity, and J the Jacobson radical of R. In [4], M. Harada has considered a left serial ring R satisfying a condition (*, 2) that every maximal submodule of a direct sum of any two hollow modules is also a direct sum of hollow modules, and characterized such a ring by the structure of eR for each primitive idempotent e. Further it has been shown that the condition (*, 2) is equivalent to saying that every factor module of $eJ\oplus eR$ is a direct sum of hollow modules for every primitive idempotent e. Modifying this, we here consider the following condition on a projective indecomposable right module eR over a ring R.

(A): *Every factor module of* $eR\oplus eR$ *is a direct sum of hollow modules.*

Clearly if R is a ring of right local type, then all projective indecomposable right R-modules satisfy the condition (A), and as well known ([6]), R is left serial. The purpose of this paper is to characterize left serial rings over which every projective indecomposable right module eR satisfies the condition (A) (i.e. rings R in the title (see Theorem 1 for the equivalence)) in terms of the structure of eR. Thus our result gives a generalization of rings of right local type.

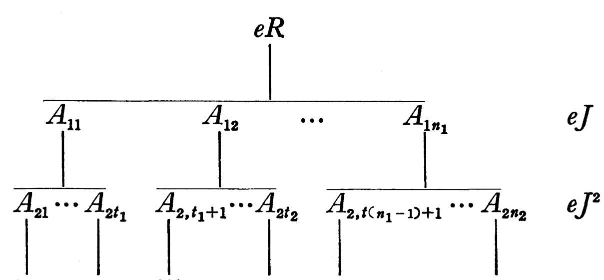
In the first section we consider various conditions equivalent to (A) (Theorem 1). In particular, the condition 4) of Theorem 1 which is described in terms of homomorphisms between factor modules of eR is frequently used later to check whether eR satisfies (A) or not. We assume in the second and third section that R is a left serial ring. In the second section we shall give some properties induced from the condition (A) to prepare the proof of the main theorem. In the third section we give the main theorem (Theorem 2). In the last section we give some examples.

The author would like to thank Professor M. Harada for his suggestion of the problem and helpful advice, and Doctor H. Asashiba for his advice and encouragement. She would also like to thank the referee for his useful comments.

Preliminaries

Throughout this paper, R is a left and right artinian ring with identity, and J is the Jacobson radical of R. Since the property of R that the condition (A) holds for all projective indecomposable right R-modules is Morita invariant, we may assume further that R is a *basic* ring. pi(R) denotes the set of all primitive idempotents of R. All modules are finitely generated unitary right R-modules. A module X is said to be *hollow* if the sum of any two proper submodules of X is a proper submodule. By the assumption that R is an artinian ring, a hollow module is precisely a local module, i.e., is isomorphic to a factor module of eR for some e in pi(R). For a module X, we put $\bar{X}:=X/XJ$, $T(X):=X\backslash XJ$, and denote by $X^{(n)}$ a direct sum of n copies of X, and by $|X|$ the length of a composition series of X. For any f and g in pi(R), we put $T(fJ^k g):=fJ^k g\backslash fJ^{k+1}g$. For division rings Δ and Δ' with $\Delta \geqq \Delta'$, we use the symbol $[\Delta:\Delta']_l$ ($[\Delta:\Delta']_r$) to mean the dimension of Δ as a left (right) Δ'-vector space. We say that R is a *left serial* ring if as a left R-module, R is a direct sum of uniserial submodules. In the second and the third sections we assume that R is a left serial ring. In this case the following hold, which we use without any references. The first one follows from [6, Corollary 4.2], and the second one is clear from the definition of left serial rings.

Lemma 1. *If R is a left serial ring, then for every e in* pi(R) *and for every natural number j, eJ^j is a direct sum of hollow modules, and eR has a structure expressed by the following diagram:*



where each A_{ik} is a hollow module, $eJ^i=\bigoplus_{k=1}^{n_i} A_{ik}$ *and* $\begin{array}{c} X \\ | \\ \hline X_1 \cdots X_s \end{array}$ *means* $XJ=\bigoplus_{i=1}^{s} X_i$.

Lemma 2. *Suppose that R is a left serial ring. Let e, f be in* pi(R), *and $a, b \in R$. If $a=eaf$ and $b=eaf$, then there exists either some $d \in eRe$ with $a=db$*

or else some $d' \in eRe$ *with* $b=d'a$.

1. The condition (A)

We use the following two lemmas from Sumioka [6] to prove Theorem 1.

Lemma 3 ([6], Lemma 1.3]). *Let* M_i $(i=1, 2, 3)$ *and* T *be submodules of a module* M *such that* $M=M_1+(M_2\oplus M_3)$ *and* $T=M_1\cap(M_2\oplus M_3)$, *and* π_3: $T\to M_3$ *the restriction map of the projection* $M_2\oplus M_3\to M_3$. *Then* π_3 *is extended to a homomorphism* $M_1\to M_3$ *if and only if* $M=(M_1'+M_2)\oplus M_3$ *for some submodule* M_1' *of* M.

Lemma 4 ([6, Lemma 2.1]). *Let* S *be a simple module and* $L_1, \cdots, L_n$ *hollow modules of length* $\geqq 2$ *and* $0\to S\xrightarrow{\alpha}\sum_{i=1}^{n} L_i\xrightarrow{\beta} M\to 0$ *an exact sequence with each* α_i *a monomorphism, where* $\alpha=(\alpha_1, \cdots, \alpha_n)^T$, *and* $n\geqq 2$. *Then* M *is decomposable if and only if the identity map* 1_{S_j} *of* S_j *is extended to a homomorphism*: $\bigoplus_{i\neq j} L_i\to L_j$ *for some* j, $1\leqq j\leqq n$, *where* $S_j:=(\bigoplus_{i\neq j} L_i)\cap L_j$.

Now we state the theorem in this section.

Theorem 1. *For* e *in* pi(R), *the following four statements are equivalent.*

1) (A): *Every factor module of* $eR\oplus eR$ *is a direct sum of hollow modules.*

2) *Every factor module of* $eR^{(n)}$ *is a direct sum of hollow modules for each natural number* n.

3) *If* M *is an* R-*module such that* $M/MJ\simeq \overline{eR}^{(n)}$ *for some* n, *then* $M\simeq \bigoplus_{i=1}^{n} eR/X_i$, *where each* X_i *is a submodule of* eR.

4) *Let* C_i *and* D_i $(i=1, 2)$ *be submodules in* eR *such that* $eR\geqq C_i>D_i$. *If* f: $C_1/D_1\to C_2/D_2$ *is an isomorphism, and* C_1/D_1 *is simple, then* f *or* f^{-1} *is extended to some homomorphism from* eR/D_1 *to* eR/D_2 *or one from* eR/D_2 *to* eR/D_1, *respectively.*

Proof. First, we introduce the following conditions 1′) and 2′) which are useful in proving the theorem:

1′) *Let* S *be simple,* X, Y *be submodules of* eR, *and assume that the following sequence is exact:*

$$0\to S\to eR/X\oplus eR/Y\to M\to 0\,.$$

Then $M\simeq eR/X'\oplus eR/Y'$ *for some submodules* X' *and* Y' *of* eR.

2′) *Let* n *be any natural number,* S *a simple module, and* X_i *a submodule of* eR *for each* $i(1\leqq i\leqq n)$. *Consider an exact sequence:*

$$0\to S\to \bigoplus_{i=1}^{n} eR/X_i\to M\to 0\,.$$

Then $M \simeq \bigoplus_{i=1}^{n} eR/Y_i$ *for some submodules* $Y_i (1 \leqq i \leqq n)$ *of* eR.

The proof proceeds as follows: 2)⇔3) and 1) ⇒ 1′) ⇒ 4) ⇒ 2′) ⇒ 2) ⇒ 1).

2) ⇒ 3): It follows from $M/MJ \simeq \overline{eR}^{(n)}$ that M is isomorphic to a factor module of $eR^{(n)}$. So by 2), $M \simeq \bigoplus_{i=1}^{n} eR/X_i$ for some $X_i < eR$ $(1 \leqq i \leqq n)$.

3) ⇒ 2): Let M be a factor module of $eR^{(n)}$. Then $M/MJ \simeq \overline{eR}^{(k)}$ for some $k \leqq n$, so by 3), M is a direct sum of hollow modules.

1) ⇒ 1′): This is clear from the fact that the module M in 1′) is an epimorphic image of $eR \oplus eR$.

1′) ⇒ 4): Let C_i and D_i be submodules with $eR \geqq C_i \geqq D_i (i=1, 2)$. We assume that a homomorphism $f: C_1/D_1 \rightarrow C_2/D_2$ is an isomorphism and C_1/D_1 is simple. To show that the assertion 4) holds, we may assume that $C_i < eR (i=1, 2)$. We consider an exact sequence:

$$0 \rightarrow C_1/D_1 \xrightarrow{(i_1, i_2 f)^{\mathrm{T}}} eR/D_1 \oplus eR/D_2 \rightarrow X \rightarrow 0\,,$$

where $i_j (j=1, 2)$ is the inclusion $C_j/D_j \rightarrow eR/D_j$, and X is the cokernel of the homomorphism $(i_1, i_2 f)^{\mathrm{T}}$. Then i_1 and $i_2 f$ are monomorphisms and not epimorphisms since $C_i < eR (i=1, 2)$. By 1′), $X \simeq eR/X' \oplus eR/Y'$ for some X', $Y' < eR$, so X is decomposable. Then by Lemma 4, $(i_2 f)(i_1^{-1}) = f$ (or $i_1(i_2 f)^{-1} = f^{-1}$) is extended to some homomorphism $eR/D_1 \rightarrow eR/D_2$ (or $eR/D_2 \rightarrow eR/D_1$).

4) ⇒ 2′): We shall show the assertion by induction on n. When $n=1$, the assertion is trivial. In the case that $n=2$, consider an exact sequence:

$$0 \rightarrow S \overset{\alpha}{\rightarrow} eR/X \oplus eR/Y \rightarrow M \rightarrow 0\,,$$

where S is simple, $\alpha(s) := \alpha_1(s) + \alpha_2(s)$ $(\alpha_1(s) \in eR/X, \alpha_2(s) \in eR/Y)$ for any $s \in S$, and X', $Y' \leqq eR$. Here we may assume that α_1 and α_2 are monomorphisms. If $\operatorname{im} \alpha \nleqq \operatorname{rad}(eR/X \oplus eR/Y)$, then $\operatorname{im} \alpha$ is a direct summand of $eR/X \oplus eR/Y$ since $\operatorname{im} \alpha$ is simple, thus the assertion holds. So we may assume that $\operatorname{im} \alpha \leqq \operatorname{rad}(eR/X \oplus eR/Y)$ and $|eR/X|$, $|eR/Y| \geqq 2$. Put $T_1/X := \alpha_1(S)$ and $T_2/Y := \alpha_2(S)$ for some T_1 and $T_2 \leqq eR$. Then T_1/X is simple and $\alpha_2 \alpha_1^{-1}: T_1/X \rightarrow T_2/Y$ is an isomorphism. By 4), $\alpha_2 \alpha_1^{-1}$(or $(\alpha_2 \alpha_1^{-1})^{-1} = \alpha_1 \alpha_2^{-1}$) is extended to a homomorphism: $eR/X \rightarrow eR/Y$ (or $eR/Y \rightarrow eR/X$). Hence by Lemma 4, M is decomposable and $M \simeq eR/X' \oplus eR/Y'$ for some X', $Y' \leqq eR$. Finally we assume that the assertion holds for $n-1 (\geqq 2)$. Suppose that the following sequence is exact:

$$0 \rightarrow S \rightarrow \bigoplus_{i=1}^{n} (eR/X_i) \overset{\varphi}{\rightarrow} M \rightarrow 0\,,$$

where S is simple. Put $L_i := \varphi(eR/X_i)$ for each $i (1 \leqq i \leqq n)$. Then by [6, Section

2], we have $M=L_1+(\bigoplus_{i=2}^{n} L_i)$ and $T:=L_1\cap(\bigoplus_{i=2}^{n} L_i)\simeq S$. Let $p: \bigoplus_{i=2}^{n} L_i \to L_n$ be the canonical projection. Then $p|_T$ (or $(p|_T)^{-1}$) is extended to a homomorphism: $L_1\to L_n$ (or $L_n\to L_1$). In the case that $p|_T$ is extended to some $\Phi: L_1\to L_n$, we put $L_1':=\{x-\Phi(x)\,|\,x\in L_1\}$. Then by Lemma 3, we obtain $M=L_1+(\bigoplus_{i=2}^{n} L_i)=(L_1'+\bigoplus_{i=2}^{n-1} L_i)\oplus L_n$. Put $M':=L_1'+(\bigoplus_{i=2}^{n-1} L_i)$ and $S':=L_1'\cap(\bigoplus_{i=2}^{n-1} L_i)$, then S' is simple. Noting that L_1' and each $L_i(1\leqq i\leqq n-1)$ are factor modules of eR, the following exact sequence implies that $M'\simeq\bigoplus_{i=1}^{n-1}(eR/Y_i)$ for some $Y_i\leqq eR$ by induction hypothesis:

$$0\to S'\overset{\alpha'}{\to} L_1'\oplus\bigoplus_{i=2}^{n-1} L_i\overset{\beta}{\to} M'\to 0,$$

where $\alpha'=(\alpha_1',\cdots,\alpha_{n-1}')^{\mathrm{T}}$, α_1' is the inclusion map and $\alpha_i':=p_i|_{S'}$ (p_i is the canonical projection) $(2\leqq i\leqq n-1)$; and $\beta=(\beta_1,\cdots,\beta_{n-1})$, $-\beta_1$ and $\beta_i(2\leqq i\leqq n-1)$ are the inclusion maps. Hence $M=M'\oplus L_n\simeq\bigoplus_{i=1}^{n-1}(eR/Y_i)\oplus eR/X_n$. The remaining case is proved similarly.

2′) ⇒ 2): Put $D:=\{H\,|\,H\simeq\bigoplus_{i=1}^{n} eR/X_i$ for some n and for some $X_i\leqq eR\}$. Then we have only to show that D is closed under factor modules. Thus it suffices to verify the following for all $m\geqq 1$, by induction on m: For each $H\in D$ and for each $X\leqq H$, $|X|=m$ implies $H/X\in D$. When $m=1$, this follows from 2′). Let $m\geqq 2$. Take $0\neq Y<X$. Then $|Y|<m$ and $|X/Y|<m$, which imply $H/X\simeq(H/Y)/(X/Y)\in D$ by induction hypothesis.

2)⇒1) is trivial. Q.E.D.

We shall characterize later the structure of eR satisfying the condition (A), i.e., the conditions of the above theorem, using mainly 4) of it.

2. Properties of eR with (A)

From now on we assume that R is a left serial ring. Further throughout this section, we assume that e is a fixed primitive idempotent, and eR satisfies the condition (A). Then we have several properties similar to ones in [4] as follows, by using 4) of the theorem.

Proposition 1 ([4, Proposition 1]). *If eR/eJ^i is uniserial and $eJ^i=A_{i1}\oplus A_{i2}\oplus\cdots\oplus A_{ip}$ for some i, where $p\geqq 3$ and each $A_{ik}(1\leqq k\leqq p)$ is a hollow module, then we have $A_{i1}\simeq A_{i2}\simeq\cdots\simeq A_{ip}$, and each A_{ik} is simple.*

Proposition 2 ([4, Proposition 5]). *If eR/eJ^i is uniserial and $eJ^i=A_{i1}\oplus A_{i2}$, where A_{i1} and A_{i2} are hollow modules, then each A_{ik} is uniserial.*

We put $\Delta := eRe/eJe$ and $\Delta(A) := \{\bar{x} \in \Delta \mid x'A \leqq A, \bar{x}' = \bar{x}$ for some x' in $eRe\}$, where A is a hollow submodule of eR and $\bar{x}$ is the coset of x in Δ. Then Δ is a division ring and $\Delta(A)$ is a division subring of Δ (see [4]). In the case that $eJ^i = A_{i1} \oplus A_{i2} \oplus \cdots \oplus A_{ip} (p \geqq 2)$, we put $\Delta(A_{i1}) = \Delta_i$. Now we consider the case $p \geqq 3$ in more detail. Since $eJe(eJ^i) = eJ^{i+1} = 0$, we have $xc = \bar{x}c$ for any $c \in eJ^i$ and any $x \in eRe$. Proposition 1 shows that each $A_{ik} (1 \leqq k \leqq p)$ is simple, so we may put $A_{i1} = aR$ and $a = eaf$ for some f in $\mathrm{pi}(R)$. Then for any $b (\neq 0)$ in A_{i1}, $b = ebf$ since A_{i1} is simple and R is basic. Noting here that R is left serial, there exists some $x \in eRe \backslash eJe$ with $b = xa = \bar{x}a$. Here $\bar{x}$ is in Δ_i since $xA_{i1} = xaR = bR = A_{i1}$, whence b is in $\Delta_i a$. Thus $A_{i1} = \Delta_i a$. For each $k (1 \leqq k \leqq p)$, put $A_{ik} = a_k R$. Similarly taking a_k instead of b, we have $a_k = ea_k f$, $a_k = x_k a = \bar{x}_k a$ for some $x_k \in eRe \backslash eJe$ and $A_{ik} = \bar{x}_k aR = \bar{x}_k \Delta_i a$. Using this fact, we obtain

Lemma 5. *Suppose $eJ^i = A_{i1} \oplus \cdots \oplus A_{ip} (p \geqq 3)$, and let $A_{i1} = aR$. Put $\mathcal{L}(eJ^i) :=$ the lattice of submodules of eJ^i and $\mathcal{L}(\Delta) :=$ the lattice of subspaces of Δ_{Δ_i}. Then we have a bijection $\alpha : \mathcal{L}(\Delta) \to \mathcal{L}(eJ^i)$ defined by $\alpha(V) := Va$ for every $V \in \mathcal{L}(\Delta)$. Further α preserves and reflects the linear independence, i.e., for any $\{V_i\} \subseteqq \mathcal{L}(\Delta)$, $\{V_i\}$ is independent if and only if so is $\{V_i a\}$.*

Proof. Since $VaR = V\Delta_i a = Va$ for any $V \in \mathcal{L}(\Delta)$, α is well defined. It is easy to show that α preserves and reflects the linear independence. To show that α is a surjection, let T be any submodule $(\neq 0)$ of eJ^i. Then T is expressed as $T = X_1 \oplus \cdots \oplus X_t$ with $X_k \simeq A_{i1} (1 \leqq k \leqq t)$. So we have $X_k = \delta_k A_{i1} = \delta_k \Delta_i a$ for some δ_k in Δ by the consideration above. Hence $T = (\bigoplus_{k=1}^{n} \delta_k \Delta_i) a$ since α reflects the independence. Thus α is a surjection. α being an injection is immediate from the fact that $ya = 0 (y \in \Delta)$ implies $y = 0$. Q.E.D.

Lemma 5 implies the following (see [4] for $p = 2$).

Proposition 3 ([4, Proposition 2]). *It holds $[\Delta : \Delta_i]_r = |\overline{eJ^i}|$ except for the case that $eJ^i = A_i \oplus B_i$ and $A_i \not\simeq B_i$, where A_i and B_i are hollow modules (in this exceptional case, we have $\Delta = \Delta_i$).*

We consider the following condition (#) on Δ as a right Δ_i-vector space.

(#) *Let V_1 and V_2 be subspaces of Δ_{Δ^i} and v_1 and v_2 be elements of Δ satisfying $|V_1| \leqq |V_2|$ and $v_1 \Delta_i \cap V_1 = 0 = v_2 \Delta_i \cap V_2$. Then there exists $\bar{x}$ in Δ such that $xV_1 \leqq V_2$ and $xv_1 \equiv v_2$ (mod V_2).*

The following is immediate from Lemma 5.

Proposition 4. *If $|\overline{eJ^i}| \geqq 3$, then the following are equivalent.*

(1) *Let $eJ^i \geqq T_1 > T_2$, $eJ^i \geqq S_1 > S_2$, and T_1/T_2 be simple, and $f : T_1/T_2 \to S_1/S_2$ be an isomorphism. Then f is extended to a homomorphism: $eR/T_2 \to eR/S_2$.*

(2) Δ *and* Δ_i *satisfy the condition* (♯).

Hence in particular, if eR satisfies the condition (A), *then* Δ *and* Δ_i *satisfy the condition* (♯).

By [4, Lemma 5], the following holds.

Proposition 5. *Let* $\Delta \geqq \Delta_i$ *be division rings. If* Δ *and* Δ_i *satisfy the condition* (♯), *then* $[\Delta : \Delta_i]_l \leqq 2$. *In particular, if eR satisfies the condition* (A), *then* $[\Delta : \Delta_i]_l \leqq 2$.

3. The structure of *eR* with (A)

Also in this section, we assume that R is left serial. Using Propositions 1 and 2, eR with the condition (A) has one of the following structures.

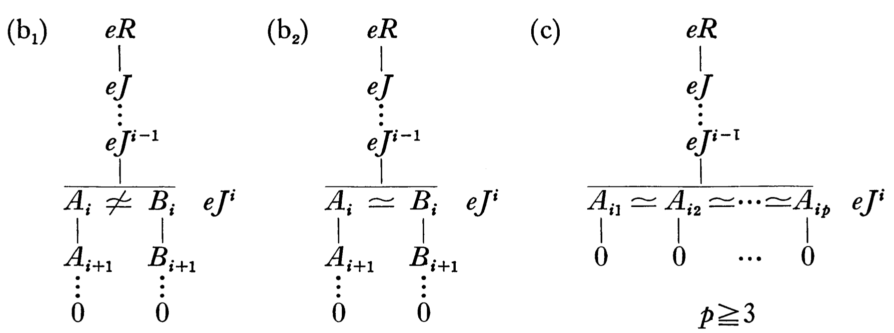
(a) *eR is a uniserial module.*

(b_1) *For some natural number i,* eR/eJ^i *is uniserial and* $eJ^i = A_{i1} \oplus A_{i2}$, *where* A_{i1} *and* A_{i2} *are uniserial modules which are not isomorphic to each other.*

(b_2) *For some natural number i,* eR/eJ^i *is uniserial and* $eJ^i = A_{i1} \oplus A_{i2}$, *where* $A_{i1} \simeq A_{i2}$ *are uniserial modules.*

(c) *For some natural number i,* eR/eJ^i *is uniserial and* $eJ^i = A_{i1} \oplus \cdots \oplus A_{ip}$ $(p \geqq 3)$, *where* $A_{i1} \simeq A_{i2} \simeq \cdots \simeq A_{ip}$ *are simple modules.*

Thus we can illustrate the structures (b_1), (b_2) and (c) as follows.



Now we state the main theorem.

Theorem 2. *Let R be a left serial ring. The following are equivalent for each* $e \in \mathrm{pi}(R)$.

1) *eR satisfies the condition* (A).

2) *eR has one of the structures* (a); (b_1); (b_2) *with* $[\Delta : \Delta_i]_l = 2$; *and* (c) *with the condition* (♯) *for* Δ *and* Δ_i.

The proof of 1)$\Rightarrow$2) is already done. So we show that each of the conditions in 2) implies the condition 1). This is immediate from Proposition 4 in the case of the structure (c), and in the case of (a) this follows from Lemma

2 and Theorem 1 (see case (i) in Lemma 8). For the proof of the other cases, we divide the argument to some lemmas.

Lemma 6. *In the diagram* (b_1), *the following statements hold.*

1) *For any* j *and* k *with* $i \leqq j \leqq k$ *and* $k-j<i$, *we have* $\bar{A}_j \not\simeq \bar{B}_k$.
2) *For any* j *and* k *with* $i \leqq j<k$ *and* $k-j<i$, *we have* $\bar{A}_j \not\simeq \bar{A}_k$.

Proof. 1) For $j (i \leqq j)$, we have $\bar{A}_j \not\simeq \bar{B}_j$ from [3, Lemma 3]. Next suppose that $\bar{A}_j \simeq \bar{B}_k$ for some j, k with $i \leqq j<k$ and $k-j<i$. Put $A_j=a_jR$, $B_k=b_kR$ and $a_j=ea_jf$, $b_k=eb_kf$ for some f in pi(R). Then there exists d in $T(eJ^{k-j})$ such that $da_j=b_k$. So $eJ^{k-j}e \neq 0$, and there exists an epimorphism: $eR \to eJ^{k-j}$. This epimorphism induces an epimorphism: $eJ^{i-(k-j)} \to eJ^i$. Thus we have $\overline{eJ^{i-(k-j)}} \simeq \overline{eJ^i}$. But this is a contradiction since $i>i-(k-j)$. We conclude that $\bar{A}_j \not\simeq \bar{B}_k$.

2) Suppose that $\bar{A}_j \simeq \bar{A}_k$ for some j, k with $i \leqq j<k$ and $0<k-j<i$. Then we may put $A_j=a_jR$ and $A_k=a_kR$. So there exists d in $eJ^{k-j}e$ such that $da_j=a_k$, thus $eJ^{k-j}e \neq 0$ and this yields the similar contradiction as in 1). Q.E.D.

Lemma 7. *In the diagram* (b_2), *we have that* $\bar{A}_j \not\simeq \bar{B}_k$ *for each* j, k *with* $i \leqq j<k$.

Proof. Suppose that $\bar{A}_j \simeq \bar{B}_k$ in the case $k-j<i$. Put $A_j=a_jR$ and $B_k=b_kR$, and $a_j=a_jg$, $b_k=b_kg$ for some g in pi(R). Then there exists d in $T(eJ^{k-j}e)$ such that $da_j=b_k$, and this yields the contradiction similar to that in the proof of Lemma 6. Next suppose that $\bar{A}_j \simeq \bar{B}_k$ in the case $k-j \geqq i$. Then there exists a_j, b_k and d as above. It follows from $k-j \geqq i$ that $d=d_1+d_2$ for some d_1 in A_i and d_2 in B_i. So $b_k=da_j=d_1a_j+d_2a_j$ and b_k is in B_i, thus $b_k=d_2a_j$, where d_2 in $T(B^{k-j})$. Then for b_j with $B_j=b_jR$, we have that $b_j=b_jg$ and $d_2b_j \neq 0$ is in $T(B_k)$ by $\bar{A}_j \simeq \bar{B}_k$. There exists r in $T(gRg)$ such that $d_2b_jr=b_k$, and hence $T(eJ^jg) \ni d_2(a_j-b_jr)=b_k-b_k=0$, a contradiction. Q.E.D.

Using Lemmas 6 and 7, we show the implication 2)$\Rightarrow$1) of Theorem 2 as the following two lemmas.

Lemma 8. *Let the diagram* (b_1) *be the structure of eR. Then eR satisfies the condition* (A).

Proof. Let C_j and D_j be submodules of eR such that $eR \geqq C_j>D_j$ and C_j/D_j is simple for $j=1, 2$, and f: $C_1/D_1 \to C_2/D_2$ be an isomorphism. We may assume that $C_j=c_jR+D_j$ for some c_j in $C_j (j=1, 2)$ satisfying $f(c_1+D_1)=c_2+D_2$, and $c_1g=c_2g$ for some $g \in$ pi(R).

(i) In the case where both c_1 and c_2 are in $T(eJ^t)$ for some $t<i$, there exists a unit x in eRe such that $xc_1=c_2$. Then x_l (the left side multiplication of x) induces f.

(ii) Suppose that c_1 is in $T(eJ^t)$ for some $t<i$ and $eJ^i \geqq C_2>D_2$ Then there exists some x in eJe such that $xc_1=c_2$. So $xD_1=xC_1J=xc_1J<C_2J<D_2$, whence x_l induces f

iii) In the case that $eJ^i \geqq C_j > D_j$ for each $j=1, 2$.

First, we show that for any C, D such that $D<C\leqq eJ^i$ and C/D is simple, there exists a unit x in eRe satisfying

$$\text{(P)}: \qquad \begin{aligned} xC &= A_{t-1}\oplus B_s \quad (\text{or } xC = A_t\oplus B_{s-1}) \\ xD &= A_t\oplus B_s \quad \text{where} \quad t, s\geqq i\,. \end{aligned}$$

For a module $X\leqq eJ^i$, put $X^{(1)}=\pi_{A_i}(X)$, $X^{(2)}=\pi_{B_i}(X)$, $X_{(1)}=X\cap A_i$ and $X_{(2)}=X\cap B_i$, where $\pi_{A_i}: eJ^i\to A_i$ and $\pi_{B_i}: eJ^i\to B_i$ are the canonical projections. Then it is easy to see that $X_{(j)}\leqq X^{(j)}$ $(j=1, 2)$ and $X^{(1)}/X_{(1)}\simeq X^{(2)}/X_{(2)}$. Now, if $D^{(1)}=D_{(1)}$, then $D^{(2)}=D_{(2)}$, and we can take $x=e$ for x in (P). Thus we may assume $D_{(1)}<D^{(1)}$. Then by the above, $D_{(2)}<D^{(2)}$ and $D^{(1)}/D_{(1)}\simeq D^{(2)}/D_{(2)}$. Since R is left serial, there exists some $\delta\in eJe$ (by Lemma 6) such that either $\delta D^{(1)}=D^{(2)}$ and $\delta D_{(1)}=D_{(2)}$; or $\delta D^{(2)}=D^{(1)}$ and $\delta D_{(2)}=D_{(1)}$. We may assume that the former holds. There exists a unique s such that $D_{(2)}\leqq eJ^s\leqq D$. Then $D=(e+\delta)\,D^{(1)}\oplus D_{(2)}=(e+\delta)\,D^{(1)}+eJ^s$. Noting that $u:=e+\delta$ is a unit in eRe and $u^{-1}\,eJ^s=eJ^s$, we have $u^{-1}D=D^{(1)}+u^{-1}\,eJ^s=D^{(1)}+eJ^s=D^{(1)}\oplus D_{(2)}$. Put C': $=u^{-1}C$, $D':=u^{-1}D$. When $C'_{(1)}=C'^{(1)}$, we can take $x=u^{-1}$. So suppose that $C'_{(1)}<C'^{(1)}$. Then $C'_{(1)}=D^{(1)}$, $C'_{(2)}=D_{(2)}$ and $C'^{(1)}/C'_{(1)}\simeq C'^{(2)}/C'_{(2)}$ is simple because so is $C'/D'\simeq C/D$. By an argument similar to one for D, we can take some $\omega\in eJe$ such that $\omega C'^{(1)}=C'^{(2)}$ and $\omega C'_{(1)}=C'_{(2)}$. Hence putting $y:=e+\omega$, we have $C'=yC'^{(1)}\oplus C'_{(2)}$ and $y^{-1}C'=C'^{(1)}\oplus C'_{(2)}$. It follows from $C'_{(1)}=D^{(1)}$ and $C'_{(2)}=D_{(2)}$ that $\omega D^{(1)}=D_{(2)}$. Then $D'=D^{(1)}\oplus D_{(2)}=yD^{(1)}\oplus D_{(2)}$, whence $y^{-1}D'=D^{(1)}\oplus D_{(2)}$. Consequently, we can take $x=y^{-1}\,u^{-1}$.

Next, we consider the case that C_j and D_j $(j=1, 2)$ have the following forms: $C_j=A_{t_j-1}\oplus B_{s_j}$ (or $C_j=A_{t_j}\oplus B_{s_j-1}$) and $D_j=A_{t_j}\oplus B_{s_j}$. Considering the structure of (b_1) and $C_1/D_1\simeq C_2/D_2$, we see that the possible cases are the following.

$$\begin{aligned} (\alpha)\quad & C_1 = A_{t_1-1}\oplus B_{s_1}\,, & D_1 &= A_{t_1}\oplus B_{s_1} & \\ & C_2 = A_{s_2}\oplus B_{t_2-1}\,, & D_2 &= A_{s_2}\oplus B_{t_2} & (t_2-t_1\geqq i) \\ (\beta)\quad & C_1 = A_{t-1}\oplus B_{s_1}\,, & D_1 &= A_t\oplus B_{s_1} & \\ & C_2 = A_{t-1}\oplus B_{s_2}\,, & D_2 &= A_t\oplus B_{s_2} & \\ (\gamma)\quad & C_1 = A_{t_1-1}\oplus B_{s_1}\,, & D_1 &= A_{t_1}\oplus B_{s_1} & \\ & C_2 = A_{t_2-1}\oplus B_{s_2}\,, & D_2 &= A_{t_2}\oplus B_{s_2} & (t_2-t_1\geqq i) \end{aligned}$$

In the case (α), we put $f(a_1+D_1)=b_2+D_2$, $A_{t_1-1}=a_1R$ and $B_{t_2-1}=b_2R$. There exists some d in $eJ^{t_2-t_1}\,e$ such that $da_1=b_2$. Since $t_2-t_1\geqq i$, we have $d=d_1+d_2$ for some d_1 in A_i and d_2 in B_i, and $b_2=da_1=d_1a_1+d_2a_1$. Then $b_2=d_2a_1$. Hence $d_2B_i=0$. Indeed if not, for some $x\in T(A_i)$ and some $y\in T(B_i)$, both d_2x and d_2y are in $T(B_{i+t_2-t_1})$ and non-zero. Thus, $\bar{A}_i\simeq\bar{B}_i$, a contradiction. Thus $(\mathrm{d}_2)_l$

induces f. The similar argument works for the cases (β) and (γ).

Finally in the general case, there exist units x and y in eRe for C_j and D_j ($j=1, 2$) as in (P). Using the isomorphism f, we put

$$f': (A_{t_1-1}\oplus B_{s_1})/(A_{t_1}\oplus B_{s_1}) \overset{x_l^{-1}}{\to} C_1/D_1 \overset{f}{\to} C_2/D_2 \overset{y_l}{\to} (A_{t_2-1}\oplus B_{s_2})/(A_{t_2}\oplus B_{s_2})$$

and apply the argument above. Q.E.D.

Lemma 9. *Let the diagram* (b_2) *be the structure of eR, and assume* $[\Delta:\Delta_i]_l=2$. *Then eR satisfies the condition* (A).

Proof. Let C_j and D_j be submodules of eR such that $eR\geqq C_j>D_j$ and C_j/D_j are simple for $j=1, 2$, and $f: C_1/D_1\to C_2/D_2$ be an isomorphism. Then $C_j=c_jR+D_j$ for some c_j in $C_j (j=1, 2)$, where we may assume that $f(c_1+D_1)=c_2+D_2$ and $c_1=c_1g$, $c_2=c_2g$ for some $g\in \mathrm{pi}(R)$.

The proof similar to that of Lemma 8 works in the following two cases:

(i) both c_1 and c_2 are in $T(eJ^t)$ $(t<i)$.

(ii) c_1 is in $T(eJ^t)$ $(t\leqq i)$ and $eJ^i\geqq C_2>D_2$.

So we show only the following case:

(iii) $eJ^i\geqq C_j>D_j$ for both $j=1, 2$.

We have that for any C and D with $eJ^i\geqq C>D$, there exists a unit x in eRe such that $xC=A_{t-1}\oplus B_s>xD=A_t\oplus B_s$ (or $xC=A_t\oplus B_{s-1}>xD=A_t\oplus B_s$) for some $t, s\geqq i$ (the proof is in Lemma 8). Further we have that for $C=A_k\oplus B_r$ $(k, r\geqq i)$, there exists a unit y in eRe such that $yC=A_r\oplus B_k$. So we may assume that $C_j>D_j$ are of the following form:

$$C_1=A_{t_1-1}\oplus B_{s_1}, \quad D_1=A_{t_1}\oplus B_{s_1}$$
$$C_2=A_{t_2-1}\oplus B_{s_2}, \quad D_2=A_{t_2}\oplus B_{s_2}.$$

It follows from $C_1/D_1\simeq C_2/D_2$ that $t_1=t_2=t$.

(α) In the case that $t\leqq\max(s_1, s_2)$. We may assume $s_1\geqq s_2$. Let $C_1=c_1R+D_1$ and $c_1R=A_{t-1}$. Then there exists a unit z in eRe such that $f(c_1+D_1)=zc_1+D_2$. It follows from $zD_1\leqq A_t\oplus B_{s_1}\leqq D_2$ that z_l induces f.

(β) In the case that $t>\max(s_1, s_2)$. We may assume $s_1\geqq s_2$. Let $B_{t-1}=bR$ and $\delta B_i=A_i$. Then $A_{t-1}=\delta bR$ and $f(\delta b+D_1)=\delta wb+D_2$ for some w in eRe with $wB_{t-1}=B_{t-1}$, i.e. $\overline{w}$ is in Δ_i. Since $[\Delta:\Delta_i]_l=2$, there exist $\bar{y}_1$ and $\bar{y}_2$ in Δ_i such that $\bar{\delta}\overline{w}=\bar{y}_1+\bar{y}_2\bar{\delta}$. So we have $\delta w=y_1+y_2\delta+j$ for some j in eJe, whence $y_2\delta b=(\delta w-y_1-j)\,b\equiv\delta wb (\mathrm{mod}\ D_2)$, since y_1b is in $B_{t-1}\leqq D_2$ and jb is in $A_t\oplus B_t\leqq D_2$. Then $f(\delta b+D_1)=y_2(\delta b)+D_2$ and $y_2D_1\leqq A_t\oplus B_{s_1}\leqq D_2$. So we have that $(y_2)_l$ induces f. Q.E.D.

Remark. If R is a finite dimensional algebra over a field, then $[\Delta:\Delta_i]_r=[\Delta:\Delta_i]_l$ holds. So for a primitive idempotent e, if eR satisfies the condition

(A), it follows from $[\Delta\colon \Delta_i]_l \leqq 2$ (by Proposition 5) that $[\Delta\colon \Delta_i]_r \leqq 2$. Hence eR never has the structure (c). Further suppose that R is a finite dimensional algebra over an algebraically closed field. Then we have $[\Delta\colon \Delta_i]_r = [\Delta\colon \Delta_i]_l = 1$. Hence eR has the structure (a) or (b_1).

4. Examples

Here we give some examples of left serial rings having projective indecomposable modules with structures (b_1), (b_2), and (c) which satisfy the condition (A).

EXAMPLE 1. Let k be a field and put

$$R := \begin{pmatrix} k & k & k & k \\ 0 & k & k & k \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}, \qquad e := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then every projective indecomposable R-module satisfies the condition (A) and eR has the structure (b_1). Note that R is not of right local type (Cf. [6]).

EXAMPLE 2. Let $K \leqq L$ be fields with $[L\colon K]=2$. Put

$$R := \begin{bmatrix} L & L & L \\ 0 & L & L \\ 0 & 0 & K \end{bmatrix} \quad \text{and} \quad e := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then eR has the structure (b_2) and satisfies the condition on the left dimension in Theorem 2. Also in this case every projective indecomposable module satisfies the condition (A) but R is not of right local type (Cf. [6]).

EXAMPLE 3 (Asashiba [1]). Let F and G be division rings and M an (F, G)-bimodule having the dimension sequence (3, 1, 2, 2, 1) (see Dowbor, Ringel and Simson [2]). The existence of such an M follows from Schofield [5, section 13] and [2, Proposition 1]. Then $R := \begin{bmatrix} F & M \\ 0 & G \end{bmatrix}$ has exactly 5 non-isomorphic indecomposable modules and $[M\colon G]_r = 3$, say $M_G = A_1 \oplus A_2 \oplus A_3$ with each $A_i \simeq G_G$. Put $e_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $e_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then we can identify $e_1 J_R = M_G$. Since the set $S := \{e_2R, e_1R, e_1R/A_1, e_1R/(A_1 \oplus A_2), e_1R/e_1J\}$ consists of 5 non-isomorphic local modules, S is a complete set of representatives of isomorphism classes of indecomposable R-modules. Thus R is of right local type. Hence every projective indecomposable R-module satisfies the condition (A). In particular so does e_1R. Further since e_1J is isomorphic to a direct sum of three copies of a simple module, e_1R has the structure (c) and satisfies the condition (♯).

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