

A CHARACTERIZATION OF CONDITIONAL EXPECTATIONS FOR $L_\infty(X)$ -VALUED FUNCTIONS

RYOHEI MIYADERA

(Received October 6, 1986)

Introduction. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, (S, X, λ) a measure space and E a Banach space. We consider constant-preserving contractive projections of $L_1(\Omega, \mathcal{A}, \mu; E)$ into itself. If $E = \mathbb{R}$ or E is a strictly convex Banach space, then it is known (Ando [1], Douglas [2] and Landers and Rogge [4]) that such operators coincide precisely with the conditional expectation operators. If $E = L_1(X, S, \lambda)$, the author [5] proved that such operators which are translation invariant coincide with the conditional expectation operators. In this paper we deal with the case when $E = L_\infty(X, S, \lambda)$. If $E = \mathbb{R}^2$ with the norm $\|(x, y)\|_{\mathbb{R}^2} = |x| \vee |y|$, then such operators can be expressed as a linear combination of two conditional expectation operators. On the other hand if $E = L_\infty(X, S, \lambda)$ and $E \neq \mathbb{R}^2$ with the norm $\|(x, y)\|_{\mathbb{R}^2} = |x| \vee |y|$, then such operators coincide with the conditional expectation given some σ -subalgebra.

1. Definitions and lemmas. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and (X, S, λ) a measure space. Let $S^+ = \{K: K \in S \text{ and } \lambda(K) > 0\}$ and $L_\infty(X, S, \lambda)$ the class of essentially bounded measurable functions on (X, S, λ) . Let $E = L_\infty(X, S, \lambda)$, then E is a Banach space with the norm defined by $\|a\|_E = \text{esssup}_{x \in X} |a(x)|$ for each $a \in L_\infty(X, S, \lambda)$. Let $L_1(\Omega, \mathcal{A}, \mu, E)$ be the class of E -valued Bochner integrable functions on $(\Omega, \mathcal{A}, \mu)$ with the norm defined by

$$\|f\|_L = \int \|f(\omega)\|_E d\mu(\omega) \quad \text{for each } f \in L_1(\Omega, \mathcal{A}, \mu, E).$$

For the definitions and properties of Bochner integral, see Hille and Phillips [3].

DEFINITION 1. For a σ -subalgebra \mathcal{B} of \mathcal{A} , a function g is called the conditional expectation of f given \mathcal{B} if g is weakly measurable with respect to \mathcal{B} , and

$$\int_B g d\mu = \int_B f d\mu \quad \text{for each } B \in \mathcal{B},$$

where the integral is the Bochner integral. We denote by $f^{\mathcal{B}}$ the conditional

expectation of f given \mathcal{B} . We shall denote by \mathcal{R} the class of real numbers. For each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ we define $(\varphi \cdot a) = \varphi(\omega) \cdot a$ for each $\omega \in \Omega$.

DEFINITION 2. Let P be a linear operator of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. P is said to be *contractive* if

$$\|P\| = \sup \{ \|P(f)\|_L : f \in L_1(\Omega, \mathcal{A}, \mu, E) \text{ and } \|f\|_L = 1 \} = 1,$$

P is *constant-preserving* if $P(l_\Omega \cdot a) = l_\Omega \cdot a$ for each $a \in E$ and P is called a *projection* if $P \circ P = P$, where l_Ω is the characteristic function of E .

Lemma 1.1. For each $f \in L_1(\Omega, \mathcal{A}, \mu, E)$ the conditional expectation $f^{\mathcal{B}}$ of f given \mathcal{B} exists uniquely up to almost every-where and the conditional expectation operator $(\cdot)^{\mathcal{B}}$ is a constant-preserving contractive projection for each σ -subalgebra \mathcal{B} of \mathcal{A} .

For the proof see Schwartz [6].

By the definition of conditional expectation $(\varphi \cdot a)^{\mathcal{B}} = \varphi^{\mathcal{B}} \cdot a$ for each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ and $a \in E$.

Lemma 1.2. If Q is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself, then for each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ with $0 \leq \varphi \leq 1$ and $a \in E$ there exists a μ -null set N such that

$$\|a\|_E - \|Q(\varphi \cdot a)(\omega)\|_E = \|a - Q(\varphi \cdot a)(\omega)\|_E \quad \text{for each } \omega \in \Omega - N.$$

For the proof see Miyadera [5].

Lemma 1.3. Let Q be a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. If, for each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ and for each nonzero element a of E , there exists $\varphi' \in L_1(\Omega, \mathcal{A}, \mu, R)$ such that $Q(\varphi \cdot a) = \varphi' \cdot a$, then there exists a σ -subalgebra \mathcal{B} of \mathcal{A} such that $Q(f)$ is the conditional expectation of f given \mathcal{B} for each $f \in L_1(\Omega, \mathcal{A}, \mu, E)$. In particular each constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself is the conditional expectation given some σ -subalgebra.

For the proof see Miyadera [5].

Lemma 1.4. Let $T \in S^+$ and $A \in \mathcal{A}$. Then there exists a μ -null set N such that for each $\omega \in \Omega - N$

$$|Q(l_A \cdot l_T)(\omega)| \leq 1 \quad (\text{a.e.x.})$$

and

$$0 \leq Q(l_A \cdot l_T)(\omega) \cdot l_T \leq 1 \quad (\text{a.e.x.})$$

Proof. By Lemma 1.2 there exists a μ -null set N such that

$$\|l_T\|_E - \|Q(l_A \cdot l_T)(\omega)\|_E = \|l_T - Q(l_A \cdot l_T)(\omega)\|_E$$

for each $\omega \in \Omega - N$, and hence for each $\omega \in \Omega - N$,

$$\|Q(l_A \cdot l_T)(\omega)\|_E \leq 1$$

and

$$\|l_T - Q(l_A \cdot l_T)(\omega)\|_E \leq 1.$$

Therefore by the definition of $\|\cdot\|_E$ we have for each $\omega \in \Omega - N$,

$$|Q(l_A \cdot l_T)(\omega)| \leq 1 \quad (\text{a.e.x})$$

and

$$0 \leq Q(l_A \cdot l_T)(\omega) \cdot l_T \leq 1 \quad (\text{a.e.x}).$$

Lemma 1.5. *Let $A \in \mathcal{A}$ and $T, T' \in S^+$ and $T \cap T' = \phi$. Then*

$$\int Q(l_A \cdot l_T) \cdot l_{T'} d\mu = 0 \quad (\text{a.e.x}).$$

Proof. Since Q is constant-preserving and contractive,

$$\begin{aligned} 1 &= \int \|l_A \cdot l_T + l_\Omega \cdot l_{T'}\|_E d\mu \geq \int \|Q(l_A \cdot l_T + l_\Omega \cdot l_{T'})\|_E d\mu \\ &= \int \|Q(l_A \cdot l_T) + l_\Omega \cdot l_{T'}\|_E d\mu = \int \|Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}\|_E d\mu \\ &\geq \left\| \int (Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}) d\mu \right\|_E. \end{aligned}$$

Similarly

$$\begin{aligned} 1 &= \int \|l_A \cdot l_T - l_\Omega \cdot l_{T'}\|_E d\mu \\ &\geq \int \|Q(l_A \cdot l_T - l_\Omega \cdot l_{T'})\|_E d\mu = \int \|Q(l_A \cdot l_T) - l_\Omega \cdot l_{T'}\|_E d\mu \\ &\geq \left\| \int (Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}) d\mu \right\|_E. \end{aligned}$$

Therefore we have proved that

$$1 \geq \left\| \int (Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}) d\mu \right\|_E$$

and

$$1 \geq \left\| \int (Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}) d\mu \right\|_E.$$

Since

$$\int Q(l_A \cdot l_T) l_{T'} d\mu = 0 \quad \text{on } (T')^c$$

and

$$\int (l_A \cdot l_{T'}) d\mu = l_{T'},$$

we have

$$\int Q(l_A \cdot l_T) \cdot l_{T'} d\mu = 0 \quad (\text{a.e.x}).$$

Lemma 1.6. *Let $A \in \mathcal{A}$ and $T \in S^+$. Suppose that there exists $T', T'' \in S^+$ such that $T' \cap T'' = \emptyset$ and $T' \cup T'' = T^c$, then there exists a μ -nullset N such that $Q(l_A \cdot l_T)(\omega) \cdot l_{T^c} = 0$ (a.e.x) for each $\omega \in \Omega - N$.*

Proof. Since Q is constant-preserving and contractive,

$$\begin{aligned} 1 &= \int \|l_A \cdot l_T + l_{\Omega} \cdot l_{T''} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E d\mu \\ &\geq \int \|Q(l_A \cdot l_T + l_{\Omega} \cdot l_{T''} + (-1)^k l_{\Omega} \cdot l_{T'})\|_E d\mu \\ &= \int \|Q(l_A \cdot l_T) + l_{\Omega} \cdot l_{T''} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E d\mu \\ &\geq \int \| (Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}) \vee \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E d\mu \\ &\geq \int \|Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}\|_E d\mu \wedge \int \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E d\mu \\ &\geq \int \| (Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}) d\mu \|_E \wedge \int \| (Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}) d\mu \|_E \\ &= 1. \end{aligned}$$

Here we also used Lemma 1.5 that

$$\int Q(l_A \cdot l_T) \cdot l_{T'} d\mu = 0$$

and

$$\int Q(l_A \cdot l_T) \cdot l_{T''} d\mu = 0.$$

We have proved that

$$1 = \int (\|Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}\|_E \vee \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E) d\mu$$

and

$$\|Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}\|_E = \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E \quad (\text{a.e.x}).$$

Therefore we have

$$1 = \int (\|Q(l_A \cdot l_T) \cdot l_{T'} + l_{\Omega} \cdot l_{T'}\|_E \vee \|Q(l_A \cdot l_T) \cdot l_{T'} - l_{\Omega} \cdot l_{T'}\|_E) d\mu$$

Since

$$\begin{aligned} & \|Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}\|_E \vee \|Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}\|_E \geq 1, \\ & \|Q(l_A \cdot l_T)(\omega) \cdot l_{T'} + l_{T'}\|_E = \|Q(l_A \cdot l_T)(\omega) \cdot l_{T'} - l_{T'}\|_E = 1 \quad (\text{a.e.x}). \end{aligned}$$

Similarly we can prove that

$$\|Q(l_A \cdot l_T)(\omega) \cdot l_{T''} + l_{T''}\|_E = \|Q(l_A \cdot l_T)(\omega) \cdot l_{T''} - l_{T''}\|_E = 1 \quad (\text{a.e.x}).$$

Therefore there exists a μ -nullset N such that

$$Q(l_A \cdot l_T)(\omega) \cdot l_{T' \cup T''} = 0 \quad (\text{a.e.x}).$$

2. A characterization of conditional expectation for $L_\infty(X)$ -valued function

Theorem 1. *If there exist pairwise disjoint elements X_1, X_2 and X_3 of S^+ such that $X_1 \cup X_2 \cup X_3 = X$, then a constant-preserving contractive projection Q of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself is a conditional expectation operator given some σ -sub-algebra.*

Proof. Let $A \in \mathcal{A}$ and $T \in S^+$ and $T \subset X_i$. Then by Lemma 1.6 there exists a μ -nullset N such that for each $\omega \in \Omega - N$

$$(1) \quad Q(l_A \cdot l_T)(\omega) \cdot l_{T^c} = 0 \quad (\text{a.e.x})$$

and

$$(2) \quad Q(l_{A^c} \cdot l_T)(\omega) \cdot l_{T^c} = 0 \quad (\text{a.e.x}).$$

Since Q is constant-preserving

$$(3) \quad Q(l_A \cdot l_T) + Q(l_{A^c} \cdot l_T) = Q(l_\Omega \cdot l_T) = l_\Omega \cdot l_T.$$

Since Q is constant-preserving and contractive

$$\begin{aligned} 1 &= \mu(A) + \mu(A^c) = \int (\|l_A \cdot l_T\|_E + \|l_{A^c} \cdot l_T\|_E) d\mu \\ &\geq \int (\|Q(l_A \cdot l_T)\|_E + \|Q(l_{A^c} \cdot l_T)\|_E) d\mu \\ &\geq \int \|Q(l_A \cdot l_T) + Q(l_{A^c} \cdot l_T)\|_E d\mu = \int \|l_\Omega \cdot l_T\|_E d\mu = 1. \end{aligned}$$

Therefore there exists a μ -nullset N' such that for each $\omega \in \Omega - N'$

$$\|Q(l_A \cdot l_T)(\omega)\|_E + \|Q(l_{A^c} \cdot l_T)(\omega)\|_E = 1.$$

This together with (1), (2) and (3), implies that for each $\omega \in \Omega - (N \cup N')$ there exists a real number $k(\omega)$ such that $Q(l_A \cdot l_T)(\omega) = k(\omega) \cdot l_T$. Obviously $k(\cdot) \in L_1(\Omega, \mathcal{A}, \mu, R)$. Since Q is linear, $k(\cdot)$ is independent of the choice of T . Let

$E_i = \{a: a \in E, a(x) = 0 \text{ for each } x \in X_i^c\}$ ($i = 1, 2$ and 3). Q is linear and continuous, and hence for each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ and $a \in E_i$, there exists $\varphi' \in L_1(\Omega, \mathcal{A}, \mu, E_i)$ such that $Q(\varphi \cdot a) = \varphi' \cdot a$. Therefore by Lemma 1.3 there exists a σ -subalgebra \mathcal{B}_i of \mathcal{A} such that $Q(f) = f^{\mathcal{B}_i}$ for each $f \in L_1(\Omega, \mathcal{A}, \mu, E_i)$. We shall prove that $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$. Let $B \in \mathcal{B}_1$. Then

$$\begin{aligned} \int l_B d\mu &= \int l_B \cdot \|l_{X_1} + l_{X_2}\|_E d\mu = \int \|l_B \cdot l_{X_1} + l_B \cdot l_{X_2}\|_E d\mu \\ &\geq \int \|Q(l_B \cdot l_{X_1} + l_B \cdot l_{X_2})\|_E d\mu = \int \|Q(l_B \cdot l_{X_1}) + Q(l_B \cdot l_{X_2})\|_E d\mu \\ &= \int \|(l_B)^{\mathcal{B}_1} \cdot l_{X_1} + (l_B)^{\mathcal{B}_2} \cdot l_{X_2}\|_E d\mu = \int \|l_B \cdot l_{X_1} + (l_B)^{\mathcal{B}_2} \cdot l_{X_2}\|_E d\mu \\ &= \int (l_B \vee (l_B)^{\mathcal{B}_2}) d\mu. \end{aligned}$$

Hence $l_B(\omega) = l_B(\omega) \vee (l_B)^{\mathcal{B}_2}(\omega)$, which implies that $l_B(\omega) = (l_B)^{\mathcal{B}_2}(\omega)$. Since $\|(l_B)^{\mathcal{B}_2}\|_L = \|l_B\|_L$, $l_B = (l_B)^{\mathcal{B}_2}$. Since B is an arbitrary element of \mathcal{B}_1 , we have proved that $\mathcal{B}_1 \subset \mathcal{B}_2$. Similarly we can prove that $\mathcal{B}_2 \subset \mathcal{B}_3$ and $\mathcal{B}_3 \subset \mathcal{B}_1$, which imply that $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$. Write $\mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$, then $Q(f) = f^{\mathcal{B}}$ for each $f \in L_1(\Omega, \mathcal{A}, \mu, E)$.

3. A characterization of a constant-preserving contractive projection of for R^2 -valued functions. If there do not exist pairwise disjoint element X_1, X_2 and X_3 such that $X_1 \cup X_2 \cup X_3 = X$ and X_1, X_2 and $X_3 \in S^+$, then $E \cong R^2$ with the norm $\|(x, y)\|_E = |x| \vee |y|$ or $E \cong R$ with the norm $\|x\|_E = |x|$. Douglas [2] showed that a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself is a conditional expectation given some σ -subalgebra. Therefore our next aim is to consider the case that $E \cong R^2$ with the norm $\|(x, y)\|_E = |x| \vee |y|$. Note that for each $f \in L_1(\Omega, \mathcal{A}, \mu, R^2)$ there exist $f_1, f_2 \in L_1(\Omega, \mathcal{A}, \mu, R)$ such that $f(\omega) = (f_1(\omega), f_2(\omega))$.

Lemma 3.1. *Suppose that $E = R^2$ and Q is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. If $f \in L_1(\Omega, \mathcal{A}, \mu, R)$ with $0 \leq f(\omega) \leq 1$ (a.e. ω) and $Q((f, f)) = (f_1, f_2)$ and $Q((f, -f)) = (g_1, g_1)$, then $f_2 = f_2$ and $g_1 = -g_2$.*

Proof. By Lemma 1.2 there exists μ -nullsets N_1 and N_2 such that for each $\omega \in \Omega - N_1$

$$\|(1, 1)\|_E - \|Q((f, f))(\omega)\|_E = \|(1, 1) - Q((f, f))(\omega)\|_E$$

and for each $\omega \in \Omega - N_2$

$$\|(1, -1)\|_E - \|Q((f, -f))(\omega)\|_E = \|(1, -1) - Q((f, -f))(\omega)\|_E.$$

Therefore we have for each $\omega \in \Omega - N_1$

$$1 - (|f_1(\omega)| \vee |f_2(\omega)|) = |1 - f_1(\omega)| \vee |1 - f_2(\omega)|$$

and for each $\omega \in \Omega - N_2$

$$1 - (|g_1(\omega)| \vee |g_2(\omega)|) = |1 - g_1(\omega)| \vee |1 - g_2(\omega)|$$

and hence $f_1(\omega) = f_2(\omega)$ for each $\omega \in \Omega - N_1$ and $g_1(\omega) = -g_2(\omega)$ for each $\omega \in \Omega - N_2$.

Theorem 2. *Let $E = R^2$ with the norm $\|(x, y)\|_E = |x| \vee |y|$. Then Q is a constant-preserving contractive projection if and only if there exist σ -subalgebras \mathcal{B} and \mathcal{C} of \mathcal{A} such that*

$$Q((f, g)) = (1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + f^{\mathcal{C}} - g^{\mathcal{C}}), 1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + g^{\mathcal{C}} - f^{\mathcal{C}})).$$

Proof. Suppose that Q is a constant-preserving contractive projection. Then by Lemma 3.1 we can get two operators Q_1 and Q_2 of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself such that for each $f \in L_1(\Omega, \mathcal{A}, \mu, R)$,

$$Q((f, f)) = (Q_1(f), Q_1(f)) \quad \text{and} \quad Q((f, -f)) = (Q_2(f), -Q_2(f)).$$

Since Q is a constant-preserving contractive projection, Q_1 and Q_2 are constant-preserving contractive projections. Therefore by Lemma 1.3 there exist σ -subalgebra \mathcal{B} and \mathcal{C} such that

$$Q_1 = (\cdot)^{\mathcal{B}} \quad \text{and} \quad Q_2 = (\cdot)^{\mathcal{C}}.$$

Then

$$\begin{aligned} Q((f, g)) &= Q(1/2(f+g) + 1/2(f-g), 1/2(f+g) - 1/2(f-g)) \\ &= (Q_1(1/2(f+g)) + Q_2(1/2(f-g)), Q_1(1/2(f+g)) - Q_2(1/2(f-g))) \\ &= ((1/2(f^{\mathcal{B}} + g^{\mathcal{B}})) + (1/2(f^{\mathcal{C}} - g^{\mathcal{C}})), (1/2(f^{\mathcal{B}} + g^{\mathcal{B}})) - (1/2(f^{\mathcal{C}} - g^{\mathcal{C}}))) \\ &= (1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + f^{\mathcal{C}} - g^{\mathcal{C}}), 1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + g^{\mathcal{C}} - f^{\mathcal{C}})). \end{aligned}$$

On the other hand let \mathcal{B} and \mathcal{C} be σ -subalgebras of \mathcal{A} and

$$Q((f, g)) = (1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + f^{\mathcal{C}} - g^{\mathcal{C}}), 1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + g^{\mathcal{C}} - f^{\mathcal{C}})).$$

Since $(\cdot)^{\mathcal{B}}$ and $(\cdot)^{\mathcal{C}}$ are constant-preserving projections, Q is a constant-preserving projection. In the following we denote

$$\{\omega: f^{\mathcal{B}} + g^{\mathcal{B}} \geq 0\} \quad \text{by} \quad \{f^{\mathcal{B}} + g^{\mathcal{B}} \geq 0\}$$

and

$$\{\omega: f^{\mathcal{C}} - g^{\mathcal{C}} < 0\} \quad \text{by} \quad \{f^{\mathcal{C}} - g^{\mathcal{C}} < 0\}, \quad \text{etc.}$$

It holds that

$$\begin{aligned}
\|Q((f, g))\|_L &= \int \|Q((f, g))\|_E d\mu \\
&= \int |(1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+f^{\mathcal{C}}-g^{\mathcal{C}}) \vee |1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+g^{\mathcal{C}}-f^{\mathcal{C}})|) d\mu \\
&= \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\} \cap \{f^{\mathcal{C}}-g^{\mathcal{C}} \geq 0\}} 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+f^{\mathcal{C}}-g^{\mathcal{C}}) d\mu + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}=0\} \cap \{f^{\mathcal{C}}-g^{\mathcal{C}}<0\}} 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+g^{\mathcal{C}}-f^{\mathcal{C}}) d\mu \\
&\quad + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\} \cap \{f^{\mathcal{C}}-g^{\mathcal{C}} \geq 0\}} 1/2(-f^{\mathcal{B}}-g^{\mathcal{B}}+f^{\mathcal{C}}-g^{\mathcal{C}}) d\mu + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\} \cap \{f^{\mathcal{C}}-g^{\mathcal{C}}<0\}} 1/2(-f^{\mathcal{B}}-g^{\mathcal{B}}+g^{\mathcal{C}}-f^{\mathcal{C}}) d\mu \\
&= \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}} \geq 0\}} 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}) d\mu + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\}} 1/2(-f^{\mathcal{B}}-g^{\mathcal{B}}) d\mu \\
&= \int_{\{f^{\mathcal{C}}-g^{\mathcal{C}} \geq 0\}} 1/2(f^{\mathcal{C}}-g^{\mathcal{C}}) d\mu + \int_{\{g^{\mathcal{C}}-f^{\mathcal{C}} \geq 0\}} 1/2(g^{\mathcal{C}}-f^{\mathcal{C}}) d\mu \\
&= \int |1/2(f^{\mathcal{B}}+g^{\mathcal{B}})| d\mu + \int |1/2(f^{\mathcal{C}}-g^{\mathcal{C}})| d\mu \\
&= \int |1/2(f+g)| d\mu + \int |1/2(f-g)| d\mu \\
&\leq \int (|f| \vee |g|) d\mu = \int \|(f, g)\|_E d\mu .
\end{aligned}$$

Therefore Q is contractive.

Acknowledgement. The author would like to thank Professors Tsuyoshi Ando, Hirokichi Kudo and Teturo Kamae for their helpful suggestions. Section 3 is suggested by Professor Kamae.

References

- [1] T. Ando: *Contractive projections in L_p -spaces*, Pacific J. Math. **17** (1966), 395–405.
- [2] R.G. Douglas: *Contractive projections on an L_1 -space*, Pacific J. Math. **15** (1965), 443–462.
- [3] E. Hille and R.S. Phillips: *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., 1957.
- [4] D. Landers and L. Rogge: *Characterization of conditional expectation operators for Banach-valued functions*, Proc. Amer. Math. Soc. **81** (1981), 107–110.
- [5] R. Miyadera: *Characterizations of conditional expectations for $L_1(X)$ -valued functions*, Osaka J. Math. **23** (1986), 313–324.
- [6] L. Schwarz: *Disintegration on measures*, Tata Institute of Fundamental Research, 1976.

Department of Physics
School of Science
Kwansei Gakuin University
Nishinomiya, Hyogo 662
Japan

