Akamatsu, T.
Osaka J. Math.
25 (1988), 71-104

ON SOME CLASSES OF NON-HYPOELLIPTIC SECOND ORDER PARTIAL DIFFERENTIAL OPERATORS

TOYOHIRO AKAMATSU

(Received September 20, 1986)

1. Introduction

Let ω be an open set in R^n, $n\geqq 1$, containing the origin and let $T>0$. This paper is concerned with the non-hypoellipticity of differential operators of second order of the form:

$$L=\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i\,\partial x_j}+\sum_{k=1}^{n} b_k(x,t)\frac{\partial}{\partial x_k}+c(x,t)+\frac{\partial}{\partial t}, \tag{1.1}$$

where $(x,t)=(x_1,\cdots,x_n,t)\in\Omega=\omega\times(-T,T)$.

We assume that

(1.2) all coefficients of L are complex-valued C^∞ functions defined in Ω,

$$a_{ij}(x,t)=a_{ji}(x,t)\,(1\leqq i,j\leqq n),\quad (x,t)\in\Omega. \tag{1.3}$$

We put

$$a(x,t,\xi)=\sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\,\xi_j,\quad (x,t,\xi)\in\Omega\times R^n. \tag{1.4}$$

The operator L is said to be hypoelliptic in Ω if for any open subset U of Ω and any $u\in\mathcal{D}'(U)$, $Lu\in C^\infty(U)$ implies $u\in C^\infty(U)$, and is said to be globally hypoelliptic in Ω if $u\in\mathcal{D}'(\Omega)$ and $Lu\in C^\infty(\Omega)$ imply $u\in C^\infty(\Omega)$.

In order that L is not hypoelliptic, Re $a(x,t,\xi)$ must vanish at some point, say $(0,0,\xi^o)$, $\xi^o\neq 0$ (cf. [8]). We give a sufficient condition for L not to be globally hypoelliptic or to be non-hypoelliptic in any open neighborhood of the origin mainly in terms of the behavior of Re $a(x,t,\xi^o)$ along the straight line $x\equiv 0$ through the origin or in terms of that of $a(x,t,\xi^o)$ along the integral curve $(x(t),t)$ of the vector field $\sum_{k=1}^{n} b_k(x,t)\,\partial/\partial x_k+\partial/\partial t$ through the origin according as the coefficients of L are complex valued or real valued. In these results we require that Re $a(0,t,\xi^o)$ or $a(x(t),t,\xi^o)$ changes its sign from plus to minus when t increases across 0 and vanishes at $t=0$ exactly to some odd order.

We review related known results on the (non-) hypoellipticity of operators

of second order with C^∞ coefficients defined in an open set $\tilde{\Omega}$ of $R^n (n \geqq 2)$:

$$P = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k(x) \frac{\partial}{\partial x_k} + c(x) .$$

Let $a(x, \xi) = \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j$ be the characteristic form of P.

In [5] Hörmander proved that

(H) if P is hypoelliptic in $\tilde{\Omega}$ and $a_{ij}(x)$ $(1 \leqq i, j \leqq n)$ are real valued, then for any $x \in \tilde{\Omega}$ the quadratic form: $\xi \to a(x, \xi)$ is semi-definite.

The quadratic form $a(x, \xi)$ may change its sign when x varies in $\tilde{\Omega}$.

Kannai [6] treated the operator $\partial/\partial t - t\partial^2/\partial x^2$, $x \in R^1$, and proved that it is not hypoelliptic in any open neighborhood of the origin. This fact is due to the change of sign of $-t$ from plus to minus near $t=0$, and it motivates us to investigate the relation between the hypoellipticity of P and the change of sign of the quadratic form $a(x, \xi)$.

Subsequently Zuily [12] generalized the Kannai's result and proved that if all coefficients of P are real analytic in $\tilde{\Omega}$, $\sum_{i,j=1}^{n} |a_{ij}(x)| + \sum_{k=1}^{n} |b_k(x)| \neq 0$ for all $x \in \tilde{\Omega}$ and if P is hypoelliptic there, then

(1.5) for every $x_0 \in \tilde{\Omega}$ there exist an open neighborhood V_0 of x_0 and an analytic function $\phi(x)$ defined in V_0 such that $a(x, \xi) = \phi(x) \sum_{i,j=1}^{n} \alpha_{ij}(x) \xi_i \xi_j \equiv \phi(x) \alpha(x, \xi)$, $(x, \xi) \in V_0 \times R^n$ where $\alpha_{ij}(x)$ $(1 \leqq i, j \leqq n)$ are real analytic functions defined in V_0 and the quadratic form $a(x, \xi)$ is non-negative

and

$$\sum_{k=1}^{n} b_k(x) \frac{\partial \phi}{\partial x_k}(x) \geqq 0 \quad \text{for any} \quad x \in V_0 \cap \phi^{-1}(0) . \tag{1.6}$$

The case where the coefficients of P are real C^∞ functions was studied by the authors of [3], [7] and [2]. Let $\tilde{\Omega}^+ = \mathrm{Int}\, \{x \in \tilde{\Omega} \mid a(x, \xi) \geqq 0, {}^\forall \xi \in R^n\}$, $\tilde{\Omega}^- = \mathrm{Int}\, \{x \in \tilde{\Omega} \mid a(x, \xi) \leqq 0, {}^\forall \xi \in R^n\}$ and $N = \tilde{\Omega} \cap \partial\tilde{\Omega}^+ = \tilde{\Omega} \cap \partial\tilde{\Omega}^-$. Under the assumption that N is an $(n-1)$-dimensional manifold of class C^k, $k \geqq 4$, Lanconelli [7] proved that if

$$\sum_{k=1}^{n} b_k(x_0) \nu_k(x_0) < 0 \quad \text{at some point} \quad x_0 \in N , \tag{1.7}$$

where $\nu(x_0) = (\nu_1(x_0), \cdots, \nu_n(x_0))$ is an interior normal to $\tilde{\Omega}^+$ at x_0, then P is not hypoelliptic in any open neighborhood of x_0, more precisely, for any sufficiently small open neighborhood V_0 of x_0 there exists a function $u \in C^0(V_0) \backslash C^4(V_0)$ such that $Pu = 0$ in V_0. This is an extension of the Zuily's result stated above and Theorem 1 (3) in Beals-Fefferman [3]. Here we note that the condition (1.7) means that the quadratic form $a(x, \xi)$ changes from the non-negative form to the non-positive form along the integral curve of the vector field $\sum_{k=1}^{n} b_k(x) \partial/\partial x_k$

through x_0.

Amano [2] proved, under some additional assumptions, that if (1.7) holds, then there exist an open neighborhood U of x_0 and a function u of the class $L^\infty(U)$ such that $Pu=0$ in U and $(x_0, \nu(x_0))$ or $(x_0, -\nu(x_0))\in WF_A(u)$.

In all of their works it is assumed that

(a) all coefficients of P are real valued,

(b) N is an $(n-1)$-dimensional manifold of class C^k, $k\geqq 4$,

(c) $\sum_{k=1}^n b_k(x_0)\,\nu_k(x_0)<0$, $x_0\in N$.

In this paper we give a sufficient condition for non-hypoellipticity of operators L defined by (1.1) which are special forms of P but do not necessarily satisfy (a), (b) or (c).

For the case where the coefficients of L are complex valued we obtain

Theorem 1.1. *Assume that there exist* $\xi^0=(\xi_1^0, \cdots, \xi_n^0)\neq 0$, *a real number* $\alpha<0$ *and an odd integer* $q>0$ *such that for sufficiently small* t

$$\text{(A.1)}\qquad \operatorname{Re} a(0, t, \xi^0) = \alpha t^q + O(t^{q+1}),$$

$$\text{(A.2)}\qquad a_{ij}(0, t) = O(t^q) \quad \textit{and} \quad b_k(0, t) = O(t^q) \quad (1\leqq i, j, k\leqq n),$$

$$\text{(A.3)}\qquad |\operatorname{grad}_x a_{ij}(0, t)| = O(t^{(q+1)/2}) \quad \textit{and} \quad |\operatorname{grad}_x b_k(0, t)| = O(t^{(q+1)/2}) \quad (1\leqq i, j, k\leqq n),$$

where $|\cdot|$ *denotes the Euclidean norm.*
Then, L is not globally hypoelliptic in any open neighborhood $U\subset\Omega$ containing the origin.

We shall prove Theorem 1.1 in sections 2, 3 and 4 by applying the usual asymptotic method with some modifications.

EXAMPLE 1.1. According to the result of Zuily [12] or Lanconelli [7] stated above, the operator $(x^2-t^3)\,\partial^2/\partial x^2+\partial/\partial t$, $(x, t)\in R^2$, is not hypoelliptic in any open subset of $R^2\setminus\{0\}$ intersecting the set $\{(x, t)\,|\,x^2=t^3\}$ and so, by definition of hypoellipticity, it is not hypoelliptic in any open neighborhood of 0. Moreover, by Theorem 1.1 it is not globally hypoelliptic in any open neighborhood of 0. Note that (b) does not hold at 0 for this operator.

EXAMPLE 1.2. Let $L_1=(x-t^3)(x^2+t^4)\,\partial^2/\partial x^2+\partial/\partial t$, $(x, t)\in R^2$. Then, $N=\{(x, t)\in R^2\,|\,x=t^3\}$ is a C^∞ manifold of dimension 1 and $(1, 0)$ is the interior normal to $\Omega^+=\{(x, t)\in R^2\,|\,x>t^3\}$ at $(0, 0)\in N$. It is easy to check that (c) does not hold at $(0, 0)$ for L_1. However, by Theorem 1.1 L_1 is not globally hypoelliptic in any open neighborhood of $(0, 0)$.

In the following two theorems the coefficients of L are assumed to be real valued in Ω, and $(x(t), t)=(x_1(t), \cdots, x_n(t), t)$ denotes the integral curve of the

vector field $\sum_{k=1}^{n} b_k(x, t)\,\partial/\partial x_k+\partial/\partial t$ through 0.

Theorem 1.2. *Assume that*

(B) *there exist* $\xi^o=(\xi_1^o, \cdots, \xi_n^o)\neq 0$, *a real number* $\alpha<0$ *and an odd integer* $q>0$ *such that for sufficiently small* t

$$a(x(t), t, \xi^o) = \alpha t^q+O(t^{q+1}) .$$

Then L *is not hypoelliptic in any open neighborhood of* 0.

We shall prove Theorem 1.2 in section 6 using the results of section 5. The outline of the proof is as follows. Suppose that (B) holds and L is hypoelliptic in some open neighborhood U of 0. Then there exist open neighborhoods V and $\tilde{V}$ of 0 (V, $\tilde{V}\subset U$), and a diffeomorphism from V to $\tilde{V}$ which transforms L to an operator $\tilde{L}$ defined in $\tilde{V}$ satisfying all the assumptions in Theorem 1.1. Then, $\tilde{L}$ is not hypoelliptic in $\tilde{V}$ which is a contradiction, because hypoellipticity is invariant under diffeomorphisms.

As an application of Theorem 1.2 and the Hörmander's theorem (H) we obtain the following theorem which will be proved in section 7.

Theorem 1.3. *Assume that the coefficients of* L *except* c *are analytic in* Ω *and* L *is hypoelliptic there. Let* $(T_1, T_2),-T\leqq T_1<T_2\leqq T$, *be the domain of definition of the curve* $(x(t), t)$. *Then one of the following three properties holds.*

(i) $a(x(t), t, \xi)\leqq 0$ *for all* $(t, \xi)\in(T_1, T_2)\times R^n$,

(ii) $a(x(t), t, \xi)\geqq 0$ *for all* $(t, \xi)\in(T_1, T_2)\times R^n$,

(iii) *there exists* T_o, $T_1<T_o<T_2$, *such that* $a(x(t), t, \xi)\leqq 0$ *for all* $(t, \xi)\in(T_1, T_o]\times R^n$, *and* $a(x(t), t, \xi)\geqq 0$ *for all* $(t, \xi)\in[T_o, T_2)\times R^n$.

REMARK. If the coefficients of L are functions of the variable t only, the analyticity condition in Theorem 1.3 is unnecessary. For the proof see [1].

NOTATION. For $x=(x_1, \cdots, x_n)\in R^n$ and a multi-index of non-negative integers $\alpha=(\alpha_1, \cdots, \alpha_n)$ we use the notation:

$$|x| = (x_1^2+\cdots+x_n^2)^{1/2},\ x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n},\quad |\alpha| = \alpha_1+\cdots+\alpha_n .$$

In sections 2, 3 and 4 we shall use the notation:

$$\partial_i = \frac{\partial}{\partial x_i}\ (1\leqq i\leqq n),\quad \partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n},\quad \partial_t^m = \frac{\partial^m}{\partial t^m}\ (m = 0, 1, \cdots) .$$

2. Derivation of ordinary differential equations

The proof of Theorem 1.1 will be based on the following lemma due to [10, Lemma 1.1].

Lemma 2.1. *Let U be an open subset of Ω such that $U \ni 0$ and suppose that L is globally hypoelliptic in U. Then for any positive integer m_1 and any compact subset K_1 of U, there exist another positive integer m_2, another compact subset K_2 of U and a positive constant C such that*

$$|f|_{m_1,K_1} \leqq C(|Lf|_{m_2,K_2} + |f|_{0,K_2}) \quad \text{for all} \quad f \in C^\infty(U),$$

where $|f|_{m,K} = \sup_{(x,t)\in K} \sum_{|\alpha|+j\leqq m} |\partial_x^\alpha \partial_t^j f(x,t)|$ *for a non-negative integer m and a compact subset K of U.*

Taking $m_1=1$ and $K_1=\{0\}$ in the above lemma we have

Corollary 2.1. *Under the same assumptions as in Lemma 2.1 there exist a positive integer M, a compact subset K of U and a positive constant C such that*

$$|\mathrm{grad}_x f(0,0)| \leqq C(|Lf|_{M,K} + |f|_{0,K}) \quad \text{for all} \quad f \in C^\infty(U). \tag{2.1}$$

By Corollary 2.1, to prove Theorem 1.1 it suffices to show that

(#) under the assumptions of Theorem 1.1, the inequality (2.1) is not valid for any choice of M, K and C.

To this end we shall determine functions $g_\lambda(x,t)=w_\lambda(x,t)\exp(-\lambda v_\lambda(x,t))$ with $\lambda>0$ as a parameter so that (2.1) does not hold for $f=\chi_\lambda g_\lambda$ as $\lambda\to+\infty$ where $\chi_\lambda(\lambda>0)$ are cut-off functions defined in a neighborhood of 0.

Now we choose real numbers ε, ρ, d, κ and μ such that

$$0<\varepsilon<\frac{1}{2}, \tag{2.2}$$

$$0<\rho<1, \tag{2.3}$$

$$d=-(1-\rho)(q-1+\varepsilon)/(q+1+\varepsilon), \tag{2.4}$$

$$\kappa=2(1-\rho)/(q+1+\varepsilon), \tag{2.5}$$

$$-d<\mu<(1-\rho-d)/2. \tag{2.6}$$

Note that we can choose μ satisfying (2.6), because $(1-\rho-d)/2+d=(1-\rho)/(q+1+\varepsilon)>0$. For every multi-index of non-negative integers $p=(p_1,\cdots,p_n)$ we put

$$\rho(p)=\begin{cases} 0 & (p=0) \\ \rho+(|p|-1)d & (|p|\geqq 1). \end{cases} \tag{2.7}$$

Let N be a positive integer satisfying

$$3(M+2)+(\mu-(1-\rho-d)/2)(N+1)<0, \tag{2.8}$$

where M is the integer which appeared in Corollary 2.1. Such an integer

exists by (2.6).

Furthermore we put

$$v_\lambda(x,t)=\sum_{|p|\leqq N}\lambda^{-\rho(p)}v_{\lambda,p}(t)\,x^p\,, \tag{2.9}$$

$$w_\lambda(x,t)=\sum_{|p|\leqq N}\lambda^{\mu|p|}w_{\lambda,p}(t)\,x^p \tag{2.10}$$

with $\lambda>0$ as a parameter and undetermined functions $v_{\lambda,p}(t)$ and $w_{\lambda,p}(t)$ ($|p|\leqq N$) to be infinitely differentiable in a neighborhood of 0. Then we obtain

$$\begin{aligned}\exp(\lambda v_\lambda)\,L(w_\lambda\exp(-\lambda v_\lambda)) &= -\lambda w_\lambda\{\partial_t v_\lambda-\lambda\sum_{i,j=1}^n a_{ij}\,\partial_i v_\lambda\,\partial_j v_\lambda+\sum_{i,j=1}^n a_{ij}\,\partial_i\partial_j v_\lambda\\ &\quad+\sum_{k=1}^n b_k\,\partial_k v_\lambda\}\\ &\quad+\partial_t w_\lambda+\sum_{i,j=1}^n a_{ij}\,\partial_i\partial_j w_\lambda-2\lambda\sum_{i,j=1}^n a_{ij}\,\partial_i v_\lambda\,\partial_j w_\lambda\\ &\quad+\sum_{k=1}^n b_k\,\partial_k w_\lambda+cw_\lambda\,.\end{aligned} \tag{2.11}$$

We want to express the Taylor expansion of the right-hand side of (2.11) with respect to x at $x=0$ and make the coefficients of x^p, $|p|\leqq N$, to be equal to 0. To this end we must make some preparations.

We can write for $i,j=1,\cdots,n$

$$\begin{aligned}\partial_i v_\lambda(x,t)&=\sum_{|p|\leqq N-1}\lambda^{-\rho(p+e_i)}(p_i+1)\,v_{\lambda,p+e_i}(t)\,x^p\,,\\ \partial_i\partial_j v_\lambda(x,t)&=\sum_{|p|\leqq N-2}\lambda^{-\rho(p+e_i+e_j)}(p_i+\delta_{ij}+1)\,(p_j+1)\,v_{\lambda,p+e_i+e_j}(t)\,x^p\,,\\ \partial_i w_\lambda(x,t)&=\sum_{|p|\leqq N-1}\lambda^{\mu(|p|+1)}(p_i+1)\,w_{\lambda,p+e_i}(t)\,x^p\,,\\ \partial_i\partial_j w_\lambda(x,t)&=\sum_{|p|\leqq N-2}\lambda^{\mu(|p|+2)}(p_i+\delta_{ij}+1)\,(p_j+1)\,w_{\lambda,p+e_i+e_j}(t)\,x^p\,,\end{aligned}$$

where $e_i=(0,\cdots,\overset{i}{1},\cdots,0)$ $(1\leqq i\leqq n)$ and δ_{ij} is the Kronecker's delta. Take sufficiently small $\delta>0$ so that

$$U_\delta=\{(x,t)\in R^{n+1}\mid\ |x|<\delta,\ |t|<\delta\}\subset U\,. \tag{2.12}$$

By the Taylor's formula we can write in U_δ

$$\begin{aligned}a_{ij}(x,t)&=\sum_{|p|\leqq N}a_{ij}^p(t)\,x^p+\sum_{|p|=N+1}a_{ij}^p(x,t)\,x^p\,,\quad i,j=1,\cdots,n\,,\\ b_k(x,t)&=\sum_{|p|\leqq N}b_k^p(t)\,x^p+\sum_{|p|=N+1}b_k^p(x,t)\,x^p\,,\quad k=1,\cdots,n\,,\\ c(x,t)&=\sum_{|p|\leqq N}c^p(t)\,x^p+\sum_{|p|=N+1}c^p(x,t)\,x^p\,.\end{aligned}$$

From the assumptions (A.2) and (A.3) of Theorem 1.1 it follows that for $i,j=1,\cdots,n$ and sufficiently small t

$$a_{ij}^0(t) = O(t^q) \quad \text{and} \quad a_{ij}^p(t) = O(t^{(q+1)/2}) \ (|p| = 1), \tag{2.13}$$

$$b_k^0(t) = O(t^q) \quad \text{and} \quad b_k^p(t) = O(t^{(q+1)/2}) \ (|p| = 1). \tag{2.14}$$

We put

$$a_{ij}^{(N+1)}(x, t) = \sum_{|p|=N+1} a_{ij}^p(x, t)\, x^p, \quad i, j = 1, \cdots, n, \tag{2.15}$$

$$b_k^{(N+1)}(x, t) = \sum_{|p|=N+1} b_k^p(x, t)\, x^p, \quad k = 1, \cdots, n, \tag{2.16}$$

$$c^{(N+1)}(x, t) = \sum_{|p|=N+1} c^p(x, t)\, x^p, \tag{2.17}$$

$$L_{N+1} = \sum_{i,j=1}^{n} a_{ij}^{(N+1)}(x, t)\, \partial_i \partial_j + \sum_{k=1}^{n} b_k^{(N+1)}(x, t)\, \partial_k. \tag{2.18}$$

We denote by $V=(v_p)_{|p|\leq N}$ and $W=(w_p)_{|p|\leq N}$ vectors whose components are complex numbers v_p and $w_p (|p| \leqq N)$ where $p=(p_1, \cdots, p_n)$ is a multi-index of non-negative integers. We also denote by $|V|$ and $|W|$ the Euclidean norms of V and W respectively. Now, for $\lambda>0$ and multi-indices of non-negative integers $p=(p_1, \cdots, p_n)$ $(|p| \leqq N)$, we define functions $\phi_{\lambda,p}(V, t)$ of V and t, and $\psi_{\lambda,p}(V, W, t)$ of V, W and t as follows:

$$\begin{aligned}
&\phi_{\lambda,p}(V, t) \\
&= \sum_{i,j=1}^{n} \sum_{\substack{\alpha+\beta+\gamma=p \\ |\beta|,|\gamma|\leqq N-1}} \lambda^{1+\rho(p)-\rho(\beta+e_i)-\rho(\gamma+e_j)}(\beta_i+1)\,(\gamma_j+1)\, a_{ij}^{\alpha}(t) \times \\
&\quad \times v_{\beta+e_i}\, v_{\gamma+e_j} \\
&\quad - \sum_{i,j=1}^{n} \sum_{\substack{\alpha+\beta=p \\ |\beta|\leqq N-2}} \lambda^{\rho(p)-\rho(\beta+e_i+e_j)}(\beta_i+\delta_{ij}+1)\,(\beta_j+1)\, a_{ij}^{\alpha}(t)\, v_{\beta+e_i+e_j} \\
&\quad - \sum_{k=1}^{n} \sum_{\substack{\alpha+\beta=p \\ |\beta|\leqq N-1}} \lambda^{\rho(p)-\rho(\beta+e_k)}(\beta_k+1)\, b_k^{\alpha}(t)\, v_{\beta+e_k},
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
&\psi_{\lambda,p}(V, W, t) \\
&= -\sum_{i,j=1}^{n} \sum_{\substack{\alpha+\beta=p \\ |\beta|\leqq N-2}} \lambda^{2\mu-\mu|\alpha|}(\beta_i+\delta_{ij}+1)\,(\beta_j+1)\, a_{ij}^{\alpha}(t)\, w_{\beta+e_i+e_j} \\
&\quad + 2\sum_{i,j=1}^{n} \sum_{\substack{\alpha+\beta+\gamma=p \\ |\beta|,|\gamma|\leqq N-1}} \lambda^{1+\mu-\mu(|\alpha|+|\beta|)-\rho(\beta+e_i)}(\beta_i+1)\,(\gamma_j+1)\, a_{ij}^{\alpha}(t) \times \\
&\quad \times v_{\beta+e_i}\, w_{\gamma+e_j} \\
&\quad - \sum_{k=1}^{n} \sum_{\substack{\alpha+\beta=p \\ |\beta|\leqq N-1}} \lambda^{\mu-\mu|\alpha|}(\beta_k+1)\, b_k^{\alpha}(t)\, w_{\beta+e_k} - \sum_{\alpha+\beta=p} \lambda^{-\mu|\alpha|}\, c^{\alpha}(t)\, w_{\beta}.
\end{aligned} \tag{2.20}$$

Finally we put

$$V_\lambda(t) = (v_{\lambda,p}(t))_{|p|\leq N}, \tag{2.21}$$

$$W_\lambda(t) = (w_{\lambda,p}(t))_{|p|\leq N}, \tag{2.22}$$

where $v_{\lambda,p}$ and $w_{\lambda,p}(|p|\leqq N)$ are the functions which appeared in the right-hand sides of (2.9) and (2.10) respectively.

Under the above preparations we can now rewrite (2.11) in the form:

$$\begin{aligned}(2.23)\qquad &\exp(\lambda v_\lambda)\, L(w_\lambda \exp(-\lambda v_\lambda))\\ &= -\lambda w_\lambda \sum_{|p|\leqq N}\left\{\frac{d}{dt}v_{\lambda,p}(t)-\phi_{\lambda,p}(V_\lambda(t),t)\right\}\lambda^{-\rho(p)}x^p\\ &\quad+\sum_{|p|\leqq N}\left\{\frac{d}{dt}w_{\lambda,p}(t)-\psi_{\lambda,p}(V_\lambda(t),W_\lambda(t),t)\right\}\lambda^{\mu|p|}x^p\\ &\quad+\sum_{i=1}^{5}R_{\lambda,i}(x,t)\,,\end{aligned}$$

where

$$\begin{aligned}(2.24)\qquad &R_{\lambda,1}(x,t)\\ &= \lambda^2 w_\lambda(x,t)\sum_{\substack{N+1\leqq|p|\leqq 3N-2\\ |\beta|+|\gamma|\leqq|p|+2\\ 1\leqq|\beta|,|\gamma|\leqq N}} r^{(1)}_{p,\beta,\gamma}(t)\,\lambda^{-\rho(\beta)-\rho(\gamma)}\,v_{\lambda,\beta}(t)\times\\ &\quad\times v_{\lambda,\gamma}(t)\,x^p\,,\end{aligned}$$

$$\begin{aligned}(2.25)\qquad &R_{\lambda,2}(x,t)\\ &= \lambda w_\lambda(x,t)\sum_{\substack{N+1\leqq|p|\leqq 2N-1\\ 1\leqq|\beta|\leqq N}} r^{(2)}_{p,\beta}(t)\,\lambda^{-\rho(\beta)}\,v_{\lambda,\beta}(t)\,x^p\,,\end{aligned}$$

$$\begin{aligned}(2.26)\qquad &R_{\lambda,3}(x,t)\\ &= \sum_{\substack{N+1\leqq|p|\leqq 2N\\ |\beta|\leqq N}} r^{(3)}_{p,\beta}(t)\,\lambda^{\mu|\beta|}\,w_{\lambda,\beta}(t)\,x^p\,,\end{aligned}$$

$$\begin{aligned}(2.27)\qquad &R_{\lambda,4}(x,t)\\ &= \lambda\sum_{\substack{N+1\leqq|p|\leqq 3N-2\\ |\beta|+|\gamma|\leqq|p|+2\\ 1\leqq|\beta|,|\gamma|\leqq N}} r^{(4)}_{p,\beta,\gamma}(t)\,\lambda^{-\rho(\beta)+\mu|\gamma|}\,v_{\lambda,\beta}(t)\,w_{\lambda,\gamma}(t)\,x^p\,,\end{aligned}$$

$$\begin{aligned}(2.28)\qquad &R_{\lambda,5}(x,t)\\ &= \lambda^2 w_\lambda(x,t)\sum_{i,j=1}^{n}a^{(N+1)}_{ij}(x,t)\,\partial_i v_\lambda(x,t)\,\partial_j v_\lambda(x,t)\\ &\quad-\lambda w_\lambda(x,t)\,L_{N+1}v_\lambda(x,t)-2\lambda\sum_{i,j=1}^{n}a^{(N+1)}_{ij}(x,t)\,\partial_i v_\lambda(x,t)\times\\ &\quad\times\partial_j w_\lambda(x,t)+L_{N+1}w_\lambda(x,t)+c^{(N+1)}(x,t)\,w_\lambda(x,t)\,,\end{aligned}$$

and where $r^{(1)}_{p,\beta,\gamma}$, $r^{(2)}_{p,\beta}$, $r^{(3)}_{p,\beta}$ and $r^{(4)}_{p,\beta,\gamma}$ are all linear combinations of a^α_{ij}, b^α_k and c^α $(|\alpha|\leqq N,\ 1\leqq i,j,k\leqq n)$.

In order that the coefficients of $x^p(|p|\leqq N)$ in the right-hand side of (2.23) may be equal to 0, we define $V_\lambda(t)=(v_{\lambda,p}(t))_{|p|\leqq N}$ as the solution of the following system of ordinary differential equations with $\lambda>0$ as a parameter:

$$\frac{d}{dt} v_{\lambda,p}(t) = \phi_{\lambda,p}(V_\lambda(t), t) \quad (|p| \leqq N), \tag{2.29}$$

with initial conditions

$$\begin{cases} v_{\lambda,0}(0) = 0, \\ v_{\lambda,e_i}(0) = \sqrt{-1}\,\xi_i^0 \quad (1 \leqq i \leqq n), \\ v_{\lambda,e_i+e_j}(0) = \delta_{ij} \quad (1 \leqq i, j \leqq n), \\ v_{\lambda,p}(0) = 0 \quad (3 \leqq |p| \leqq N). \end{cases} \tag{2.30}$$

and with $V_\lambda(t)$ thus defined, we define $W_\lambda(t) = (w_{\lambda,p}(t))_{|p| \leqq N}$ as the solution of the following system of linear ordinary differential equations with $\lambda > 0$ as a parameter:

$$\frac{d}{dt} w_{\lambda,p}(t) = \psi_{\lambda,p}(V_\lambda(t), W_\lambda(t), t) \quad (|p| \leqq N), \tag{2.31}$$

with initial conditions

$$\begin{cases} w_{\lambda,0}(0) = 1, \\ w_{\lambda,p}(0) = 0 \quad (1 \leqq |p| \leqq N). \end{cases} \tag{2.32}$$

Here we note that the domain of definition of $V_\lambda(t)$ depends on $\lambda > 0$ and hence so does that of $W_\lambda(t)$. In the next section we shall determine the domains of definitions of $V_\lambda(t)$ and $W_\lambda(t)$ and estimate the functions $v_\lambda(x, t)$, $w_\lambda(x, t)$ and the function:

$$R_\lambda(x, t) = \sum_{i=1}^{5} R_{\lambda,i}(x, t). \tag{2.33}$$

3. Estimates of v_λ, w_λ and R_λ

At first we determine the domain of definition of $V_\lambda(t)$.

Proposition 3.1. *Let δ be the constant in* (2.12).

(i) *There exists a constant r_0, $0 < r_0 < \delta$, independent of $\lambda > 1$ such that for every $\lambda > 1$, the differential equation* (2.29) *with initial conditions* (2.30) *has a unique solution $V_\lambda(t) = (v_{\lambda,p}(t))_{|p| \leqq N}$ defined in the interval $|t| \leqq r_0 \lambda^{-\kappa}$.*

(ii) *There exists a constant $C > 0$ independent of $\lambda > 1$ such that for all $\lambda > 1$*

$$\sup_{|t| \leqq r_0 \lambda^{-\kappa}} |V_\lambda(t)| \leqq C, \tag{3.1}$$

$$\sup_{|t| \leqq r_0 \lambda^{-\kappa}} \left|\frac{d}{dt} V_\lambda(t)\right| \leqq C\lambda^\kappa. \tag{3.2}$$

Proof. Throughout the proof we shall denote by C positive constants inde-

pendent of $\lambda>1$. Taking into account that the functions $\phi_{\lambda,p}(V,t)$ $(\lambda>0,\ |p|\leqq N)$ defined by (2.19) are independent of the variable v_0, we denote by $\tilde{V}=(v_p)_{1\leqq|p|\leqq N}$ a vector whose components are complex numbers $v_p(1\leqq|p|\leqq N)$ and in relation to (2.21) we put $\tilde{V}_\lambda(t)=(v_{\lambda,p}(t))_{1\leqq|p|\leqq N}$. Then, $\phi_{\lambda,p}(V,t)=\phi_{\lambda,p}(\tilde{V},t)$ and we divide (2.29) and (2.30) into two parts:

$$
(3.3)\qquad \begin{cases} \dfrac{d}{dt}v_{\lambda,0}(t) \\ \quad=\phi_{\lambda,0}(\tilde{V}_\lambda(t),t) \\ \quad=\lambda^{1-2\rho}\displaystyle\sum_{i,j=1}^{n}a_{ij}^0(t)\,v_{\lambda,e_i}(t)\,v_{\lambda,e_j}(t) \\ \qquad-\lambda^{-\rho-d}\displaystyle\sum_{i,j=1}^{n}a_{ij}^0(t)\,(1+\delta_{ij})\,v_{\lambda,e_i+e_j}(t) \\ \qquad-\lambda^{-\rho}\displaystyle\sum_{k=1}^{n}b_k^0(t)\,v_{\lambda,e_k}(t)\quad\text{(by (2.19) and (2.8))}, \\ v_{\lambda,0}(0)=0 \end{cases}
$$

and

$$
(3.4)\qquad \begin{cases} \dfrac{d}{dt}v_{\lambda,p}(t)=\phi_{\lambda,p}(\tilde{V}_\lambda(t),t)\quad(1\leqq|p|\leqq N), \\ v_{\lambda,e_i}(0)=\sqrt{-1}\,\xi_i^0\quad(1\leqq i\leqq n), \\ v_{\lambda,e_i+e_j}(0)=\delta_{ij}\quad(1\leqq i,j\leqq n), \\ v_{\lambda,p}(0)=0\quad(3\leqq|p|\leqq N). \end{cases}
$$

We can slove (3.3) by quadrature and (3.4) is an equation with unknown functions $v_{\lambda,p}(t)$ $(1\leqq|p|\leqq N)$.

Put $\tilde{V}_0=(v_{0,p})_{1\leqq|p|\leqq N}$ with $v_{0,e_i}=\sqrt{-1}\,\xi_i^0\,(1\leqq i\leqq n)$, $v_{0,e_i+e_j}=\delta_{ij}(1\leqq i,j\leqq n)$ and $v_{0,p}=0\,(3\leqq|p|\leqq N)$ and put $D_{\lambda,\kappa}=\{(\tilde{V},t)\,|\,\tilde{V}-\tilde{V}_0|\leqq 1$ and $|t|\leqq\delta\lambda^{-\kappa}\}$. From (2.19) and (2.8) it follows that

$$
\begin{aligned}
&|\phi_{\lambda,p}(\tilde{V},t)| \\
&\leqq C\sum_{i,j=1}^{n}\sum_{\substack{\alpha+\beta+\gamma=p\\ |\beta|,|\gamma|\leqq N-1}}\lambda^{1-\rho+(|\alpha|-1)d}\,|a_{ij}^{\alpha}(t)|+C\sum_{i,j=1}^{n}\sum_{\substack{\alpha+\beta=p\\ |\beta|\leqq N-2}}\lambda^{(|\alpha|-2)d} \\
&\quad\times|a_{ij}^{\alpha}(t)|+C\sum_{k=1}^{n}\sum_{\substack{\alpha+\beta=p\\ |\beta|\leqq N-1}}\lambda^{(|\alpha|-1)d}\,|b_k^{\alpha}(t)|,\ 1\leqq|p|\leqq N,\ (\tilde{V},t)\in D_{\lambda,\kappa}.
\end{aligned}
$$

Since $d<0$ by (2.4), we have for $\lambda>1$

$$
\begin{aligned}
&|\phi_{\lambda,p}(\tilde{V},t)| \\
&\leqq C\sum_{i,j=1}^{n}(\lambda^{1-\rho-d}\,|a_{ij}^0(t)|+\sum_{|\alpha|=1}\lambda^{1-\rho}\,|a_{ij}^{\alpha}(t)|)+C\lambda^{1-\rho+d} \\
&\quad+C\sum_{i,j=1}^{n}(\lambda^{-2d}\,|a_{ij}^0(t)|+\sum_{|\alpha|=1}\lambda^{-d}\,|a_{ij}^{\alpha}(t)|) \\
&\quad+C\sum_{k=1}^{n}\lambda^{-d}\,|b_k^0(t)|+C,\ 1\leqq|p|\leqq N,\ (\tilde{V},t)\in D_{\lambda,\kappa}.
\end{aligned}
$$

Hence, using (2.13) and (2.14) we have for $\lambda>1$

$$\begin{aligned}&\sup_{(\tilde{V},t)\in D_{\lambda,\kappa}}|\phi_{\lambda,p}(\tilde{V},t)|\\&\leqq C(\lambda^{1-\rho-d-\kappa q}+\lambda^{1-\rho-\kappa(q+1)/2}+\lambda^{1-\rho+d}+\lambda^{-2d-\kappa q}\\&\quad+\lambda^{-d-\kappa(q+1)/2}+\lambda^{-d-\kappa q}+1)\,.\end{aligned}$$

From (2.2)–(2.5) we see that all exponents of λ in the right-hand side of the above inequality are less than or equal to κ and so

$$\sup_{(\tilde{V},t)\in D_{\lambda,\kappa}}|\phi_{\lambda,p}(\tilde{V},t)|\leqq C\lambda^{\kappa}\quad\text{for all}\quad\lambda>1\,,\quad 1\leqq|p|\leqq N\,. \tag{3.5}$$

Put $\Phi_\lambda(\tilde{V},t)=(\phi_{\lambda,p}(\tilde{V},t))_{1\leqq|p|\leqq N}$. By (3.5) we have with another constant $C>0$ independent of $\lambda>1$

$$\sup_{(\tilde{V},t)\in D_{\lambda,\kappa}}|\Phi_\lambda(\tilde{V},t)|\leqq C\lambda^{\kappa}\,. \tag{3.6}$$

Let $r_0=\min(\delta,\ C^{-1})$. Then, from (3.6) and the fundamental theorem on existence of solutions of systems of ordinary differential equations, it follows that (3.4) has a unique solution $\tilde{V}_\lambda(t)$ defined in the interval $|t|\leqq r_0\lambda^{-\kappa}$ such that $(\tilde{V}_\lambda(t),t)\in D_{\lambda,\kappa}$. Therefore (i) has been proved and we have

$$\sup_{|t|\leqq r_0\lambda^{-\kappa}}|\tilde{V}_\lambda(t)|\leqq C\,. \tag{3.7}$$

Furthermore by (3.4) and (3.5) it holds that

$$\sup_{|t|\leqq r_0\lambda^{-\kappa}}\left|\frac{d}{dt}\tilde{V}_\lambda(t)\right|\leqq C\lambda^{\kappa}\,. \tag{3.8}$$

It remains to estimate $v_{\lambda,0}(t)$ and $\frac{d}{dt}v_{\lambda,0}(t)$. Using (3.3), (2.13), (2.14) and (3.7), we easily have for $\lambda>1$

$$\sup_{|t|\leqq r_0\lambda^{-\kappa}}\left|\frac{d}{dt}v_{\lambda,0}(t)\right|\leqq C(\lambda^{1-2\rho}+\lambda^{-\rho-d}+\lambda^{-\rho})\,\lambda^{-\kappa q}\,. \tag{3.9}$$

On the other hand, from (2.2)–(2.5) it follows that

$$\max(1-2\rho,\ -\rho-d,\ -\rho)-\kappa q<1-\rho-\kappa q<\kappa\,.$$

Hence by (3.9) we have for $\lambda>1$

$$\sup_{|t|\leqq r_0\lambda^{-\kappa}}\left|\frac{d}{dt}v_{\lambda,0}(t)\right|\leqq C\lambda^{\kappa}\,. \tag{3.10}$$

Since $v_{\lambda,0}(0)=0$ by (3.3), we have by (3.10)

$$\sup_{|t|\leqq r_0\lambda^{-\kappa}}|v_{\lambda,0}(t)|\leqq C\,. \tag{3.11}$$

From (3.7) and (3.11) we obtain (3.1), and from (3.10) and (3.8) we obtain (3.2). Q.E.D.

With $v_{\lambda,p}(t)$ ($|p| \leqq N$) thus defined, we define $v_\lambda(x, t)$ by (2.9) in $R^n \times [-r_0\lambda^{-\kappa}, r_0\lambda^{-\kappa}]$. Next proposition will play an essential role when we shall prove (♯) in the next section.

Proposition 3.2. *There exist constants $r_1(0<r_1<r_0)$ and $C>0$ both independent of $\lambda>1$ such that for all sufficiently large λ*

$$\lambda \operatorname{Re} v_\lambda(x, t) \geqq C\lambda^{2(1-\rho)} t^{q+1} + \frac{1}{2}\lambda^{1-\rho-d}|x|^2, \quad |x| \leqq \lambda^{-\mu}, \quad |t| \leqq r_1\lambda^{-\kappa}. \tag{3.12}$$

Proof. We note that $t^{q+1} \geqq 0$ because q is a positive odd integer. Throughout the proof we shall denote by C_1 and C_2 positive constants independent of $\lambda>1$, and use the notation: $' = \frac{d}{dt}$.

By (2.30) we can write

$$v_{\lambda,e_i}(t) = \sqrt{-1}\,\xi_i^0 + v'_{\lambda,e_i}(\theta_i t)\, t, \quad 0<\theta_i<1, \quad i = 1, \cdots, n.$$

Then by (3.3) it holds that

$$\begin{aligned} v'_{\lambda,0}(t) = &-\lambda^{1-2\rho}\sum_{i,j=1}^{n} a_{ij}^0(t)\,\xi_i^0\,\xi_j^0 \\ &+A_\lambda(t)+B_\lambda(t)+C_\lambda(t), \quad |t| \leqq r_0\lambda^{-\kappa}, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} A_\lambda(t) = \lambda^{1-2\rho}\sum_{i,j=1}^{n} a_{ij}^0(t)\,(2\sqrt{-1}\,\xi_i^0\, v'_{\lambda,e_j}(\theta_j t)\, t + \\ + v'_{\lambda,e_i}(\theta_i t)\, v'_{\lambda,e_j}(\theta_j t)\, t^2), \end{aligned}$$

$B_\lambda(t)$ = the second term of the right-hand side of the differential equation in (3.3),

$C_\lambda(t)$ = the third term of the right-hand side of the same equation as above.

On the other hand, from (2.13), (2.14) and Proposition 3.1 (ii) it follows that

$$\begin{aligned} &|A_\lambda(t)| \leqq C_1(\lambda^{1-2\rho+\kappa}|t|^{q+1} + \lambda^{1-2\rho+2\kappa}|t|^{q+2}), \quad |t| \leqq r_0\lambda^{-\kappa}, \\ &|B_\lambda(t)| \leqq C_1\lambda^{-\rho-d}|t|^q, \quad |t| \leqq r_0\lambda^{-\kappa}, \\ &|C_\lambda(t)| \leqq C_1\lambda^{-\rho}|t|^q, \quad |t| \leqq r_0\lambda^{-\kappa}. \end{aligned}$$

Hence, integrating both sides of (3.13) from 0 to t, and using the initial condition of (3.3) and the hypothesis (A.1) of Theorem 1.1, we have

$$\begin{aligned} &\operatorname{Re} v_{\lambda,0}(t) \\ &\geqq C_2\lambda^{1-2\rho} t^{q+1} - C_1(\lambda^{1-2\rho+\kappa}|t|^{q+2} + \lambda^{1-2\rho+2\kappa}|t|^{q+3}) \\ &\quad - C_1\lambda^{-\rho-d} t^{q+1} - C_1\lambda^{-\rho} t^{q+1}, \end{aligned} \tag{3.14}$$

for all $\lambda>1$ and $|t|\leqq r_0\lambda^{-\kappa}$.

When $|p|=1$, we have by (2.29), (2.19), (2.7) and (3.1)

$$\begin{aligned}|v'_{\lambda,p}(t)|&\leqq C_1\sum_{i,j=1}^{n}\sum_{|\alpha|\leqq 1}(\lambda^{1-\rho+(|\alpha|-1)d}+\lambda^{(|\alpha|-2)d})|a_{ij}^{\alpha}(t)|\\&\quad+C_1\sum_{k=1}^{n}\sum_{|\alpha|\leqq 1}\lambda^{(|\alpha|-1)d}|b_k^{\alpha}(t)|\end{aligned}$$

for all $\lambda>1$ and $|t|\leqq r_0\lambda^{-\kappa}$, and so from (2.13) and (2.14) it follows that

$$\begin{aligned}|v'_{\lambda,p}(t)|&\leqq C_1(\lambda^{1-\rho-d}+\lambda^{-2d})|t|^q+C_1(\lambda^{1-\rho}+\lambda^{-d})|t|^{(q+1)/2}\\&\quad+C_1(\lambda^{-d}|t|^q+|t|^{(q+1)/2})\\&\leqq C_1(\lambda^{1-\rho-d}|t|^q+\lambda^{1-\rho}|t|^{(q+1)/2}),\end{aligned}$$

for all $\lambda>1$ and $|t|\leqq r_0\lambda^{-\kappa}$, because $0<-d<-2d<1-\rho-d$ by (2.2)–(2.4). Hence, since $\operatorname{Re} v_{\lambda,p}(0)=0$ ($|p|=1$) by (2.30), we have for $|p|=1$

$$\begin{aligned}&|\operatorname{Re} v_{\lambda,p}(t)|\\&=\Big|\int_0^t \operatorname{Re} v'_{\lambda,p}(s)\,ds\Big|\leqq C_1(\lambda^{1-\rho-d}t^{q+1}+\lambda^{1-\rho}|t|^{1+(q+1)/2}),\end{aligned}$$

for all $\lambda>1$ and $|t|\leqq r_0\lambda^{-\kappa}$.
Hence, noting that $\rho(p)=\rho$ ($|p|=1$) by (2.7), we have

$$\begin{aligned}&\Big|\sum_{|p|=1}\lambda^{-\rho(p)}\operatorname{Re} v_{\lambda,p}(t)\,x^p\Big|\\&\quad\leqq C_1(\lambda^{1-2\rho-d}t^{q+1}+\lambda^{1-2\rho}|t|^{1+(q+1)/2})|x|,\end{aligned}\tag{3.15}$$

for all $\lambda>1$, $x\in R^n$ and $|t|\leqq r_0\lambda^{-\kappa}$.

Since $v_{\lambda,e_i+e_j}(0)=\delta_{ij}(i,j=1,\cdots,n)$ by (2.30), we have by (3.2)

$$|\operatorname{Re} v_{\lambda,e_i+e_j}(t)|=\Big|\int_0^t \operatorname{Re} v'_{\lambda,e_i+e_j}(s)\,ds\Big|\leqq C_1\lambda^{\kappa}|t|\quad\text{if}\quad i\neq j,$$

and

$$\operatorname{Re} v_{\lambda,e_i+e_i}(t)=1+\int_0^t \operatorname{Re} v'_{\lambda,e_i+e_i}(s)\,ds\geqq 1-C_1\lambda^{\kappa}|t|,$$

for all $\lambda>1$ and $|t|\leqq r_0\lambda^{-\kappa}$. Hence, noting that $\rho(p)=\rho+d$ ($|p|=2$) by (2.7), we have

$$\sum_{|p|=2}\lambda^{-\rho(p)}\operatorname{Re} v_{\lambda,p}(t)\,x^p\geqq(1-C_1\lambda^{\kappa}|t|)\lambda^{-\rho-d}|x|^2,\tag{3.16}$$

for all $\lambda>1$, $x\in R^n$ and $|t|\leqq r_0\lambda^{-\kappa}$.

When $3\leqq|p|\leqq N$, we have by (2.7) and (3.1)

$$|\lambda^{-\rho(p)}\,\mathrm{Re}\,v_{\lambda,p}(t)\,x^p| \leqq C_1\lambda^{-\rho-(|p|-1)d}|x|^{|p|}$$
$$= C_1\lambda^{-\rho-d}|x|^2\lambda^{-(|p|-2)d}|x|^{|p|-2}$$
$$\leqq C_1\lambda^{-\rho-d}|x|^2\lambda^{-(|p|-2)(d+\mu)}$$

for all $\lambda>1$, $|x|\leqq\lambda^{-\mu}$ and $|t|\leqq r_0\lambda^{-\kappa}$.
Hence, since $d+\mu>0$ by (2.6), it follows that

$$|\sum_{3\leqq|p|\leqq N}\lambda^{-\rho(p)}\,\mathrm{Re}\,v_{\lambda,p}(t)\,x^p| \leqq C_1\lambda^{-\rho-2d-\mu}|x|^2, \tag{3.17}$$

for all $\lambda>1$, $|x|\leqq\lambda^{-\mu}$ and $|t|\leqq r_0\lambda^{-\kappa}$.

From (2.9) and (3.14)–(3.17) it follows that

$$\mathrm{Re}\,v_\lambda(x,t)\geqq C_2\lambda^{1-2\rho}t^{q+1}\{1-C_1C_2^{-1}(\lambda^\kappa|t|+\lambda^{2\kappa}|t|^2+\lambda^{-1+\rho-d}+\lambda^{-1+\rho})\}$$
$$-C_1(\lambda^{1-2\rho-d}t^{q+1}+\lambda^{1-2\rho}|t|^{1+(q+1)/2})|x|$$
$$+(1-C_1\lambda^\kappa|t|-C_1\lambda^{-d-\mu})\lambda^{-\rho-d}|x|^2,$$

for all $\lambda>1$, $|x|\leqq\lambda^{-\mu}$ and $|t|\leqq r_0\lambda^{-\kappa}$.
Now we choose a sufficiently small real number r_1 such that

$$0<r_1<r_0,\ 1-C_1C_2^{-1}(r_1+r_1^2)\geqq\frac{1}{2}\quad\text{and}\quad 1-C_1r_1\geqq\frac{3}{4}.$$

Then, if $|t|\leqq r_1\lambda^{-\kappa}$

$$1-C_1C_2^{-1}(\lambda^\kappa|t|+\lambda^{2\kappa}|t|^2)\geqq\frac{1}{2}\quad\text{and}\quad 1-C_1\lambda^\kappa|t|\geqq\frac{3}{4}.$$

Hence

$$\mathrm{Re}\,v_\lambda(x,t)\geqq C_2\lambda^{1-2\rho}t^{q+1}\{\frac{1}{2}-C_1C_2^{-1}(\lambda^{-1+\rho-d}+\lambda^{-1+\rho})\}$$
$$-C_1(\lambda^{1-2\rho-d}t^{q+1}+\lambda^{1-2\rho}|t|^{1+(q+1)/2})|x|$$
$$+(\frac{3}{4}-C_1\lambda^{-d-\mu})\lambda^{-\rho-d}|x|^2,$$

for all $\lambda>1$, $|x|\leqq\lambda^{-\mu}$ and $|t|\leqq r_1\lambda^{-\kappa}$.

On the other hand, $\max(-1+\rho-d, -1+\rho, -d-\mu)<0$ by (2.2)–(2.4) and (2.6), and so $\lim_{\lambda\to\infty}\lambda^{-1+\rho-d}=\lim_{\lambda\to\infty}\lambda^{-1+\rho}=\lim_{\lambda\to\infty}\lambda^{-d-\mu}=0$. Therefore there exists a sufficiently large $\lambda_1>1$ such that for all $\lambda>\lambda_1$, $|x|\leqq\lambda^{-\mu}$ and $|t|\leqq r_1\lambda^{-\kappa}$

$$\mathrm{Re}\,v_\lambda(x,t)\geqq\frac{1}{8}C_2\lambda^{1-2\rho}t^{q+1}+\frac{1}{2}\lambda^{-\rho-d}|x|^2+\frac{1}{8}\lambda^{-\rho-d}|x|^2$$
$$-C_1(\lambda^{1-2\rho-d}t^{q+1}+\lambda^{1-2\rho}|t|^{1+(q+1)/2})|x|$$
$$+\frac{1}{8}C_2\lambda^{1-2\rho}t^{q+1}$$
$$\equiv\frac{1}{8}C_2\lambda^{1-2\rho}t^{q+1}+\frac{1}{2}\lambda^{-\rho-d}|x|^2+Q(|x|,t,\lambda).$$

Here we consider $Q(|x|, t, \lambda)$ as a quadratic function of $|x|$. Then, to prove Proposition 3.2 it suffices to show that the discriminant of $Q(|x|, t, \lambda) \leqq 0$ for all $|t| \leqq r_1 \lambda^{-\kappa}$ and $\lambda > \lambda_1$ if λ_1 is chosen sufficiently large.

When $|t| \leqq r_1 \lambda^{-\kappa}$ it holds that
the discriminant of Q

$$\begin{aligned}
&\leqq 2C_1^2(\lambda^{2-4\rho-2d} t^{2q+2} + \lambda^{2-4\rho}|t|^{q+3}) - \frac{1}{16} C_2 \lambda^{1-3\rho-d} t^{q+1} \\
&= \left\{2C_1^2(\lambda^{1-\rho-d} t^{q+1} + \lambda^{1-\rho+d} t^2) - \frac{1}{16} C_2\right\} \lambda^{1-3\rho-d} t^{q+1} \\
&\leqq \left\{2C_1^2(\lambda^{1-\rho-d-\kappa(q+1)} r_1^{q+1} + \lambda^{1-\rho+d-2\kappa} r_1^2) - \frac{1}{16} C_2\right\} \lambda^{1-3\rho-d} t^{q+1}.
\end{aligned}$$

On the other hand, from (2.2)–(2.5) we see that

$$\begin{aligned}
&1-\rho-d-\kappa(q+1) = 2(1-\rho)(\varepsilon-1)/(q+1+\varepsilon) < 0 \quad \text{and} \\
&1-\rho+d-2\kappa = -2(1-\rho)/(q+1+\varepsilon) < 0.
\end{aligned}$$

Hence $\lim_{\lambda\to\infty} \lambda^{1-\rho-d-\kappa(q+1)} = \lim_{\lambda\to\infty} \lambda^{1-\rho+d-2\kappa} = 0$ and so the desired estimate has been proved. Q.E.D.

In order to estimate derivatives of $v_\lambda(x, t)$ we prepare the following lemma.

Lemma 3.1. *For every non-negative integer m there exists a constant $C_m > 0$ independent of $\lambda > 1$ such that*

$$\sup_{|t| \leqq r_0 \lambda^{-\kappa}} \left| \frac{d^m}{dt^m} V_\lambda(t) \right| \leqq C_m \lambda^{2(1-\rho)m} \quad \text{for all} \quad \lambda > 1. \tag{3.18}$$

Proof. The proof is by induction on m. For the case $m=0$ (3.18) is nothing but (3.1). Assuming that (3.18) is vaild for all m less than or equal to a non-negative integer m_0, we shall prove that (3.18) is valid for $m = m_0 + 1$.

From (2.7) and (2.2)–(2.4) it follows that

$$\begin{aligned}
&1+\rho(p)-\rho(\beta+e_i)-\rho(\gamma+e_j) \leqq 1-\rho-d < 2(1-\rho) \quad \text{if} \quad |\beta|+|\gamma| \leqq |p|, \\
&\rho(p)-\rho(\beta+e_i+e_j) \leqq -2d < 2(1-\rho) \quad \text{if} \quad |\beta| \leqq |p|, \\
&\rho(p)-\rho(\beta+e_k) \leqq -d < 1-\rho < 2(1-\rho) \quad \text{if} \quad |\beta| \leqq |p|.
\end{aligned}$$

Taking (2.19) into account we differentiate both sides of (2.29) m_0-times. Then, from the above inequalities and the hypothesis of induction we see that (3.18) is vlaid for $m = m_0 + 1$. Q.E.D.

Proposition 3.3. *Let $h = (h_1, \cdots, h_n)$ be a multi-index of non-negative integers and m be a non-negative integer. Then, there exists a constant $C_{h,m} > 0$ independent of $\lambda > 1$ such that*

$$|\partial_x^h \partial_t^m v_\lambda(x,t)| \leqq C_{h,m} \lambda^{2(1-\rho)(|h|+m)} (1+\lambda^{1-\rho-d}|x|^2)^{N/2} \tag{3.19}$$

for all $\lambda>1$, $x\in R^n$ *and* $|t|\leqq r_0\lambda^{-\kappa}$.

Proof. Using (2.9) we write

$$\partial_x^h \partial_t^m v_\lambda(x,t) = \sum_{|p|\leqq N} \lambda^{-\rho(p)} \frac{d^m}{dt^m} v_{\lambda,p}(t)\, \partial_x^h(x^p)\,.$$

Let $\alpha=(\alpha_1,\cdots,\alpha_n)$ be a multi-index of non-negative integers. If $|\alpha|\geqq|h|$ we have with a constant $C_{h,\alpha}$ depending only on h and α

$$\begin{aligned}|\partial_x^h(x^\alpha)| &\leqq C_{h,\alpha}|x|^{(|\alpha|-|h|)}\\ &= C_{h,\alpha}\lambda^{-(1-\rho-d)(|\alpha|-|h|)/2}(\lambda^{1-\rho-d}|x|^2)^{(|\alpha|-|h|)/2}\\ &\leqq C_{h,\alpha}\lambda^{-(1-\rho-d)(|\alpha|-|h|)/2}(1+\lambda^{1-\rho-d}|x|^2)^{|\alpha|/2}\end{aligned}$$

and if $|\alpha|<|h|$ we have $\partial_x^h(x^\alpha)=0$. Therefore, since $(1-\rho-d)/2<1-\rho<2(1-\rho)$ by (2.4) and (2.3), we have

$$|\partial_x^h(x^\alpha)| \leqq C_{h,\alpha}\lambda^{2(1-\rho)|h|-(1-\rho-d)|\alpha|/2}(1+\lambda^{1-\rho-d}|x|^2)^{|\alpha|/2} \tag{3.20}$$

for all $\lambda>1$ and $x\in R^n$.
Hence, from Lemma 3.1 and (3.20) it follows that

$$\begin{aligned}|\partial_x^h \partial_t^m v_\lambda(x,t)| \leqq{}& C_{h,m}\lambda^{2(1-\rho)(|h|+m)}(1+\lambda^{1-\rho-d}|x|^2)^{N/2}\times\\ &\times\sum_{|p|\leqq N}\lambda^{-\rho(p)-(1-\rho-d)|p|/2}\end{aligned}$$

for all $\lambda>1$, $x\in R^n$ and $|t|\leqq r_0\lambda^{-\kappa}$.

On the other hand, by (2.7) and (2.2)–(2.4)

$$\begin{aligned}-\rho(p)-(1-\rho-d)|p|/2 &= \begin{cases}0 & (p=0)\\ -\rho+d-(1-\rho+d)|p|/2 & (|p|\geqq 1)\end{cases}\\ &\leqq\begin{cases}0 & (p=0)\\ -\rho+d & (|p|\geqq 1)\end{cases}\end{aligned}$$

$\leqq 0$ and so we obtain (3.19). Q.E.D.

Since $V_\lambda(t)=(v_{\lambda,p}(t))_{|p|\leqq N}$ has been defined in the interval $|t|\leqq r_0\lambda^{-\kappa}$ in Proposition 3.1, we can also define $W_\lambda(t)=(w_{\lambda,p}(t))_{|p|\leqq N}$ as the solution of (2.31) and (2.32) in the same interval, because by virtue of (2.20), (2.31) is a linear equation with respect to the unknown functions $w_{\lambda,p}(t)$ $(|p|\leqq N)$. With $w_{\lambda,p}(t)$ $(|p|\leqq N)$ thus defined, we define $w_\lambda(x,t)$ by (2.10) in $R^n\times[-r_0\lambda^{-\kappa}, r_0\lambda^{-\kappa}]$. In order to estimate derivatives of $w_\lambda(x,t)$ in this region we prepare the following lemma.

Lemma 3.2. *For every non-negative integer* m, *there exists a constant* $C_m>0$

independent of $\lambda>1$ *such that*

$$\sup_{|t|\leqq r_0\lambda^{-\kappa}}\left|\frac{d^m}{dt^m}W_\lambda(t)\right|\leqq C_m\lambda^{2(1-\rho)m} \quad \textit{for all} \quad \lambda>1\,. \tag{3.21}$$

Proof. The proof is by induction on m. Firstly we shall prove (3.21) for $m=0$. We denote by C_0 positive constants independent of $\lambda>1$. By (2.31), (2.20) and (3.1)

$$\begin{aligned}\left|\frac{d}{dt}W_\lambda(t)\right|\leqq C_0|W_\lambda(t)|\Big\{&\sum_{i,j=1}^n\sum_{|\alpha|\leqq N}|a_{ij}^\alpha(t)|\lambda^{2\mu-\mu|\alpha|}\\&+\sum_{i,j=1}^n\sum_{|\alpha|+|\beta|\leqq N}|a_{ij}^\alpha(t)|\lambda^{1+\mu-\mu(|\alpha|+|\beta|)-\rho(\beta+e_i)}\\&+\sum_{k=1}^n\sum_{|\alpha|\leqq N}|b_k^\alpha(t)|\lambda^{\mu-\mu|\alpha|}+\sum_{|\alpha|\leqq N}|c^\alpha(t)|\lambda^{-\mu|\alpha|}\Big\}\,,\end{aligned}$$

for all $\lambda>1$ and $|t|\leqq r_0\lambda^{-\kappa}$. Since $\mu>0$ by (2.2)–(2.4) and (2.6), and $1+\mu-\mu(|\alpha|+|\beta|)-\rho(\beta+e_i)=1-\rho+\mu(1-|\alpha|)-(\mu+d)|\beta|\leqq 1-\rho+\mu(1-|\alpha|)$ by (2.7) and (2.6), it follows from (2.13) and (2.14) that

$$\begin{aligned}\left|\frac{d}{dt}W_\lambda(t)\right|\leqq C_0|W_\lambda(t)|(&\lambda^{2\mu-\kappa q}+\lambda^{\mu-\kappa(q+1)/2}+\lambda^{1-\rho+\mu-\kappa q}\\&+\lambda^{1-\rho-\kappa(q+1)/2}+\lambda^{1-\rho-\mu}+\lambda^{\mu-\kappa q}+1)\,,\end{aligned}$$

for all $\lambda>1$ and $|t|\leqq r_0\lambda^{-\kappa}$ where we have estimated $a_{ij}^\alpha(t)$ $(2\leqq|\alpha|\leqq N, i,j=1,\cdots,n)$, $b_k^\alpha(t)$ $(1\leqq|\alpha|\leqq N, k=1,\cdots,n)$ and $c^\alpha(t)$ $(|\alpha|\leqq N)$ from above by positive constants.

From (2.2)–(2.6) we see that all exponents of λ in the above inequality are smaller than κ. Hence

$$\left|\frac{d}{dt}W_\lambda(t)\right|\leqq C_0\lambda^\kappa|W_\lambda(t)| \quad \text{for all} \quad \lambda>1 \quad \text{and} \quad |t|\leqq r_0\lambda^{-\kappa}\,.$$

Therefore from (2.32) and the Gronwall's inequality we obtain (3.21) for $m=0$.

Secondly, assuming that (3.21) is valid for all m less than or equal to a non-negative integer m_0, we shall prove that (3.21) is valid for $m=m_0+1$.

From (2.2)–(2.7) it follows that

$$\begin{aligned}&2\mu-\mu|\alpha|\leqq 2\mu\leqq 1-\rho-d<2(1-\rho)\,,\\&1+\mu-\mu(|\alpha|+|\beta|)-\rho(\beta+e_i)\leqq 1-\rho+\mu<2(1-\rho)\,,\\&\mu-\mu|\alpha|\leqq\mu<2(1-\rho)\,,\\&-\mu|\alpha|\leqq 0<2(1-\rho)\,.\end{aligned}$$

Note that the second inequality follows from the inequality already appeared in the first pragraph in this proof. Taking (2.20) into account we differentiate both sides of (2.31) m_0-times. Then, by the above inequalities, Lemma 3.1 and

the hypothesis of induction, we obtain (3.21) for $m=m_0+1$. Q.E.D.

Proposition 3.4. *Proposition 3.3 holds for $w_\lambda(x,t)$ instead of $v_\lambda(x,t)$.*

Proof. The proof is similar to that of Proposition 3.3. The difference consists in the fact that we use (2.10), (2.31) and Lemma 3.2 instead of (2.9), (2.29) and Lemma 3.1 respectively, and use the following inequality:

$\mu|p|-(1-\rho-d)|p|/2\leqq 0$, which follows from (2.6). Q.E.D.

Let $R_\lambda(x,t)$ be the function defined by (2.33). We consider this as the remainder term of the right-hand side of (2.23).

Proposition 3.5. *Let $h=(h_1,\cdots,h_n)$ be a multi-index of non-negative integers and m be a non-negative integer. Then there exists a constant $C_{h,m}>0$ independent of $\lambda>1$ such that*

$$|\partial_x^h\partial_t^m R_\lambda(x,t)|\leqq C_{h,m}\lambda^{2+2(1-\rho)(|h|+m+2)}\times \times\lambda^{(\mu-(1-\rho-d)/2)(N+1)}(1+\lambda^{1-\rho-d}|x|^2)^{2N+1/2} \tag{3.22}$$

for all $\lambda>1$, $x\in R^n$ and $|t|\leqq r_0\lambda^{-\kappa}$.

Proof. For the sake of brevity we put $\nu=1-\rho-d$. From (2.24)–(2.28) and (2.15)–(2.18), we obtain the following five inequalities by using the Leibniz formula, (3.20), Lemma 3.1, Proposition 3.3, Lemma 3.2 and Proposition 3.4.

$$|\partial_x^h\partial_t^m R_{\lambda,1}(x,t)| \leqq C_{h,m}\lambda^{2+2(1-\rho)(|h|+m)}\sum_{\substack{N+1\leqq|\beta|\leqq 3N-2\\ |\beta|+|\gamma|\leqq|\beta|+2\\ 1\leqq|\beta|,|\gamma|\leqq N}}\lambda^{-\rho(\beta)-\rho(\gamma)-\nu|\beta|/2}\times(1+\lambda^\nu|x|^2)^{(N+|\beta|)/2},$$

$$|\partial_x^h\partial_t^m R_{\lambda,2}(x,t)| \leqq C_{h,m}\lambda^{1+2(1-\rho)(|h|+m)}\sum_{\substack{N+1\leqq|\beta|\leqq 2N-1\\ 1\leqq|\beta|\leqq N}}\lambda^{-\rho(\beta)-\nu|\beta|/2}\times(1+\lambda^\nu|x|^2)^{(N+|\beta|)/2},$$

$$|\partial_x^h\partial_t^m R_{\lambda,3}(x,t)| \leqq C_{h,m}\lambda^{2(1-\rho)(|h|+m)}\sum_{\substack{N+1\leqq|\beta|\leqq 2N\\ |\beta|\leqq N}}\lambda^{\mu|\beta|-\nu|\beta|/2}(1+\lambda^\nu|x|^2)^{|\beta|/2},$$

$$|\partial_x^h\partial_t^m R_{\lambda,4}(x,t)| \leqq C_{h,m}\lambda^{1+2(1-\rho)(|h|+m)}\sum_{\substack{N+1\leqq|\beta|\leqq 3N-2\\ |\beta|+|\gamma|\leqq|\beta|+2\\ 1\leqq|\beta|,|\gamma|\leqq N}}\lambda^{-\rho(\beta)+\mu|\gamma|-\nu|\beta|/2}\times(1+\lambda^\nu|x|^2)^{|\beta|/2},$$

$$|\partial_x^h\partial_t^m R_{\lambda,5}(x,t)| \leqq C_{h,m}\lambda^{2+2(1-\rho)(|h|+m+2)-\nu(N+1)/2}(1+\lambda^\nu|x|^2)^{2N+1/2} +C_{h,m}\lambda^{1+2(1-\rho)(|h|+m+2)-\nu(N+1)/2}(1+\lambda^\nu|x|^2)^{(3N+1)/2}$$

$$+C_{h,m}\lambda^{1+2(1-\rho)(|h|+m+2)-\nu(N+1)/2}(1+\lambda^{\nu}|x|^2)^{(3N+1)/2}$$
$$+C_{h,m}\lambda^{2(1-\rho)(|h|+m+2)-\nu(N+1)/2}(1+\lambda^{\nu}|x|^2)^{N+1/2}$$
$$+C_{h,m}\lambda^{2(1-\rho)(|h|+m)-\nu(N+1)/2}(1+\lambda^{\nu}|x|^2)^{N+1/2},$$

for all $\lambda>1$, $x\in R^n$ and $|t|\leqq r_0\lambda^{-\kappa}$, where we have estimated the right-hand side of (2.28) term by term.

On the other hand, from (2.2)–(2.4), (2.6) and (2.7) it follows that

if $N+1\leqq|p|$, $|\beta|+|\gamma|\leqq|p|+2$ and $1\leqq|\beta|, |\gamma|$, then
$$-\rho(\beta)-\rho(\gamma)-\nu|p|/2$$
$$=-2\rho-(|\beta|+|\gamma|-2)d-\nu|p|/2\leqq-2\rho+(\mu-\nu/2)|p|<(\mu-\nu/2)\times$$
$$\times(N+1),$$

if $N+1\leqq|p|$ and $1\leqq|\beta|\leqq N$, then $-\rho(\beta)-\nu|p|/2$
$$=-\rho-(|\beta|-1)d-\nu|p|/2<-|\beta|d-\nu|p|/2<(\mu-\nu/2)|p|$$
$$\leqq(\mu-\nu/2)(N+1),$$

if $N+1\leqq|p|$ and $|\beta|\leqq N$, then $\mu|\beta|-\nu|p|/2<(\mu-\nu/2)|p|$
$$\leqq(\mu-\nu/2)(N+1),$$

and

if $N+1\leqq|p|$, $1\leqq|\beta|$ and $|\beta|+|\gamma|\leqq|p|+2$, then
$$-\rho(\beta)+\mu|\gamma|-\nu|p|/2\leqq-\rho+\mu(|\beta|-1)+\mu|\gamma|-\nu|p|/2$$
$$<1-\rho+\mu(|\beta|+|\gamma|-2)-\nu|p|/2\leqq1-\rho+(\mu-\nu/2)|p|$$
$$\leqq1-\rho+(\mu-\nu/2)(N+1).$$

Hence we obtain (3.22). Q.E.D.

4. Proof of (♯)

Using the results of section 3, we shall prove (♯). Let r_1 be the constant determined in Proposition 3.2. Taking $\chi_1(x)\in C_0^{\infty}(R^n)$ and $\chi_2(t)\in C_0^{\infty}(R^1)$ such that $\chi_1(x)=1(|x|\leqq\frac{1}{2})$, $=0(|x|\geqq1)$ and $\chi_2(t)=1(|t|\leqq\frac{1}{2})$, $=0(|t|\geqq1)$, we put

$$f_{\lambda}(x,t)=\chi_1(\lambda^{\mu}x)\,\chi_2\left(\frac{\lambda^{\kappa}t}{r_1}\right)w_{\lambda}(x,t)\exp(-\lambda v_{\lambda}(x,t)). \tag{4.1}$$

Since $r_1<r_0$ by Proposition 3.2 and $v_{\lambda}(x,t)$ and $w_{\lambda}(x,t)$ are C^{∞} functions defined in $R^n\times[-r_0\lambda^{-\kappa}, r_0\lambda^{-\kappa}]$, we see that $f_{\lambda}(x,t)\in C_0^{\infty}(R^{n+1})$. In what follows we shall prove that (2.1) does not hold for f_{λ} as $\lambda\to+\infty$.

By (2.9), (2.10), (2.30) and (2.32) it holds that $w_{\lambda}(x,0)=1$ and $v_{\lambda}(x,0)=\sqrt{-1}\lambda^{-\rho}\sum_{i=1}^{n}\xi_i^0x_i+\lambda^{-\rho-d}|x|^2$. Hence, $\mathrm{grad}_x f_{\lambda}(0,0)=-\sqrt{-1}\lambda^{1-\rho}\xi^0$. Since $1-\rho>0$ by (2.3) and $\xi^0\neq0$, we obtain

$$\lim_{\lambda\to+\infty} |\mathrm{grad}_x f_\lambda(0,0)| = +\infty . \tag{4.2}$$

Let $\eta=(\eta_1, \cdots, \eta_n)$ be a multi-index of non-negative integers and m be a non-negative integer. Using (4.1), (2.23), (2.29), (2.31) and (2.33) we can write with a function $r_\lambda^{\eta,m}\in C_0^\infty(R^{n+1})$

$$\begin{aligned}\partial_x^\eta \partial_t^m (Lf_\lambda) &= \chi_1(\lambda^\mu x)\chi_2\left(\frac{\lambda^\kappa t}{r_1}\right)\partial_x^\eta \partial_t^m (R_\lambda \exp(-\lambda v_\lambda)) \\ &\quad + r_\lambda^{\eta,m}\exp(-\lambda v_\lambda) \\ &\equiv F_\lambda^{\eta,m} + G_\lambda^{\eta,m} .\end{aligned} \tag{4.3}$$

By the Leibniz formula we can write

$$\begin{aligned}&\partial_x^\eta \partial_t^m (R_\lambda \exp(-\lambda v_\lambda)) \\ &= \sum_{k=0}^{|\eta|+m} \sum_{\substack{h_0+h_1+\cdots+h_k=\eta \\ m_0+m_1+\cdots+m_k=m}} C_{k,h_0,\cdots,m_k}^{\eta,m}\, \partial_x^{h_0}\partial_t^{m_0} R_\lambda \times \partial_x^{h_1}\partial_t^{m_1}(-\lambda v_\lambda)\times\cdots\times \\ &\quad \times \partial_x^{h_k}\partial_t^{m_k}(-\lambda v_\lambda)\times\exp(-\lambda v_\lambda) ,\end{aligned}$$

where $C_{k,h_0,\cdots,m_k}^{\eta,m}$ is a positive constant depending only on η, m, k, h_j and m_j $(0\leqq j\leqq k)$. On the other hand, it is clear that $|x|\leqq\lambda^{-\mu}$ and $|t|\leqq r_1\lambda^{-\kappa}$ on supp $F_\lambda^{\eta,m}$ and so by Proposition 3.2 we have

$$-\lambda \operatorname{Re} v_\lambda(x,t)\leqq -\frac{1}{2}\lambda^{1-\rho-d}|x|^2$$

for all sufficiently large $\lambda>0$ and $(x,t)\in\mathrm{supp}\ F_\lambda^{\eta,m}$. Hence by Proposition 3.3 and Proposition 3.4 it holds that

$$\begin{aligned}|F_\lambda^{\eta,m}(x,t)| &\leqq C_1\sum_{k=0}^{|\eta|+m}\lambda^{k+2+2(1-\rho)(|\eta|+m+2)}\times\lambda^{(\mu-(1-\rho-d)/2)(N+1)}\times \\ &\quad\times(1+\lambda^{1-\rho-d}|x|^2)^{2N+1/2+kN/2}\exp\left(-\frac{1}{2}\lambda^{1-\rho-d}|x|^2\right) \\ &\leqq C_2\lambda^{3(|\eta|+m+2)}\times\lambda^{(\mu-(1-\rho-d)/2)(N+1)} ,\end{aligned}$$

for all sufficiently large $\lambda>0$ and $(x,t)\in R^{n+1}$, where C_1 and C_2 are positive constants independent of $\lambda>1$. Hence by (2.8) we have

$$\lim_{\lambda\to+\infty}\sup_{(x,t)\in R^{n+1}}|F_\lambda^{\eta,m}(x,t)| = 0 \quad \text{if} \quad |\eta|+m\leqq M . \tag{4.4}$$

Next we shall estimate $G_\lambda^{\eta,m}$. It is clear that for all $\lambda>1$

$$|x|\leqq\lambda^{-\mu}\leqq 1 \quad\text{and}\quad |t|\leqq r_1\lambda^{-\kappa}<r_0\lambda^{-\kappa} \quad\text{on}\quad \mathrm{supp}\ r_\lambda^{\eta,m}$$

and

$$|x|\geqq\frac{1}{2}\lambda^{-\mu} \quad\text{or}\quad |t|\geqq\frac{r_1}{2}\lambda^{-\kappa} \quad\text{on}\quad \mathrm{supp}\ r_\lambda^{\eta,m} ,$$

since $\mu>0$ by (2.6) and (2.4), and $r_1<r_0$ by Proposition 3.2. Hence by (2.9), (2.10), Lemma 3.1 and Lemma 3.2 there exist constants K_1 and $K_2>0$ independent of $\lambda>1$ such that

$$\sup_{(x,t)\in R^{n+1}} |r_\lambda^{\eta,m}(x,t)| \leqq K_1 \lambda^{K_2} \quad \text{for all} \quad \lambda>1 .$$

On the other hand, letting $\rho_0=\min(1-\rho-d-2\mu,\ 2(1-\rho)-\kappa(q+1))$, we have from Proposition 3.2 that

$$-\lambda \operatorname{Re} v_\lambda(x,t) \leqq -K_3 \lambda^{\rho_0}$$

for all sufficiently large λ and $(x,t)\in\operatorname{supp} r_\lambda^{\eta,m}$ where $K_3>0$ is a constant independent of $\lambda>1$. Hence

$$\sup_{(x,t)\in R^{n+1}} |G_\lambda^{\eta,m}(x,t)| \leqq K_1 \lambda^{K_2} \exp(-K_3 \lambda^{\rho_0}) .$$

Since $\rho_0>0$ by (2.6), (2.5), (2.2) and (2.3), it holds that

$$\lim_{\lambda\to+\infty} \sup_{(x,t)\in R^{n+1}} |G_\lambda^{\eta,m}(x,t)| = 0 . \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$\lim_{\lambda\to+\infty} |Lf_\lambda|_{M,K} = 0 . \tag{4.6}$$

Finally, since $|x| \leqq \lambda^{-\mu}$ and $|t| \leqq r_1 \lambda^{-\kappa}$ on $\operatorname{supp} f_\lambda$, it follows from Proposition 3.2 and Proposition 3.4 that there exists a constant $K_4>0$ independent of $\lambda>1$ such that

$$|f_\lambda(x,t)| \leqq K_4(1+\lambda^{1-\rho-d}|x|^2)^{N/2} \exp\left(-\frac{1}{2}\lambda^{1-\rho-d}|x|^2\right),$$

for all sufficiently large λ and $(x,t)\in R^{n+1}$. Hence

$$|f_\lambda|_{0,K} \quad \text{is bounded as} \quad \lambda\to+\infty . \tag{4.7}$$

(4.2), (4.6) and (4.7) imply that (2.1) does not hold for $f=f_\lambda$ which completes the proof of Theorem 1.1.

5. Two lemmas about real C^∞ functions

In the following two preliminary lemmas for the proof of Theorem 2.2, we shall denote by Ω_0 an open subset in R^{n+1} containing 0.

Lemma 5.1. *Let $f(x,t)$, $(x,t)\in\Omega_0$, be a real C^∞ function and assume that*

(5.1) *there exist $\alpha<0$ and an odd integer $q>0$ such that $f(0,t)=\alpha t^q+O(t^{q+1})$ for sufficiently small t,*

(5.2) *there exist* $\beta \neq 0$, *an integer* $k(1 \leqq k \leqq n)$ *and a non-negative integer* m *such that* $0 \leqq m < (q+1)/2$ *and* $\dfrac{\partial f}{\partial x_k}(0, t) = \beta t^m + O(t^{m+1})$ *for sufficiently small* t.

Then there exists $(x_0, t_0) \in \Omega_0$ *satisfying*

$$f(x_0, t_0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial t}(x_0, t_0) < 0 . \tag{5.3}$$

Proof. Without loss of generality we may suppose that $n=1$. Using the Taylor's formula we expand $f(x, t)$ with respect to x at $x=0$. Then by (5.1) and (5.2) we can write in a sufficiently small neighborhood $V_\delta = \{(x, t) \mid |x| \leqq \delta, |t| \leqq \delta\}$ $(\delta > 0)$ of 0,

$$f(x, t) = \alpha(t) \{t^q + \beta(t) t^m x + \gamma(x, t) x^2\} , \tag{5.4}$$

where

$$\alpha(t) < 0 \quad \text{and} \quad \beta(t) \neq 0 \quad \text{in} \quad |t| \leqq \delta . \tag{5.5}$$

Since $q > 0$ is an odd integer we see from (5.2) that

$$2m+1 \leqq q . \tag{5.6}$$

Put $\theta(t) = -2\beta(t)^{-1} t^{q-m}$, $|t| \leqq \delta$. Then $\theta(t) \in C^\infty(|t| \leqq \delta)$ and $\theta(0) = 0$ by (5.6) and so $|\theta(t)| \leqq \delta$ in $|t| \leqq \varepsilon$ if we take $\varepsilon (0 < \varepsilon < \delta)$ sufficiently small. Therefore if $|t| \leqq \varepsilon$ we can substitute $\theta(t)$ for x in (5.4) and we have

$$f(\theta(t), t) = -\alpha(t) t^q \{1 - 4\gamma(\theta(t), t) \beta(t)^{-2} t^{q-2m}\} , \quad |t| \leqq \varepsilon .$$

Since $q - 2m \geqq 1$ by (5.6), it follows from (5.5) that for sufficiently small $\varepsilon > 0$, $f(\theta(t), t) > 0$ if $0 < t < \varepsilon$. On the other hand, by (5.4) and (5.5), $f(0, t) < 0$ if $0 < t < \varepsilon$. Hence, by the intermediate value theorem, for every $t \in (0, \varepsilon)$ there exists $x(t)$ satisfying

$$f(x(t), t) = 0 \tag{5.7}$$

and

$$|x(t)| < |\theta(t)| = 2|\beta(t)|^{-1} t^{q-m} . \tag{5.8}$$

Differentiating both sides of (5.4) with respect to t, and substituting $x(t)$ for x, we have by (5.5) and (5.7)

$$\frac{\partial f}{\partial t}(x(t), t) = \alpha(t) \{q t^{q-1} + g(t)\} , \quad 0 < t < \varepsilon , \tag{5.9}$$

where $g(t) = \dfrac{d\beta}{dt}(t) t^m x(t) + m\beta(t) t^{m-1} x(t) + \dfrac{\partial \gamma}{\partial t}(x(t), t) x(t)^2$.

By (5.8) there exists a constant $C>0$ such that

$$|g(t)| \leqq Ct^q+2mt^{q-1}+Ct^{2(q-m)}, \quad 0<t<\varepsilon$$

and so, since $\alpha(t)<0$ by (5.5), it follows from (5.9) that

$$\frac{\partial f}{\partial t}(x(t), t) \leqq \alpha(t)\{qt^{q-1}-Ct^q-2mt^{q-1}-Ct^{2(q-m)}\}, \quad 0<t<\varepsilon .$$

Hence, noting that $q-2m \geqq 1$ and $2(q-m)>q$ by (5.6), we have

$$\frac{\partial f}{\partial t}(x(t), t)<0, \quad 0<t<\varepsilon \tag{5.10}$$

with $\varepsilon>0$ sufficiently small.

From (5.7) and (5.10) we obtain (5.3). Q.E.D.

Lemma 5.2. *Let $f(x, t)$ and $g(x, t)$ be real C^∞ functions defined in Ω_0. Assume that*

(5.11) *there exists $\delta>0$ such that $f(x, t)\,g(x, t) \geqq 0$ if $|x| \leqq \delta$ and $|t| \leqq \delta$,*

(5.12) *there exist a real number $\alpha \neq 0$ and an odd integer $q>0$ such that for sufficiently small t*

(i) $f(0, t) = \alpha t^q+O(t^{q+1})$,

(ii) $g(0, t) = O(t^q)$,

(iii) $|\mathrm{grad}_x f(0, t)| = O(t^{(q+1)/2})$.

Then

$$|\mathrm{grad}_x g(0, t)| = O(t^{(q+1)/2}) . \tag{5.13}$$

Proof. Without loss of generality we may suppose that $n=1$. The proof is by contradiction. So suppose that (5.13) does not hold. Then there exist a real number $\gamma \neq 0$ and an integer m such that

$$\frac{\partial g}{\partial x}(0, t) = \gamma t^m+O(t^{m+1}) \tag{5.14}$$

and

$$0 \leqq m<(q+1)/2 . \tag{5.15}$$

By (5.11) we have in $V_\delta = \{(x, t) \mid |x| \leqq \delta, \ |t| \leqq \delta\}$

$$\begin{aligned} 0 \leqq & \{f(0, t)+\frac{\partial f}{\partial x}(0, t)\,x+O(x^2)\}\{g(0, t)+\frac{\partial g}{\partial x}(0, t)\,x+O(x^2)\} \\ & = f(0, t)\,g(0, t)+f(0, t)\frac{\partial g}{\partial x}(0, t)\,x+f(0, t)\,O(x^2) \end{aligned} \tag{5.16}$$

$$+\frac{\partial f}{\partial x}(0,t)\,g(0,t)\,x+\frac{\partial f}{\partial x}(0,t)\,\frac{\partial g}{\partial x}(0,t)\,x^2+\frac{\partial f}{\partial x}(0,t)\,O(x^3)$$
$$+g(0,t)\,O(x^2)+\frac{\partial g}{\partial x}(0,t)\,O(x^3)+O(x^4)\,.$$

Using (5.12) (ii) we write with a real constant β

$$g(0,t)=\beta t^q+O(t^{q+1})\,. \tag{5.17}$$

Let c be the real number satisfying

$$\alpha\beta+\alpha\gamma c=-1\,. \tag{5.18}$$

This is possible because $\alpha\gamma\neq 0$. Since q is a positive odd integer, we see from (5.15) that

$$2m\leqq q-1 \quad (\text{hence } m+1\leqq q-m) \tag{5.19}$$

and so we can substitute ct^{q-m} for x in (5.16) if we take t sufficiently small.

Then from (5.12) (i), (5.17), (5.12) (iii) and (5.14) it follows that for sufficiently small t

$$\begin{aligned}0\leqq\{&\alpha\beta t^{2q}+O(t^{2q+1})\}+\{\alpha\gamma ct^{2q}+O(t^{2q+1})\}+O(t^{3q-2m})\\&+O(t^{(q+1)/2+2q-m})+O(t^{(q+1)/2+2q-m})+O(t^{(q+1)/2+3q-3m})\\&+O(t^{3q-2m})+O(t^{3q-2m})+O(t^{4q-4m})\,.\end{aligned}$$

By (5.19) it is easy to see that $\min(3q-2m,\ (q+1)/2+2q-m,\ (q+1)/2+3q-3m,\ 4q-4m)\geqq 2q+1$. Hence, using (5.18) we obtain for sufficiently small t

$$0\leqq(\alpha\beta+\alpha\gamma c)\,t^{2q}+O(t^{2q+1})=-t^{2q}+O(t^{2q+1})$$

which is a contradiction. Q.E.D.

6. Proof of Theorem 1.2

At first we state two lemmas due to Zuily [12].

Lemma 6.1. *Let Ω_1 be an open subset in R^n, $n\geqq 1$, and*

$$P_1=\phi(x)\sum_{i,j=1}^{n}\alpha_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j}+\sum_{k=1}^{n}\beta_k(x)\frac{\partial}{\partial x_k}+\gamma(x)\,,\quad x\in\Omega_1$$

be a partial differential operator of order 2 with real coefficients belonging to $C^\infty(\Omega_1)$. Suppose that P_1 is hypoelliptic in Ω_1. Then

$$\sum_{i,j=1}^{n}\alpha_{ij}(x)\frac{\partial\phi}{\partial x_i}(x)\frac{\partial\phi}{\partial x_j}(x)=0\quad\text{if}\quad\phi(x)=0\,.$$

This lemma is due to Theorem II.1 (iii) of [12]. Although in [12], analyticity of the coefficients of P_1 is assumed, we can apply the method of proof there to the C^∞ case without modification.

Lemma 6.2. *Let Ω_2 be an open set in R^{n+1}, $n \geqq 1$, and*

$$P_2 = \frac{\partial}{\partial t} + t\left\{\sum_{i,j=1}^{n} b_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + t\sum_{i=1}^{n} b_{in+1}(x,t)\frac{\partial^2}{\partial x_i \partial t} + b_{n+1n+1}(x,t)\, t^2 \frac{\partial^2}{\partial t^2} + \sum_{i=1}^{n} \beta_i(x,t)\frac{\partial}{\partial x_i}\right\} + \gamma(x,t), \quad (x,t)\in\Omega_2$$

be a partial differential operator of order 2 with real coefficients belonging to $C^\infty(\Omega_2)$. Assume that there exist $(x_0, 0, \tilde{\xi}_0)\in R^n\times R^1\times R^{n+1}$, an open neighborhood V_0 of x_0, $\varepsilon>0$, a conic open neighborhood Γ_0 of $\tilde{\xi}_0$ and a constant $C>0$ such that $V_0\times(-\varepsilon,\varepsilon)\subset\Omega_2$ and

$$\sum_{i,j=1}^{n} b_{ij}(x,t)\,\xi_i\,\xi_j \leqq -C|\xi|^2 \quad \text{for all} \quad (x,t,\tilde{\xi})\in V_0\times(-\varepsilon,\varepsilon)\times\Gamma_0,$$

where $\tilde{\xi}=(\xi_1,\cdots,\xi_n,\xi_{n+1})$. Then P_2 is not hypoelliptic in $V_0\times(-\varepsilon,\varepsilon)$.

We can find a proof of this lemma in p. 117–p. 120 of [12] where we take $l=k=0$ and replace $n-1$ with n.

By Lemma 5.1, Lemma 6.1, Lemma 6.2 and the Hörmander's theorem (H) mentioned in the introduction, we obtain

Lemma 6.3. *Let Ω_3 be an open subset in R^{n+1}, $n\geqq 1$, and*

$$P_3 = \sum_{i,j=1}^{n} p_{ij}(x,t)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{k=1}^{n} q_k(x,t)\frac{\partial}{\partial x_k} + r(x,t) + \frac{\partial}{\partial t}, \quad (x,t)\in\Omega_3,$$

be a partial differential operator of order 2 with real coefficients belonging to $C^\infty(\Omega_3)$. Assume that

(6.1) *P_3 is hypoelliptic in Ω_3,*

(6.2) *if $p_{ij}(x,t)=0$ for all $i,j=1,\cdots,n$, then $q_k(x,t)=0$ for all $k=1,\cdots,n$,*

(6.3) *there exist a real number $\alpha<0$ and an odd integer $q>0$ such that $p_{11}(0,t) = \alpha t^q + O(t^{q+1})$.*

Then

$$|\mathrm{grad}_x\, p_{11}(0,t)| = O(t^{(q+1)/2}). \tag{6.4}$$

Proof. The proof is by contradiction. Assume that (6.4) does not hold.

Then we can apply Lemma 5.1 to $p_{11}(x, t)$ and so there exists $(x_o, t_o)\in\Omega_3$ such that

$$p_{11}(x_o, t_o) = 0 \quad \text{and} \quad \frac{\partial p_{11}}{\partial t}(x_o, t_o)<0\,. \tag{6.5}$$

By (6.1) and the Hörmander's theorem (H)

(6.6) the quadratic form: $\xi = (\xi_1, \cdots, \xi_n) \to \sum_{i,j=1}^{n} p_{ij}(x, t)\,\xi_i\,\xi_j$ is semi-definite for all $(x, t)\in\Omega_3$.

Hence

$$0\leqq p_{ij}(x, t)^2\leqq p_{ii}(x, t)\,p_{jj}(x, t)\,,\quad i, j = 1, \cdots, n\,,\quad (x, t)\in\Omega_3\,. \tag{6.7}$$

Especially we have

$$0\leqq p_{1j}(x, t)^2\leqq p_{11}(x, t)\,p_{jj}(x, t)\,,\quad j = 1, \cdots, n\,,\quad (x, t)\in\Omega_3\,. \tag{6.8}$$

Let ω be an open neighborhood of (x_o, t_o) such that

$$\frac{\partial p_{11}}{\partial t}(x, t)<0\,,\quad (x, t)\in\omega\,. \tag{6.9}$$

From now on in the proof we shall take ω as a sufficiently small neighborhood of (x_o, t_o) if necessary. Let (x', t') be any point in ω satisfying $p_{11}(x', t')=0$. Then by (6.8) $p_{11}(x, t)\,p_{jj}(x, t)$ $(1\leqq j\leqq n)$ attain their minimums at (x', t') and so

$$\begin{aligned} 0 &= \frac{\partial p_{11}}{\partial t}(x', t')\,p_{jj}(x', t')+p_{11}(x', t')\,\frac{\partial p_{jj}}{\partial t}(x', t') \\ &= \frac{\partial p_{11}}{\partial t}(x', t')\,p_{jj}(x', t')\,,\quad j = 1, \cdots, n\,. \end{aligned}$$

Hence $p_{jj}(x', t')=0(1\leqq j\leqq n)$ by (6.9) and it follows from (6.7) that $p_{ij}(x', t')=0$ $(1\leqq i, j\leqq n)$. Furthermore by (6.2) $q_k(x', t')=0(1\leqq k\leqq n)$. Hence by (6.9) and the implicit function theorem we can write with real-valued functions $\alpha_{ij}, \beta_k\in C^\infty(\omega)$ $(1\leqq i, j, k\leqq n)$

$$p_{ij}(x, t) = p_{11}(x, t)\,\alpha_{ij}(x, t)\,,\quad q_k(x, t) = p_{11}(x, t)\,\beta_k(x, t),\ i, j, k = 1, \cdots, n\,,$$
$$(x, t)\in\omega\,.$$

Hence

$$P_3 = p_{11}\sum_{i,j=1}^{n}\alpha_{ij}\,\frac{\partial^2}{\partial x_i\,\partial x_j}+p_{11}\sum_{k=1}^{n}\beta_k\,\frac{\partial}{\partial x_k}+r+\frac{\partial}{\partial t}\,,\quad (x, t)\in\omega\,. \tag{6.10}$$

Since $\alpha_{11}(x, t)=1$ in ω, it follows from (6.6) that

$$\sum_{i,j=1}^{n} \alpha_{ij}(x,t)\,\xi_i\,\xi_j \geqq 0\,, \quad (x,t,\xi)\in\omega\times R^n\,. \tag{6.11}$$

Hence

$$\text{if} \quad \sum_{i,j=1}^{n} \alpha_{ij}(x,t)\,\xi_i\,\xi_j = 0 \quad \text{then} \quad \frac{\partial}{\partial\xi_k}\left(\sum_{i,j=1}^{n} \alpha_{ij}(x,t)\,\xi_i\,\xi_j\right) = 2\sum_{i=1}^{n} \alpha_{ik}(x,t)\,\xi_i = 0\,, \quad k=1,\cdots,n\,. \tag{6.12}$$

On the other hand, from (6.10) and Lemma 6.1 it follows that

$$\text{if} \quad p_{11}(x,t)=0,\ (x,t)\in\omega\,, \quad \text{then} \quad \sum_{i,j=1}^{n} \alpha_{ij}(x,t)\,\frac{\partial p_{11}}{\partial x_i}(x,t)\times \times\frac{\partial p_{11}}{\partial x_j}(x,t) = 0\,. \tag{6.13}$$

Hence by (6.12)

$$\text{if} \quad p_{11}(x,t)=0,\ (x,t)\in\omega\,, \quad \text{then} \quad \sum_{i=1}^{n} \alpha_{ik}(x,t)\,\frac{\partial p_{11}}{\partial x_i}(x,t) = 0\,, \quad k=1,\cdots,n\,. \tag{6.14}$$

It is clear that (6.11) is valid for ξ replaced with $\mathrm{grad}_x\, p_{11}(x,t)$ and so combining this fact with (6.9) and (6.13), and using the implicit function theorem, we see that there exists a real-valued function $a(x,t)\in C^\infty(\omega)$ such that

$$\sum_{i,j=1}^{n} \alpha_{ij}(x,t)\,\frac{\partial p_{11}}{\partial x_i}(x,t)\,\frac{\partial p_{11}}{\partial x_j}(x,t) = p_{11}(x,t)^2\,a(x,t)\,. \tag{6.15}$$

Analogously we see from (6.14) that there exist real-valued functions $b_k(x,t)\in C^\infty(\omega)$ $(1\leqq k\leqq n)$ such that

$$\sum_{i=1}^{n} \alpha_{ik}(x,t)\,\frac{\partial p_{11}}{\partial x_i}(x,t) = p_{11}(x,t)\,b_k(x,t)\,, \quad k=1,\cdots,n\,. \tag{6.16}$$

Now we make the change of variables:

$$\Phi:\begin{cases} y = x \\ s = p_{11}(x,t)\,. \end{cases} \tag{6.17}$$

By (6.5) Φ is a diffeomorphism from ω to an open neighborhood $\tilde{\omega}$ of $(x_o,0)\in R^{n+1}_{y,s}$. From (6.10), (6.15) and (6.16) we see that P_3 is transformed by Φ to the following operator:

$$\tilde{P}_3 = s\sum_{i,j=1}^{n} \alpha_{ij}\left(\frac{\partial}{\partial y_i}+\frac{\partial p_{11}}{\partial x_i}\frac{\partial}{\partial s}\right)\left(\frac{\partial}{\partial y_j}+\frac{\partial p_{11}}{\partial x_j}\frac{\partial}{\partial s}\right) \tag{6.18}$$

$$+s\sum_{k=1}^{n}\beta_k\left(\frac{\partial}{\partial y_k}+\frac{\partial p_{11}}{\partial x_k}\frac{\partial}{\partial s}\right)+\frac{\partial p_{11}}{\partial t}\frac{\partial}{\partial s}+r$$
$$=\left(\frac{\partial p_{11}}{\partial t}+sA\right)\frac{\partial}{\partial s}+s\left\{\sum_{i,j=1}^{n}\alpha_{ij}\frac{\partial^2}{\partial y_i\,\partial y_j}+2s\sum_{j=1}^{n}b_j\frac{\partial^2}{\partial y_j\,\partial s}\right.$$
$$\left.+s^2a\frac{\partial^2}{\partial s^2}+\sum_{k=1}^{n}\beta_k\frac{\partial}{\partial y_k}\right\}+r,\quad (y,s)\in\tilde{\omega},$$

where A is a real-valued function belonging to $C^\infty(\tilde{\omega})$. By (6.9) there exist an open neighborhood $V_0\subset R^n_y$ of x_0 and real numbers ε, $C>0$ such that

$$\left(\frac{\partial p_{11}}{\partial t}+sA\right)^{-1}\leqq -C,\quad (y,s)\in V_0\times(-\varepsilon,\varepsilon)\subset\tilde{\omega}. \tag{6.19}$$

For $\delta>0$ put $\Gamma_\delta=\{\tilde{\xi}\in R^{n+1}\mid \delta\xi_1>|\xi'|\}$ where $\tilde{\xi}=(\xi_1,\cdots,\xi_n,\xi_{n+1})$ and $\xi'=(\xi_2,\cdots,\xi_{n+1})$. Then Γ_δ is a conic open neighborhood of $(1,0,\cdots,0)$. Since $\alpha_{11}(\Phi^{-1}(y,s))=\alpha_{11}(x,t)=1$ in $\tilde{\omega}$, taking V_0 and ε, $\delta>0$ sufficiently small we have

$$\sum_{i,j=1}^{n}\alpha_{ij}\xi_i\xi_j\geqq\frac{1}{2}\xi_1^2\geqq\frac{1}{4}|\tilde{\xi}|^2,\quad (y,s,\tilde{\xi})\in V_0\times(-\varepsilon,\varepsilon)\times\Gamma_\delta. \tag{6.20}$$

Let $R=\left(\frac{\partial p_{11}}{\partial t}+sA\right)^{-1}\tilde{P}_3$, $(y,s)\in V_0\times(-\varepsilon,\varepsilon)$. Then R is hypoelliptic in $V_0\times(-\varepsilon,\varepsilon)$. On the other hand, from (6.18)–(6.20) we easily see that R satisfies all the assumptions of Lemma 6.2 and so R is not hypoelliptic in $V_0\times(-\varepsilon,\varepsilon)$ which is a contradiction. Q.E.D.

Now we begin to prove Theorem 1.2. Suppose that L satisfies the hypothesis (B) of Theorem 1.2 and L is hypoelliptic in some open neighborhood U of 0. We shall show that this yields a contradiction.

Step 1. We shall show that there exist an integer $i_0(1\leqq i_0\leqq n)$, an odd integer $q_0>0$ and a real number $\alpha_0<0$ such that

$$a_{i_0i_0}(x(t),t)=\alpha_0t^{q_0}+O(t^{q_0+1}) \tag{6.21}$$

and

$$a_{ij}(x(t),t)=O(t^{q_0}),\quad i,j=1,\cdots,n, \tag{6.22}$$

where $(x(t),t)$ is the integral curve of the vector field $\sum_{k=1}^{n}b_k(x,t)\,\partial/\partial x_k+\partial/\partial t$ through the origin.

Let $e_k=(0,\cdots,\overset{k}{1},\cdots,0)$ $(1\leqq k\leqq n)$. Since L is hypoelliptic in $U\ni 0$ by hypothesis, it follows from the Hörmander's theorem (H) that for sufficiently small $\delta>0$, the quadratic form: $R^2\ni(w,z)\to a(x(t),t,w\xi^0+ze_k)=w^2a(x(t),t,\xi^0)+2wz\sum_{i=1}^{n}a_{ik}(x(t),t)\xi_i^0+z^2a_{kk}(x(t),t)$ is semi-definite for all t, $|t|\leqq\delta$. Hence

$$0 \leqq a(x(t), t, \xi^0)\, a_{kk}(x(t), t), \quad |t| \leqq \delta, \quad k = 1, \cdots, n. \tag{6.23}$$

Hence from the hypothesis (B) of Theorem 1.2 and (6.23) we see that for every $k(1 \leqq k \leqq n)$, $a_{kk}(x(t), t)=0$ at $t=0$ in infinite order or there exist $\alpha_k<0$ and an odd integer $q_k>0$ such that $a_{kk}(x(t), t)=\alpha_k t^{q_k}+O(t^{q_k+1})$. In the first case we put $q_k=+\infty$. Let $q_0=\min(q_1, \cdots, q_n)$. Again by the Hörmander's theorem (H) the quadratic form: $R^2 \ni (w, z) \to a(x(t), t, we_i+ze_j)=a_{ii}(x(t), t)+2wza_{ij}(x(t), t)+z^2 a_{jj}(x(t), t)$ is semi-definite for $|t| \leqq \delta$, $i, j=1, \cdots, n$, and so we have

$$a_{ij}(x(t), t)^2 \leqq a_{ii}(x(t), t)\, a_{jj}(x(t), t), \quad |t| \leqq \delta, \quad i, j = 1, \cdots, n. \tag{6.24}$$

From the hypothesis (B) of Theorem 1.2 and (6.24) we see that $q_0<+\infty$ and taking i_0 such that $q_{i_0}=q_0$ we obtain (6.21) and (6.22) with $\alpha_0=\alpha_{i_0}$.

Step 2. Renumbering the variables $x_i(1 \leqq i \leqq n)$ we may suppose that $i_0=1$. Then (6.21) can be written as

$$a_{11}(x(t), t) = \alpha_0 t^{q_0}+O(t^{q_0+1}), \tag{6.25}$$

where $\alpha_0<0$ and $q_0>0$ is an odd integer.

Let $(x(y, t), t)=(x_1(y, t), \cdots, x_n(y, t), t)$ be the integral curve of the vector field $\sum_{k=1}^{n} b_k(x, t)\, \partial/\partial x_k+\partial/\partial t$ through $(y, 0)=(y_1, \cdots, y_n, 0)$, i.e., let $(x_1(y, t), \cdots, x_n(y, t))$ be the solution of the ordinary differential equations with y as a parameter:

$$\frac{d}{dt} x_k(t) = b_k(x(t), t), \quad x_k(0) = y_k, \quad k = 1, \cdots, n. \tag{6.26}$$

Now we make the change of variables:

$$\Psi: \begin{cases} x = x(y, s) \\ t = s. \end{cases} \tag{6.27}$$

Since $x(y, 0)=y$ by (6.26), Ψ is a diffeomorphism from an open set $\tilde{V}$, $(0, 0) \in \tilde{V} \subset R^{n+1}_{y,s}$, to an open set V, $(0, 0) \in V \subset U$. Let

$$\Psi^{-1}: \begin{cases} y = y(x, t) \\ s = t. \end{cases} \tag{6.28}$$

be the inverse of Ψ. Since $y_k=y_k(x(y, s), s)$ $(1 \leqq k \leqq n)$ it follows from (6.26) that

$$0 = \frac{\partial}{\partial s}(y_k(x(y, s), s)) = \sum_{i=1}^{n} b_i(x(y, s), s) \frac{\partial y_k}{\partial x_i}(x(y, s), s)+\frac{\partial y_k}{\partial t}(x(y, s), s).$$

Hence in the new variables y and s, L can be written in the form:

$$\tilde{L} = \sum_{i,j=1}^{n} a_{ij}\Big(\sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}\Big)\Big(\sum_{l=1}^{n} \frac{\partial y_l}{\partial x_j}\frac{\partial}{\partial y_l}\Big) + \sum_{i=1}^{n} b_i\Big(\sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}\Big) + c + \sum_{k=1}^{n} \frac{\partial y_k}{\partial t}\frac{\partial}{\partial y_k} + \frac{\partial}{\partial s}$$

$$= \sum_{k,l=1}^{n} \tilde{a}_{kl}(y, s)\frac{\partial^2}{\partial y_k \partial y_l} + \sum_{l=1}^{n} \tilde{b}_l(y, s)\frac{\partial}{\partial y_l} + \tilde{c}(y, s) + \frac{\partial}{\partial s} \tag{6.29}$$

where for $k, l=1, \cdots, n$

$$\tilde{a}_{kl}(y, s) = \sum_{i,j=1}^{n} a_{ij}(x(y, s), s)\frac{\partial y_k}{\partial x_i}(x(y, s), s)\frac{\partial y_l}{\partial x_j}(x(y, s), s), \tag{6.30}$$

$$\tilde{b}_l(y, s) = \sum_{i,j,k=1}^{n} a_{ij}(x(y, s), s)\frac{\partial y_k}{\partial x_i}(x(y, s), s)\frac{\partial}{\partial y_k}\Big(\frac{\partial y_l}{\partial x_j}(x(y, s), s)\Big), \tag{6.31}$$

and

$$\tilde{c}(y, s) = c(x(y, s), s). \tag{6.32}$$

Since the Jacobi matrix $\frac{\partial y}{\partial x}(x(y, s), s)$ is non-singular in $\tilde{V}$, it follows from (6.30) that if $\tilde{a}_{kl}(y, s)=0$ for all $k, l=1, \cdots, n$ then $a_{ij}(x(y, s), s)=0$ for all $i, j=1, \cdots, n$. Hence, also by (6.30)

$$\text{if } \tilde{a}_{kl}(y, s) = 0 \text{ for all } k, l = 1, \cdots, n \text{ then } \tilde{b}_l(y, s) = 0 \text{ for all } l = 1, \cdots, n. \tag{6.33}$$

Step 3. From definition of $x(y, t)$ it is clear that $(x(0, t), t)=(x(t), t)$=the integral curve of the vector field $\sum_{k=1}^{n} b_k(x, t)\,\partial/\partial x_k+\partial/\partial t$ through 0. On the other hand, $\partial y_i/\partial x_j(x, 0)=\delta_{ij}$, $i, j=1, \cdots, n$, since $y(x, 0)=x$ by (6.26) and (6.28). Hence by (6.30)

$$\tilde{a}_{11}(0, s) = \sum_{i,j=1}^{n} a_{ij}(x(s), s)\,(\delta_{1i}+O(s))\,(\delta_{1j}+O(s))$$

and so from (6.25) and (6.22) it follows that

$$\tilde{a}_{11}(0, s) = \alpha_o s^{q_o}+O(s^{q_o+1}), \tag{6.34}$$

where $\alpha_o<0$ and $q_o>0$ is an odd integer. Moreover by (6.22) and (6.30)

$$\tilde{a}_{kl}(0, s) = O(s^{q_o}), \quad k, l = 1, \cdots, n. \tag{6.35}$$

Since $\tilde{L}$ is hypoelliptic in $\tilde{V}$ it follows from Lemma 6.3, (6.33) and (6.34) that

$$|\operatorname{grad}_y \tilde{a}_{11}(0, s)| = O(s^{(q_o+1)/2}) \tag{6.36}$$

and it follows from the Hörmander's theorem (H) that

(6.37) the quadratic form: $\xi \to \tilde{a}(y, s, \xi) = \sum_{k,l=1}^{n} \tilde{a}_{kl}(y, s)\, \xi_k\, \xi_l$ is semi-definite for all $(y, s) \in \tilde{V}$.

Hence

$$0 \leqq \tilde{a}_{11}(y, s)\, \tilde{a}_{ll}(y, s),\ (y, s) \in \tilde{V},\quad l = 1, \cdots, n. \tag{6.38}$$

Then by Lemma 5.2, (6.38), (6.34), (6.35) with $k=l$ and (6.36)

$$|\operatorname{grad}_y \tilde{a}_{ll}(0, s)| = O(s^{(q_0+1)/2}),\quad l = 1, \cdots, n. \tag{6.39}$$

By (6.37) the quadratic form: $R^2 \ni (w, z) \to \tilde{a}(y, s, we_1 + ze_i + ze_j) = w^2\, \tilde{a}_{11}(y, s) + 2wz(\tilde{a}_{1i}(y, s) + \tilde{a}_{1j}(y, s)) + z^2(\tilde{a}_{ii}(y, s) + 2\tilde{a}_{ij}(y, s) + \tilde{a}_{jj}(y, s))$ is semi-definite for all $(y, s) \in \tilde{V}$. Hence

$$0 \leqq \tilde{a}_{11}(y, s)\,(\tilde{a}_{ii}(y, s) + 2\tilde{a}_{ij}(y, s) + \tilde{a}_{jj}(y, s)),\ (y, s) \in \tilde{V},\quad i, j = 1, \cdots, n. \tag{6.40}$$

Hence by Lemma 5.2, (6.40) and (6.34)–(6.36)

$$|\operatorname{grad}_y \tilde{a}_{ii}(0, s) + 2 \times \operatorname{grad}_y \tilde{a}_{ij}(0, s) + \operatorname{grad}_y \tilde{a}_{jj}(0, s)| = O(s^{(q_0+1)/2}),\quad i, j = 1, \cdots, n. \tag{6.41}$$

Combining this with (6.39) we have

$$|\operatorname{grad}_y \tilde{a}_{kl}(0, s)| = O(s^{(q_0+1)/2}),\quad k, l = 1, \cdots, n. \tag{6.42}$$

From (6.31) and (6.30) we see that $\tilde{b}_l(y, s)$ $(1 \leqq l \leqq n)$ are linear combinations of $\tilde{a}_{kj}(y, s)$ $(1 \leqq k, j \leqq n)$, because the Jacobi matrix $\dfrac{\partial y}{\partial x}(x(y, s), s)$ is non-singular. Hence by (6.35) and (6.42) we have

$$\tilde{b}_l(0, s) = O(s^{q_0}),\quad l = 1, \cdots, n \tag{6.43}$$

and

$$|\operatorname{grad}_y \tilde{b}_l(0, s)| = O(s^{(q_0+1)/2}),\quad l = 1, \cdots, n. \tag{6.44}$$

From (6.34), (6.35) and (6.42)–(6.44) we see that $\tilde{L}$ satisfies all the assumptions of Theorem 1.1 with $\xi^0 = (1, 0, \cdots, 0)$, $\alpha = \alpha_0$ and $q = q_0$. Hence $\tilde{L}$ is not hypoelliptic in $\tilde{V}$. On the other hand, it is obvious that $\tilde{L}$ is hypoelliptic in $\tilde{V}$, because L is hypoelliptic in V. This is a contradiction.

7. Proof of Theorem 1.3

Step 1. We shall show that

(7.1) there exists no point $(t_1, \xi_1, t_2, \xi_2) \in (T_1, T_2) \times (R^n \setminus \{0\}) \times (T_1, T_2) \times (R^n \setminus \{0\})$ such that $t_1 < t_2$, $a(x(t_1), t_1, \xi_1) > 0$ and $a(x(t_2), t_2, \xi_2) < 0$.

Suppose that (7.1) does not hold, and let (t_1, ξ_1, t_2, ξ_2) be a point in $(T_1, T_2) \times (R^n \setminus \{0\}) \times (T_1, T_2) \times (R^n \setminus \{0\})$ satisfying

$$t_1 < t_2,\ a(x(t_1), t_1, \xi_1) > 0 \quad \text{and} \quad a(x(t_2), t_2, \xi_2) < 0 . \tag{7.2}$$

Let

$$s_0 = \sup \{ s \in [t_1, T_2) \mid a(x(t), t, \xi_1) \geqq 0 \quad \text{in} \quad [t_1, s] \} . \tag{7.3}$$

Then by (7.2) and definition of s_0

$$t_1 < s_0 \leqq T_2 , \tag{7.4}$$

$$a(x(t), t, \xi_1) \geqq 0 \quad \text{if} \quad t_1 \leqq t < s_0 . \tag{7.5}$$

Since $a(x(t_1), t_1, \xi_1) \neq 0$ by (7.2), and $a(x(t), t, \xi_1)$ is a real analytic function of t in (T_1, T_2), there exists no open subinterval of (T_1, T_2) where $a(x(t), t, \xi_1)$ vanishes identically. Hence, by (7.5), for any t, $t_1 \leqq t < s_0$, there exists a sequence $\{\tau_k\}_{k=1}^{\infty}$ such that $\lim_{k \to \infty} \tau_k = t$ and $a(x(\tau_k), \tau_k, \xi_1) > 0$, $k = 1, 2, \cdots$. Hence by the Hörmander's theorem (H)

$$a(x(\tau_k), \tau_k, \xi) \geqq 0 \qquad \text{for all} \quad \xi \in R^n , \quad k = 1, 2, \cdots .$$

Hence letting $k \to \infty$ we have

$$a(x(t), t, \xi) \geqq 0 \qquad \text{for all} \quad (t, \xi) \in [t_1, s_0) \times R^n$$

and so, since $a(x(t_2), t_2, \xi_2) < 0$ by (7.2), we see that

$$s_0 < t_2 . \tag{7.6}$$

On the other hand, by definition of s_0 and (7.5), for any $\varepsilon > 0$ there exists $t_\varepsilon \in (s_0, s_0 + \varepsilon)$ such that $a(x(t_\varepsilon), t_\varepsilon, \xi_1) < 0$. Combining this with (7.4)–(7.6) and the analyticity of $a(x(t), t, \xi_1)$ with respect to t, we see that there exist $\alpha < 0$ and an odd integer $q > 0$ such that $a(x(t), t, \xi_1) = \alpha (t - s_0)^q + O((t - s_0)^{q+1})$. Then by Theorem 1.2, L is not hypoelliptic in any open neighborhood of $(x(s_0), s_0)$ which contradicts to the hypoellipticity of L in Ω. Thus we have proved (7.1).

Step 2. Supposing that neither (i) nor (ii) hold we shall prove that (iii) is valid. Then there exist $(t_1, \xi_1, t_2, \xi_2) \in (T_1, T_2) \times (R^n \setminus \{0\}) \times (T_1, T_2) \times (R^n \setminus \{0\})$ such that

$$a(x(t_1), t_1, \xi_1) > 0 , \tag{7.7}$$

$$a(x(t_2), t_2, \xi_2) < 0 . \tag{7.8}$$

Hence from (7.1) it follows respectively that

$$a(x(t), t, \xi) \geqq 0 \quad \text{for all} \quad (t, \xi) \in [t_1, T_2) \times R^n , \tag{7.9}$$

$$a(x(t), t, \xi) \leqq 0 \quad \text{for all} \quad (t, \xi) \in (T_1, t_2] \times R^n . \tag{7.10}$$

Let

$$T_0 = \inf \{ s \in (T_1, T_2) \mid a(x(t), t, \xi) \geqq 0 \quad \text{in} \quad [s, T_2) \times R^n \} \tag{7.11}$$

and

$$T_0' = \sup \{ s \in (T_1, T_2) \mid a(x(t), t, \xi) \leqq 0 \quad \text{in} \quad (T_1, s] \times R^n \} . \tag{7.12}$$

From (7.8), (7.9) and definition of T_0 we see that $t_2 < T_0 \leqq t_1$. From (7.7), (7.10) and definition of T_0' we see that $t_2 \leqq T_0' < t_1$. Hence to prove (iii) it suffices to show that $T_0 = T_0'$.

Firstly suppose that $T_0 < T_0'$. Then by definitions of T_0 and T_0', $a(x(t), t, \xi) = 0$ for all $(t, \xi) \in (T_0, T_0') \times R^n$. Hence $a(x(t), t, \xi) = 0$ for all $(t, \xi) \in (T_1, T_2) \times R^n$, because $a(x(t), t, \xi)$ is an analytic function of t. But this contradicts to (7.7), and so $T_0 \geqq T_0'$.

Secondly suppose that $T_0 > T_0'$. Then by definitions of T_0 and T_0', there exist $(\tau_1, \eta_1), (\tau_2, \eta_2) \in (T_1, T_2) \times R^n$ such that

$$T_0' < \tau_2 < \tau_1 < T_0, \ a(x(\tau_1), \tau_1, \eta_1) < 0 \quad \text{and} \quad a(x(\tau_2), \tau_2, \eta_2) > 0 .$$

But this contradicts to (7.1), and so $T_0 \leqq T_0'$. Thus we have proved that $T_0 = T_0'$.

References

[1] T. Akamatsu: *Hypoellipticity of some degenerate parabolic operators,* Proc. Fac. Sci. Tokai Univ. **20** (1985), 13–33.

[2] K. Amano: *Stochastic representation and singularities of solutions of second order equations with semidefinite characteristic form,* Trans. Amer. Math. Soc. **286** (1984), 295–312.

[3] R. Beals and C. Fefferman: *On hypoellipticity of second order operators,* Comm. Partial Differential Equations **1** (1976), 73–85.

[4] B. Helffer and C. Zuily: *Non-hypoellipticité d'une classe d'opérateurs différentiels,* C.R. Acad. Sci. Paris **277** (1973), 1061–1063.

[5] L. Hörmander: *Hypoelliptic second order differential equations,* Acta Math. **119** (1967), 147–171.

[6] Y. Kannai: *An unsolvable hypoelliptic differential operator,* Israel J. Math. **9** (1971), 306–315.

[7] E. Lanconelli: *Un'osservazione sugli operatori differenziali ipoellittici del secondo ordine,* Boll. Un. Mat. Ital. **15** (1978), 470–480.

[8] S. Mizohata: *Hypoellipticité des équations paraboliques,* Bull. Soc. Math. France **85** (1957), 15–50.

[9] O.A. Oleinik and E.V. Radkevič: *On the local smoothness of weak solutions and hypoellipticity of differential second order equations,* Russian Math. Surveys **26** (1971), 139–156.

[10] F. Treves: *Hypoelliptic partial differential equations of principal type with analytic coefficients,* Comm. Pure Appl. Math. **23** (1970), 637–651.

[11] F. Treves: *Hypoelliptic partial differential equations of principal type,* Comm. Pure Appl. Math. **24** (1971), 631–670.

[12] C. Zuily: *Sur l'hypoellipticité des opérateurs différentiels du second ordre a coefficients réels,* J. Math. Pures Appl. (9) **55** (1976), 99–125.

Department of Mathematics
Faculty of Science
Tokai University
Hiratsuka, Kanagawa 259–12
Japan