Bernardi, E.
Osaka J. Math.
25 (1988), 19-31

PROPAGATION OF SINGULARITIES FOR HYPERBOLIC OPERATORS WITH MULTIPLE INVOLUTIVE CHARACTERISTICS

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(Received July 22, 1986)

0. Introduction

In this paper we will give a result on propagation of C^∞ singularities generalizing previous results of R.B. Melrose and G.A. Uhlmann [8]; we consider pseudodifferential operators whose principal symbol vanishes at order $m\geqq 2$ on an involutive manifold. Explicity we shall assume:

(i) Let X be a C^∞ manifold of dimension n and let Σ be a C^∞ closed conic, non radial, involutive submanifold of codimension $d\geqq 2$ in $T^*(X)\setminus\{0\}$, the cotangent bundle minus the zero section.

We therefore have, denoting by ω and $\sigma=d\omega$ the canonical 1 and 2 forms in the symplectic manifold $T^*(X)$, $\gamma\in\Sigma\Rightarrow T_\gamma(\Sigma)^\sigma\subset T_\gamma(\Sigma)$ where with $T_\gamma(\Sigma)^\sigma$ we denote the dual with respect to the bilinear form σ. When Σ is given by $\{\gamma\in T^*(X)\backslash 0\,|\, q_1(\gamma)=\cdots=q_d(\gamma)=0\}$ where $q_j\in C^\infty(T^*(X)\setminus\{0\}), j=1,\cdots,d$ are positevely homogeneous of degree one and for any $\gamma\in\Sigma$, $dq_j(\gamma)$ and $\omega(\gamma)$ are linearly independent one forms, then we have $\{q_i, q_j\}\ (\gamma)=0$ where $\{q_i, q_j\}$ denotes as usual the Poisson bracket between q_i and q_j. Frobenius Theorem then gives that Σ is locally foliated of dimension d by the flow out of the Hamiltonian fields of the q_j. The leaf through $\gamma^0\in\Sigma$, whose tangent space in γ^0 is $T_{\gamma^0}(\Sigma)^\sigma$ will be denoted by F_{γ^0}. Moreover for any $\gamma\in\Sigma$ the bilinear form σ induces an isomorphism

$$J_\sigma\colon\ T_\gamma(T^*(X)\backslash 0)/T_\gamma(\Sigma)\to T^*_\gamma(F_\gamma)\,.$$

(ii) Let $\varphi\in C^\infty(X)$ real valued and $\tilde{\varphi}=\varphi\circ\pi$ where π from $T^*(X)$ to X is the canonical projection.

Let $P(x, D_x)$ be a classical properly supported pseudodifferential operator of order $m+k$ in X, $m\in\boldsymbol{N}$, $k\in\boldsymbol{R}$. Let P_{m+k} be its principal symbol. We assume: P is hyperbolic with respect to the level surfaces of φ([5]), the Hamiltonian field of $\tilde{\varphi}$, $H_{\tilde{\varphi}}$ is transversal to Σ and P_{m+k} vanishes exactly of order m on Σ.

(iii) (Microlocal Levi Condition) ([9])

Microlocally near every point $\gamma^0\in\Sigma$ in a neighborhood of which Σ is given as in (i):

$$P(x, D_x) \equiv \sum_{|\alpha| \leq m} A_\alpha(x, D_x) Q_1^{\alpha_1}(x, D_x) \cdots Q_d^{\alpha_d}(x, D_x) \tag{0.1}$$

where $Q_1, \cdots, Q_d$ are first order pseudodifferential operators with principal symbol $q_1, \cdots, q_d$, and A_α are pseudodifferential operators of order k. Here $A \equiv B$ if there exists $\Gamma \ni \gamma^0$ such that for any $v \in \mathcal{E}'(X)$ $WF(v) \subset \Gamma \Rightarrow (A-B) v \in C^\infty(X)$.
It is well known that P induces on F_{γ^0} a differential operator P^0 homogeneous of order m in the fibers of $T^*(F_{\gamma^0})$: for its principal symbol one has:

$$P^0_{\gamma^0;m}(v) = \lim_{t \to 0} t^{-m} P_{m+k}(\gamma^0 + tv) \tag{0.2}$$

P^4 is hyperbolic with respect to $J_\sigma(H_\varphi(\gamma^0)) = N(\gamma^0)$. Finally we shall assume here that:
(iv) for any $\gamma \in \Sigma$, P^0_γ is strictly hyperbolic with respect to $N(\gamma)$.
Now denoting by Γ_γ the component of $N(\gamma)$ in the complement of $\{v \in T^*(F_\gamma) | P^0_\gamma(v) = 0\}$ and by $(\Gamma_\gamma)^0$ the (euclidean) polar of Γ_γ, ([2]) let $E^+(\gamma)$ $(E^-(\gamma))$ be the forward (backward) emission from γ along the field of cones $(\Gamma_\gamma)^0$, cfr. (4.11). Then the result of our paper is given in the following:

Theorem. *Let P satisfy assumption* (i)-(iv). *Let $v \in \mathcal{D}'(X)$ and $\gamma^0 \in \Sigma \setminus WF(Pv)$. If there exists a conic neiborhood Γ of γ^0, and a choice of sign $+$ or $-$ such that*:

$$\Gamma \cap WF(v) \cap (E^\pm(\gamma^0) \setminus (\gamma^0)) = \phi$$

Then γ^0 does not belong to $WF(v)$.

REMARKS.

(i) R.B. Melrose and G.A. Uhlmann proved the theorem when $m=2$ (and $d \geq 3$: if $d=2$ see [10] for the construction of a microlocal parametrix). In that case assumption (iii) reduces to the Levi condition that the subprincipal symbol of P vanishes on Σ ([3], [6]). Always in the case of double characteristics similar results have been obtained by R. Lascar [7] and Ivrii [6]. (Ivrii's results which are proved by means of microlocal energy estimates are more general for Σ may have symplectic components also, see however [1] for a precise formulation of some results). In the case of $m \geq 2$ and involutive characteristics only, assuming an ellipticity and a Calderon's type uniqueness condition for the operator induced on the leaf J. Sjostrand [9] proved that in general there is propagation in any direction on the leaf. The results in [7] are proved essentialy by means of Carleman estimates on the leaf and techniques formerly developed in [9]. For the construction of a parametrix in case of multiple characteristics (of constant multiplicity) see also the work of Chazarain [12].

(ii) Here we shall prove the theorem by constructing as in [8] a microlocal parametrix for the operator and the diffusion result will be clear from direct inspection. We want to point out that under assumptions (i)-(iv) P behaves like a principal type operator outside Σ and moreover null bicharacteristics starting outside Σ do not have limit points on Σ ([6]). As in [8] simple examples of operators satisfying (i)-(iv) are provided by taking $X=X_1\times X_2$, dim $X_1=d$ and P a strictly hyperbolic operator of order m in X_1 extended trivially in X. As in [8] however this is not a microlocal model of the general case.

(iii) The proof is given in four steps: 1) we reduce the operator to a standard simpler form using the invariance under canonical transformation and conjugation via Fourier Integral Operator of the assumptions; 2) we solve Hamilton-Jacobi equations in polar coordinates; 3) we construct a microlocal parametrix for the Cauchy problem; 4) we finally compute the WF's and conclude.

It is a pleasure to thank A. Bove and C. Parenti for useful conversations and helpful encouragement.

1. Some preparations

Let $\gamma^0\in\Sigma=\{\gamma\in T^*(X)\backslash 0 \mid q_1(\gamma)=\cdots=q_d(\gamma)=0\}$ as in assumption (i). Then $T_\gamma(\Sigma)=[H_{q_1}(\gamma),\cdots,H_{q_d}(\gamma)]^\sigma$. By (ii) there exists $j\in\{1,\cdots,d\}$ such that $\sigma(H_{q_j}(\gamma),H_\varphi(\gamma))\neq 0$. Let us consider $\Sigma'=\{\gamma\in T^*(X)\backslash 0 \mid q_1(\gamma)=\cdots=q_d(\gamma)=0,\ \varphi(\gamma)=0\}$. We have $\operatorname{rank}(\sigma)_{|\Sigma'}=2n-d-1-\dim\operatorname{Ker}(M)$, where M is the matrix:

$$\begin{pmatrix} \{q_i,q_j\}_{|\Sigma',i,j=1,\cdots,d}; & \{q_i,\varphi\}_{|\Sigma',i=1,\cdots,d} \\ \{q_i,\varphi\}_{|\Sigma',i=1\cdots,d} & 0 \end{pmatrix}$$

It is obvious that M has rank 2, therefore rank $(\sigma)_{|\Sigma'}=2(n-d)$. Then ([5] Th. 21.2.4) there is a canonical homogeneous transformation sending Σ' in $\{(x,\xi)\in T^*(\boldsymbol{R}^n)\backslash\{0\} \mid \xi_{n-d+1}=\cdots=\xi_n=x_n=0\}$. Since Σ is involutive, it is sent into $\{(x,\xi)\in T^*(\boldsymbol{R}^n)\backslash\{0\} \mid \xi_{n-d+1}=\cdots=\xi_n=0\}$ and φ in these new canonical coordinates is sent to x_n. Setting $\boldsymbol{R}^n\ni x=(x',x'',x_n)\in\boldsymbol{R}^{n-d}\times\boldsymbol{R}^{d-1}\times\boldsymbol{R}$ and $x'''=(x'',x_n)$, Σ is then given by $\xi'''=0$. Let F be a Fourier Integral operator elliptic in γ^0, of order zero and such that, with $D_j=D_{xj}$ $F^{-1}Q_jF=D_{j+n-d}+r_j(x,D_x)$, $j=1,\cdots,d$ with r_j of order zero. Assumption (iii) and Lemma 0.1 in the first chapter of [7] now give that microlocally near $\gamma^0=(x=0;\xi'=(0,\cdots,0,1),\xi''=0,\xi_n=0)$:

$$P(x,D_x)\equiv\sum_{|\alpha|\leq m}A_\alpha(x,D_x)\,D_{n-d+1}^{\alpha_1}\cdots D_n^{\alpha_d}$$

where now A_α are pseudodifferential operators of order zero. After composition with a pseudodifferential operator of order zero in view of the hyperbolicity of P we can assume that the complete symbol of P is given by:

$$p(x,\xi)=\xi_n^m+\sum_0^m{}_j\Big[\sum_{|\alpha''|\leq j}{}_{\alpha'',j}A^0_{\alpha'',j}(x,\xi)\,(\xi'')^{\alpha''}\Big]\,\xi_n^{m-j} \tag{1.1}$$

Now we have $\partial_{\xi_n}^h p_m(\delta^0)=\delta_{hm}\, m!$ where δ_{hm} is the Kronecker symbol and $h\in\{1, \cdots, m\}$. So by using a pseudodifferential version of the Malgrange preparation theorem:

$$P(x, D_x)\cong Q(x, D_x)\, [D_n^m+\sum_1^m{}_j\, E_j(x, D_x, D_{x''})\, D_n^{m-j}] \tag{1.2}$$

in a conic neighborhood of γ^0 and Q elliptic at γ^0. Comparison of (1.1) with (1.2) and composition with a parametrix of Q finally gives that near γ^0 with different $A_{\alpha,j}$:

$$p(x, \xi) = \xi_n^m+\sum_1^m{}_j\, [\sum_{|\alpha''|\leq j} A_{\alpha'',j}(x, \xi', \xi'')\, (\xi'')^{\alpha''}]\, \xi_n^{m-j}\,. \tag{1.3}$$

In these coordinates the leaf F_{γ^0} through $\gamma^0\in\Sigma$, $\gamma^0=(x^0, \xi'^0, \xi''=0, \xi_n=0)$ is given by:

$$F_{\gamma^0}= \{(x, \xi)\in T^*(\boldsymbol{R}^n)\setminus\{0\}\mid x' = x'^0, \xi' = \xi'^0, \xi'' = 0, \xi_n = 0\}\ .$$

The principal symbol of the operator P^0 is:

$$p^0_{(x'^0,\xi^0)}(x''', \xi'', \xi_n) = \xi_n^m+\sum_1^m{}_j\, [\sum_{|\alpha''|=j} A^0_{\alpha'',j}(x'^0, x''', \xi'^0, 0)\, (\xi'')^{\alpha''}]\, \xi_n^{m-j}\,. \tag{1.4}$$

Therefore assumption (iv) requires that $p^0_{(x',\xi)}(x''', \xi'', \xi_n)=0$ has m real distinct roots ξ_n.

Let us now study the local structure of Char $(P)\setminus\Sigma$. We shall assume $d>2$. We introduce polar coordinates near Σ: $\xi''=\rho\omega$, $\rho\in[0, +\infty[$, $\omega\in S^{d-2}$. The principal symbols of P and P^0 are then given by:

$$p_m = \xi_n^m+\sum_1^m{}_j\, [\sum_{|\alpha''|=j} A^0_{d'',j}(x, \xi', \rho\omega)\, (\omega)^{\alpha''}]\, \rho^j\xi_n^{m-j} \tag{1.5}$$

$$p_m^0 = \xi_n^m+\sum_1^m{}_j\, [\sum_{|\alpha''|=j} A^0_{\alpha'',j}(x, \xi', 0)\, (\omega)^{\alpha''}]\, \rho^j\xi_n^{m-j} \tag{1.6}$$

Let us blow up again singularities at $\xi_n=\rho=0$, $u=\xi_n/\rho$:

$$p_m = u^m+\sum_1^m{}_j\, [\sum_{|\alpha''|=j} A^0_{\alpha'',j}(x, \xi', \rho\omega)\, (\omega)^{\alpha''}]\, u^{m-j} \tag{1.5$'$}$$

$$p_m^0 = u^m+\sum_1^m{}_j\, [\sum_{|\alpha''|=j} A^0_{\alpha'',j}(x, \xi', 0)\, (\omega)^{\alpha''}]\, u^{m-j} \tag{1.6$'$}$$

By Rouche's Theorem and assumption (iv) we have that $p_m=0$ has for positive and sufficiently small ρm real zeros $\mathrm{u}_h=\rho u_h$ and:

$$u_h\neq u_k \qquad \text{if} \quad h\neq k\,. \tag{1.7}$$

This shows that in Char $(P)\backslash\Sigma$ near Σ, P is of principal type and Char $(P)\backslash\Sigma$ has m local components intersecting over Σ. Moreover p_m is there factorized as:

$$(1.8) \qquad p_m = \prod_1^m {}_h\, q_h \text{ where } q_h = \xi_n - |\xi''|\, u_h(x, \xi', \xi'') .$$

By considering the Hamilton systems for one of these factors one gets from Gronwall's Lemma:

$$(1.9) \qquad |\xi''(\gamma_1)| \leqq M(\gamma_1, \gamma_2)\, |\xi''(\gamma_2)| .$$

γ_1, γ_2 belonging to the same null bicharacteristic of p_m. This proves that the simple Hamiltonian flow in Char $(P)\backslash\Sigma$ has no limit point in Σ (see [6], Proposition 0.3, (ii)).
Finally the case $d=2$ is treated in the same way with $\omega=\pm 1$. Moreover in the following we will always deal with $d>2$, leaving the trivial extensions $d=2$ to the reader.

2. The eikonal equation

As in [8] and already in (1.5), (1.6) we introduce polar coordinates taking near Σ: $\xi''=\rho\omega$, $\rho\in[0, +\infty[$, $\omega\in S^{d-2}$. We want to solve:

$$(2.1) \qquad P_m(x, \nabla_x\varphi(x)) = 0 , \quad \varphi(x', x'', 0; \omega, \rho, \eta') = \rho\langle\omega, x''\rangle + \langle\eta', x'\rangle$$

In order to use Hamilton-Jacoby theory let us look for φ of the form:

$$(2.2) \qquad \varphi(y', y'', y_n; \omega, \rho, \eta') = \langle\eta', y'\rangle + \rho\psi(y, \omega, \rho, \eta')$$

ψ homogeneous of degree zero in (ρ, ω). Then (2.1) goes into:

$$(2.3) \qquad (\partial_n\psi)^m + \sum_1^m {}_j \Big[\sum_{|\alpha''|=j} A^0_{\alpha'',j}(y, \eta'+\rho\nabla_{y'}\psi, \rho\nabla_{y''}\psi)\,(\nabla_{y''}\psi)^{\alpha''}\Big]\,(\partial_n \psi)^{m-j} = 0$$
$$\psi(y', y'', 0; \omega, \rho, \eta') = \langle\omega, y''\rangle$$

Let us denote by q_m the Hamiltonian function in (2.3):

$$q_m(y', y'', y_n, \xi', \xi'', \xi_n, \rho, \eta') =$$
$$(\xi_n)^m + \sum_1^m {}_j \Big[\sum_{|\alpha''|=j} A^0_{\alpha'',j}(y, \eta'+\rho\xi', \rho\xi'')\,(\xi'')^{\alpha''}\Big]\,(\xi_n)^{m-j}$$

Therefore:

$$q_m(y', y'', y_n, \xi', \xi'', \xi_n, 0, \eta') = p^0_{m,(y',\eta')}(y'', y_n, \xi'', \xi_n)$$

Now the equation:

$$0 = q_m(0, 0, 0, \xi', \xi'' = \omega, \xi_n, 0, \eta') =$$
$$(\xi_n)^m + \sum_1^m {}_j [\sum_{|\alpha''|=j} A^0_{\alpha'',j}(0, \eta', 0)\,(\omega)^{\alpha''}]\,(\xi_n)^{m-j} = p^0_{m,(0,\eta')}(0, \omega, \xi_n)$$

has m real distinct roots ξ_n as $\omega \neq 0$. If $0<\rho$ is sufficiently small then $q_m(y', y'', y_n, \xi', \xi'', \xi_n, \rho, \eta')=0$ has m real distinct roots $(\xi_n)^h$, $h=1, \cdots, m$ by Rouchè's Theorem and the hyperbolicity assumption. By theorem 6.4.5 in [4] there exists m functions $\psi_h(y, \omega, \rho, \eta')$ C^∞ in a conic neighborhood of $(y'=0, y''=0, y_n=0, \rho=0, \omega, \eta=(0, \cdots, 1))$ such that:

$$(2.3)' \qquad \begin{aligned} &q_m(y', y'', y_n, \nabla_{y'}\psi_h, \nabla_{y''}\psi_h, \partial_n\psi_h, \rho, \eta') = 0\\ &\psi_h(y', y'', 0; \omega, \rho, \eta') = \langle\omega, y''\rangle\\ &\partial_n\psi_h(0, 0, 0; \omega, \rho, \eta') = (\xi_n)^h(\omega, \rho, \eta'), h = 1, \cdots, m \end{aligned}$$

From $(2.3)'$ if $|y|<\delta$, $\rho<\delta|\eta'|$, $|\eta'-(0, \cdots, 1)|\eta'||<\delta|\eta'|$, $\omega\in S^{d-2}$ and $0<\delta$ sufficiently small we have m C^∞ functions $\varphi_h(y, \omega, \rho, \eta')=\langle\eta', y'\rangle+\rho\psi_h(y, \omega, \rho, \eta')$ solutions of equation (2.1).

3. Microlocal Cauchy problem

In this section we want to solve the following microlocal Cauchy problem:

$$(3.1) \qquad \begin{aligned} Pv &= D_n^m v + \sum_1^m {}_j [\sum_{|\alpha''|\leq j} A_{\alpha'',j}(x, D_{x'}, D_{x''})\,D_{x''}^{\alpha''}]\,D_n^{m-j}(v) \equiv 0\\ D_n^h v(x', x'', x_n &= 0) \equiv \delta_{h,m-1}\,\delta(x', x'') \quad h\in\{0, \cdots, m-1\} \end{aligned}$$

microlocally near $\gamma^0=(x=0; \xi'=(0, \cdots, 0, 1), \xi''=0, \xi_n=0)$, where $\delta_{h,m-1}$ denotes the Kronecker symbol.
Let us look for v as a sum of oscillotary integrals:

$$(3.2) \qquad v = \sum_1^m {}_j\, I_{\varphi_j}(a_j)$$

where $I_{\varphi_j}(a_j)\,(x)=\int_{S^{d-2}}\int_0^{+\infty}\int_{R^{n-d}} \exp(i\varphi_j(x, \omega, \rho, \eta'))\,a_j(x, \omega, \rho, \eta')\,d\eta'\,d\rho d\omega$ the φ_j's are the phase functions found in section 2 and a_j are classical symbols to be determined.
Let us recall that:

$$(3.4) \qquad e^{-i\varphi}P(e^{+i\varphi}a_j) =$$
$$\sum_{\alpha\geqslant 0} 1/\alpha!\,\partial_\xi^\alpha P(x, \nabla_x\varphi)\,D_z^\alpha\{\exp(i\varphi_2(x, \omega, \rho, \eta', z))\,a_j(z, \omega, \rho, \eta')\}\,|_{x=z}$$

where

$$\varphi_2(x, \omega, \rho, \eta', z)=\varphi(z, \omega, \rho, \eta')-\varphi(x, \omega, \rho, \eta')+\langle x-z, \nabla_x\varphi(x, \omega, \rho, \eta')\rangle.$$

Now:

$$(3.5) \qquad Pv = \sum_1^m {}_j I_{\varphi_j}(b_j)$$

with

$$\begin{aligned}(3.6) \quad b_j = {} & P_m(x, \nabla_x\varphi_j)\, a_j + P_{m-1}(x, \nabla_x\varphi_j)\, a_j + \cdots + \\ & \sum_1^n {}_h\, \partial_{\xi_h} P_m(x, \nabla_x\varphi_j)\, D_{x_h} a_j + \sum_1^n {}_h\, \partial_{\xi_h} P_{m-1}(x, \nabla_x\varphi_j)\, D_{x_h} a_j + \cdots + \\ & (-i/2) \sum_1^n {}_{hk}\, \partial^2_{\xi_h\xi_k} P_m(x, \nabla_x\varphi_j)\, (\partial^2_{x_h x_k}\varphi_j)\, a_j + \\ & (1/2) \sum_1^n {}_{hk}\, \partial^2_{\xi_k\xi_k} P_m(x, \nabla_x\varphi_j)\, D^2_{x_h x_k} a_j + \\ & (-i/2) \sum_1^\infty {}_t \sum_1^n {}_{hk}\, \partial^2_{\xi_h\xi_k} P_{m-t}(x, \nabla_x\varphi_j)\, (\partial_{x_h x_k}\varphi_j)\, a_j + \\ & \quad + (1/2) \sum_1^\infty {}_t \sum_1^n {}_{hk}\, \partial^2_{\xi_h\xi_k} P_{m-t}(x, \nabla_x\varphi_j)\, D^2_{x_h x_k} a_j + \\ & \quad + \sum_{|\alpha|\geqq 3} {}_\alpha (1/\alpha!)\, \partial^\alpha_\xi P(x, \nabla_x\varphi_j)\, D^\alpha_z \{\exp(i\varphi_{2,j}(x, \omega, \rho, \eta', z)) \\ & \quad\; a_j(z, \omega, \rho, \eta')\}_{|x=z}\,.\end{aligned}$$

Since φ_j solves equation (2.1) we have:

$$\begin{aligned}(3.7) \quad b_j = {} & \sum_1^n {}_h\, \partial_{\xi_k} P_m(x, \nabla_x\varphi_j)\, D_{x_h} a_j + \\ & (P_{m-1}(x, \nabla_x\varphi_j) - (+i/2) \sum_1^n {}_{hk}\, \partial^2_{\xi_h\xi_k} P_m(x, \nabla_x\varphi_j)\, \partial^2_{x_h x_k}\varphi_j)\, a_j + R_j(a_j)\end{aligned}$$

with $R_j(a_j)$ easily determined from (3.6). Now from (1.3) and the form of φ_j, setting:

$$(3.8) \qquad \alpha_{hj} = \partial_{\xi_h} P_m(x, \nabla_x\varphi_j(x, \omega, \rho, \eta'))\, j = 1, \cdots, m;\ h = 1, \cdots, n$$

we have:

$$(3.9) \qquad \alpha_{nj} = \rho^{m-1}\, \partial_{\xi_n} q_m(y', y'', y_n, \nabla_{y'}\psi_j, \nabla_{y''}\psi_j, \partial_n\psi_j, \rho, \eta')$$

From the discussion in section 2 $\alpha_{nj} = \rho^{m-1}\tilde{\alpha}_{nj}$ with $\tilde{\alpha}_{nj}(x, \omega, \rho, \eta') \neq 0$ in a conic neighborhood of $\gamma^0 \in \Sigma$ for every $j = 1, \cdots, m$.
If $1 \leqq h \leqq n-d$:

$$(3.10) \quad \alpha_{hj} = \sum_1^m {}_t \Big[\sum_{|\alpha''|=t} \partial_{\xi_h} A^0_{\alpha'',t}(y, \eta' + \rho\nabla_{y'}\psi_j, \rho\nabla_{y''}\psi_j)\, (\nabla_{y''}\psi_j)^{\alpha''} \Big] (\partial_n\psi_j)^{m-t} \rho^m$$

If $n-d+1 \leqq h \leqq n-1$ then $\alpha_{hj} = \rho^{m-1}\tilde{\alpha}_{hj}$ where $\tilde{\alpha}_{hj}$ is a similar although slightly more involved expression as (3.10). Now:

$$\begin{aligned}(3.11) \qquad & P_{m-1}(x, \nabla_x\varphi_j) - (-i/2) \sum_1^n {}_{hk}\, \partial^2_{\xi_h\xi_k} P_m(x, \nabla_x\varphi_j)\, \partial^2_{x_h x_k}\varphi_j = \\ & \quad = \rho^{m-1}\, \tilde{b}_j(x, \omega, \rho, \eta')\end{aligned}$$

which follows from an easy but tedious calculation. Therefore if $Pv\equiv 0$ then b_j has to be ~ 0. From (3.6) we get:

$$R_j=\sum_{2}^{\infty}{}_t\,\rho^{m-t}R_{j,t}(x,D_x,\rho) \tag{3.12}$$

Let L_j be the first order differential operator:

$$L_j=\tilde{\alpha}_{nj}D_{x_n}+\sum_{1}^{n-1}{}_h\,\tilde{\alpha}_{hj}D_{x_h}+\tilde{b}_j \tag{3.13}$$

The transport equations $b_j\sim 0$ then become:

$$L_j(a_j)+\sum_{2}^{\infty}{}_t\,\rho^{1-t}R_{j,t}(x,D_x,\rho)\,a_j\cong 0 \tag{3.14}$$

Let us consider the initial conditions in (3.1). First recall that all the φ_j coincide at $x_n=0$ with $\varphi_0(y',y'',\omega,\rho,\eta')=\langle\eta',y'\rangle+\rho\langle\omega,y''\rangle$. Moreover as in [8] microlocally near $\gamma^0=(x=0;\ \xi'=(0,\cdots,0,1),\ \xi''=0,\ \xi_n=0)$ the Dirac delta is represented by:

$$\begin{aligned}(3.15)\quad \delta(x',x'')\equiv v_0(x',x'')&=\\ &=1/(2\pi)^{n-1}\int_{S^{d-2}}\int_0^{+\infty}\int_{\boldsymbol{R}^{n-d}}\exp(i\varphi_0(x',x'',\omega,\rho,\eta')\,\rho^{d-2}\sigma_1(\rho/\delta|\eta'|)\\ &\quad d\eta'\,d\rho d\omega\end{aligned}$$

with $\sigma_1\in C^\infty(\boldsymbol{R})$, $\sigma_1(t)=1$ if $|t|\leqq 1/2$, $\sigma_1(t)=0$ if $|t|\geqq 1$.
Then we obtain:

$$\begin{aligned}(3.16)\qquad &\exp(i\varphi_0)\,(a_1+\cdots+a_m)|_{x_n=0}=0\\ &\exp(i\varphi_0)\,((\partial_{x_n}a_1+\cdots+\partial_{x_n}a_m)|_{x_n=0}+(\rho D_{x_n}\psi_1)|_{x_n=0}\\ &\qquad a_1|_{x_n=0}+\cdots+(\rho D_{x_n}\psi_m)|_{x_n=0}\,a_m|_{x_n=0}=0\end{aligned}$$

$\exp(i\varphi_0)$ $[(\rho D_{x_n}\psi_1)^{m-1}|_{x_n=0}\,a_1|_{x_n=0}+\cdots+(\rho D_{x_n}\psi_m)^{m-1}|_{x_n=0}\,a_m|_{x_n=0}+$terms with powers of ρ strictly lower than $m-1]=\exp(i\varphi_0)\,1/(2\pi)^{n-1}\rho^{d-2}\sigma_1(\rho/\delta|\eta'|)$
The last equation suggests that a_j should be of the form:

$$a_j(x,\omega,\rho,\eta')=\sum_{0}^{d-m-1}{}_k\,\rho^k a_{jk}(x,\omega,\rho,\eta')+\sum_{1}^{\infty}{}_k\,\rho^{-k}b_{jk}(x,\omega,\rho,\eta') \tag{3.17}$$

if $d-m-1>0$ which will be the case treated first a_{jk} and b_{jk} here are homogeneous of degree zero in (ρ,η').
From (3.15) we have:

$$\begin{bmatrix}1\cdots & & \cdots 1\\ (D_{x_n}\psi_1)|_{x_n=0} & \cdots & (D_{x_n}\psi_m)|_{x_n=0}\\ \cdots\cdots & & \cdots\cdots\\ (D_{x_n}\psi_1)^{m-1}|_{x_n=0} & \cdots & (D_{x_n}\psi_m)^{m-1}|_{x_n=0}\end{bmatrix}\cdot\begin{bmatrix}a_{1d-m-1}|_{x_n=0}\\ \vdots\\ a_{md-m-1}|_{x_n=0}\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 1/(2\pi)^{n-1}\sigma_1\end{bmatrix} \tag{3.18}$$

Since $\partial_n\psi|_{x_n=0}$ is the j-th root of $q_m(0, 0, 0, \xi', \xi''=\omega, \xi_n, \rho, \eta')=0$ and for $j\in\{1, \cdots, m\}$ all these roots are distinct, the linear system has a unique solution giving initial data at $x_n=0$ for a_{jd-m-1}. On the other hand by ordering (3.13) according to descending powers of ρ we have:

$$(3.19) \qquad \begin{aligned} L_j(a_{jd-m-1}) &= 0 \\ a_{jd-m-1}|_{x_n=0} &= \text{data} \end{aligned}$$

As $\tilde{\alpha}_{nj}\neq 0$ from (3.9), (3.19) has a (local) unique C^∞ solution a_{jd-m-1}, $j\in\{1, \cdots, m\}$.

Let us solve for a_{jp} with $p<d-m-1$. (3.13) then yields:

$$(3.20) \qquad \begin{aligned} 0\cong{}&\rho^{d-m-1} L_j(a_{jd-m-1})+\rho^{d-m-2}(L_j(a_{jd-m-2})+R_{j,2}(a_{jd-m-1}))+ \\ &\rho^{d-m-3}(L_j(a_{jd-m-3})+R_{j,3}(a_{jd-m-1})+R_{j,2}(a_{jd-m-2})+\cdots+ \\ &\rho(L_j(a_{j1})+\sum_{2}^{d-m-1}{}_k R_{j,k}(a_{jk}))+L_j(a_{j0})+\sum_{1}^{d-m-1}{}_k R_{j,k+1}(a_{jk})+ \\ &\rho^{-1}(L_j(b_{j1})+\sum_{0}^{d-m-1}{}_k R_{j,k+2}(a_{jk}))+ \\ &\rho^{-2}(L_j(b_{j2})+\sum_{0}^{d-m-1}{}_k R_{j,k+3}(a_{jk})+R_{j,2}(b_{j1}))+\cdots \end{aligned}$$

From (3.16) we have, with V_{hj} the element of place (h, j) in the matrix (3.18):

$$(3.21) \qquad \rho^h \sum_{1}^{m}{}_j V_{hj}\, a_j+\sum_{1}^{h}{}_t \rho^{h-t}\, S_{t,h}(a_j)\cong\delta_{hm-1}\, 1/(2\pi)^{n-1}\, \rho^{d-2}\, \sigma_1(\rho/\delta\,|\eta'|)$$

where $S_{t,h}$ are differential operators with coefficients homogeneous of degree zero in (ρ, η'). Inserting (3.17) in (3.21):

$$(3.22) \qquad \begin{aligned} &\sum_{0}^{d-m-1}{}_k \rho^{h+k}(\sum_{1}^{m}{}_j V_{hj}\, a_{jk})+\sum_{1}^{\infty}{}_k \rho^{h-k}(\sum_{1}^{m}{}_j V_{hj}\, b_{jk})+\sum_{0}^{d-m-1}{}_k \sum_{1}^{h}{}_t \rho^{h+k-t}\, S_{t,h}(a_{jk})+ \\ &\sum_{1}^{\infty}{}_k \sum_{1}^{h}{}_t \rho^{h-k-t}\, S_{t,h}(b_{jk})\cong\delta_{hm-1}\, 1/(2\pi)^{n-1}\, \rho^{d-2}\, \sigma_1(\rho/\delta\,|\eta'|) \end{aligned}$$

For instance let us find the initial conditions for a_{jd-m-2}:

$$(3.23) \qquad \rho^{h+d-m-2}(\sum_{1}^{m}{}_j V_{hj}\, a_{jd-m-2}+S_{1,h}(a_{jd-m-1})) = 0 \text{ at } x_r = 0,\ h\in\{0, \cdots, m-1\}.$$

(3.23) together with:

$$(3.24) \qquad L_j(a_{jd-m-2})+R_{j,2}(a_{jd-m-1}) = 0$$

enables us to find a (local) C^∞ solution a_{jd-m-2}, $j\in\{1, \cdots, m\}$. And so on for all a_{jp} $p\in\{1, \cdots, d-m-1\}$.

As far as b_{j1} is concerned:

$$(3.24) \qquad \rho^{h-1}(\sum_{1}^{m}{}_j V_{hj}\, b_{j1})+\rho^{h-1}\sum_{0}^{d-m-1}{}_k \sum_{1}^{h}{}_t\, \delta_{tk+1} S_{t,h}(a_{jk})=0,\ h\in\{0, \cdots, m-1\} \text{ at } x_n=0.$$

$$L_j(b_{j1})+\sum_{0}^{d-m-1}{}_k R_{j,k+2}(a_{jk})=0 \tag{3.25}$$

Therefore it is possible to find all a_{jk} and b_{jk}.
By Taylor's formula:

$$b_{jk}(y, \omega, \rho, \eta')=\sum_{0}^{k-1}{}_h \rho^h b_{jhk}(y, \omega, \eta')+\rho^k b_{jk}(y, \omega, \rho, \eta') \tag{3.26}$$

b_{jhk}, (b_{jk}) homogeneous in $\eta'((\rho, \eta'))$ of degree $-h$ $(-k)$. Hence:

$$a_j=\sum_{0}^{d-m-1}{}_k \rho^k a_{jk}(x, \omega, \rho, \eta')+\sum_{k<0}{}_k \{\sum_{1}^{-k}{}_t \rho^{-t} a_{jkt}(x, \omega, \eta')+ \\ +\mu_{jk}(x, \omega, \rho, \eta')\}\ . \tag{3.27}$$

Now to define the classical part $v^{(1)}$ of the solution as in [8], let us put:

$$a_j^{(1)}(x, \omega, \rho, \eta')=\sum_{0}^{d-m-1}{}_k \rho^k a_{jk}(x, \omega, \rho, \eta')+ \\ +\sum_{k<0}{}_k \mu_{jk}(x, \omega, \rho, \eta')\, \sigma_1(\rho/\delta|\eta'|)$$

and the $a_j^{(1)}$'s are supported where the phase functions are defined.

To construct the "singular" part of the solution $v^{(2)}$, for any $j\in\{1, \cdots, m\}$ and for any $h\geqq 1$ choose symbols using the standard asymptotics $a_{jh}^{(2)}\in S^0(\boldsymbol{R}^n\times S^{d-2}\times \boldsymbol{R}^{n-d})$ such that $a_{jh}^{(2)}\sim\sum_k a_{jkh}\sigma_1$. In the same way select $a_j^{(2)}\in S^{-1}(\boldsymbol{R}^+; S^0)$, $j\in\{1, \cdots, m\}$ (for a definition of $S^m(\boldsymbol{R}^+; S^{m'}(\boldsymbol{R}^n\times S^{d-2}\times \boldsymbol{R}^{n-d}))$ see [8] page 577) such that

$$a_j^{(2)}-\sum_{1}^{N}{}_k \rho^{-k} a_{jk}^{(2)}\in S^{-N-1}(\boldsymbol{R}^+; S^0) \quad \text{for any} \quad N\geqq 1 \tag{3.28}$$

If $v=v^{(1)}+v^{(2)}=\sum_{1}^{m}{}_j I_\varphi(a_j^{(1)})+\sum_{1}^{m}{}_j I_\varphi(a_j^{(2)})$ let us calculate Pv: (3.5) and the preceeding constuction show that

$$Pv=I_\psi(c)=\int_{\boldsymbol{R}^{n-d}} e^{i\langle x', \xi'\rangle} c(x, \xi')\, d\xi' \tag{3.29}$$

where $c\in S^0(\boldsymbol{R}^n\times \boldsymbol{R}^{n-d})$. It is easily verified that $\partial_n^h v|_{x_n=0}=I_\psi(b_h)+\delta_{hm-1} v_0$ with $b_h\in S^0(\boldsymbol{R}^n\times \boldsymbol{R}^{n-d})$, $h\in\{0, \cdots, m-1\}$
Now let us look for a solution of:

$$Pv=I_\psi(c) \quad \text{and} \quad \partial_n^h v|_{x_n=0}=I_\psi(b_h)\,, \quad h\in\{0, \cdots, m-1\}\ . \tag{3.30}$$

v will be searched of the form $I_\psi(a^{(3)})$. Now:

$$P(I_\psi(a^{(3)}))=I_\psi(D_n^m a^{(3)}+\sum_{1}^{m}{}_j [\sum_{|\alpha''|\leq j} A_{\alpha,j}(x', x''', \xi', 0)\,(D_{x''})^{\alpha''}]\, D_n^{m-j} a^{(3)}+Ma^{(3)})$$

Because of the strict hyperbolicity assumption (iv) and the fact M: $S^k\rightarrow S^{k-1}$ we begin by solving:

$$(3.31)_0 \qquad P_\gamma(a_0^{(3)}) = c \text{ and } \partial_n^h a_0^{(3)}|_{x_n=0} = b_h\,, \quad h\in\{0, \cdots, m-1\}$$

with $a_0^{(3)}\in S^0(\boldsymbol{R}^n\times\boldsymbol{R}^{n-d})$, $\gamma=(x', x''', \xi', 0)\in\Sigma$.
Then by recursively solving:

$$(3.31)_j \qquad P_\gamma(a_j^{(3)}) = -Ma_{j+1}^{(3)}\in S^j \text{ and } \partial_n^h a^{(3)}|_{x_n=0} = 0\,,$$
$$j = -1, -2, -3, \cdots h\in\{0, \cdots, m-1\}$$

we can select $a^{(3)}\sim\sum_{j<0} a^{(3)}$ and it is obvious that
$(P_\gamma+M)\,(a^{(3)})\in S^{-\infty}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-d})$ and the choice can be made in order to have $\delta_n^h a^{(3)}|_{x_n=0}=b_h$, $h\in\{0, \cdots, m-1\}$
Finally setting $u=v^{(1)}+v^{(2)}-I_\psi(a^{(3)})$ we have $Pu=0$ and $\partial_n^h u|_{x_n=0}=\delta_{hm-1}v_0\equiv\delta_{hm-1}\,\delta(x', x'')$, $h\in\{0, \cdots, m-1\}$, which is the solution of (3.1) if $d-m-1>0$. If $d-m-1\leqq 0$ choosing at once

$$(3.32) \qquad a_j(x, \omega, \rho, \eta') = \sum_{1+m-d}^{\infty}{}_k\, \rho^{-k}\, a_{jk}(x, \omega, \rho, \eta')$$

the proof goes as before and this gives in any case the solution of (3.1).

4. Propagation of singularities

First of all the construction of section 3 can be repeated uniformly when $s\in]-\delta, \delta[$ and we find:

$$(4.1) \qquad E^{(s)}: \mathcal{D}'(\boldsymbol{R}^{n-1}) \to \mathcal{E}'(\boldsymbol{R}^n),\ P\,E^{(s)} \equiv 0 \text{ near } (\gamma^0, \gamma^{0'})\,,$$
$$\partial_n^h E^{(s)}|_{x_n=s} \equiv \delta_{hm-1}\, Id \text{ near } (\gamma^0, \gamma^{0'})\ h\in\{0, \cdots, m-1\}$$

where $\gamma^{0'}=(x=0; \xi'=(0, \cdots, 0, 1), \xi''=0)$.
We shall now reason when $d-m-1>0$ remarking that the case $d-m-1\leqq 0$ is dealt with exactly in the same way.

$$(4.2) \qquad (E^{(s)}f)\,(x) = (E^{(s)}_{(1)}+E^{(s)}_{(2)}+E^{(s)}_{(3)})\,(f)\,(x)$$

where we have set:

$$(4.3) \qquad (E^{(s)}_{(1)})\,(f)\,(x) = \sum_1^m{}_j \int_{S^{d-2}}\int_0^{+\infty}\int_{\boldsymbol{R}^{n-d}} \exp(i\varphi_j(x, s, \omega, \rho, \eta')$$
$$-\langle z', \eta'\rangle-\rho\langle\omega, z''\rangle)$$
$$\times a_j^{(1)}(x, s, \omega, \rho, \eta')f(z', z'')\,dz'\,dz''\,d\eta'\,d\rho d\omega$$

$$(4.4) \qquad (E^{(s)}_{(2)})\,(f)\,(x) = \sum_1^m{}_j \int_{S^{d-2}}\int_0^{+\infty}\int_{\boldsymbol{R}^{n-d}} \exp(i\varphi_j(x, s, \omega, \rho, \eta')$$
$$-\langle z', \eta'\rangle-\rho\langle\omega, z''\rangle)$$
$$\times a_j^{(2)}(x, s, \omega, \rho, \eta')f(z', z'')\,dz'\,dz''\,d\eta'\,d\rho d\omega$$

(4.5) $\quad (E^{(s)}_{(3)})(f)(x)=\int_{\boldsymbol{R}^{n-d}}\int_{\boldsymbol{R}^{n-1}} e^{j\langle x'-z',\eta'\rangle}\, a^{(3)}(x,s,\eta')f(z',z'')\,dz'\,dz''\,d\eta'$

where in (4.3) $a^{(1)}_j\in S^{d-m-1}$, in (4.4) $a^{(2)}_j\in S^{-1}(\boldsymbol{R}^+;S^0)$ and in (4.5) $a^{(3)}\in S^0$. By Duhamel's principle let us put:

$$E_+(f)(x)=-\int_{-\delta}^{x_n}((E^{(s)}\circ\gamma_{(s)}\circ A)(f)(x)\,ds,\ \delta>0 \tag{4.6}$$

with conesupp(A) compactly supported near $\gamma^{0'}$ and $\gamma_{(s)}u=u$ restricted to $x_n=s$. Since $\gamma^{0'}\notin\{(x,\xi)\,|\,x_n=s,\ \xi'=0,\ \xi''=0,\ \xi_n=0\}$ $\gamma_{(s)}\circ A$ is well defined and it is clear that:

$$P\,E_+(f)\equiv f \text{ near } \gamma^0 \tag{4.7}$$

Now the rules to compute wave fronts sets given ([4]):

$$WF(E^{(s)}f)\,|_{x=s}\subset[\bigcup_1^m{}_j\,K_j^+\cup\tilde{K}_F^+]\circ(i_s^*)^{-1}(WF(f)) \tag{4.8}$$

where $f\in\mathcal{D}'(\boldsymbol{R}^{n-1})$, $(i_s^*)^{-1}(WF(f))=\{(z',z'',s,\zeta',\zeta'',\zeta_n)\,|\,\zeta_n\in\boldsymbol{R},\ (z',z'',\zeta',\zeta'')\in WF(f)\}$, K_j^+ denotes $\exp(tH_{q_j})$ (cfr. (1.8)), $t\geqq 0$ and $\tilde{K}_F^+$ is the relation defined by:

(4.9) $(\rho,\bar{\rho})\in\tilde{K}_F^+$ if and only if ρ and $\bar{\rho}$ belong to same leaf F_ρ through $\rho\in\Sigma$ and $x_n(\rho)\geqq x_n(\bar{\rho})$. Then we have

$$WF(E^+)\subset\bigcup_1^m{}_j\,K_j^+\cup\tilde{K}_F^+ \tag{4.10}$$

It is clear that we will also have an other microlocal parametrix E^- satisfying (4.7) and (4.10) with the reversed time orientation. We now want to be more precise on Σ. Let us introduce as in [8] the following relations:

(4.11) $K_F^+\ni(\rho,\bar{\rho})$ if and only if ρ and $\bar{\rho}\in F_\rho\subset\Sigma$ and $\exists\gamma\colon[0,1]\to F_\rho$ Lipschitz continous curve $\gamma(0)=\rho$, $\gamma(1)=\bar{\rho}$ and $\dot{\gamma}(t)\in(\Gamma_\gamma)^0$ a.e.

$\partial K_F^+\ni(\rho,\bar{\rho})$ if and only if there exists a null bicharacteristics of the operator P^0 induced on the leaf F_ρ that joints ρ with $\bar{\rho}$.
∂K_F^+ is the boundary of K_F^+, see e.g. Duistermaat [2]. Let now $\gamma^0\in K\subset\Sigma\times\Sigma\backslash\partial K_F^+$, K compact, $U\Supset K$ open neighborhood of K. Denoting by $k(x,z',z'')$ the kernel of $E^{(0)}$ we have in U' (proj. of U at $s=0$).

$$k(x,z',z'')=\int e^{i\langle x'-z',\xi\rangle}\,a(x,z'',\xi')\,d\xi' \tag{4.12}$$

This is clear since the term in (4.5) is already of this type and in (4.3), (4.4) at $\rho=0$ the phase is stationary exactly on null bicharacteristics of P^0. Therefore integrating by parts in $d\rho\,d\omega$ (4.12) follows. Now $k(x,z',z'')$ solves $Pk=0$ and $\partial_n^h k\,|_{x_n=0}=\delta_{hm-1}\,\delta(x'-z',x''-z'')$ $h\in\{0,\cdots,m-1\}$. Therefore a has to be a

solution of $P_\gamma(a)=-Ma$ and $\partial_n^h a|_{x_n=0}\in S^{-\infty}$, if $x'\neq z'$ and $x''\neq z''$ $h\in\{0,\cdots, m-1.\}$
Since P_γ is strictly hyperbolic we obtain that a is still in $S^{-\infty}$ outside the set obtained by emanating from $(z', z'', 0; \eta', 0, 0)$ along curves defined in K_F^+. Finally:

$$WF(E^{\pm}f)\subset K^{\pm}\circ WF(f) \tag{4.13}$$

Where $K^{\pm}$ is the generalized flow as defined in [11]. Passing to a parametrix for ${}^tP$, the transpose of P, and microlocalizing near γ^0 now ends in a standard way the proof of the theorem.

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