Kannai, Y.
Osaka J. Math.
25 (1988), 1–18

EXISTENCE AND SMOOTHNESS FOR CERTAIN DEGENERATE PARABOLIC BOUNDARY VALUE PROBLEMS

YAKAR KANNAI

(Received July 21, 1986)

1. In [6], S. Ito has considered the following parabolic initial-boundary value problem:

$$\frac{\partial u}{\partial t}=\sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i(x, t)\frac{\partial u}{\partial x_i}+c(x, t)u \tag{1.1}$$

in $\Omega\times R_+$ with $\Omega\subset R^n$,

$$\alpha(x, t)\frac{\partial u}{\partial n}(x, t)+\beta(x, t)u(x, t)=f(x, t) \qquad \text{on} \quad \partial\Omega\times R_+ \tag{1.2}$$

where $\partial/\partial n$ is the derivation in the direction of outer co-normal and

$$\alpha(x, t)\geq 0,\ \beta(x, t)\geq 0,\ \alpha(x, t)+\beta(x, t)=1\,, \tag{1.3}$$

and

$$u(x, 0)=u_0(x) \qquad \text{for} \quad x\in\Omega\,. \tag{1.4}$$

He solved the problem by constructing explicitly a fundamental solution. We wish to apply instead the well-known method of reducing boundary value problems to pseudo-differential problems on the boundary [5, Chapter XX for the elliptic case, and [3,10] for the parabolic case]. In [7] we have analyzed in this manner the corresponding degenerate elliptic boundary value problem and established the hypoellipticity of the appropriate pseudo-differential boundary operator; in fact, under assumptions such as (1.3), one easily sees that this operator satisfies the condition for the existence of a parametrix with symbol in a suitable space $S^m_{\rho,\delta}(\partial\Omega)$. The parabolic case, to be considered in the sequel, differs from the elliptic case in two respects: (i) The boundary operator induced by (1.2) is no longer invertible in an $S^m_{\rho,\delta}(\partial\Omega\times R_+)$ space - one needs weighted classes of symbols as introduced in [1] (in fact vector weights are required; (ii) The manifold $\partial\Omega\times R_+$ is a manifold *with boundary* $\partial\Omega\times\{0\}$ and a Cauchy problem for a pseudo-differential equation has to be solved. We deal with these matters in sections 2 and 3, respectively, and establish existence, uniqueness and smoothness of solutions to the problem (1.1), (1.2) and (1.4) in section 4.

One advantage of our approach is that we can consider cases where α and β are not required to be real. The zeros of a can even be simple when regarded as function of t (but have to be multiple as functions of x). We could also discuss non-cylindrical domains (as in [10] and [11]) and higher order operators (there the conditions become however much more involved). We do not treat here the delicate question of necessity, dealt with in [8] (compare [7, p. 327]) for an example of an unusual problem).

A substantial part of the research described here was carried out while I was visiting Japan. I am very much indebted to Prof. S. Mizohata, who in his gentle way encouraged me to consider this problem; to Professors M. Ikawa, C. Iwasaki and K. Taniguchi, for helpful conversations; to Prof. H. Tanabe, for being my host; and to the Japan Society for the Promotion of Science, for supporting my visit to Japan.

2. In this section we shall prove the hypoellipticity of certain degenerate pseudo-differential operators of (weighted) order 1 defined on open subsets of cylindrical manifolds of the form $H\times R$ where H is a d-dimensional manifold.

Denote the general point of $H\times R$ by $(x, t)=(x_1, \cdots, x_d, t)$, the dual by $(\xi, \tau)=(\xi_1, \cdots, \xi_d, \tau)$, (where we have fixed a certain collection of coordinate systems on H) and set

$$\lambda(\xi, \tau) = (1+|\xi|^4+|\tau|^2)^{1/4} = (1+(\sum_{j=1}^{d} \xi_j^2)^2+|\tau|^2)^{1/4} \tag{2.1}$$

where a specific Riemannian metric has been chosen on the fibers of $T^*(H)$. For an open subset X of $H\times R$, let $S^m_{\lambda,\rho,\delta}(X)$ denote the class of all C^∞ functions $a(x, t, \xi, \tau)$ on $X\times R^{d+1}$ such that for every compact $K\subset X$ and each pair of d-dimensional multi-indices α,β and two non-negative integers α_{d+1}, β_{d+1} there exists a constant $C=C(\alpha, \beta, \alpha_{d+1}, \beta_{d+1}, K, a)$ such that

$$|a^{(\alpha, \alpha_{d+1})}_{(\beta, \beta_{d+1})}(x, t, \xi, \tau)| \leq C\lambda^{m-|\alpha|\rho-2\alpha_{d+1}\rho+|\beta|\delta+2\beta_{d+1}\delta} \tag{2.2}$$

where ρ and δ are fixed constants satisfying $1\geq\rho>\delta\geq 0$ and $|\alpha|=\sum_{j=1}^{d}\alpha_j$, $|\beta|=\sum_{j=1}^{d}\beta_j$, and $a^{(\alpha, \alpha_{d+1})}_{(\beta, \beta_{d+1})}=\partial_\xi^\beta\partial_\tau^{\alpha_{d+1}}D_x^\beta D_t^{\beta_{d+1}}a$. Note that $S^m_{\lambda,\rho,\delta}$ coincides with the (Beals) class defined by the weight vectors $\Phi_1=\cdots=\Phi_d=\lambda^\rho$, $\Phi_{d+1}=\lambda^{2\rho}$, $\varphi_1=\cdots=\varphi_d=\lambda^{-\delta}$, $\varphi_{d+1}=\lambda^{-2\delta}$. The following is a generalization of Theorem 3.1 in [7].

Theorem 2.1. *Let*

$$p(x, t, \xi, \tau) = a(x, t)e(x, t, \xi, \tau)+b(x, t, \xi, \tau) \tag{2.3}$$

where

$$a(x, t) \in C^\infty(X) \quad and \quad a(y, t) = 0 \tag{2.4}$$

for $(y, t) \in X$ *implies* $D_{x_j} a(y, t) = 0$, $1 \leq j \leq d$. *Assume further that* $b \in S^0_{\lambda,\rho,\delta}$, $e \in S^1_{\lambda,\rho,\delta}$ *where* $\delta \leq \frac{1}{2}$, *that for every compact subset* K *of* X *and* α, β, α_{d+1}, β_{d+1} *there exists a constant* $C = C(\alpha, \alpha_{d+1}, \beta, \beta_{d+1}, K)$ *such that for all* $(x, t) \in K$,

$$|e^{(\alpha, \alpha_{d+1})}_{(\beta, \beta_{d+1})}(x, t, \xi, \tau)| \leq C(1 + |e(x, t, \xi, \tau)|)\lambda^{-|\alpha|\rho - 2\alpha_{d+1}\rho + |\beta|\delta + 2\beta_{d+1}\delta} \tag{2.5}$$

and that there exist positive constants c, ε, *such that*

$$|a(x, t)e(x, t, \xi, \tau)| + |b(x, t, \xi, \tau)| \geq c \tag{2.6}$$

if $\lambda \geq c$, *and*

$$|\arg [a(x, t)e(x, t, \xi, \tau)/b(x, t, \xi, \tau)]| \leq \pi - \varepsilon \tag{2.7}$$

if the argument in (2.7) *is well defined and* $\lambda \geq c$.

Then there exists a positive constant C_1 *such that*

$$|p(x, t, \xi, \tau)| \geq C_1 \qquad if \quad \lambda \geq c \tag{2.8}$$

and for every α, β, α_{d+1}, β_{d+1} *and compact sbuset* K *of* X *there exists a constant* $C_2 = C_2(\alpha, \alpha_{d+1}, \beta, \beta_{d+1}, K)$ *such that for all* $(x, t) \in K$ *and* ξ, τ *such that* $\lambda \geq c$,

$$|p^{(\alpha, \alpha_{d+1})}_{(\beta, \beta_{d+1})}(x, t, \xi, \tau)| \leq C_2 |p(x, t, \xi, \tau)| \lambda^{-|\alpha|\rho - 2\alpha_{d+1}\rho + (|\beta|/2) + \beta_{d+1}} \tag{2.9}$$

Proof. We shall denote by C any constant which is independent of x, t, ξ, τ if $(x, t) \in K$ where K is a compact subset of X and $\lambda(\xi, \tau) \geq c$. Note first that

$$|b(x, t, \xi, \tau)| \leq C\,|p(x, t, \xi, \tau)| \,. \tag{2.10}$$

In fact, (2.10) is certainly true if $b(x, t, \xi, \tau) = 0$ or $a(x, t)e(x, t, \xi, \tau) = 0$. Otherwise

$$b = ([ae/b] + 1)^{-1} p$$

where by (2.7) $|[ae/b] + 1|$ is bounded away from zero. Hence the triangle inequality $|ae| \leq |p| + |b|$ implies also that

$$|a(x, t)e(x, t, \xi, \tau)| \leq C\,|p(x, t, \xi, \tau)|. \tag{2.11}$$

The estimate (2.8) now follows from (2.6) along with (2.10) and (2.11).

To prove (2.9), note first that $b \in S^0_{\lambda,\rho,\delta}$, $\delta \leq \frac{1}{2}$, and (2.8) imply that if $\lambda \geq c$ then

$$\begin{aligned} |b^{(\alpha, \alpha_{d+1})}_{(\beta, \beta_{d+1})}| &\leq C\lambda^{-|\alpha|\rho - 2\alpha_{d+1}\rho + |\beta|\delta + 2\beta_{d+1}\delta} \\ &\leq C\,|p|\lambda^{-|\alpha|\rho - 2\alpha_{d+1}\rho + |\beta|\delta + 2\beta_{d+1}\delta} \\ &\leq C\,|p|\lambda^{-|\alpha|\rho - 2\alpha_{d+1}\rho + (|\beta|/2) + \beta_{d+1}} \end{aligned} \tag{2.12}$$

Thus we have to estimate only $(ae)^{(\alpha,\alpha_{d+1})}_{(\beta,\beta_{d+1})}$. By Leibniz's rule,

$$
\begin{aligned}
&(ae)^{(\alpha,\alpha_{d+1})}_{(\beta,\beta_{d+1})}(x,t,\xi,\tau)=\\
&=\sum_{\substack{\gamma\le\beta\\ \gamma_{d+1}\le\beta_{d+1}}}\binom{\beta}{\gamma}\binom{\beta_{d+1}}{\gamma_{d+1}}a_{(\gamma,\gamma_{d+1})}(x,t)e^{(\alpha,\alpha_{d+1})}_{(\beta-\gamma,\beta_{d+1}-\gamma_{d+1})}(x,t,\xi,\tau)\\
&=\sum_{\substack{\gamma\le\beta\\ \gamma_{d+1}\le\beta_{d+1}}}\Big(\sum_{|\gamma|+2\gamma_{d+1}=0}+\sum_{|\gamma|+2\gamma_{d+1}=1}+\sum_{|\gamma|+2\gamma_{d+1}\geqq2}\Big)\\
&=\sum_0+\sum_1+\sum_2
\end{aligned}
\tag{2.13}
$$

By (2.5), (2.8), and (2.11),

$$
\begin{aligned}
|\sum_0|&=|a(x,t)|\,|e^{(\alpha,\alpha_{d+1})}_{(\beta,\beta_{d+1})}(x,t,\xi,\tau)|\\
&\le C\,|a(x,t)|(1+|e(x,t,\xi,\tau)|)\lambda^{-|\alpha|\rho-2\alpha_{d+1}\rho+|\beta|\delta+2\beta_{d+1}\delta}\\
&\le C(1+|a(x,t)e(x,t,\xi,\tau)|)\lambda^{-|\alpha|-2\alpha_{d+1}\rho+|\beta|\delta+2\beta_{d+1}\delta}\\
&\le C\,|p(x,t,\xi,\tau)|\lambda^{-|\alpha|\rho-2\alpha_{d+1}\rho+(\beta/2)+\beta_{d+1}}
\end{aligned}
\tag{2.14}
$$

if $\lambda\ge c$ and $(x,t)\in K$. If $|\gamma|+2\gamma_{d+1}=1$ then $|\gamma|=1$, $\gamma_{d+1}=0$ and we use the following essentially well-known

Fact. If $a(x,t)\in C^2(X)$ and $a(y,t)=0$ for $(y,t)\in X$ implies $D_{x_j}a(y,t)=0$, $1\le j\le d$, then for every compact subset K of X there exists a constant $C(K)$ such that for all $(x,t)\in K$ and $1\le j\le d$,

$$|D_{x_j}a(x,t)|\le C\,|a(x,t)|^{1/2}. \tag{2.15}$$

The fact can be proved exactly as lemma 3.2 in [7] with t appearing as a parameter. It follows from the fact and from (2.5) that if $(x,t)\in K$ then

$$
\begin{aligned}
|\sum_1|&\le C\sum_{j=1}^{d}|D_{x_j}a(x,t)|\sum_{|\mu|=|\beta|-1}|e^{(\alpha,\alpha_{d+1})}_{(\mu,\beta_{d+1})}(x,t,\xi,\tau)|\\
&\le C\,|a(x,t)|^{1/2}(1+|e(x,t,\xi,\tau)|)\lambda^{-|\alpha|\rho-2\alpha_{d+1}\rho+(|\beta|-1)\delta+\beta_{d-1}\delta}.
\end{aligned}
\tag{2.16}
$$

But $a(x,t)$ is bounded on K, and if $\lambda\ge c$ then by (2.8) and (2.11)

$$|a(x,t)|^{1/2}|e(x,t,\xi,\tau)|^{1/2}\le(1+|(ae)(x,t,\xi,\tau)|)/2\le C\,|p(x,t,\xi,\tau)|. \tag{2.17}$$

Inserting (2.17) in (2.16), recalling that $e\in S^{1}_{\lambda,\rho,\delta}$ and $\delta\le\frac{1}{2}$ and noting that $|\beta|\ge1$, we find that

$$
\begin{aligned}
|\sum_1|&\le C\,|p|\lambda^{-|\alpha|\rho-2\alpha_{d+1}\rho+(|\beta|/2)+\beta_{d+1}}\\
&\quad+C|p|\lambda^{1/2-|\alpha|\rho-2\alpha_{d+1}\rho+(|\beta|-1)/2+\beta_{d+1}}\\
&\le C|p|\lambda^{-|\alpha|\rho-2\alpha_{d+1}\rho+(|\beta|/2)+\beta_{d+1}}.
\end{aligned}
\tag{2.18}
$$

Using (2.5) once more, we see that if $\lambda \geq c$, $(x, t) \in K$, then

$$|\sum_2| \leq C(1+|e|)\lambda^{-|\alpha|\rho-2\alpha_{d+1}\rho+|\beta|\delta-|\gamma|\delta+2\beta_{d+1}\delta-2\gamma_{d+1}\delta}$$

$$\leq C\lambda^{1-|\alpha|\rho-2\alpha_{d+1}\rho+|\beta|\delta-|\gamma|\delta+2\beta_{d+1}\delta-2\gamma_{d+1}\delta}. \tag{2.19}$$

But $\delta \leq \frac{1}{2}$, $|\gamma| \leq |\beta|$, $\gamma_{d+1} \leq \beta_{d+1}$, and for $\sum_2$, $|\gamma|+2\gamma_{d+1} \geq 2$. Hence

$$(|\beta|+2\beta_{d+1}-|\gamma|-2\gamma_{d+1})\left(\delta-\frac{1}{2}\right)-\frac{1}{2}(|\gamma|+2\gamma_{d+1})+1 \leq 0$$

or

$$1+|\beta|\delta-|\gamma|\delta+2\beta_{d+1}\delta-2\gamma_{d+1}\delta \leq \frac{|\beta|}{2}+\beta_{d+1}. \tag{2.20}$$

Applying (2.8) and using (2.20) in the exponent of (2.19) we obtain the estimate

$$|\sum_2| \leq C|p(x, t, \xi, \tau)|\lambda^{-|\alpha|\rho-2\alpha_{d+1}\rho+(|\beta|/2)+\beta_{d+1}}. \tag{2.21}$$

The conclusion (2.9) now follows from (2.12), (2.13), (2.14), (2.18) and (2.21).

As in [1], [4] and [9], we denote by $L^m_{\lambda,\rho,\delta}(X)$ the class of (pseudo-differential) operators P such that for every $f \in C_0^\infty(X)$ there exists a symbol $p_f(x, t, \xi, \tau) \in S^m_{\lambda,\rho,\delta}(X)$ such that for all $\varphi \in C_0^\infty(X)$

$$P(f\varphi)(x, t) = (2\pi)^{-(d+1)} \iint e^{i(x\cdot\xi+t\tau)} p_f(x, t, \xi, \tau)\hat{\varphi}(\xi, \tau) d\xi d\tau. \tag{2.22}$$

Note that, as usual, the symbol of $P \in L^m_{\lambda,\rho,\delta}(X)$ is unique modulo $S^{-\infty}_{\lambda,\rho,\delta} = \bigcap_{m=0}^{\infty} S^{-m}_{\lambda,\rho,\delta}(X)$. In an analogy to (a special case of) theorem 4.2 in [4], we have

Theorem 2.2. *Let $P \in L^m_{\lambda,\rho,\delta}(X)$ and assume that the symbol $p(x, t, \xi, \tau)$ of P satisfies the inequalities* (2.8) *and*

$$|p^{(\alpha,\alpha_{d+1})}_{(\beta,\beta_{d+1})}(x, t, \xi, \tau)| \leq C_2|p(x, t, \xi, \tau)|\lambda^{-|\alpha|\rho-2\alpha_{d+1}\rho+|\beta|\delta+2\beta_{d+1}\delta} \tag{2.9'}$$

with $\delta < \rho$. Then P has a (right and left) parametrix $E \in L^0_{\lambda,\rho,\delta}(X)$ and P^* is hypoelliptic.

Proof. By theorem 8.7 of [1], the operator P^* is also in $L^m_{\lambda,\rho,\delta}(X)$ and its symbol $p^\sharp(x, t, \xi, \tau)$ has an asymptotic expansion

$$p^\sharp(x, t, \xi, \tau) \sim \sum_{(\mu,\mu_{d+1})} (\mu!)^{-1}(\mu_{d+1}!)^{-1}\bar{p}^{(\mu,\mu_{d+1})}_{(\mu,\mu_{d+1})} \tag{2.23}$$

in the sense that for any positive integer N,

$$p^\sharp - \sum_{|\mu|+2\mu_{d+1}<N} (\mu!)^{-1}(\mu_{d+1}!)^{-1}\bar{p}^{(\mu,\mu_{d+1})}_{(\mu,\mu_{d+1})} \in S^{m-N(\rho-\delta)}_{\lambda,\delta,\rho}. \tag{2.24}$$

Choosing $N > (\rho-\delta)^{-1}$ and using (2.9′) we infer first that for λ sufficiently large

$|p^\sharp|$ is bounded away from zero. Differentiating (2.24) we see that for all α, α_{d+1}, β_{d+1}, β,

$$|p^{\sharp(\alpha,\alpha_{d+1})}_{(\beta,\beta_{d+1})}-\sum_{|\mu|+2\mu_{d+1}<N}(\mu!)^{-1}(\mu_{d+1}!)^{-1}\tilde{p}^{(\mu+\alpha,\mu_{d+1}+\alpha_{d+1})}_{(\mu+\beta,\mu_{d+1}+\beta_{d+1})}|$$
$$\leq C\lambda^{m-N((\rho-\delta)-|\alpha|\rho-2\alpha_{d+1}\rho+|\beta|\rho+2\beta_{d+1}\delta}.$$

By (2.9') and the fact that $|\tilde{p}|$ is bounded away from zero for large λ we deduce that (2.9') holds also for $p^\sharp$ replacing p. Hence we have to construct only a right parametrix for P (for the adjoint of a right parametrix for P^* is a left parametrix for P). If $A\in L^m_{\lambda,\rho,\delta}(X)$, $B\in L^{m'}_{\lambda,\rho,\delta}(X)$ and if one of them is properly supported, then by theorem 8.4 of [1] their composition AB is in $L^{m+m'}_{\lambda,\rho,\delta}(X)$ and the symbol $a\circ b$ of AB has the asymptotic expansion

$$a\circ b\sim\sum_{\alpha,\alpha_{d+1}}(\alpha!)^{-1}(\alpha_{d+1}!)^{-1}a^{(\alpha,\alpha_{d+1})}b_{(\alpha,\alpha_{d+1})}\tag{2.25}$$

in the sense that for any positive integer N,

$$a\circ b-\sum_{|\alpha|+2|\alpha_{d+1}|<N}(\alpha!)^{-1}(\alpha_{d+1}!)^{-1}a^{(\alpha,\alpha_{d+1})}b_{(\alpha,\alpha_{d+1})}\in S^{m+m'-N(\rho-\delta)}_{\lambda,\rho,\delta}.\tag{2.26}$$

Moreover, by the (local analog) of theorem 4.13 of [1], given a sequence $a_j\in S^{m-j(\rho-\delta)}_{\lambda,\rho,\delta}(X)$, $j=0, 1, \cdots$, there exists a properly supported operator $A\in L^m_{\lambda,\rho,\delta}(X)$ with symbol $a\in S^m_{\lambda,\rho,\delta}(X)$ such that $a\sim\sum_{j=0}^\infty a_j$ (i.e., for every N,

$$a-\sum_{j<N}a_j\in S^{m-N(\rho-\delta)}_{\lambda,\rho,\delta}(X)).\tag{2.27}$$

Hence it suffices to construct a symbol $e(x, t, \xi, \tau)\in S^0_{\lambda,\rho,\delta}(X)$ such that

$$p\circ e-1\in S^{-\infty}_{\lambda,\rho,\delta}(X)\tag{2.28}$$

and this will be achieved by first solving the recursion formulas

$$p(x, t, \xi, \tau)e_0(x, t, \xi, \tau)\sim 1,$$
$$p(x, t, \xi, \tau)e_{j+1}(x, t, \xi, \tau)\sim p(x, t, \xi, \tau)e_j(x, t, \xi, \tau)-p\circ e_j \qquad \text{for } j\geq 0,\tag{2.29}$$

and then setting $e\sim\sum_{j=0}^\infty e_j$ (we say that $a\sim b$ if $a-b\in S^{-\infty}_{\lambda,\rho,\delta}(X)$).

Define $e_0(x, t, \xi, \tau)=p(x, t, \xi, \tau)^{-1}$ if $\lambda(\xi, \tau)$ is sufficiently large and extend the definition so that $e_0\in C^\infty(X\times R^{d+1})$. Then e_0 is bounded and $pe_0=1$ for λ sufficiently large. More generally

$$|e_0{}^{(\alpha,\alpha_{d+1})}_{(\beta,\beta_{d+1})}(x, t, \xi, \tau)|\leq|C|e_0|\lambda^{-\rho|\alpha|-2\rho\alpha_{d+1}+|\beta|\delta+2\beta_{d+1}\delta}$$
$$\leq C\lambda^{-\rho|\alpha|-2\rho\alpha_{d+1}+|\beta|\delta+2\beta_{d+1}\delta}\tag{2.30}$$

for λ sufficiently large. In fact, we can prove (2.30) by induction over $|\alpha|+\alpha_{d+1}+|\beta|+\beta_{d+1}$ (for $\alpha=\alpha_{d+1}=\beta=\beta_{d+1}=0$ it is just the boundedness of

e_0). Differentiation of the equation $p(x, t, \xi, \tau)e_0(x, t, \xi, \tau)=1$ gives, when λ is large,

$$pe_{0(\beta,\,\beta_{d+1})}^{(\alpha,\,\alpha_{d+1})} = \sum C_{\mu,\mu_{d+1},\nu,\nu_{d+1}} p_{(\nu,\,\nu_{d+1})}^{(\mu,\,\mu_{d+1})} e_{0(\beta-\nu,\,\beta_{d+1}-\nu_{d+1})}^{(\alpha-\mu,\,\alpha_{d+1}-\mu_{d+1})}$$

where $C_{\mu,\mu_{d+1},\nu,\nu_{d+1}}$ are constants and the summation is extended over multi-indices μ, μ_{d+1}, ν, ν_{d+1} with $\mu\leq\alpha$, $\mu_{d+1}\leq\alpha_{d+1}$, $\nu\leq\beta$, $\nu_{d+1}\leq\beta_{d+1}$ and $|\mu+\nu|+\mu_{d+1}+\nu_{d+1}>0$. Using (2.8), (2.9') and the induction hypothesis, we obtain (2.30). We claim that more generally

$$e_j(x, t, \xi, \tau)\in S_{\lambda,\rho,\delta}^{-j(\rho-\delta)}(X) \qquad \text{holds for all } j\geq 0\,, \tag{2.31}$$

or that for all α, α_{d+1}, β, β_{d+1}, $(x, t)\in K$ and λ sufficiently large,

$$|e_{j(\beta,\,\beta_{d+1})}^{(\alpha,\,\alpha_{d+1})}(x, t, \xi, \tau)|\leq C\lambda^{-j(\rho-\delta)-|\alpha|\rho-2\alpha_{d+1}\rho+|\beta|\delta+2\beta_{d+1}\delta}. \tag{2.32}$$

We prove (2.31) by induction on j. Note first that the recursion formula defining e_{j+1} (for large $\lambda(\xi, \tau)$) may be written (for any positive integer N) in the form

$$pe_{j+1} = -\sum_{0<|\gamma|+2\gamma_{d+1}<N} (\gamma!)^{-1}(\gamma_{d+1}!)^{-1}p^{(\gamma,\gamma_{d+1})}e_{j(\gamma,\gamma_{d+1})}+r_{j,N} \tag{2.33}$$

where $r_{j,N}\in S_{\lambda,\rho,\delta}^{1-(N+j)(\rho-\delta)}(X)$. Choosing $N>(\rho-\delta)^{-1}+1$ we obtain $r_{j,N}\in S_{\lambda,\rho,\delta}^{-(j+1)(\rho-\delta)}(X)$, and applying (2.8) and (2.9') we deduce that (2.32) is valid for $\alpha=\alpha_{d+1}=\beta=\beta_{d+1}=0$. We now use induction over increasing $|\alpha+\beta|+\alpha_{d+1}+\beta_{d+1}$. Differentiation of (2.33) gives

$$\begin{aligned} pe_{j+1(\beta,\,\beta_{d+1})}^{(\alpha,\,\alpha_{d+1})} &= \sum C_{\mu,\mu_{d+1},\nu,\nu_{d+1}} p_{(\nu,\,\nu_{d+1})}^{(\mu,\,\mu_{d+1})} e_{j+1(\alpha-\nu,\,\beta_{d+1}-\nu_{d+1})}^{(\alpha-\mu,\,\alpha_{d+1}-\mu_{d+1})} \\ &-\sum C'_{\mu,\mu_{d+1},\nu,\nu_{d+1}} \sum_{0<|\gamma|+2\gamma_{d+1}<N} p_{(\nu,\,\nu_{d+1})}^{(\gamma+\mu,\,\gamma_{d+1}+\mu_{d+1})} e_{j(\gamma+\beta-\nu,\,\gamma_{d+1}+\beta_{d+1}-\nu_{d+1})}^{(\alpha-\mu,\,\alpha_{d+1}-\mu_{d+1})} \\ &+r_{j,N(\beta,\,\beta_{d+1})}^{(\alpha,\,\alpha_{d+1})} \end{aligned} \tag{2.34}$$

where the $C_{\mu,\mu_{d+1},\nu,\nu_{d+1}}$ and the $C'_{\mu,\mu_{d+1},\nu,\nu_{d+1}}$ are constants, the summation in the first sum in the right hand side of (2.34) is extended over multi-indices μ, μ_{d+1}, ν, ν_{d+1} such that $\mu\leq\alpha$, $\nu\leq\beta$, $\mu_{d+1}\leq\alpha_{d+1}$, $\nu_{d+1}\leq\beta_{d+1}$ and $|\mu+\nu|+\mu_{d+1}+\nu_{d+1}>0$, and the second summation is extended over all $\mu\leq\alpha$, $\nu\leq\beta$, $\mu_{d+1}\leq\alpha_{d+1}$, $\nu_{d+1}\leq\beta_{d+1}$. Applying (2.9') and the induction hypothesis (2.32) for lower derivatives of e_{j+1}, (2.9') and the induction hypothesis (2.32) for e_j, and the assumption on $r_{j,N}$, and dividing by p in accordance with (2.8), we conclude that the first sum, second sum, and third term in (2.34), respectively, are each bounded by a constant times $\lambda^{-(j+1)(\rho-\delta)-|\alpha|\rho-2\alpha_{d+1}\rho+|\beta|\delta+2\beta_{d+1}\delta}$ and (2.32) is proved.

From Theorem 2.1 and 2.2 we obtain at once

Corollary 2.3. *If the symbol p of the operator $P \in L^1_{\lambda,\rho,\delta}(X)$ satisfies the assumptions of Theorem* 2.1, *then P is hypoelliptic and has a* (*right and left*) *parametrix $E \in L^0_{\lambda,\rho,1/2}$.*

In section 4 we will use the following fact which was established in the proof of Theorem 2.2.

Proposition 2.4. *Let a, b, e satisfy the assumptions of Theorem* 2.1. *Let e_0 be equal to $[a(x, t)e(x, t, \xi, \tau)+b(x, t, \xi, \tau)]^{-1}$ for $\lambda(\xi, \tau)$ large. Then*

$$(ae+b)\circ e_0-(ae+b)e_0 \in S^{-(\rho-1/2)}_{\lambda,\rho,1/2}(X)\,. \tag{2.35}$$

The relation (2.35) follows from (2.29) (with $p=ae+b$) and (2.33), where we also use (2.9), (2.30), and $|p|\,|e_0|=1$.

3. We shall derive here results concerning existence and uniqueness of solutions of a Cauchy problem for a pseudo-differential equation in a cylindrical domain. Note that the construction of a right parametrix in section 2 is not enough for obtaining existence (even in the case of a compact manifold without boundary), as the cokernel, while finite dimensional in the compact case, might nevertheless be non-trivial. Moreover, the pseudo-local character of the operator under consideration calls for some care in the consideration of problems involving domains with boundaries. It turns out that both difficulties are overcome via the realization of the fact that the operators which figure in the analysis of parabolic equations are Volterra operators [2], [9], [10], [11]. To introduce and discuss those latter operators we need several further definitions.

Given a d-dimensional manifold H and a real number T, set $H_T=\{(x, t)\in H\times R\colon t<T\}$. In [1] weighted Sobolev spaces were introduced, and their properties investigated, for general weight vectors. For our purposes here a simpler representation is possible (compare also [9], [11]): for all $s\in R$, denote by $H^s(R^{d+1})$ the space of temperate distributions u such that $\hat{u}$ is a function and the norm

$$\|u\|_s^2 = \int_{R^{d+1}} \lambda^{2s}(\xi, \tau)\,|\hat{u}(\xi, \tau)|^2\, d\xi d\tau \tag{3.1}$$

is finite, and by $H^s_+(R^{d+1})$ the closed subspace of $H^s(R^{d+1})$ consisting of those u for which $u(x, t)=0$ if $t<0$. If $H\subset R^d$ we can define $H^s(H\times R)$ and $H^s_+(H\times R)$ as the spaces of the restrictions of elements of $H^s(R^{d+1})$ and $H^s_+(R^{d+1})$ respectively, with the norms

$$\|u\|_s = \inf \|U\|_s$$

where the infimum is extended over all extensions U of u. If H is a compact manifold without boundary then one can define $H^s(H\times R)$ and $H^s_+(H\times R)$ using appropriate systems of local cooradinates and partitions of unity. Given a compact subset σ of H, we denote by $H^s_\sigma(H\times R)$ $(H^s_{+\sigma}(H\times R))$ the closed subspace of $H^s(H\times R)(H^s_+(H\times R))$ consisting of distributions with support in $\sigma\times R$. Given a positive number T, we define $H^s(H_T)(H^s_+(H_T), H^s_\sigma(H_T), H^s_{+\sigma}(H_T))$ as the space of distributions $u\in D'(H_T)$ which are extendible to $H\times R$ such that an extension U of u is an element of $H^s(H\times R)(H^s_+(H\times R), H^s_\sigma(H\times R), H^s_{+\sigma}(H\times R))$ and

$$||u||_s=\inf||U||_s$$

where the infimum is extended over all such extensions.

In conformity to the usual terminology in system theory, we say that an operator $P\in L^m_{\lambda,\rho,\delta}(X)$ is *causal if*, for all $t_0\in R$ and all $\varphi\in C^\infty_0(X)$ such that $\varphi(x, t)=0$ for $t<t_0$, we have $(P\varphi)(x, t)=0$ for $t<t_0$. Properly supported causal operators extend naturally to $H^s_+(H_T)$ (with values in $D'(H_T)$), and all causal operators extend to $H^s_{+\sigma}(H_T)$.

Proposition 3.1. *Let $P\in L^m_{\lambda,\rho,\delta}(X)$, and suppose that for every $f\in C^\infty_0(X)$, the symbol $p_f(x, t, \xi, \tau)$ can be continued analytically to the half-plane* $\mathrm{Im}\,\tau<0$ *so that* (i) *$p_f(x, t, \xi, \tau)$ is continuous in $X\times R^d\times\{\tau\colon \mathrm{Im}\,\tau\leq 0\}$*, (ii) *there exists a constant μ (which might depend on f) and for every $(x, t)\in X$ there exists a constant C such that*

$$|p_f(x, t, \xi, \tau)|\leq C\lambda^\mu(\xi, \tau)\qquad \textit{for}\quad \xi\in R^d,\ \mathrm{Im}\,\tau\leq 0\,. \tag{3.2}$$

Then P is causal.

Proof. Let $\varphi\in C^\infty_0(X)$ and $\varphi(x, t)=0$ for $t\leq t_0$ for a fixed $t_0\in R$. Choose $f\in C^\infty_0(X)$ such that $\mathrm{supp}\,\varphi\subset\{(x, t)\colon f(x, t)=1\}$. Then for every $(x, t)\in X$ we have

$$(P\varphi)(x, t)=P(f\varphi)(x, t)=(2\pi)^{-(d+1)}\iint e^{i(x\cdot\xi+t\tau)}p_f(x, t, \xi, \tau)\hat{\phi}(\xi, \tau)d\xi\, d\tau\,. \tag{3.3}$$

The Paley-Wiener theorem implies that for every $N>0$ there exists $C_N>0$ such that for all $\xi\in R^d$ and complex numbers $\tau=\tau_1+i\tau_2$ with $\tau_2\leq 0$ we have

$$|\hat{\phi}(\xi, \tau_1+i\tau_2)|\leq C_N\lambda^{-N}(\xi, \tau)e^{t_0\tau_2}\,. \tag{3.4}$$

By (3.2), (3.4) and Cauchy's theorem we can move the τ integration in (3.3) to a line $\mathrm{Im}\,\tau=\tau_2$ for a fixed $\tau_2<0$ and obtain the estimate

$$|(P\varphi)(x, t)|\leq CC_N(2\pi)^{-(d+1)}\iint\lambda^{\mu-N}(\xi, \tau)e^{(t_0-t)\tau_2}d\xi\, d\tau\,. \tag{3.5}$$

If we choose $N>\mu+d+2$ and let $\tau_2\to-\infty$ we find that $(P\varphi)(x, t)=0$ if $t<t_0$.

A causal operator $P\in L^m_{\lambda,\rho,\delta}(X)$ is called a Volterra operator if $m<0$. We have (compare [9], [11]) the following:

Proposition 3.2. *Let $P\in L^m_{\lambda,\rho,\delta}(H_T)$ be a properly supported Volterra operator, and let $\varphi\in C_0^\infty(H)$, $g\in H^s_+(H_T)$, $s\in R$. Then there exists a unique $f\in \bigcap_{0<t<T} H^s_+(H_t)$ such that*

$$f+\varphi Pf=g\,. \tag{3.6}$$

Proof. Theorems 9.5 and 9.10 of [1] and $m<0$ imply that for every $\psi\in C_0^\infty(-\infty, T)$, the operator $\psi(t)\varphi(x)P$ is a compact operator in the Hilbert space $H^s_+(H_T)$. We solve first the equation

$$f+\psi\varphi Pf=g\,. \tag{3.7}$$

It suffices to show that the homogeneous equation $f+\psi\varphi Pf=0$ has only the trivial solution in $H^s_+(H_T)$. For any real number a, the operator Λ_0^a, given by

$$(\Lambda_0^a u)(x, t)=(2\pi)^{-1}\int(\tau-i)^{a/2}u^\wedge(x, \tau)e^{it\tau}d\tau \tag{3.8}$$

where $u^\wedge(x, \tau)$ is the partial Fourier transform of u with respect to the t variable, is a causal operator in $L^a_{\lambda,\rho,\delta}$. If $a<-m$ then $P_1=\Lambda_0^a\psi\varphi P$ is a Volterra operator.

We turn now to estimate the norm of $\psi\varphi P$ in $H^s_+(H_t)$ for $0<t<T$. For this we note that $\psi\varphi Pu\in H^s_{+\sigma}(H_t)$ with $\sigma=\operatorname{supp}\varphi$ and that a norm in $H^s_{+\sigma}(H_t)$ may be given by the formula $||u||_{s,t}=||L^sU||_{0,t}$ where U is an extension of u (supported at $\sigma\times\{t\geq 0\}$) and L^s is given by

$$(L^sU)(x, t)=(2\pi)^{-(d+1)}\iint e^{i(x\cdot\xi+t\tau)}(\xi^2+\tau i+1)^{s/2}\hat{U}(\xi, \tau)d\xi\,d\tau \tag{3.9}$$

for the analyticity of $(\xi^2+i\tau+1)^{s/2}$ in the half-plane $\operatorname{Im}\tau>0$ implies that the restriction of L^sU to H_t does not depend on the extension U. Note also that L^s and Λ_0^a commute, and for $w\in H^0_+(H\times R)$, and $a>0$,

$$(\Lambda_0^{-a}w)(x, t)=\int_0^t e^{-(t-t')}(t-t')^{a/2-1}w(x, t')dt'$$

so that

$$\begin{aligned}||\Lambda_0^{-a}w||^2_{0,t}&\leq\int_0^t||\int_0^{t_1}(t_1-t')^{a/2-1}w(x, t')dt'||^2_{L^2(H)}dt_1\\ &\leq Ct^a||w||^2_{0,t}\,.\end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned}||\varphi\psi Pu||_{s,t}&=||L^s\varphi\psi Pu||_{0,t}=||\Lambda_0^{-a}L^sP_1u||_{0,t}\\ &\leq Ct^{a/2}||L^sP_1u||_{0,t}\leq C_1t^{a/2}||P_1u||_{s,t}\leq C_2t^{a/2}||\psi\varphi Pu||_{s+a,t}\\ &\leq C_3t^{a/2}||u||_{s+a+m,t}\leq C_3t^{a/2}||u||_{s,t}\,.\end{aligned} \tag{3.11}$$

It follows from (3.11) that the solution $f(x, t)$ of the equation $f+\psi\varphi Pf=0$ is equal to the zero element of $H_+^s(H_{T_0})$ if $0<T_0^{a/2}<C_3^{-1}$. Iteration of the argument proves that $f\equiv 0$ in H_t. By Fredholm's alternative, the equation (3.7) is uniquely solvable. Moreover, (3.11) implies that if $g(x, t)\equiv 0$ for $0\leq t<t_0$ then $f(x, t)\equiv 0$ for $0<t<t_0$. Let now $\psi_i\in C_0^\infty(-\infty, T)$ be such that $\psi_i(t)\equiv 1$ in a neighborhood of $t\in[0, t_i]$, $i=1, 2$ for $0<t_1<t_2<T$, and let f_i be the unique solution of $f_i+\psi_i\varphi Pf_i=g$, $i=1, 2$. Then $f_1(x, t)=f_2(x, t)$ for $t<t_1$. Thus we can construct $f(x, t)$ (the solution of (3.6)) to be the common value of the solutions of (3.7) for those ψ which are identically equal to 1 in a neighborhood of $[0, t]$. By causality of P, f defined in this way satisfies (3.6).

Combining Propositions 3.1 and 3.2 with some arguments from the proof of Theorem 2.2, we can deduce solvability of the equation $Pu=f$ in $H_+^s(H_t)$, say, if the symbol $p(x, t, \xi, \tau)$ satisfy the conditions of Proposition 3.1 and Theorem 2.2 and *the symbol* $p(x, t, \xi, \tau)$ *never vanishes for* $\xi\in R^d$, $\operatorname{Im}\tau\leq 0$. We will not give the details here, as a stronger result will be proved in the next section (in the process of proving Theorem 4.3).

4. Let Ω be an open, bounded subset of R^n (or of an n-dimensional smooth manifold) with a C^∞ boundary $\partial\Omega$. Let L be a second order parabolic operator,

$$Lu(x, t) = \frac{\partial u}{\partial t} - \sum_{i,j=1}^n a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^n b_i(x, t)\frac{\partial u}{\partial x_i} + c(x, t)u \tag{4.1}$$

where a_{ij}, b_i, $c\in C^\infty(\bar{\Omega}\times\bar{R}_+)$ for $1\leq i, j\leq n$, and the elliptic part of L is strongly elllptic, i.e.,

$$\operatorname{Re}\Big(\sum_{i,j=1}^n a_{ij}(x, t)\xi_i\xi_j\Big)>0 \tag{4.2}$$

for all $(x, t)\in\bar{\Omega}\times\bar{R}_+$ and $\xi\in R^n\setminus\{0\}$. This is equivalent to L being parabolic (compare [10], [11]): For all $\xi\in R^n$ and $\tau\in C$ with $\operatorname{Im}\tau\leq 0$, $|\xi|+|\tau|>0$,

$$i\tau+\sum_{k,l=1}^n a_{kl}(x, t)\xi_k\xi_l\neq 0\,. \tag{4.3}$$

Boundary value problems for L have been studied in [3] and in [10] via pseudo-differential operators on the boundary; we introduce some further notation in order to summarize the information needed for our purposes.

Let H and X be as in section 2. (We write down explicitly the definition for $X\subset R^d$, with obvious modifications if H is a manifold.) Denote by C^- the set of complex numbers with non-positive imaginary part, and by $S^m(X)$ the class of functions $p(x, t, \xi, \tau)\in C^\infty(X\times R^{d+1})$ which are the boundary values of functions defined and continuous on $X\times R^d\times C^-$ and holomorphic in τ for $\operatorname{Im}\tau<0$, and such that there exists a sequence of functions $p_j(x, t, \xi, \tau)$, $j=0, 1, \cdots$, satisfying

(i) $p_j \in C^\infty(X\times R^{d+1}\setminus\{0\})$, p_j is the boundary values of a function defined and continuous on $X\times(R^d\times C^-\setminus\{0, 0\})$ and holomorphic in τ for $\operatorname{Im}\tau<0$,

(ii) $p_j(x, t, \lambda\xi, \lambda^2\tau)=\lambda^{m-j}p_j(x, t, \xi, \tau)$ for all $\lambda>0$, $(x, t)\in X$, $(\xi, \tau)\in R^d\times C^-\setminus\{0, 0\}$.

(iii) For every compact subset K of X, pair of d-dimensional multi-indices α, β and two non-negative integers α_{d+1}, β_{d+1} and for every positive integer N there exists a constant C such that

$$(4.4)\quad |p^{(\alpha,\alpha_{d+1})}_{(\beta,\beta_{d+1})}(x, t, \xi, \tau)-\sum_{j=0}^{N-1}p^{(\alpha,\alpha_{d+1})}_{j(\beta,\beta_{d+1})}(x, t, \xi, \tau)|\leq C[\lambda(\xi, \tau)]^{m-N-|\alpha|-2\alpha_{d+1}}$$

whenever $(x,t, \xi, \tau)\in K\times R^{d+1}$ and $\lambda(\xi, \tau)\geq 1$.

Note that $S^m(X)$ is a subset of $S^m_{\lambda,1,0}(X)$. It was proved in [10] that p_m is invariant under diffeomorphisms of H.

Let N be a C^∞ unit vector field defined in a neighborhood of $\partial\Omega$, transversal to $\partial\Omega$ and pointing to the exterior of $\bar{\Omega}$ (this includes as special cases the unit normal or conormal vector fields). If $u\in C^2(\bar{\Omega}_T)$ we denote by $\gamma_0 u$ the restriction of u to $(\partial\Omega)_T$ and by $\gamma_1 u$ the restriction of Nu to $(\partial\Omega)_T$.

Proposition 4.1. (i) *The linear maps γ_j extend continuously to bounded linear maps of $\{u\in H^s_+(\Omega_T), Lu=0$ in $\Omega_T\}$ into $H^{s-j-(1/2)}_+((\partial\Omega)_T)$ for $j=0, 1$, $s\in R$;* (ii) *For every $s\in R$ and $w\in H^s_+((\partial\Omega)_T)$ there exists a unique $u\in H^{s+(1/2)}_+(\Omega_T)$ such that $Lu=0$ in Ω_T and $\gamma_0 u=w$;* (iii) *The linear map Q: $w\in H^s_+((\partial\Omega)_T)\rightarrow\gamma_1 u\in H^{s-1}_+((\partial\Omega)_T)$ (where u is the solution of $Lu=0$ in Ω_T and $\gamma_0 u=w$) is a pseudo-differential operator in $L^1_{\lambda,1,0}((\partial\Omega)_T)$ with symbol $q\in S^1((\partial\Omega)_T)$. The leading term $q_1(x', t, \xi', \tau)$ of q can be computed at an element (x', t, ξ', τ) of the cotangent bundle $T^*(\partial\Omega\times R)$ with $(\xi', \tau)\neq(0, 0)$, $x'\in\partial\Omega$, $t<T$, as follows: Choose a local coordinate system in an R^{n+1} neighborhood of (x', t) in which Ω is given by $x_n>0$, $\partial\Omega=\{(x', 0)\}$ where $x'=(x_1, \cdots, x_{n-1})$ and $N=-\partial/\partial x_n$. Then q_1 is the solution of the quadratic equation*

$$(4.5)\quad \sum_{j,k=1}^{n-1}a_{jk}(x', t)\xi'_j\xi'_k+i\sum_{j=1}^{n-1}(a_{nj}(x', t)+a_{jn}(x', t))q_1\xi'_j-a_{nn}(x', t)q_1^2+i\tau=0$$

which satisfies $\operatorname{Re} q_1>0$.

Proof. Statement (i) is a special case of Proposition (29) in [10]. Statement (ii) is the well-known solvability of the mixed initial-boundary value problem for parabolic equations with vanishing Cauchy data at $t=0$ (and can be proved by the results of sections 2 and 3 above, compare also [2], [3] and [10]). Statement (iii) follows from the discussion in page 69 of [10]. In fact, the function $e^{i(x'\cdot\xi')-q_1x_n}$ is a solution of the "frozen" differential equation, and is bounded in $x_n\geq 0$ if $\operatorname{Re} q_1>0$. The assumption (4.2) that $\sum^n_{i,j=1}a_{ij}(x, t)\frac{\partial^2}{\partial x_i\partial x_j}$

is strongly elliptic implies that the equation (4.5) has exactly one root with a positive real part. Hence the symbol of the Calderon projector consists of projecting the Cauchy data γ_0, γ_1 at $x_n=0$ onto the subspace generated by the Cauchy data of $e^{i\langle x'\cdot\xi'\rangle - q_1 x_n + i\tau}$.

Let now $a(x, t)$ and $b(x, t)$ be (complex-valued) C^∞ functions defined on $\partial\Omega\times\bar{R}_+$ and consider the degenerate initial boundary value problem

$$(4.6)\qquad Lu = g(x, t) \quad \text{in} \quad \Omega\times R_+$$

$$(4.7)\qquad a(x, t)(Nu)(x, t)+b(x, t)u(x, t) = f(x, t) \quad \text{on} \quad \partial\Omega\times R_+$$

$$(4.8)\qquad u(x, 0) = u_0(x) \quad \text{in} \quad \Omega\,,$$

where $g(x, t)\in C^\infty(\bar{\Omega}\times\bar{R}_+)$, $f(x, t)\in C^\infty(\partial\Omega\times\bar{R}_+)$ and $u_0(x)\in C^\infty(\bar{\Omega})$.

We assume that the data satisfy the following compatibility conditions (compare [8]):

$$(4.9)\qquad \frac{\partial^k f}{\partial t^k}(x, 0) = \frac{\partial^k}{\partial t^k}[aN(u)+bu](x, 0)\,, \qquad x\in\partial\Omega,\ k = 0, 1, \cdots$$

where the right hand side is computed explicitly by successive use of Leibniz rule, (4.6) and (4.8). Note that it suffices to solve (4.6)–(4.8) in $\Omega\times[0, T)$ for some $T>0$, for one can then obtain the solution for all $t\in R_+$ by iterations. Let $h(x, t)\in C^\infty(\partial\Omega\times R_+)$ satisfy the compatibility conditions for the mixed Cauchy-Dirichlet problem (i.e., for $b\equiv 1$, $a\equiv 0$ and h replacing f) and let $v(x, t)$ be the solution of the problem

$$(4.10)\qquad Lv = g(x, t) \quad \text{in} \quad \Omega\times R_+$$

$$(4.11)\qquad v(x, t) = h(x, t) \quad \text{on} \quad \partial\Omega\times R_+$$

$$(4.12)\qquad v(x, 0) = u_0(x) \quad \text{in} \quad \Omega\,.$$

Replacing u by $u-v$ and noting that the mixed Cauchy-Dirichlet problem is well-posed, we see that we may, as we will from now on, consider the problem (4.6)–(4.8) in the case that $g(x, t)\equiv 0$ in $\bar{\Omega}\times R_+$ and $u_0(x)\equiv 0$ in Ω (f being modified by subtracting $a(Nv)+bv$). Then the restriction of the problem (4.6)–(4.8) to $\Omega\times[0, T)$ is equivalent by Proposition 4.1 (and after extending $a(x, t)$ and $b(x, t)$ to $\partial\Omega\times(-\infty, T)$) to the following problem: Find $w\in C^\infty((\partial\Omega)_T)$ such that $w\equiv 0$ for $t<0$ and

$$(4.13)\qquad a(x, t)(Qw)(x, t)+b(x, t)w(x, t) = f(x, t) \quad \text{in} \quad (\partial\Omega)_T$$

(Note that the compatibility conditions and $u_0=0$, $g=0$ imply that $\frac{\partial^k f}{\partial t^k}(x, t)=0$ for $x\in\partial\Omega$, $t=0$, $k=0, 1, \cdots$.) By Proposition 4.1 (iii), (4.13) is a pseudo-differential equation on $(\partial\Omega)_T$ of the form $p(x, t, D_x, D_t)u(x, t)=f(x, t)$ where

$$p(x, t, \xi, \tau) = a(x, t)q(x, t, \xi, \tau)+b(x, t)\,. \tag{4.14}$$

We wish to apply the results of sections 2 and 3 (with $d=n-1$, $H=\partial\Omega$, $X=(\partial\Omega)_T$). To do so we assume that $a(x, t)$ satisfies (2.4), i.e., the zeros of a are multiple relative to x. We assume also that there exists a positive ε such that

$$|\arg[a(x, t)q_1(x, t, \xi, \tau)/b(x, t)]|\leq\pi-\varepsilon \tag{4.15}$$

for $(x, t, \xi, \tau)\in T^*(\partial\Omega\times R_+)$, $(\xi, \tau)\neq 0$ such that the argument is well defined.

Proposition 4.2. *If a and b satisfy* (2.4), (4.15) *and do not vanish simultaneously, then the symbol $p(x, t, \xi, \tau)$ defined in* (4.14) *satisfies the assumptions of Theorem* 2.1, *the operator $aQ+b$ has a* (*right and left*) *parametrix and is hypoelliptic on the boundary $\partial\Omega\times R$ and the solutions of the initial boundary value problem* (4.6)–(4.8) *are C^∞ smooth.*

Proof. By Proposition 4.1, Theorems 2.1 and 2.2, and Corollary 2.3, it suffices to prove the first assertion. Set $\rho=1$, $\delta=0$, $H=\partial\Omega$, $e=q$, $d=n-1$, and $X=(\partial\Omega)_T$. By definition

$$\begin{aligned} e(x, t, \xi, \tau) &= q_1(x, t, \xi, \tau)+(q-q_1)(x, t, \xi, \tau) = \\ &= q_1(x, t, \xi, \tau)+r(x, t, \xi, \tau) \end{aligned} \tag{4.16}$$

where $r\in S^0(X)$ and q_1 is given by (4.5), so that

$$q_1(x, t, \xi, \tau) = \lambda(\xi, \tau)q_1(x, t, \xi\lambda^{-1}(\xi, \tau), \tau\lambda^{-2}(\xi, \tau)) \tag{4.17}$$

and $\mathrm{Re}\, q_1>0$. Hence there exists a constant $c>0$ such that

$$|e(x, t, \xi, \tau)|\geq c\lambda(\xi, \tau) \tag{4.18}$$

if $(x, t)\in\partial\Omega\times[0, T)$ and $\lambda(\xi, \tau)\geq c$. It follows that

$$\begin{aligned} |e^{(\alpha, \alpha_{d+1})}_{(\beta, \beta_{d+1})}(x, t, \xi, \tau)| &\leq C\lambda^{1-|\alpha|-2\alpha_{d+1}} \\ &\leq Cc^{-1}|e(x, t, \xi, \tau)|\lambda^{-|\alpha|-2\alpha_{d+1}} \end{aligned} \tag{4.19}$$

which is equivalent to (2.5). The inequality (2.6) follows from the non-vanishing of $|a|+|b|$ and (4.18). By (4.16), (4.17) and (4.18),

$$\lim_{\lambda(\xi,\tau)\to\infty}[\arg q_1(x, t, \xi, \tau)-\arg e(x, t, \xi, \tau)] = 0 \tag{4.20}$$

uniformly in $\partial\Omega\times[0, T]$. Thus (2.7) follows from (4.15).

Proposition 4.2 provides the extension to the parabolic case of results obtained in sections 2 and 3 of [7] in the elliptic case. To get existence (not just modulo a finite dimensional subspace) and uniqueness we strengthen the assumption (4.15) to: There exists a positive ε such that

$$(4.21)\qquad |\arg[a(x, t)q_1(x, t, \xi, \tau)/b(x, t)]| \leq \pi-\varepsilon$$

for $(x, t, \xi, \tau) \in T^{*-}(\partial\Omega \times R)$ (where $T^{*-}(\partial\Omega \times R)$ is the set of complex cotangent vectors such that ξ is real, $\tau \in C^-$), $(\xi, \tau) \neq (0, 0)$ and the argument is well-defined.

Theorem 4.3. *Let a and b satisfy* (2.4) *and* (4.21) *and assume further that a and b do not vanish simultaneously. Then the initial boundary value problem* (4.6)–(4.8) *has a unique solution in* $C^\infty(\bar{\Omega} \times \bar{R}_+)$.

Proof. By Propositions 4.1 and 4.2 and using the same argument as in the beginning of the proof of Proposition 4.2 we see that it suffices to prove that for every $T>0$, $s \in R$ and $f \in H^s_+((\partial\Omega)_T)$ there exists a unique element $w \in \bigcap_{0<t<T} H^s_+((\partial\Omega)_t)$ such that

$$(4.22)\qquad aQw+bw=f \qquad \text{on} \quad (\partial\Omega)_T\,.$$

Note that the uniqueness theorem for the mixed Cauchy-Dirichlet problem for the operator L in $\Omega \times R_+$ implies that the operator Q is causal. (Hence the operator $aQ+b$ is also causal.) Set

$$(4.23)\qquad \tilde{q}(x, t, \xi, \tau) = q_1(x, t, \xi, \tau-i)\,.$$

Then $\tilde{q} \in C^\infty((\partial\Omega)_T \times R^{n-1} \times R)$, and is holomorphic in τ for $\operatorname{Im}\tau<0$ (actually for $\operatorname{Im}\tau<1$). For every pair of $(n-1)$ dimensional multi-indices α, β and every pair of non-negative integers α_n, β_n, and for *real* ξ, τ such that $\lambda(\xi, \tau)$ is large enough, we expand $q_{1(\beta,\beta_n)}^{(\alpha,\alpha_n)}(x, t, \xi, \tau-i)$ according to Taylor's theorem and obtain the formula

$$(4.24)\qquad \begin{aligned} \tilde{q}_{(\beta,\beta_n)}^{(\alpha,\alpha_n)}(x, t, \xi, \tau) &= \sum_{j=0}^{N-1} \frac{(-i)^j}{j!} \frac{\partial^j}{\partial\tau^j} q_{1(\beta,\beta_n)}^{(\alpha,\alpha_n)}(x, t, \xi, \tau) \\ &+ \frac{(-i)^N}{(N-1)!} \int_0^1 (1-s)^{N-1} \frac{\partial^N}{\partial\tau^N} q_{1(\beta,\beta_n)}^{(\alpha,\alpha_n)}(x, t, \xi, \tau-is)ds \end{aligned}$$

By homogeneity,

$$(4.25)\qquad \begin{aligned} \frac{\partial^N}{\partial\tau^N} q_{1(\beta,\beta_n)}^{(\alpha,\alpha_n)}(x, t, \xi, \tau-is) &= \lambda(\xi, \tau)^{1-|\alpha|-2|\alpha_n|-2N}\,. \\ &\cdot \frac{\partial^N}{\partial\tau^N} q_{1(\beta,\beta_n)}^{(\alpha,\alpha_n)}(x, t, \xi\lambda^{-1}(\xi, \tau), (\tau-is)\lambda^{-2}(\xi, \tau)) \end{aligned}$$

The argument of the right hand side $q_{1(\beta,\beta_n)}^{(\alpha,\alpha_n+N)}$ of (4.25) lies in a compact subset of $T^{*-}(\partial\Omega \times R)$ if $0 \leq s \leq 1$, $\lambda(\xi, \tau) \geq 1+\delta$ for $\delta>0$, and $(x, t) \in K$ where K is compact subset of $\partial\Omega \times R$. Hence (4.4) is satisfied, $\tilde{q} \in S^1(\partial\Omega)$, and $\tilde{q}-q(=-\tilde{q}_1+q_1-q) \in S^0_{\lambda,1,0}((\partial\Omega)_T)$. By (4.21),

$$(4.26) \qquad |\arg[a(x, t)\tilde{q}(x, t, \xi, \tau)/b(x, t)]| \leq \pi - \varepsilon$$

for all $(x, t, \xi, \tau) \in ((\partial\Omega)_T) \times R^{n-1} \times C^-)$ such that $a(x, t) \neq 0$, $b(x, t) \neq 0$. By assumption, a and b do not vanish simultaneously, so that (2.6) is satisfied with $\tilde{q}=e$. Hence (as in (2.8))

$$(4.27) \qquad |a(x, t)\tilde{q}(x, t, \xi, \tau) + b(x, t)| \geq C_1$$

for all $(x, t, \xi, \tau) \in (\partial\Omega)_T \times R^{n-1} \times C^-$. Note that the analog of (2.9) holds for $a\tilde{q}+b$ as the "elliptic" symbol q satisfies (2.5) for real τ and $\rho=1$, $\delta=0$. Arguing as in the proof of Theorem 2.2 (with $\rho=1$, $\delta=\frac{1}{2}$), we conclude that the symbol

$$(4.28) \qquad r(x, t, \xi, \tau) = \frac{1}{a(x, t)\tilde{q}(x, t, \xi, \tau) + b(x, t)}$$

is an element of $S^0_{\lambda,1,1/2}((\partial\Omega)_T)$, and by (4.27) and Proposition 3.1 the pseudo-differential operator $R \in L^0_{\lambda,1,1/2}((\partial\Omega)_T)$ (given by the symbol r) is causal, and the symbol of $(aQ+b)R-I$ is given by

$$(4.29) \qquad \begin{aligned} (aq+b)\circ r - 1 &= [(a\tilde{q}+b)\circ r - (a\tilde{q}+b)r] + a(q-\tilde{q})r \\ &\quad + [a(q-\tilde{q})\circ r - a(q-\tilde{q})r] = p_1 + p_2 + p_3 \,. \end{aligned}$$

By proposition 2.4 (here $e_0=r$) $p_1 \in S^{-1/2}_{\lambda,1,1/2}((\partial\Omega)_T)$. By (4.28) $ar\tilde{q} = 1 - br \in S^0_{\lambda,1,1/2}((\partial\Omega)_T)$. But the principal symbol q_1 of $\tilde{q}$ is invertible of degree 1. Hence $ar \in S^{-1}_{\lambda,1,1/2}((\partial\Omega)_T)$, and the fact that $q-\tilde{q} \in S^0_{\lambda,1,0}((\partial, \Omega)_T)$ implies that $p_2 \in S^{-1}_{\lambda,1,1/2}((\partial\Omega)_T)$. By (2.26) $p_3 \in S^{-1/2}_{\lambda,1,1/2}((\partial\Omega)_T)$. It follows that the operator $V=(aQ+b)R-I$ is a Volterra operator, and by Proposition 3.2 (where we put $\varphi \equiv 1$ by compactness of $\partial\Omega$) the equation

$$(4.30) \qquad (I+V)v = f$$

has a unique solution $v \in \bigcap_{0<t<T} H^s_+((\partial\Omega)_t)$ for all f in $H^s_+((\partial\Omega)_T)$. Thus $w=Rv \in \bigcap_{0<t<T} H^s_+((\partial\Omega)_t)$ is a solution of (4.22). By a similar argument (or by passing to adjoints) we can show that the operator $V'=R(aQ+b)-I$ is a Volterra operator. Hence there is only one solution w for the equation

$$(4.31) \qquad (I+V')w = R(aQ+b)w = Rf$$

in the appropriate space.

EXAMPLE 1. Let the coefficients of Lu be real-valued, let $a_{ij}=a_{ji}$ and let N be the conormal vector field. Then (in a coordinate system where Ω is defined by $x_n>0$) we have (compare (1.5) in [6])

$$Nu = -\sum_{j=1}^{n} a_{jn}\frac{\partial u}{\partial x_j}$$

and if we choose coordinates so as to have $N=-\partial/\partial x_n$ it follows that $a_{jn}=\delta_{jn}$ on the boundary. It follows from (4.5) that in this case $q_1=\sqrt{a+ib}$ where a is non-negative if $\xi'\in R^{n-1}$, $\tau\in C^-$. Hence $|\arg q_1|\leq\frac{\pi}{4}$, and (4.21) is satisfied whenever the functions a and b do not vanish simultaneously and there exists a positive ε such that $|\arg[u(x,t)/b(x,t)]|\leq\frac{3\pi}{4}-\varepsilon$ for $(x,t)\in\partial\Omega\times R$ such that $a\neq 0$, $b\neq 0$. This condition is satisfied if a, $b\geq 0$ (the case considered in [6] and [8]), but the non-negativity or even the reality of a and b are irrelevant–one can take e.g. a real (with odd multiple roots) and b purely imaginary. Moreover, the condition (2.4) restricts the zeros of a to be multiple only with respect to the space variables; the zeros can be simple with respect to t (intuitively, the order of the zero with respect to the variable in X should be higher than the order of the symbol with respect to the dual variable, and q_1 is of order $\frac{1}{2}$ with respect to τ). Thus $a=t-t_0$ and $b=i$ $(0<t_0\leq T)$ or $a=x^k\,(t-t_0)$ for $k\geq 2$ and $b=i$ satisfy all conditions.

EXAMPLE 2. In general it follows from the homogeneity of q_1 and the fact that $\operatorname{Re} q_1>0$ on the compact set where $\lambda(\xi,\tau)=1$ that there exists a positive ε such that $|\arg q_1|\leq\frac{\pi}{2}-\frac{3\varepsilon}{2}$. Thus (4.21) can be satisfied if $|\arg(a/b)|\geq\frac{\pi}{2}+\frac{\varepsilon}{2}$. Once more a and b do not have to be real and non-negative.

REMARK 1. The methods of this paper work equally well if we consider non-cylindrical domains (compare [10] and [11]).

REMARK 2. Using similar methods we can allow b to be a pseudo-differential operator of order zero (and not just a multiplication operator), if we impose appropriate ellipticity conditions near the zeros of $a(x,t)$.

REMARK 3. Extending our theory to systems, we could treat degenerate boundary value problems for higher order operators. The conditions become however much more complicated.

REMARK 4. One can define the concept of a hypoelliptic boundary value problem in analogy to the definition in [7] and show that problems satisfying the conditions of Proposition 4.2 are actually hypoelliptic.

References

[1] R. Beals: *A general calculus of pseudo-differential operators*, Duke Math J. **42** (1975), 1–42.

[2] E.B. Fabes and N.M. Riviere: *Systems of parabolic equations with uniformly continuous coefficients,* J. Analyse Math. **17** (1966), 305–335.

[3] E.B. Fabes and M. Jodeit, Jr.: *Boundary value problems for second-order parabolic equations,* Proc. A.M.S. Symp. on Singular Integrals (1966), 82–105.

[4] L. Hörmander: Pseudo-differential operators and hypoelliptic equations, Proc. A.M.S. Symp. on Singular Integrals (1966), 138–183.

[5] L. Hörmander: The analysis of linear partial differential operators, Vol. III, Springer Verlag, Berlin, 1985.

[6] S. Itô: *Fundamental solutions of parabolic differential equations and boundary value problems,* Japan J. Math. **27** (1957), 55–102.

[7] Y. Kannai: *Hypoellipticity of certain degenerate elliptic boundary value problems,* Trans. Amer. Math. Soc. **217** (1976), 311–328.

[8] S. Mizohata: *On the well-posed singular boundary value problems for heat operators,* Journées "Equations Aux Dérivées Partielles", Saint-Jean-de-Monts, (1983), Conférence no. 9.

[9] A. Piriou: *Une classe d'opérateurs pseudo-différentiels du type de Volterra,* Ann. Inst. Fourier, Grenoble **20**, 1 (1970), 77–94.

[10] A. Piriou: *Problèmes aux limites généraux pour des opérateurs differentiel paraboliques dans un domaine borné,* Ann. Inst. Fourier, Grenoble **21**, 1 (1971), 59–78.

[11] M.I. Višik and G.I. Èskin: *Parabolic convolution equations in a bounded region,* Mat. Sb. **71** (113) (1966), 162–190. English Translation: A.M.S. Translations (2) Vol. 95, 1970, 131–161.

Department of Theoretical Mathematics
The Weizmann Institute of Science
Rehovot 76100, Israel