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A FLUCTUATION THEOREM ASSOCIATED WITH CAUCHY PROBLEMS FOR STATIONARY RANDOM OPERATORS

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1. Introduction

Let $\{H^p; p\in\boldsymbol{R}\}$ be a family of separable real Hilbert spaces which are modeled on the Sobolev spaces on a compact manifold without boundary. Consider a stationary process $L(\omega, t)$ on a probability space $(\Omega, \mathcal{F}, P)$ with values in a certain class of linear operators on $H^{-\infty}=\bigcup_p H^p$, which are modeled on pseudo-differential operators. Denote by L the mean operator of $L(\omega, t)$. We assume that the following abstract Cauchy problems are 'well-posed':

$$(1.1)\qquad \begin{cases} \dfrac{du(t)}{dt}=L\left(\omega, \dfrac{t}{\varepsilon}\right)u(t) \\ u(0)=u_0\in H^p\,, \end{cases}$$

and

$$(1.2)\qquad \begin{cases} \dfrac{du(t)}{dt}=Lu(t) \\ u(0)=u_0\in H^p\,. \end{cases}$$

The aim of this paper is to investigate the fluctuation of $u^\varepsilon(\omega, t)$ around $u^0(t)$ where $u^\varepsilon(\omega, t)$ and $u^0(t)$ are the solutions of (1.1) and (1.2) respectively. Precisely, let $C([0, T]\to H^q)$ be the space of all continuous functions on $[0, T]$ with values in H^q, for $q\in\boldsymbol{R}$. Under the assumption (A.I), (A.II), and (A.III) in Section 2, we show that for any $T>0$, the stochastic process $X^\varepsilon(\omega, t)=\dfrac{u^\varepsilon(\omega, t)-u^0(t)}{\sqrt{\varepsilon}}$ converges weakly to a Gaussian process $X^0(\omega, t)$ in the sense of distribution on $C([0, T]\to H^q)$ for any $q\leq p-\alpha$, where α is determined by the assumptions.

A mathematical motivation of this paper was taken from Khas'minskii's work [8]. We summarize his work here. Let $F(\omega, t, x)$ be a strongly mixing process which is a twice differentiable vecter field on $\boldsymbol{R}^d$ for each ω and t. Let $F(x)$ be the vector field defined as the mean of the process $F(\omega, t, x)$ in some sense. He considered the following Cauchy problems

(1.3)
$$\begin{cases} \dfrac{dx(t)}{dt} = F\Big(\omega, \dfrac{t}{\varepsilon}, x(t)\Big) \\ x(0) = x_0 \in \boldsymbol{R}^d, \end{cases}$$

and

(1.4)
$$\begin{cases} \dfrac{dx(t)}{dt} = F(x(t)) \\ x(0) = x_0 \in \boldsymbol{R}^d \end{cases}$$

and proved the fluctuation of $x^{\varepsilon}(t)$ around $x^0(t)$ where $x^{\varepsilon}(t)$ and $x^0(t)$ are the solutions of (1.3) and (1.4) respectively. In other words, our result might be regarded as an infinite dimensional but linear version of Khas'minskii's work. In his case, the Cauchy problems (1.3) and (1.4) are always well-posed although the random functions $F(\omega, t, \cdot)$ are non-linear in general. In particular the energy estimate (2.4), which plays an imoprtant role in our case, is rather trivial in virtue of the fundamental theory of ordinary differential equations. On the contrary, in the infinite dimensional case, the Cauchy problems are well-studied only for the linear operators. Therefore we shall restrict ourselves to the linear case and consider the well-posed class $\mathcal{L}$ which will be defined in Section 2. Our strongly mixing condition (A.I) is weaker than the assumption (3.3) in [8] in virtue of the boundedness condition of the well-posed class $\mathcal{L}$ (see Remark 1 in [8, p. 222]). Khas'minskii assumed the existence of infinitesimal characteristics instead of the stationarity of the process $F(\omega, t, x)$ but the author do not know how to express those conditions reasonably in the infinite dimensional case. This is the reason why we assume that the process $L(\omega, t)$ is stationary in the sense of the assumption (A.II).

Now we mention the example of the random process $L(\omega, t)$ which satisfies our assumptions (A.I), (A.II), and (A.III). Let T^d be a d-dimensional torus and $\{\eta(\omega, t): t \in \boldsymbol{R}\}$ be a T^d-valued stationary process which satisfies the strongly mixing condition (A.I). Consider the following random operator of elliptic type

$$L(\omega, t) = \sum_{j,k=1}^{d} a_{jk}(x+\eta(\omega, t))\partial_x^{jk} + \sum_{j=1}^{d} b_j(x+\eta(\omega, t))\partial_x^j + c(x+\eta(\omega, t)).$$

Under some regularity conditions on $a_{jk}(x)$, $b_j(x)$ and $c(x)$, we can prove that for each ω, $L(\omega, \cdot)$ belongs to some well-posed class as a function on $\boldsymbol{R}$ with values in operators on the Sobolev spaces $H^p(T^d)$ ($p \in \boldsymbol{R}$) and we can prove that the random function $L(\omega, t)$ satisfies the assumptions (A.I), (A.II), and (A.III). A similar result is valid for random partial differential operators of first order. The proof of the above facts are given in Section 3. These examples are essential in the sense that they suggest the formulation of our problem and illustrate the image of the well-posed class.

It is natural to ask whether the same fluctuation theorem holds or not in the case when the process $L(\omega, t)$ takes values in partial differential operators on a non-compact manifold or a manifold with boundary. In the former case, we have obtained a similar result by use of the weighted Sobolev spaces in [11] when the manifold is just the d-dimensional Euclidean space $\boldsymbol{R}^d$.

We notice that the same problem for the second order parabolic equations is studied in [14] and related topics can be found in [1, p. 516–p. 533] and [7].

The main theorem is stated after the precise description of our problem in Section 2. Two typical examples stated above are discussed in Section 3. The other sections are devoted to the proof of the main theorem in Section 2.

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2. Statement of Theorem

First of all we define a family of abstract Sobolev spaces $H^p(p\in\boldsymbol{R})$ which are modeled on the Sobolev spaces on an orientable compact manifold without boundary. Let H^0 be a real separable Hilbert space endowed with an inner product $(\cdot,\cdot)_0$ and let Λ be a positive definite self-adjoint operator with the inverse Λ^{-1} which is assumed to be a Hilbert-Schmidt operator. Let $p>0$, put $H^p=\mathcal{D}(\Lambda^p)$: the domain of Λ^p and define a Hilbertian norm on H^p by $||u||_p=||\Lambda^p u||_0$ for $u\in H^p$. For $u\in H^0$, we define a Hilbertian norm by $||u||_{-p}=||\Lambda^{-p}u||_0$. H^{-p} is defined as the completion of H^0 by the norm $|||\cdot||_{-p}$. Then it is easy to see that H^p is continuously embedded into H^q for $p>q$ and the inclusion is a compact operator. Moreover, if $p\geqq q+1$, the inclusion is a Hilbert-Schmidt operator. Writing $H^\infty=\bigcap_{p\in\boldsymbol{R}} H^p$ and $H^{-\infty}=\bigcup_{p\in\boldsymbol{R}} H^p$, the operator Λ can be uniquely extended to the operator on $H^{-\infty}$ which is also denoted by the same letter Λ. Then the Hilbert space H^p is characterized as $H^p=\{u\in H^{-\infty};\ \Lambda^p u\in H^0\}$ and $||u||_p=||\Lambda^p u||_0$. Such a family of abstract Sobolev spaces is called a scale of Hilbert spaces in Daletskii [3].

Next, we introduce a class of time dependent operators on $H^{-\infty}$ which satisfy some conditions for the 'well-posedness' of the equations (1.1) and (1.2). In what follows, for topological spaces E_1 and E_2, $C(E_1\to E_2)$ denotes the space of all continuous mappings from E_1 into E_2.

DEFINITION. Let $H^p(p\in\boldsymbol{R})$ be a family of abstract Sobolev spaces which are defined above. Given a positive number m and families of positive numbers $\{C_p\}_{p\in\boldsymbol{R}}$ and $\{C_{T,p}\}_{T>0,p\in\boldsymbol{R}}$, we say that a function $L(\cdot)$ defined on $\boldsymbol{R}$ with values in operators on $H^{-\infty}$ belongs to the *well-posed class* $\mathcal{L}=\mathcal{L}(m, \{C_p\}_{p\in\boldsymbol{R}}, \{C_{T,p}\}_{T>0,p\in\boldsymbol{R}})$ if it satisfies the following conditions:

(1) For each $t\in\boldsymbol{R}$, $L(t)$ is a linear operator on $H^{-\infty}$ and $L(t)H^{p+m}\subset H^p$ for any $p\in\boldsymbol{R}$. Moreover, $L(\cdot)\in C(\boldsymbol{R}\to\boldsymbol{B}(H^{p+m}\to H^p))$ and

$$\sup_{t\in R}\sup_{\substack{u\in H^{p+m}\\ ||u||_{p+m}=1}} ||L(t)u||_p \leqq C_p , \tag{2.1}$$

where $B(H^{p+m}\to H^p)$ denotes the Banach space of all bounded linear operators from H^{p+m} into H^p.

(2) For any $T>0$, and for any $u_0\in H^{p+m}$, the Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = L(t)u(t) \\ u(0) = u_0 \end{cases} \tag{2.2}$$

has a solution in $C([0, T]\to H^{p+m})\cap C^1([0, T]\to H^p)$.

(3) (energy estimate). If $v(\cdot)\in C([0, T]\to H^{p+m})\cap C^1([0, T]\to H^p)$ satisfies

$$\frac{dv(t)}{dt} = L(t)v(t)+f(t) \quad \text{in} \quad H^p \tag{2.3}$$

for $f(\cdot)\in C([0, T]\to H^p)$, then we have

$$||v(t)||_p^2 \leqq C_{T,p}(||v(0)||_p^2 + \int_0^t ||f(s)||_p^2 ds) \tag{2.4}$$

for all $t\in[0, T]$.

(4) For any $s>0$, $L^s(\cdot)$ also satisfies the conditions (1), (2), and (3). Here $L^s(\cdot)$ is the operator valued function defined by $L^s(t)=L(st)$.

REMARK. Let $L(\cdot)$ belong to $\mathcal{L}$. Then for any $u_0\in H^{p+m}$ and for any $f(\cdot)\in C([0, T]\to H^{p+m})$, the Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = L(t)u(t)+f(t) \\ u(0) = u_0 \end{cases} \tag{2.5}$$

has a unique solution in $C([0, T]\to H^{p+m})\cap C^1([0, T]\to H^p)$ in virtue of the conditions (2) and (3).

We believe that if one looks at the formulation of Cauchy problems in [9], [12], and [13] he can see that our assumpitons on $\mathcal{L}$ are reasonable.

Now we add the probabilistic assumptions. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_s^t;\ -\infty\leqq s\leqq t\leqq\infty\}$ be a family of sub-σ-algebras of $\mathcal{F}$ with $\mathcal{F}_{t_1}^{s_1}\subset\mathcal{F}_{s_2}^{t_2}$ for $s_2\leqq s_1$ and $t_1\leqq t_2$.

Our first assumption is the following:

$$\alpha(s) = \sup_t \sup_{\xi,\eta} |E[\xi\eta]-E[\xi]E[\eta]| \tag{A.I}$$

decreases to 0 as s goes to ∞ and $\int_0^\infty s\alpha(s)\, ds <\infty$. Here $\sup_{\xi,\eta}$ is taken over all

$\mathcal{F}^t_{-\infty}$-measurable ξ with $|\xi|\leqq 1$ and all $\mathcal{F}^{\infty}_{t+s}$-measurable η with $|\eta|\leqq 1$. As usual $E[\cdot]$ denotes the expectation with respect to the probability measure P.

A stochastic process $\{\Phi(\omega, t);\ -\infty<t<\infty\}$ is called a strongly mixing process with mixing coefficient $\alpha(t)$ if it is $\mathcal{F}^t_t$-measurable for each t fixed. Let $\mathcal{L}=\mathcal{L}(m, \{C_p\}_{p\in\boldsymbol{R}}, \{C_{T,p}\}_{T>0,p\in\boldsymbol{R}})$ be a well-posed class. Consider a random function $L(\omega, \cdot)$ on $(\Omega, \mathcal{F}, P)$ with values in $\mathcal{L}$. Our second assumption is:

(A.II) For any $u, v\in H^{\infty}$, the real valued stochastic process $\{(L(\omega, t)u, v)_0;\ -\infty<t<\infty\}$ is a stationary and strongly mixing process with mixing coefficient $\alpha(t)$.

Then we can define the mean operator L of $L(\omega, t)$ as follows: First we have

$$E|(L(\cdot, t)u, v)_p|\leqq E||L(\cdot, t)u||_p||v||_p\leqq C_p||u||_{p+m}||v||_p \tag{2.6}$$

for any $u, v\in H^{\infty}$, from the condition (1) on $\mathcal{L}$. Thus, for any $u\in H^{\infty}$ we can define Lu as an element of H^0 such that $E(L(\cdot, t)u, v)_0=(Lu, v)_0$ for any $v\in H^{\infty}$ in virtue of Riesz' representation theorem. Obviously Lu is independent of t since $L(\omega, t)$ is stationary. Using the estimate (2.6), L can be extended uniquely to an operator on $H^{-\infty}$ which is also denoted by L. Clearly, L satisfies the condition (1) on $\mathcal{L}$ as a constant operator valued function on $\boldsymbol{R}$. But we do not know whether L belongs to $\mathcal{L}$ or not. From this point of view, our last assumption is:

(A.III) The operator L belongs to $\mathcal{L}$.

Nwo we can state our result.

Theorem. *Let* $\mathcal{L}=\mathcal{L}(m, \{C_p\}_{p\in\boldsymbol{R}}, \{C_{T,p}\}_{T>0,p\in\boldsymbol{R}})$ *be a well-posed class on the abstract Sobolev spaces* $H^p(p\in\boldsymbol{R})$ *and let* $L(\omega, t)$ *be a random function satisfying the assumptions* (A.I), (A.II), *and* (A.III). *For* $u_0\in H^{p+3m+1}$ *and* $\varepsilon>0$, *we denote by* $u^{\varepsilon}(\omega, t)$ *and* $u^0(t)$ *the solutions of the abstract Cauchy problems*:

$$\begin{cases} \dfrac{du(t)}{dt} = L\left(\omega, \dfrac{t}{\varepsilon}\right)u(t) \\ u(0) = u_0 \end{cases} \tag{2.7}$$

and

$$\begin{cases} \dfrac{du(t)}{dt} = Lu(t) \\ u(0) = u_0 \end{cases} \tag{2.8}$$

respectively. Then for any $T>0$, *we have*

$$\sup_{0\leq t\leq T} E\|u^\varepsilon(t)-u^0(t)\|_p^2\leq C\varepsilon \tag{2.9}$$

where C is a constant which is independent of ε. Moreover, the distribution of $X^\varepsilon(\omega, t)=\dfrac{u^\varepsilon(\omega, t)-u^0(t)}{\sqrt{\varepsilon}}$ converges weakly on $C([0, T]\to H^p)$ as ε goes to 0. The limit distribution coincides with the distribution of an H^{p+m}-valued continuous stochastic process $\{X^0(\omega, t); 0\leq t\leq T\}$ which satisfies the equation

$$X^0(\omega, t) = W^0(\omega, t)+\int_0^t LX^0(\omega, s)ds \qquad in \quad H^p \tag{2.10}$$

where the integration in the right hand side means the Bochner integral of an H^p-valued function on $[0, T]$. $\{W^0(\omega, t); 0\leq t\leq T\}$ is an H^{p+2m}-valued continuous stochastic process with independent increments characterized by

$$E(W^0(t), v)_{p+2m} = 0 \tag{2.11}$$

and

$$E[(W^0(t), v)_{p+2m}(W^0(s), w)_{p+2m}] = \int_0^{t\wedge s} \langle v, w\rangle(u^0(r))dr \tag{2.12}$$

for $v, w\in H^\infty$, where $\langle v, w\rangle(u)$ is given by

$$\begin{aligned}\langle v, w\rangle(u) = \int_0^\infty dtE[((L(t)-L)u, v)_{p+2m}((L(0)-L)u, w)_{p+2m} \\ +((L(0)-L)u, v)_{p+2m}(L(t)-L)u, w)_{p+2m}]\end{aligned} \tag{2.13}$$

for $u\in H^{p+3m}$ and $v, w\in H^\infty$.

In the statement of Theorem, we did not refer to the measurability of $u^\varepsilon(\omega, t)$ but it is guaranteed by the following:

Proposition 2.1. *Assume that a random function $L(\omega, t)$ with values in $\mathcal{L}$ satisfies that $(L(\omega, t)u, v)_0$ is $\mathcal{F}$-measurable for any $u, v\in H^\infty$. Then for any $u_0\in H^p$, the solution of the evolution equation*

$$\begin{cases} \dfrac{du(t)}{dt} = L(\omega, t)u(t) \\ u(0) = u_0 \end{cases} \tag{2.14}$$

is $\mathcal{F}\,|\,\mathcal{B}(C([0, T]\to H^p))$-measurable, where $\mathcal{B}(E)$ denotes the topological Borel σ-algebra of a topological space E.

Proof. Let $\mathcal{H}$ be the Hilbert space of all Hilbert-Schmidt operators from H^p into H^{p-m-1} endowed with Hilbert-Schmidt norm. We regard $\mathcal{L}$ as a topological subspace of $C([0, T]\to\mathcal{H})$. For a sequence $\{L_n\}_{n=0}^\infty\subset\mathcal{L}$ we denote by $u^n(\cdot)$, the solution of the equation $\frac{du(t)}{dt}=L_n(t)u(t),\ u(0)=u_0$. Assume that L_n converges to L_0 in $C([0, T]\to\mathscr{H})$ as $n\to\infty$, $n\geqq 1$. Then we have

$$\frac{d}{dt}(u^n(t)-u^0(t)) = L_n(t)(u^n(t)-u^0(t))+(L_n(t)-L_0(t))u^0(t)$$

in H^{p-m} and so in H^{p-m-1}. From the energy estimate (2.4), we have

$$\begin{aligned}||u^n(t)-u^0(t)||^2_{p-m-1}&\leqq C_{T,p-m-1}\int_0^t ||(L_n(s)-L_0(s))u^0(s)||^2_{p-m-1}ds\\ &\leqq C_{T,p-m-1}T\sup_{0\leqq s\leqq T}||L_n(s)-L_0(s)||^2_{HS}\cdot\sup_{0\leqq s\leqq T}||u^0(s)||^2_p\,.\end{aligned}$$

Thus the function Φ: $\mathscr{L}\ni L(\cdot)\to u(\cdot)\in C([0, T]\to H^{p-m-1})$ is continuous where $u(t)$ is the solution of $\frac{du(t)}{dt}=L(t)u(t),\ \ u(0)=u_0$. Next we show that the function $L(\cdot, \cdot)$: $\omega\in\Omega\to L(\omega, \cdot)\in\mathscr{L}$ is $\mathscr{F}|\mathscr{B}(\mathscr{L})$-measurable. In virtue of the second countability of the topological space $\mathscr{L}$, it suffices to show that $\{\omega\in\Omega;\ \sup_{0\leqq t\leqq T}||L(\omega, t)-L(t)||_{HS}<\delta\}\in\mathscr{F}$ for any $L(\cdot)\in\mathscr{L}$ and any $\delta>0$. On the other hand, we have

$$\begin{aligned}&\{\omega\in\Omega;\ \sup_{0\leqq t\leqq T}||L(\omega, t)-L(t)||_{HS}<\delta\}\\ &\quad=\bigcup_{n=1}^{\infty}\ \bigcap_{t\in Q\cap[0,T]}\{\omega;\ ||L(\omega, t)-L(t)||_{HS}\leqq\delta-\frac{1}{n}\}\end{aligned}$$

and

$$\begin{aligned}&\{\omega;\ ||L(\omega, t)-L(t)||_{HS}\leqq\delta-\frac{1}{n}\}\\ &\quad=\{\omega;\ \sum_{k=1}^{\infty}||(L(\omega, t)-L(t))e_k||^2_{p-m-1}\leqq\left(\delta-\frac{1}{n}\right)^2\}\end{aligned}$$

where $\{e_k\}_{k=1}^{\infty}$ is a complete orthonormal system of H^p. Since for any $e\in H^p$,

$$||(L(\omega, t)-L(t))e||_{p-m-1}=\sup_{\substack{v\in H^p\\ ||v||_p=1}}|((L(\omega, t)-L(t))e, v)_{p-m-1}|$$

and since H^{p-m-1} is separable, we conclude that $||(L(\omega, t)-L(t))e||_{p-m-1}$ is $\mathscr{F}$-measurable. This implies that $\{\omega;\ \sup_{0\leqq t\leqq T}||L(\omega, t)-L(t)||_{HS}<\delta\}\in\mathscr{F}$. Thus the function $L(\cdot, \cdot)$: $\omega\to L(\omega, \cdot)$ is $\mathscr{F}|\mathscr{B}(\mathscr{L})$-measurable. Since $u(\omega, \cdot)=\Phi(L(\omega, \cdot))$, $u(\omega, \cdot)$ is $\mathscr{F}|\mathscr{B}(C([0, T]\to H^{p-m-1}))$-measurable. On the other hand $u(\omega, \cdot)\in C([0, T]\to H^p)$ by the condition (2) on $\mathscr{L}$ and $C([0, T]\to H^p)$ is a Borel subset of $C([0, T]\to H^{p-m-1})$, we can see that $u(\omega, t)$ is $\mathscr{F}|\mathscr{B}(C([0, T]\to H^p))$-measurable.

3. Examples

In this section we shall give two typical examples of random functions $L(\omega, t)$ which satisfy the assumptions (A.I), (A.II), and (A.III) in the previous section (see Proposition 3.2 below).

Let M be a d-dimensional torus $\boldsymbol{R}^d/\boldsymbol{Z}^d$. As usual we identify a point in M with a point in $[0, 1)^d$ and a function on M with a function on $\boldsymbol{R}^d$ which is invariant under the action of $\boldsymbol{Z}^d$. Let $H^p(M)$ be the Sobolev space of order p on M. If we put $H^p = H^{p\beta}(M)$ for some $\beta > \frac{d}{2}$, $H^p(p \in \boldsymbol{R})$ forma family of abstract Sobolev spaces. In this case, $H^0 = H^0(M) = L^2(M)$ ($= L^2$-space with respect to the Haar measure on M) and $\Lambda = \Lambda_0^\beta$ where $\Lambda_0 = (1-\Delta)^{1/2}$ and $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_j^2}$. Let K be a positive constant and let $\{A_\alpha\}_\alpha$ be a family of positive numbers where α's are multi-indices. Put

$$\mathcal{A} = \{a(t, x);\ a(t, \cdot) \text{ ia s continuous function from } \boldsymbol{R} \text{ into } C^\infty(M \to \boldsymbol{R}) \text{ and } \sup_{t,x} |\partial_x^\alpha a(t, x)| \leqq A_\alpha \text{ for any multi-index } \alpha\},$$

where $\partial_x^\alpha = \partial^{\alpha_1 + \cdots + \alpha_d}/\partial x_1^{\alpha_1} \cdots x\partial_d^{\alpha_d}$. Consider the following classes of time dependent differential operators:

$$\mathcal{L}_1 = \{L(t) = \sum_{k=1}^d b_j(t, x)\partial_x^j + c(t, x);\ b_j(t, x) \in \mathcal{A},\ c(t, x) \in \mathcal{A}\quad j = 1, 2, \cdots, d\}$$

and

$$\mathcal{L}_2 = \{L(t) = \sum_{j,k=1}^d a_{jk}(t, x)\partial_{kx}^j + \sum_{j=1}^d b_j(t, x)\partial_x^j + c(t, x);\ a_{jk}(t, x) \in \mathcal{A},\ b_j(t, x) \in \mathcal{A},\ c(t, x) \in \mathcal{A}\quad 1 \leqq j, k \leqq d \text{ and } \inf_{t,x} \sum_{j,k=1}^d a_{ij}(t, x)\xi_j\xi_k \geqq K \sum_{j=1}^d \xi_j^2 \quad \text{for any } (\xi_1, \cdots, \xi_d) \in \boldsymbol{R}^d\}.$$

First we show the following:

Lemma 3.1. *There is a family of positive constants $\{K_p\}_{p \in \boldsymbol{R}}$ such that for any $u \in H^\infty(M)$*

$$\sup_{t \in \boldsymbol{R}} (L(t)u, u)_p \leqq K_p \|u\|_p^2 \tag{3.1}$$

holds for any $L(\cdot) \in \mathcal{L}_1 \cup \mathcal{L}_2$.

Proof. For $L(\cdot) \in \mathcal{L}_1$, we have $2(\Lambda^p L(t)u, \Lambda^p u)_0 = 2([\Lambda^p, L(t)]u, \Lambda^p u)_0 + ((L(t) + L(t)^*)\Lambda^p u, \Lambda^p u)_0$ for each $t \in \boldsymbol{R}$, where $L(t)^*$ is the formal adjoint of $L(t)$ and $[\Lambda^p, L(t)] = \Lambda^p L(t) - L(t)\Lambda^p$. In the same way as Corollary 1° and Corollary

2° of Theorem 1.7 in [9, p. 59–p. 60], we can show that $L(t)+L(t)^*$ and $[\Lambda^p, L(t)]$ are pseudo-differential operators of order 0 and $p+1-1=p$ respectively. In particular we can show that $\|(L(t)+L(t)^*)u\|_0 \leqq D_1\|u\|_0$ and $\|[\Lambda^p, L(t)]u\|_0 \leqq D_2\|u\|_p$ for $u\in H^\infty(M)$ where D_1 and D_2 are positive constants depending only on p and A_α's. Thus we have $|(L(t)u, u)_p| \leqq (D_1+2D_2)\|u\|_p^2$. This implies (3.1) for $L(\cdot)\in\mathcal{L}_1$.

For $L(\cdot)\in\mathcal{L}_2$, we consider the operator $\Lambda^p(-L(t))\Lambda^{-p}$. Then it is a second order elliptic pseudo-differential operator on M. In the same way as the proof of Gårding's inequality for an elliptic operator on $\boldsymbol{R}^d$, we can show that

$$-(\Lambda^p L(t)\Lambda^{-p}u, u)_0 \geqq D_3\|u\|_1^2 - D_4\|u\|_0^2 , \tag{3.2}$$

for any $u\in H^\infty(M)$ where D_3 and D_4 are positive constants depending only on p, K, and A_α's. For the detailed proof see Kumano-go [9, p. 54–0p. 60, p. 79–p. 81 and p. 134] or Taylor [13 Chapter II p. 55]. In the inequality (3.2), substitut; ing $\Lambda^p u$ for u we obtain $(\Lambda^p L(t)u, \Lambda^p u)_0 D_4 \leqq \|\Lambda^p u\|_0^2$. This implies (3.1) for $L(\cdot)\in\mathcal{L}_2$.

Now we can show:

Proposition 3.1. *There exist families of positive numbers* $\{C_p\}_{p\in\boldsymbol{R}}$ *and* $\{C_{T,p}\}_{T>0,p\in\boldsymbol{R}}$ *such that* $\mathcal{L}_k$ *is a subclass of the well-posed class* $\mathcal{L}(k, \{C_p\}_{p\in\boldsymbol{R}'} \{C_{T,p}\}_{T>0,p\in\boldsymbol{R}})$ *for* $k=1, 2$.

Proof. To prove the proposition, we have to verify that every $L(\cdot)\in\mathcal{L}_k$ satisfies the conditions (1), (2), (3) and (4) in the definition of the well-posed class. It is an immeadiate consequence of the Calderon-Vaillancourt theorem (see 11, p. 224]) that there is a family of positive numbers $\{C_p\}_{p\in\boldsymbol{R}}$ such that the inequality (2.1) holds for any $L(\cdot)\in\mathcal{L}_k$, $k=1, 2$. So (1) is valid for any $L(\cdot)\in\mathcal{L}$, $k=1, 2$. (4) is clear from the definition of $\mathcal{L}_k$. Next we prove the energy estimate (2.4). Suppose that for $L(\cdot)\in\mathcal{L}_k$, $v(\cdot)\in C([0, T]\to H^{p+k})\cap C^1([0, T]\to H^p)$ and $\frac{dv(t)}{dt}=L(t)v(t)+f(t)$ for $f(\cdot)\in C([0, T]\to H^p)$. Then we have $\frac{d}{dt}\|v(t)\|_p^2 = 2(L(t)v(t), v(t))_p+2(f(t), v(t))_p \leqq 2K_p\|v(t)\|_p^2+\|f(t)\|_p^2+\|v(t)\|_p^2=(2K_p+1)\times \|v(t)\|_p^2+\|f(t)\|_p^2$ in virtue of Lemma 3.1. Applying Gronwall's inequality ([13, p. 73]) to $\|v(t)\|_p^2$ we have

$$\|v(t)\|_p^2 \leqq e^{(2K_p+1)T}(\|v(0)\|_p^2+\int_0^t \|f(s)\|_p^2 ds)$$

for any $t\in[0, T]$. So we can take $e^{(2K_p+1)T}$ for $C_{T,p}$. Thus the condition (3) is verified. From the energy estimate we can show the solvability condition (2) by the standard manner in Taylor [13, Chapter IV].

Next we construct the examples of random functions $L(\omega, t)$ which satisfy the assumptions (A.I), (A.II) and (A.III). Let $\{\eta(\omega, t); -\infty<t<\infty\}$ be an $\boldsymbol{R}^d$-valued stationary and strongly mixing process with mixing coefficient $\alpha(t)$. Furthermore we assume that all of its sample paths are continuous. For fixed elements

$$L_1=\sum_{j=1}^{d} b_j(x)\partial_x^j+c(x)\in\mathcal{L}_1$$

and

$$L_2=\sum_{j,k=1}^{d} a_{jk}(x)\partial_x^{jk}+\sum_{j=1}^{d} b_j(x)\partial_x^j+c(x)\in\mathcal{L}_2$$

with coefficients which do not depend on t, we define random functions $L_1(\omega, t)$ and $L_2(\omega, t)$ taking values in $\mathcal{L}_1$ and $\mathcal{L}_2$ by

$$L_1(\omega, t)=\sum_{j=1}^{d} b_j(x+\eta(\omega, t))\partial_k^j+c(x+\eta(\omega, t))$$

and

$$L_2(\omega, t)=\sum_{j,k=1}^{d} a_{jk}(x+\eta(\omega, t))\partial_x^{jk}+\sum_{j=1}^{d} b(x+\eta(\omega, t))\partial_x^j+c(x+\eta(\omega, t))$$

respectively. Then we can prove:

Proposition 3.2. *The random function* $L_k(\omega, t)$ *satisfies the assumptions* (A.I), (A.II) *and* (A.III) *for* $k=1, 2$.

Proof. For any $u, v\in H^\infty(M)$ and $y\in\boldsymbol{R}^d$ the map $\Phi_k: y\to(L_k(y)u, v)_0$ is continuous where

$$L_1(y)=\sum_{j=1}^{d} b_j(x+y)\partial_x^j+c(x+y)$$

and

$$L_2(y)=\sum_{j,k=1}^{d} a_{jk}(x+y)\partial_x^{jk}+\sum_{j=1}^{j} b_j(x+y)\partial_x^k+c(x+y)\,.$$

By the assumption on $\eta(\omega, t)$, $(L_k(\omega, t)u, v)_0=\Phi_k(\eta(\omega, t))$ is also a stationary and strongly mixing process with mixing coefficient $\alpha(t)$ for $k=1, 2$. Thus (A.I) and (A.II) are satisfied. It remains to show (A.III). Denote by $\bar{L}_k$ the mean operator of $L_k(\omega, t)$. For each $x\in M$ fixed, put $\bar{a}_{jk}(x)=E[a_{jk}(x+\eta(\cdot, t))]$, $\bar{b}_j(x)=E[b_j(x+\eta(\cdot, t))]$ and $\bar{c}(x)=E[c(x+\eta(\cdot, t))]$ for $1\leqq j, k\leqq d$. We can easily see that

$$\bar{L}_1=\sum_{j=1}^{d} \bar{b}_j(x)\partial_x^j+\bar{c}(x)$$

and

$$\bar{L}_2 = \sum_{j,k=1}^{d} \bar{a}_{jk}(x)\partial_x^{jk} + \sum_{j=1}^{d} \bar{b}_j(x)\partial_x^j + \bar{c}(x),$$

and $\bar{L}_k$ belongs to $\mathcal{L}_k$ for $k=1, 2$. Hence (A.III) is satisfied. //

REMARKS. (1) Put $H^p = \overbrace{H^{p\beta}(M) + \cdots + H^{p\beta}(M)}^{r}$ and $\Lambda = (\overbrace{\Lambda_0^{\frac{\beta}{2}} + \cdots + \Lambda_0^{\frac{\beta}{2}}}^{r}$ Consider the following class of time dependent operators on $H^{-\infty}$.

$$\mathcal{L}_1^r = \{L(t) = \sum_{j=1}^{d} B_j(t, x) \begin{bmatrix} \partial_x^j & & O \\ & \ddots & \\ O & & \partial_x^j \end{bmatrix} + C(t, x);$$

$B_j(t, x)$ is an $r \times r$-symmetric matrix with entries in $\mathcal{A}$ and $C(t, x)$ is an $r \times r$-matrix with entries in $\mathcal{A}$.$\}$

For $u_0 = \begin{bmatrix} u_{01} \\ \vdots \\ u_{0r} \end{bmatrix} \in H^p$, we can consider the Cauchy problem for the first order symmetric hyperbolic system. $\mathcal{L}_1$ is the special case $(r=1)$ of $\mathcal{L}_1^r$.

(2) For a positive integer m consider the class

$$\mathcal{L}_{2m} = \{ L(t) = \sum_{|\alpha| \leq 2m} a_\alpha(t, x)\partial_x^\alpha;\ a_\alpha(t, x) \in \mathcal{A}$$

$$\text{and} \quad \inf_{t,x} ((-1)^{m+1} \sum_{|\alpha|=2m} a_\alpha(t, x)\xi^\alpha) \geqq K(\sum_{j=1}^{d} \xi_j^2)^m$$

$$\text{for any} \quad \xi = (\xi_1, \cdots, \xi_d) \in \boldsymbol{R}^d\},$$

where $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_d^{\alpha_d}$ for multi-index α. The operator in $\mathcal{L}_{2m}$ is called the $2m$-th order elliptic differential operator. In these cases of $\mathcal{L}_1^r$ and $\mathcal{L}_{2m}$, we can construct the same examples as in the cases of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively.

4. Auxiliary Lemmas

First of all, we give two lemmas, Lemma 4.1 and Lemma 4.2 which are concerned with strongly mixing processes. Lemma 4.1 is the basic tool in our argument and it can be proved in the same way as the proof of Lemma 2.1 in [8]. Lemma 4.2 is used in Step 1 of the proof of Theorem in the next section, and it is an immeadiate consequence of Theorem 18.2.1 and Theorem 18.3.1 in [6].

Lemma 4.1. *Let n be a positive integer. Let $\Phi_i(\omega, t)$, $i=1, 2, \cdots, 2n$ be real valued strongly mixing processes with mixing coefficient $\alpha(t)$ where expectations are zeeo for rach t and $M_i = \sup_{t,\omega} |\Phi_i(\omega, t)| < \infty$. Then there is a positive constant $C(n)$ which depends only on n and $\alpha(t)$ such that*

$$(4.1)\qquad \int_t^{t+T}\int_t^{t+T}\cdots\int_t^{t+T} ds_1 ds_2\cdots ds_{2n}\,|E[\Phi_1(s_1)\Phi_2(s_2)\cdots\Phi_{2n}(s_{2n})]| \leqq C(n)T^n M_1 M_2\cdots M_{2n} \quad \text{for any} \quad t. \qquad //$$

Lemma 4.2. *Let* $\{\Phi(\omega, t);\ -\infty<t<\infty\}$ *be a real valued stationary and strongly mixing process with mixing coefficient* $\alpha(t)$ *whose expectation is zero and* $M=\sup_{t,\omega}|\Phi(\omega, t)|<\infty$. *Then there is a positive constant C such that*

$$(4.2)\qquad \left|\frac{1}{T}E\left|\int_0^T \Phi(t)dt\right|^2-2\int_0^\infty E[\Phi(t)\Phi(0)]dt\right| \leqq \frac{CM^2}{\sqrt{T}}. \qquad //$$

In what follows we drop the letter ω if there occurs no confusion and we always assume the hypotheses of Theorem in Section 2. Recall that

$$(4.3)\qquad X^\varepsilon(t)=\frac{u^\varepsilon(t)-u^0(t)}{\sqrt{\varepsilon}}$$

where $u^\varepsilon(t)$ and $u^0(t)$ are the solutions of evolution equations (2.7) and (2.8) respectively. Since the initial data u_0 is in H^{p+3m+1}, we have $X^\varepsilon(\cdot)\in C([0, T]\to H^{p+3m+1})\cap C^1([0, T]\to H^{p+2m+1})$ and $X^\varepsilon(\cdot)$ satisfies the equation

$$(4.4)\qquad X^\varepsilon(t)=W^\varepsilon(t)+\int_0^t L\left(\frac{s}{\varepsilon}\right)X^\varepsilon(s)ds$$

as an H^{p+2m+1}-valued function on $[0, T]$. Here $W^\varepsilon(t)$ is defined by

$$(4.5)\qquad W^\varepsilon(t)=\frac{1}{\sqrt{\varepsilon}}\int_0^t\left(L\left(\frac{s}{\varepsilon}\right)-L\right)u^0(s)ds\in C^1([0, T]\to H^{p+2m+1}).$$

Let $Y^\varepsilon(t)$ be the solution of the equation

$$(4.6)\qquad \begin{cases} Y^\varepsilon(t)=W^\varepsilon(t)+\int_0^t LY^\varepsilon(s)ds \\ Y^\varepsilon(0)=0. \end{cases}$$

Since $\frac{1}{\sqrt{\varepsilon}}\left(L\left(\frac{t}{\varepsilon}\right)-L\right)u^0(t)\in C([0, T]\to H^{p+2m+1})$, the equation (4.6) has a uuiqne solution in $C([0, T]\to H^{p+2m+1})\cap C^1([0, T]\to H^{p+m+1})$ in virtue of the assumptions on the class $\mathcal{L}$. Put

$$(4.7)\qquad Z^\varepsilon(t)=X^\varepsilon(t)-Y^\varepsilon(t)\in C([0, T]\to H^{p+2m+1}).$$

Then $Z^\varepsilon(t)$ satisfies the equation

$$(4.8)\qquad Z^\varepsilon(t)=\int_0^t L\left(\frac{s}{\varepsilon}\right)Z^\varepsilon(s)ds+\int_0^t\left(L\left(\frac{s}{\varepsilon}\right)-L\right)Y^\varepsilon(s)ds$$

as an H^{p+m+1}-valued function.

In the rest of this paper, unless otherwise stated, the letter C is commonly used to denote those constants which are independent of ε, ω, and, $t\in[0, T]$.

In the following lemma we give the basic estimates for the above defined processes which guarantee that the distributions of the associated processes are tight on $C([0, T]\to H^q)$ for properly chosen q.

Lemma 4.3. *Let $\{e_k^0\}_{k=1}^\infty$ be a complete orthonormal system of H^0 which consists of the eigenvectors of Λ. Then $\{e_k^p=\Lambda^{-p}e_k^0\}_{k=1}^\infty$ becomes a complete orthonormal system of H^p. Let π_n^p: $H^p\to[e_1^0, e_2^0, \cdots, e_n^0]^\perp$ be the orthogonal projection onto the orthogonal complement $[e_1^0, e_2^0, \cdots, e_n^0]^\perp$ of the finite dimensional linear subspace of H^p generated by $e_1^0, e_2^0, \cdots, e_n^0$. Under the same hypotheses of Theorem in Section 2, we have, for any t, $t+h\in[0, T]$*

$$E\|\pi_n^{p+2m}W^\varepsilon(t+h)-\pi_n^{p+2m}W^\varepsilon(t)\|_{p+2m}^4\leqq Ch^2\Big(\sum_{k=n+1}^\infty \|e_k^{p+2m+1}\|_{p+2m}^2\Big)^2, \tag{4.9}$$

$$E\|\pi_n^{p+m}Y^\varepsilon(t+h)-\pi_n^{p+m}Y^\varepsilon(t)\|_{p+m}^4\leqq Ch^2\Big(\sum_{k=n+1}^\infty \|e_k^{p+m+1}\|_{p+m}^2\Big)^2, \tag{4.10}$$

$$E\|X^\varepsilon(t+h)-X^\varepsilon(t)\|_p^4\leqq Ch^2, \quad \text{and} \tag{4.11}$$

$$\sup_{0\leqq t\leqq T} E\Big\|\int_0^t \Big(L\Big(\frac{s}{\varepsilon}\Big)-L\Big)Y^\varepsilon(s)ds\Big\|_{p+m-2}^2\leqq C\varepsilon. \tag{4.12}$$

Proof. Proof of (4.9). Write e_j and π for e_k^{p+2m+1} and π_n^{p+2m} respectively. Put $\Phi_k(s)=((L(s)-L)u^0(s), e_k)_{p+2m+1}$ $k=1, 2, \cdots$. Then it is easy to see that $W^\varepsilon(t)$ can be written as

$$W^\varepsilon(t)=\sqrt{\varepsilon}\sum_{k=1}^\infty\Big(\int_0^{t/\varepsilon}\Phi_k(s)ds\Big)e_k. \tag{4.13}$$

Using the fact that $\{e_k\}_{k=1}^\infty$ is also a orthogonal system in H^{p+2m} we have from (4.13)

$$\begin{aligned} &E\|\pi W^\varepsilon(t+h)-\pi W^\varepsilon(t)\|_{p+2m}^4 \\ &=\varepsilon^2\sum_{k_1=n+1}^\infty\sum_{k_2=n+1}^\infty \|e_{k_1}\|_{p+2m}^2\|e_{k_2}\|_{p+2m}^2\int_{t/\varepsilon}^{t+h/\varepsilon}\int_{\varepsilon/\varepsilon}^{t+h/\varepsilon}\int_{/t\varepsilon}^{t+h/\varepsilon}\int_{t/\varepsilon}^{t+h/\varepsilon} \\ &\quad ds_1ds_2ds_3ds_4E[\Phi_{k_1}(s_1)\Phi_{k_2}(s_2)\Phi_{k_2}(s_3)\Phi_{k_2}(s_4)]. \end{aligned} \tag{4.14}$$

From the assumptions on $L(\omega, t)$ we have $|\Phi_k(s)|\leqq C\sup_{0\leqq r\leqq T}\|u^0(r)\|_{p+3m+1}$. Thus from Lemma 4.1 we obtain

$$E\|\pi W^\varepsilon(t+h)-\pi W^\varepsilon((t)\|_{p+2m}^4\leqq Ch^2\Big(\sum_{k=n+1}^\infty\|e_k\|_{p+2m}^2\Big)^2.$$

Proof of (4.10). Let $\{T(t)\}_{0\leqq t\leqq T}$ be the semi-group of linear operators on

$H^{-\infty}$ such that for each $u\in H^{p+m}$, $T(t)u\in C([0, T]\to H^{p+m})\cap C^1([0, T]\to H^p)$ denotes the unique solution of the evolution equation

$$(4.15)\qquad \begin{cases} \dfrac{du(t)}{dt}=Lu(t) \\ u(0)=u\,. \end{cases}$$

Write e_k and π for e_k^{p+m+1} and π_n^{p+m} respectively. Put

$$\Phi_k(s)=(T(t-\varepsilon s+h)(L(s)-L)u^0(\varepsilon s),\ e_k)_{p+m+1}\quad\text{and}$$
$$\Psi_k(s)=((T(t-\varepsilon s+h)-T(t-\varepsilon s))(L(s)-L)u^0(\varepsilon s),\ e_k)_{p+m+1}\,.$$

Then we can easily see that

$$(4.16)\qquad |\Phi_k(s)|\leqq C\sup_{0\leqq r\leqq T}||u^0(r)||_{p+2m+1}$$

and

$$(4.17)\qquad |\Psi_k(s)|\leqq Ch\qquad\text{for}\quad 0\leqq s\leqq\frac{T}{\varepsilon}\,,$$

since $\left(L\left(\frac{r}{\varepsilon}\right)-L\right)u^0(r)\in C([0, T]\to H^{p+2m+1})$ and

$$((T(t-s+h)-T(t-s))\left(L\left(\frac{s}{\varepsilon}\right)-L\right)u^0(s)=\int_{t-s}^{t-s+h}LT(r)\left(L\left(\frac{s}{\varepsilon}\right)-L\right))u^0(s)\,dr$$

in H^{p+m+1} for $0\leqq s\leqq t\leqq T$. On the other hand we can write

$$\begin{aligned} &Y^\varepsilon(t+h)-Y^\varepsilon(t)\\ &=\frac{1}{\sqrt{\varepsilon}}\int_t^{t+h}T(t-s+h)\left(L\left(\frac{s}{\varepsilon}\right)-L\right)u^0(s)ds\\ &\qquad+\frac{1}{\sqrt{\varepsilon}}\int_0^t(T(t-s+h)-T(t-s))\left(L\left(\frac{s}{\varepsilon}\right)-L\right)u^0(s)ds\\ &=I_1+I_2\,. \end{aligned}$$

Therefore in the same way as the proof of (4.9) we have

$$E||\pi I_1||_{p+m}^4\leqq Ch^2(\sum_{k=n+1}^{\infty}||e_k||_{p+m}^2)^2\quad\text{and}\quad E||\pi I_2||_{p+m}^4\leqq Ch^4(\sum_{k=n+1}^{\infty}||e_k||_{p+m}^2)^2\,.$$

Hence we have

$$E||\pi Y^\varepsilon(t+h)-\pi Y^\varepsilon(t)||_{p+m}^2\leqq Ch^2(\sum_{k=n+1}^{\infty}||e_k||_{p+m}^2)\,.$$

Proof of (4.11). First we show that

$$(4.18)\qquad ||X^\varepsilon(t)||_{p+m}^2\leqq C\left[\int_0^t||W^\varepsilon(s)ds||_{p+2m}^2ds+||W^\varepsilon(t)||_{p+2m}^2\right]$$

In fact, since $W^\varepsilon(\cdot)\in C^1([0, T]\to H^{p+2m})$ and $X^\varepsilon(\cdot)\in C([0, T]\to H^{p+3m})\cap C^1([0, T]\to H^{p+2m})$, we have

$$X^\varepsilon(t)-W^\varepsilon(t)=\int_0^t L(X^\varepsilon(s)-W^\varepsilon(s))ds+\int_0^t LW^\varepsilon(s)ds\,,$$

where the integration in the right hand side is the Bochner integral of an H^{p+m}-valued function on $[0, T]$. Thus from the energy estimate (2.4), we have

$$\|X^\varepsilon(t)-W^\varepsilon(t)\|_{p+m}^2\leqq C_{T,p+m}\int_0^t \|LW^\varepsilon(s)\|_{p+m}^2 ds\leqq C_{T,p+m}C_{p+m}\int_0^t \|W^\varepsilon(s)\|_{p+2m}^2 ds\,.$$

This implies (4.18). Thus we have

$$\sup_{0\leqq t\leqq T} E\|X^\varepsilon(t)\|_{p+m}^4\leqq C \tag{4.19}$$

by using the estimate (4.9) with $t=0$, $h=t$, and $n=0$. On the other hand

$$X^\varepsilon(t+h)-X^\varepsilon(t)=W^\varepsilon(t+h)-W^\varepsilon(t)+\int_t^{t+h} L\Big(\frac{s}{\varepsilon}\Big)X^\varepsilon(s)ds \qquad \text{in} \quad H^{p+2m}\,.$$

Therefore we have

$$E\|X^\varepsilon(t+h)-X^\varepsilon(t)\|_p^4\leqq 2E\|W^\varepsilon(t+h)-W^\varepsilon(t)\|_p^4+2E\|\int_t^{t+h} L\Big(\frac{s}{\varepsilon}\Big)X^\varepsilon(s)ds\|_p^4\,.$$

In virtue of (4.9) it suffices to estimate the second term in the right hand side. But we have

$$\begin{aligned}
&E\|\int_t^{t+h} L\Big(\frac{s}{\varepsilon}\Big)X^\varepsilon(s)ds\|_p^4\\
&\leqq\int_t^{t+h}\int_t^{t+h}\int_t^{t+h}\int_t^{t+h} ds_1ds_2ds_3ds_4\prod_{j=1}^4 E\|L\Big(\frac{s_j}{\varepsilon}\Big)X^\varepsilon(s_j)\|_p\\
&\leqq C\int_t^{t+h}\int_t^{t+h}\int_t^{t+h}\int_t^{t+h} ds_1ds_2ds_3ds_4\prod_{j=1}^4 E\|X^\varepsilon(s_j)\|_{p+m}\\
&\leqq Ch^4\,.
\end{aligned}$$

from (4.19) and Hölder's inequality.

Proof of (4.12). Put $p_1=p+2m+1$, $p_2=p+m-1$, $p_3=p+m-2$, $e_k=e_k^{p_1}$ and $f_k=e_k^{p_2}$ for convenience. In addition, put $\Phi_k(s, r)=(T(s-\varepsilon r)(L(r)-L)u^0(\varepsilon r), e_k)_{p_1}$ and $\Psi_{kl}(s)=((L(s)-L)e_k, f_l)_{p_2}$. Since we can write

$$Y^\varepsilon(s)=\sum_{k=1}^\infty\sqrt{\varepsilon}\Big(\int_0^{s/\varepsilon}\Phi_k(s, r)dr\Big)e_k \qquad \text{in} \quad H^{p_1}$$

and

$$(L(s)-L)e_k=\sum_{l=1}^{\infty}\Psi_{kl}(s)f_l \qquad \text{in} \quad H^{p_2},$$

we can easily see that

$$E\|\sqrt{\varepsilon}\int_0^t ds\Big(L\Big(\frac{s}{\varepsilon}\Big)-L\Big)Y^\varepsilon(s)\|_{p_3}^2$$
$$=\varepsilon^2\sum_l\|f_l\|_{p_3}^2\sum_{k_1}\sum_{k_2}\int_0^{t/\varepsilon}\int_0^{t/\varepsilon}ds_1ds_2\int_0^{s_1}\int_0^{s_2}dr_1dr_2$$
$$E[\Phi_{k_1}(s_1,r_1)\Phi_{k_2}(s_2,r_2)\Psi_{k_1l}(s_1)\Psi_{k_2l}(s_2)]\,.$$

Therefore we have

$$E\|\sqrt{\varepsilon}\int_0^t\Big(L\Big(\frac{s}{\varepsilon}\Big)-L\Big)Y^\varepsilon(s)ds\|_{p_3}^2 \tag{4.20}$$
$$\leqq\varepsilon^2\sum_l\|f_l\|_{p_3}^2\sum_{k_1}\sum_{k_2}\int_0^{t/\varepsilon}\int_0^{t/\varepsilon}\int_0^{t/\varepsilon}\int_0^{t/\varepsilon}ds_1ds_2ds_3ds_4$$
$$|E[\Phi_{k_1}(s_1,s_3)\Phi_{k_2}(s_2,s_4)\Psi_{k_1l}(s_3)\Psi_{k_2l}(s_4)]|$$

On the other hand it is easy to see that

$$|\Phi_k(s,r)|\leqq C\sup_{0\leqq t\leqq T}\|u^0(t)\|_{p_1+m}$$

and

$$|\Psi_{kl}(s)|\leqq\|e_k\|_{p_2+m}\,.$$

Thus applying Lemma 4.1 to (4.20) and recalling $p_1=p+2m+1$, $p_2=p+m-1$, $p_3=p+m-2$, $e_k=e_k^{p_1}$ and $f_k=e_k^{p_2}$ we conclude that

$$E\|\sqrt{\varepsilon}\int_0^t\Big(L\Big(\frac{s}{\varepsilon}\Big)-L\Big)Y^\varepsilon(s)ds\|_{p+m-2}^2\leqq C\sum_l\|e_l^{p+m-1}\|_{p+m-2}^2\Big(\sum_k\|e_k^{p+2m+1}\|_{p+2m-1}\Big)^2$$
$$<\infty\,.$$

since the inclusion $H^{p+2m+1}\subset H^{p+2m-1}$ is a nuclear operator and the inclusion $H^{p+m-1}\subset H^{p+m-2}$ is a Hilbert-Schmidt operator.

5. Proof of Theorem

The purpose of this section is to prove Theorem in Section 2. As before, $X^\varepsilon(t)=\dfrac{u^\varepsilon(t)-u^0(t)}{\sqrt{\varepsilon}}$, $W^\varepsilon(t)=\dfrac{1}{\sqrt{\varepsilon}}\displaystyle\int_0^t\Big(L\Big(\frac{s}{\varepsilon}\Big)-L\Big)u^0(s)ds$, and $Z^\varepsilon(t)=X^\varepsilon(t)-Y^\varepsilon(t)$ where $Y^\varepsilon(t)$ is the solution of the equation (4.6).

Now we prove Theorem. The averaging principle (2.9) follows immeadiately from the estimate (4.11). The proof of the fluctuation property is divided into following four steps: In Step 1, we show that the distribution of

$W^\varepsilon(\cdot)$ converges weakly to the distribution of $W^0(\cdot)$ on $C([0, T]\to H^{p+2m})$. In Step 2, we prove that the distribution of $Y^\varepsilon(\cdot)$ converges weakly to the distribution of $X^0(\cdot)$ on $C(0, T]\to H^{p+m})$. In Step 3, we prove the tightenss of the distributions of $X^\varepsilon(\cdot)$ on $C([0, T]\to H^p)$. And in the last step we show that the limit distribution of $X^\varepsilon(\cdot)$ coincides with the distribution of $X^0(\cdot)$.

Step 1. The estimate (4.9) of Lemma 4.3 implies the tightness of the distributions of $W^\varepsilon(\cdot)$ on $C([0, T]\to H^{p+2m})$ in virtue of Proposition 4.1 in [10]. We have to show that the distribution of $W^\varepsilon(\cdot)$ coincides with the distribution of $W^0(\cdot)$. For any finite sequence $s_1\leqq t_1<s_2\leqq t_2<\cdots<s_k\leqq t_k$ and any finite sequence $h_1, h_2, \cdots, h_n\in H^\infty$, define n-dimensional random variables Δ_j^ε, $j=1, 2, \cdots, k$ by $\Delta_j^\varepsilon=((W^\varepsilon(t_j)-W^\varepsilon(s_j), h_1)_0, \cdots, (W^\varepsilon(t_j)-W^\varepsilon(s_j), h_n)_0)$. Then for any $\xi\in\boldsymbol{R}^n$, we have

$$(5.1)\quad |E\exp(\sum_{j=1}^k i(\xi, \Delta_j^\varepsilon))-\prod_{j=1}^k E\exp(i(\xi, \Delta_j^\varepsilon))|\leqq\alpha\left(\frac{\varepsilon}{\delta}\right)\to 0, \qquad \text{as}\quad \varepsilon\to 0\,,$$

where $(\cdot, \cdot)$ denotes the Euclidean inner product on $\boldsymbol{R}^n$ and $\delta=\min_{1\leqq j\leqq k-1}(t_{j+1}-s_j)$. (5.1) guarantees that any limit process of $W^\varepsilon(\cdot)$ an has independent increments. Therefore, as in the case of the finite dimensional continuous process with independent increments, it suffices to show the following lemma to see that the limit distribution coincides with the distribution of $W^0(\cdot)$.

Lemma 5.1. *For any $v, w\in H^\infty$ and any $0\leqq s, t\leqq T$, we have*

$$(5.2)\qquad E(W^\varepsilon(t), v)_{p+2m}=0\,,$$

and

$$(5.3)\qquad \lim_{\varepsilon\to 0} E[(W^\varepsilon(t), v)_{p+2m}(W^\varepsilon(s), w)_{p+2m}]=\int_0^{t\wedge s}\langle v, w\rangle(u^0(r))dr$$

where $\langle v, w\rangle(u)$ is the same quantity as the statement of Theorem in Section 2.

Proof. The proof is similar to that of Lemma 3.1 in [8]. (5.2) is obvious. We prove the euqation (5.3). For the sake of simplicity, we assume that $v=w$, $T\geqq 1$ and $t=s=1$. Put $\Phi(r)=((L(r)-L)u^0(\varepsilon r), v)_{p+2m}$ and $\Phi_k(r)=((L(r)-L)u^0\left(\frac{k}{n}\right), v)_{p+2m}$ for $k=0, 1, \cdots, n-1$,

where n is a positive integer which will be determined later. Then we have

$$\begin{aligned}
&E[(W^\varepsilon(1), v)_{p+2m}^2]\\
&=\varepsilon\int_0^{1/\varepsilon}\int_0^{1/\varepsilon}dr_1dr_2E[\Phi(r_1)\Phi(r_2)]\\
&=\varepsilon\iint_{G_1}dr_1dr_2E[\Phi(r_1)\Phi(r_2)]+\varepsilon\iint_{G_2}dr_1dr_2E[\Phi(r_1)\Phi(r_2)]\\
&=I_1+I_2\,,
\end{aligned}$$

where $G_1=\bigcup_{k=0}^{n-1}\left\{(r_1, r_2); \frac{k}{n\varepsilon}\leqq r_1, r_2\leqq\frac{k+1}{n\varepsilon}\right\}$, and $G_2=\left[0, \frac{1}{\varepsilon}\right]\times\left[0, \frac{1}{\varepsilon}\right]\backslash G_1$. Since $|\Phi(r)|\leqq C\sup_{0\leqq s\leqq T}||u^0(s)||_{p+3m}||v||_{p+2m}$ for $0\leqq r\leqq\frac{T}{\varepsilon}$, we have

$$
\begin{aligned}
|I_2| &\leqq 2\sum_{k=0}^{n-1}\varepsilon\int_0^{k/n\varepsilon} dr_1\int_{k/n\varepsilon}^{k+1/n\varepsilon} dr_1|E[\Phi(r_1)\Phi(r_2)]| \\
&\leqq 2\sum_{k=0}^{n-1}\varepsilon\int_0^{k/n\varepsilon} dr_1\int_{k/n\varepsilon}^{k+1/n\varepsilon} dr_2\alpha(r_1-r_2)\,(\sup_{t,\omega}|\Phi(r)|)^2 \\
&\leqq Cn\varepsilon||v||_2^{p+2m},
\end{aligned}
$$

from the strong mixing property of $\Phi(r)$. Next we have

$$
\begin{aligned}
I_1 &= \varepsilon\sum_{k=0}^{n-1}\int_{k/n\varepsilon}^{k+1/n\varepsilon}\int_{k/n\varepsilon}^{k+1/n\varepsilon} dr_1dr_2E[\Phi_k(r_1)\Phi_k(r_2)] \\
&\quad+\varepsilon\sum_{k=0}^{k-1}\int_{k/n\varepsilon}^{k+1/n\varepsilon}\int_{k/n\varepsilon}^{k+1/n\varepsilon} dr_1dr_2E[\Phi(r_1)\Phi(r_2)-\Phi_k(r_1)\Phi_k(r_1)] \\
&= I_3+I_4\,.
\end{aligned}
$$

Put $\delta(n)=\sup_{\substack{|s_1-s_2|\leqq1/n \\ 0\leqq s_1,s_2\leqq T}}||u^0(s_1)-u^0(s_2)||_{p+3m}$. Then we have $|\Phi(r)-\Phi_k(r)|\leqq C\delta(n)||v||_{p+2m}$ for $0\leqq r\leqq\frac{T}{\varepsilon}$. Thus using Lemma 4.1, we can show that

$$
|I_4|\leqq Cn\varepsilon\frac{1}{n\varepsilon}\delta(n)||v||_{p+2m}^2=C\delta(n)||v||_{p+2m}^2\,.
$$

On the other hand, from Lemma 4.2 we obtain

$$
|n\varepsilon\int_{k/n\varepsilon}^{k+1/n\varepsilon}\int_{k/n\varepsilon}^{k+1/n\varepsilon} dr_1dr_2E[\Phi_k(r_1)\Phi_k(r_2)]-2\int_0^\infty E[\Phi_k(0)\Phi_k(t)dt|\leqq C\sqrt{\varepsilon n}||v||_{p+2m}^2
$$

Therefore we have

$$
|I_3-\frac{2}{n}\sum_{k=0}^{n-1}\int_0^\infty E[\Phi_k(0)\Phi_k(t)]dt|\leqq C\sqrt{\varepsilon n}||v||_{p+2m}^2\,.
$$

Hence we have

$$
|I-\frac{2}{n}\sum_{k=0}^{n-1}\int_0^\infty E[\Phi_k(0)\Phi_k(t)]dt|\leqq C(n\varepsilon+\sqrt{n\varepsilon}+\delta(n))||v||_{p+2m}^2\,.
$$

Taking $n=n(\varepsilon)$ so that $n(\varepsilon)\varepsilon\to0$ and $n(\varepsilon)\to\infty$ as $\varepsilon\to0$, and recalling the equation (2.13) we conclude that

$$
\lim_{\varepsilon\to0}E[(W^\varepsilon(1), v)]_{p+2m}^2=\int_0^1\langle v, v\rangle(u^0(r))dr\,. \qquad //
$$

Step 2. In virtue of Proposit,on 4.1 in [10], the estimate (4.10) implies the tightness of the distributions of $Y^{\varepsilon}(\cdot)$ on $C([0, T]\rightarrow H^{p+m})$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be any sequence with $\varepsilon_n\rightarrow 0$ $(n\rightarrow\infty)$ such that the distribution of $Y^{\varepsilon_n}(\cdot)$ converges weakly on $C([0, T]\rightarrow H^{p+m})$. In virtue of Skorohod's theorem (see Theorem 2.7 in [6, Chapter 1]), we may assume that $C([0, T]\rightarrow H^{p+m})$-valued random variables Y^{ε_n} converge in $C([0, T]\rightarrow H^{p+m})$ P-a.e. Thus the limit process $Y^0(\cdot)$ satisfies the equation

$$Y^0(t) = W^0(t)+\int_0^t LY^0(s)ds \qquad \text{in} \quad H^p.$$

On the other hand for $w(\cdot)\in C([0, T]\rightarrow H^{p+2m})$ consider the equation

$$\begin{cases} y(t) = w(t)+\int_0^t Ly(s)\,ds \\ y(0) = 0\,. \end{cases} \tag{5.4}$$

Let $u(\cdot)\in C([0, T]\rightarrow H^{p+m})\cap C^1([0, T]\rightarrow H^p)$ be the unique solution of the equation

$$\begin{cases} \dfrac{du(t)}{dt} = Lu(t)+Lw(t) \\ u(0) = 0\,. \end{cases} \tag{5.5}$$

It is easy to see that $y(t)=u(t)-w(t)$ is the unique solution of the equation (5.4) in $C([0, T]\rightarrow H^{p+m})\cap C^1([0, T]\rightarrow H^p)$ and $||y(t)||_p\leqq C||w(t)||_p$ for any $0\leqq t\leqq T$, in virtue of the energy estimate (2.4). Therefore the correspondence $C([0, T]\rightarrow H^{p+2m})\ni w(\cdot)\rightarrow y(\cdot)\in C([0, T]\rightarrow H^p)$ is a continuous mapping. Hence the equation (2.10) determines a unique probability distribution on $C([0, T]\rightarrow H^p)$. The proof of the second step is now complete. //

Step 3. Different from the finite dimensional case, the tightness of the distributions of $X^{\varepsilon}(\cdot)$ on $C([0, T]\rightarrow H^p)$ can not be shown directly from the estimate (4.11), but in the present case, it can be shown by the following argument. For any $\delta>0$, there is a positive constant C such that

$$P\{\omega\in\Omega;\ X^{\varepsilon}(\omega, \cdot)\in\Gamma_1\} >1-\delta\,, \tag{5.6}$$

where $\Gamma_1=\{x(\cdot)\in C([0, T]\rightarrow H^p);\ x(\cdot)$ are equi-continuous and $\sup_{0\leqq t\leqq T}||x(t)||_p\leqq C\}$. On the other hand we can show that

$$\sup_{0\leqq t\leqq T}||X^{\varepsilon}(t)||_{p+m}\leqq C\sup_{0\leqq t\leqq T}||W^{\varepsilon}(t)||_{p+2m} \tag{5.7}$$

from the estimate (4.18) in the proof of (4.11). From (5.7) and the tightness of the distributions of $W^{\varepsilon}(\cdot)$ on $C([0, T]\rightarrow H^{p+2m})$ we can see that for any $\delta>0$, there is a positive constant C such that

$$(5.8) \qquad P\{\omega\in\Omega;\ X^{\varepsilon}(\omega,\ \cdot)\in\Gamma_2\}>1-\delta\ ,$$

where $\Gamma_2=\{x(\cdot)\in C([0,\ T]\to H^{p+m}),\ \sup_{0\leq t\leq T}||x(t)||_{p+m}\leqq C\}$. Since any bounded set in H^{p+m} is relatively compact in H^p, $\Gamma_1\cap\Gamma_2$ is a relatively compact set in $C([0,\ T]\to H^p)$ in virtue of Ascoli-arzelà theorem. Hence from (5.6) and (5.8) the distributions of $X^{\varepsilon}(\cdot)$ are tight on $C([0,\ T]\to H^p)$. //

Step 4. This step is quite different from the finite dimensional case (see Khas'minskii [8]). For $q\leqq\min(p,\ p+m-2)$ let $\mathcal{H}$ be the Hilbert space of all Hilbert-Schmidt operators from H^q into H^{q-m-1} endowed with an inner product

$$(5.9) \qquad (A,\ B)_{HS}=\sum_{k=1}^{\infty}(Ae_k^q,\ Be_k^q)_{q-m-1}\ ,$$

where $\{e_k^q\}_{k=1}^{\infty}$ is a complete orthnormal system of H^q. Let $L^2([0,\ T\to\mathcal{H}])$ be the Hilbert space of all $\mathcal{H}$-valued L^2-functions functions defined on $[0,\ T]$ endowed with an inner product

$$(5.10) \qquad ((A(\cdot),\ B(\cdot)))=\int_0^T(A(t),\ B(t))_{HS}dt\ .$$

From the definition, the well-posed class $\mathcal{L}$ is contained in a closed ball $S_0\subset L^2([0,\ T]\to\mathcal{H})$ centered at 0. Since S_0 is a weakly compact set in $L^2([0,\ T]\to\mathcal{H})$, it is a compact metric space with respect to the weak topology in virtue of Theorem 3 in [4, p. 434]. For example, the metric is given by

$$(5.11) \qquad d(A(\cdot),\ B(\cdot))=\sum_{n=1}^{\infty}\frac{1}{2^n}|((A(\cdot)-B(\cdot),\ B_n(\cdot)))|$$

for $A(\cdot),\ B(\cdot)\in S_0$, where $\{B_n(\cdot)\}_{n=1}^{\infty}$ is a sequence of elements in $L^2([0,\ T]\to\mathcal{H})$ such that their linear hull is dense in $L^2([0,\ T]\to\mathcal{H})$ and $((B_n(\cdot),\ B_n(\cdot)))\leqq 1$ for $n=1,\ 2,\cdots$. In particular the sequence $\{B_n(\cdot)\}_{n=1}^{\infty}$ can be chosen so that

$$(5.12) \qquad \sum_{k=1}^{\infty}(\sup_{0\leq t\leq T}||_nB(t)e_k^q||_{q-m-1})^2\leqq 1\ .$$

In fact, let $\mathcal{K}$ denote the linear subspace of all elements $B(\cdot)$ of $C([0,\ T]\to\mathcal{H})$ such that $B(t)e_k^q=0$ for all $t\in[0,\ T]$ if k is sufficiently large. Then it is easy to see that $\mathcal{K}$ is dense in $L^2([0,\ T]\to\mathcal{H})$. Thus we can take a sequence $\{\tilde{B}_n(\cdot)\}_{n=1}^{\infty}\subset\mathcal{K}$ which is dense in $L^2([0,\ T]\to\mathcal{H})$. Therefore we can choose $B_n(\cdot)$ as

$$B_n(\cdot)=(\sum_{k=1}^{\infty}(\sup_{0\leq t\leq T}||\tilde{B}_n(t)e_k^q||_{p-m-1})^2)^{-(1/2)}\tilde{B}_n(\cdot)\ .$$

Consider the product space

$$(5.13) \qquad S=C([0,\ T]\to H^q)\times C([0,\ T]\to H^{q-m-1})\times S_0\ .$$

Put $F^\varepsilon(t)=\int_0^t (L^\varepsilon(s)-L)Y^\varepsilon(s)\,ds$ where $L^\varepsilon(t)=L\left(\frac{t}{\varepsilon}\right)$. Then the distribution of $F^\varepsilon(\cdot)$ converges weakly to the distribution of the process θ which is identically zero. In fact, since $q\leqq p$, and since we have already shown that the distributions of $Y^\varepsilon(\cdot)$ on $C([0, T]\to H^{p+m})$ are tight, it is easy to see that the distributions of $F^\varepsilon(\cdot)$ on $C([0, T]\to H^q)$ are tight. On the other hand, $\sup_{0\leqq t\leqq T} E\|F^\varepsilon(t)\|_q\to 0$ as $\varepsilon\to 0$, from the estimate (4.12). Hence we have the above fact. Next we have

$$
\begin{aligned}
&E[d(L^\varepsilon(\cdot), L)]\\
&=E\left[\sum_{n=1}^\infty \frac{1}{2^n}\left|\int_0^T \sum_{k=1}^\infty ((L^\varepsilon(t)-L)e_k^q,\ B_n(t)e_k^q)_{q-m-1}dt\right|\right]\\
&\leqq \sum_{n=1}^\infty \frac{1}{2^n}\sum_{k=1}^\infty E\left|\int_0^T ((L^\varepsilon(t)-L)e_k^q,\ B_n(t)e_k^q)_{q-m-1}dt\right|\\
&\leqq \sum_{n=1}^\infty \frac{1}{2^n}\sum_{k=1}^\infty \left(E\left|\int_0^T ((L^\varepsilon(t)-L)e_k^q,\ B_n(t)e_k^q)_{q-m-1}\right|^2\right)^{1/2}.
\end{aligned}
$$

On the other hand, we have from Lemma 4.1

$$
\begin{aligned}
&E\left|\int_0^T ((L^\varepsilon(t)-L)e_k^q,\ B_n(t)e_k^q)_{q-m-1}dt\right|^2\\
&=\varepsilon^2\int_0^{T/\varepsilon}\int_0^{T/\varepsilon} ds_1 ds_2 E[((L(s_1)-L)e_k^q,\ B_n(\varepsilon s_1)e_k^q)_{q-m-1}((L(s_2)-L)e_k^q,\ B_n(\varepsilon s_2)e_k^q)_{q-m-1}]\\
&\leqq C\varepsilon(\sup_{0\leqq t\leqq T}\|B_n(t)e_k^q\|_{q-m-1})^2\|e_k^q\|_{q-m-1}^2.
\end{aligned}
$$

Therefore it follows that $E[d(L^\varepsilon(\cdot), L)]\leqq C\varepsilon$ in virtue of Schwartz' inequality. Hence we have shown that for any sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n\to 0$ as $n\to\infty$, $L^{\varepsilon_n}(\cdot)$ converges to L (non-random) in S_0 in probability. In virtue of Theorem 12.3 in [2, p. 195], the distribution of $(F^{\varepsilon_n}, L^{\varepsilon_n})$ converges weakly to the distribution of (θ, L) on $C([0, T]\to H^{q-m-1})\times S_0$. Suppose that the distribution of $Z^{\varepsilon_n}(\cdot)$ on $C([0, T]\to H^q)$ converges weakly to the distribution of $Z^0(\cdot)$. Using Theorem 12.3 in [2] again, we can see that the distribution of $(Z^{\varepsilon_n}, F^{\varepsilon_n}, L^{\varepsilon_n})$ on S converges weakly to the distribution of (Z^0, θ, L). In virtue of Skorohod's theorem, we may assume that $(Z^{\varepsilon_n}, F^{\varepsilon_n}, L^{\varepsilon_n})$ converges to (Z^0, θ, L) in S. P-a.e.. For a while we proceed our discussion with fixing an $\omega\in\Omega$ such that $(Z^{\varepsilon_n}(\omega), F^{\varepsilon_n}(\omega), L^{\varepsilon_n}(\omega))$ converges to $(Z^0(\omega), \theta, L)$ in S. For any fixed $t\in[0, T]$ and $h\in H^{q-m-1}$ we have

$$
\begin{aligned}
&(Z^0(t), h)_{q-m-1}-\int_0^t (LZ^0(s), h)_{q-m-1}ds\\
&=(Z^0(t), h)_{q-m-1}-(Z^{\varepsilon_n}(t), h)_{q-m-1}\\
&\quad+\int_0^t (L^{\varepsilon_n}(s)Z^{\varepsilon_n}(s), h)_{q-m-1}ds+(F^{\varepsilon_n}(t), h)_{q-m-1}-\int_0^t (L^{\varepsilon_n}(s)Z^0(s), h)_{q-m-1}ds
\end{aligned}
$$

$$+\int_0^t (L^{\varepsilon_n}(s)Z^0(s), h)_{q-m-1}ds - \int_0^t (Z^0(s), h)_{q-m-1}ds\,.$$

Clearly, $|(Z^{\varepsilon_n}(t), h)_{q-m-1}-(Z^0(t), h)_{q-m-1}|\to 0$, $|(F^{\varepsilon_n}(t), h)_{q-m-1}|\to 0$, and $|\int_0^t (L^{\varepsilon_n}(s)Z^{\varepsilon_n}(s), h)_{q-m-1}ds-\int_0^t (L^{\varepsilon_n}(s)Z^0(s), h)_{q-m-1}ds|\to 0$ as $n\to\infty$. Let $\mathcal{H}'$ be the space of all Hilbert-Schmidt operators from H^{q-m-1} into H^q. We note that $L^2([0, T]\to\mathcal{H})$ and $L^2([0, T]\to\mathcal{H}')$ are isometric and isomorphic by the correspondence $L^2([0, T]\to\mathcal{H})$ $A(\cdot)\to A'(\cdot)\in L^2([0, T]\to\mathcal{H}')$ where for each t, $A'(t)=A(t)'$: the dual operator of $A(t)$. We define an element $B'(\cdot)$ of $L^2([0, T]\to\mathcal{H}')$ as follows:

$$B'(s)g=\begin{cases}\begin{cases} cZ^0(s) & \text{for } g=ch \quad (c\in\boldsymbol{R}) \\ 0 & \text{for } g\in\{ch;\, c\in\boldsymbol{R}\}^{\perp}\end{cases} & \text{if } 0\leqq s\leqq t \\ 0 & \text{if } t<s\leqq T\,.\end{cases}$$

Since $L^{\varepsilon_n}(\cdot)$ converges to L in the weak topology of $L^2([0, T]\to\mathcal{H})$ we have

$$\begin{aligned}&\int_0^t (L^{\varepsilon_n}(s)Z^0(s), h)_{q-m-1}ds\\ &=\int_0^T (B'(s)h, L^{\varepsilon_n\prime}(s)h)_q ds\to\int_0^T (B'(s)h, L'h)_q ds\\ &\qquad\qquad\qquad=\int_0^t (LZ^0(s), h)_{q-m-1}ds \qquad (n\to\infty)\,.\end{aligned}$$

Thus we have seen that

$$(Z^0(t), h)_{q-m-1}=\int_0^t (LZ^0(s), h)_{q-m-1}ds$$

for any $0\leqq t\leqq T$ and for any $h\in H^{q-m-1}$. Therefore we have

$$Z^0(t)=\int_0^t LZ^0(s)ds \qquad \text{for any} \quad 0\leqq t\leqq T \quad \text{P-a.e..}$$

Hence we have seen that $Z^0(\cdot)=\theta$. It follows that the distribution of $Z^\varepsilon(\cdot)$ on $C([0, T]\to H^q)$ converges weakly to θ. Therefore the limit distributions of $X^\varepsilon(\cdot)$ and $Y^\varepsilon(\cdot)$ coincide on $C([0, T]\to H^q)$. Hence they coincide on $C([0, T])\to H^p)$. This completes the proof of Theorem. //

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