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FUNCTION-THEORETIC PROPERTIES OF SOLUTIONS OF SEMI-LINEAR $\bar{\partial}$-EQUATIONS — SIMILARITY PRINCIPLE —

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Introduction. The present paper is concerned with the function-theoretic properties and the local existence of solutions for a system of semi-linear first order partial differential equations in a complex-valued unknown function defined on a domain G (connected open set) in $\boldsymbol{C}^n$ ($n\geqq 2$). The system of semi-linear equations is as follows:

$$\bar{\partial}_j w = f_j(z, w)\,, \quad j = 1, 2, \cdots, n\,, \tag{0}$$

where $\bar{\partial}_j=(1/2)(\partial/\partial x_j+i\partial/\partial y_j)$, $z_j=x_j+iy_j$ and $z=(z_1, z_2, \cdots, z_n)\in\boldsymbol{C}^n$. When we rewrite this system of equations by the real form, then we obtain

$$\begin{cases}\partial_{x_j} u-\partial_{y_j} v = 2p_j\,,\\ \partial_{y_j} u+\partial_{x_j} v = 2q_j\,,\end{cases}$$

where $w=u+iv$ and $f_j=p_j+iq_j$.

When Eq (0) is $\boldsymbol{R}$-linear, that is, f_j is in the form $\alpha_j w+\beta_j \bar{w}+\gamma_j$ for all j, we call it the *Vekua equation* and those functions satisfying this equation the *Vekua functions* (named after I.N. Vekua, who constructed the theory of the above functions of one variable).

Among the various studies of Vekua functions of one variable, boundary value problems are still taken up by many Russian mathematicians. The effective instruments for studying the Vekua functions of one variable are composed of three means: (a) the Cauchy-Green formula for smooth functions on the closure of open set with smooth boundary, (b) the Fredholm integral equation (with the Cauchy kernel) of the second kind, and (c) the representation theorem of Vekua functions. From all of them the similarity principle is derived, which plays an essential role in the theory of Vekua functions.

On the contrary, when we are interested in the generalization of one variable to several variables like Eq (0), the questions arise: (1) Do the solutions of Eq (0) have the similar properties to analytic functions (of several variables)? (2) Is the representation theorem of solutions valid? (3) does the existence theorem of solutions hold?

In the case of several variables, the integral formula for the smooth functions in a closed domain with smooth boundary is, in general, dependent on the shape of the domain (see [8], [9]). In addition, since Eq (0) is overdetermined, one must consider the compatibility conditions which its solutions must satisfy. Therefore it seems to be difficult to solve the questions stated above.

The article [11] is probably the first to the knowledge of the present author, which showed the existence of local solutions of the homogeneous Vekua equation of several variables and the local validity of the similarity principle of Vekua functions of several variables, under some integrability conditions in such a way that the set of zeroes of any nontrivial Vekua function is an $(n-1)$-dimensional complex manifold of $\boldsymbol{C}^n$ (if it is non void). Since then many Russian mathematicians have investigated the Vekua functions of several variables ([5], [7], [11], [14], [15], [16], [19], [20]). W. Tuschke [20] treated the function-theoretic properties of those functions satisfying certain differential inequalities. The solutions of quasi-homogeneous Eq (0) with infinite (continuous) differentiability fulfill the differential inequalities; when our equations are of class C^2, the solutions do not always satisfy the inequalities; the twice differentiability is sufficient to develop our arguments. Above all the process of obtaining the result is intrinsically different from that of Tuschke's.

The method of obtaining several function-theoretic properties similar to holomorphic functions of several variables, together with the local existence of solutions to the quasi-homogeneous Eq(0) is that, under certain assumptions, Eq(0) can be reduced either to the normal total differential equation in all variables or to the following system of equations: the $\bar{\partial}$-semi-linear equation in one variable and the normal total differential equation in the remainder of variables.

We obtain some properties similar to holomorphic functions of several variables, for example, that the set of zeroes is an analytic set of G, and that if a solution vanishes on an open subset of G, it vanishes on G. We also obtain the representation theorem of solutions under adequate conditions. Moreover we obtain the local existence theorem and hence the local similarity principle for the family of all holomorphic functions whose element is expressed by the composition of ϕ and h, where ϕ is an arbitrary holomorphic function in a variable and h is the fixed holomorphic function in several variables.

In Section 1 we shall state the complex Frobenius theorem and a theorem of Sommer, of which we make full use throughout this paper, together with the prelimilaries and notation.

In Section 2, by the standard technique we shall discuss the identity theorem and the property of the zeroes.

In Section 3 we shall consider the family of all the additional first order partial differential equations to Eq(0) and the subfamily $\boldsymbol{F}_0$ composed of those equations which have the first order terms arising from the vector fields of (1, 0)-type. We, in addition, treat the set $\mathcal{S}$ of vector fields of (1, 0)-type which corresponds to $\boldsymbol{F}_0$. $\mathcal{S}$ depends in general on each solution of Eq(0). Under the assumption that the rank of $\mathcal{S}$ is constant on an open set, we shall deal with the special holomorphic change of variables, which will play an essential role in the present paper.

In Section 4, from the special holomorphic change of variables obtained in Section 3 we shall derive several function-theoretic properties of solutions to Eq(0) similar to holomorphic functions. Furthermore, by reducing Eq(0) to the Vekua equation of one variable we shall prove the existence of holomorphic function similar to each given solution (the representation theorem).

In Section 5 we shall deal with the integrability conditions of quasi-homogeneous Eq(0) on the assumption that the special holomorphic change of variables considered in Section 3 does not depend on the solutions.

In the final section we shall treat the local solvability of quasi-homogeneous Eq(0), the validity of the local similarity principle, and the theorem of Montel type.

1. Preliminaries and notation

Our system of first order semi-linear partial differential equations in the present paper is related to the Nirenberg's work [17] and Sommer's one [18]. Therefore we intend to explain the complex Frobenius theorem. For brevity, all functions appeared in this paper are complex valued ones infinitely differentiable on their domains of definitions and when no confusion is likely we shall use the word "smooth" for "infinitely differentiable". Let us now consider a system of complex smooth vector fields $P^k(x)$, $x=(x_1, x_2, \cdots, x_d)$, $k=1, 2, \cdots, m$, defined on an open set U in $\boldsymbol{R}^d$, i.e.,

$$P^k(x) = \sum_{j=1}^d \alpha_j^k(x) \frac{\partial}{\partial x_j},$$

where α_j^k are smooth on U.

Let $\mathcal{P}$ be the system of vectors generated by $P^1, P^2, \cdots, P^m$ over $C^\infty(U)$ (i.e., each element of $\mathcal{P}$ is written in the form $\sum_{j=1}^m \beta_j P^j$, where β_j is smooth on U, $j=1, 2, \cdots, m$), and let $\tilde{\mathcal{P}}$ be the system of vector fields generated by $P^1, P^2, \cdots, P^m, \bar{P}^1, \bar{P}^2, \cdots, \bar{P}^m$ over $C^\infty(U)$. Here $\bar{P}^j$ denote the vector fields whose coefficients are the complex conjugations of coefficients of P^j. We assume that the *rank* of $\mathcal{P}$, i.e., the one of the matrix $(\alpha_j^k(x))$ is constant m and the rank of $\tilde{\mathcal{P}}$ is constant $m+r$ on U, where necessarily $r \leqq \mathrm{Min}(m, d-m)$.

Theorem 1.1 (Nirenberg's complex Frobenius theorem [17]). *The following statements are equivalent to each other.*

$$(*)\qquad \begin{cases} \textit{The commutator of any two elements in } \mathcal{P} \textit{ belongs to } \mathcal{P}, \textit{ and} \\ \textit{the commutator of any two elements in } \tilde{\mathcal{P}} \textit{ belongs to } \tilde{\mathcal{P}}. \end{cases}$$

$$(**)\qquad \begin{cases} \textit{There exist new local coordinates } y = (y_1, y_2, \cdots, y_d) \textit{ for } \boldsymbol{R}^d \\ \textit{such that } \mathcal{P} \textit{ is equivalent to the system of vector fields generated} \\ \textit{by} \\ \qquad\qquad \partial/\partial\xi_j,\ j = 1, 2, \cdots, r. \\ \textit{and} \qquad \partial/\partial y_k,\ k = 2r+1, 2r+2, \cdots, m\,, \\ \textit{where} \qquad \xi_j = y_j + i y_{r+j}\,, \quad \textit{and} \\ \qquad\qquad \partial/\partial\xi_j = (1/2)\,(\partial/\partial y_j + i\partial/\partial y_{r+j})\,. \end{cases}$$

Now we shall make a comment on Theorem 1.1 from viewpoint of the complex structure. (In what follows, unless stated otherwise, we shall agree on the index ranges $1 \leqq j, k \leqq n$.)

Let d be $2n$. For arbitrary point $x \in U$ we think of the real tangent space $T_x(U)$ $(=T_x)$ to U at x. T_x is spanned by $\partial/x_1, \cdots, \partial/\partial x_{2n}$. Let J_x^0 be the canonical complex structure of T_x, that is, an automorphism of T_x such that

$$J_x^0(\partial/\partial x_j) = \partial/\partial x_{n+j}\,, \quad J_x^0(\partial/\partial x_{n+j}) = -\partial/\partial x_j\,.$$

Now we consider the product tangent space $V_x = T_x \times T_x$ and introduce the two complex structures $\tilde{J}_x$ and $\tilde{J}_x^0$ on V_x: for $(v_1, v_2) \in V_x$

$$\tilde{J}_x\colon \tilde{J}_x(v_1, v_2) = (-v_2, v_1)$$
$$\tilde{J}_x^0\colon \tilde{J}_x^0(v_1, v_2) = (J_x^0 v_1, J_x^0 v_2)\,.$$

Then we denote by CT_x and V_x^C the real vector spaces V_x equipped with the complex structures $\tilde{J}_x$ and $\tilde{J}_x^0$, respectively. The system of vector fields $\mathcal{P}$ in Theorem 1.1 is a mapping which assigns to every point $x \in U$ an m-dimensional subspace of CT_x. Of course there is a complex structure J_x' of T_x other than J_x^0. If we write W_x^C for the V_x equipped with the complex structure $\tilde{J}_x'$ which is defined by $\tilde{J}_x'(v_1, v_2) = (J_x' v_1, J_x' v_2)$ for $(v_1, v_2) \in V_x$, then V_x^C and W_x^C are, in general, different complex vector spaces over the same real vector space V_x. Therefore it follows that the change of variables $x \to y$ in Theorem 1.1 gives us no information about complex structures.

From now on, we shall think of a special case, i.e. $d = 2n$, $r = m$. In this case we have $m \leqq n$ and $\mathcal{P} \cap \overline{\mathcal{P}} = 0$. Let $\boldsymbol{C}^n$ denote the vector space $\boldsymbol{R}^{2n}$ equipped with the canonical complex structure J: $J(x_j) = x_{n+j}$, $J(x_{n+j}) = -x_j$. And putting $z_j = x_j + i x_{n+j}$, we consider the canonical coordinates $z = (z_1, \cdots, z_n)$ as a system of complex coordinates on $\boldsymbol{C}^n$. Introducing $\partial/\partial z_j = (1/2)\,(\partial/\partial x_j - i\partial/\partial x_{n+j})$, $\partial/\partial \bar{z}_j = (1/2)\,(\partial/\partial x_j + i\partial/\partial x_{n+j})$, we denote them by ∂_j and $\bar{\partial}_j$, respectively. Then $\{\partial_j, \bar{\partial}_j\}$ is the canonical base for CT_x. And any vector field P on U, $P(x) =$

$\sum_{j=1}^{2n} \alpha_j(x)\, \partial/\partial x_j$, in terms of the canonical coordinates and base is written in the complex form

$$P(z) = \sum \beta_j(z)\, \partial_j + \gamma_j(z)\, \bar{\partial}_j .$$

By $\bar{P}(z)$ we denote the vector field $\sum \bar{\gamma}_j\, \partial_j + \bar{\beta}_j\, \bar{\partial}_j$.

Let the system of vector fields $\mathcal{P}$ in Theorem 1.1 be given by the complex form mentioned above. Let us consider the following system of equations in an unknown function u defined on U: for $P^k \in \mathcal{P}$

$$P^k u = \sum \beta_j^k\, \partial_j u + \gamma_j^k\, \bar{\partial}_j u = g^k(z, u)\,, \quad 1 \leqq k \leqq m (\leqq n)\,. \tag{1.0}$$

Then, by Theorem 1.1 we have the complex coordinates ξ such that the system of equations (1.0) is transformed into the system of equations

$$(\partial/\partial\bar{\xi}_k)\, v = h^k(\xi, v)\,, \quad 1 \leqq k \leqq m\,.$$

As is seen from the mentioned above, the transformation $z \rightarrow \xi$ is not always holomorphic. We have the following theorem connected with this, of which we shall avail ourselves later.

Theorem 1.2 (Sommer [18]). *Assume that the system $\mathcal{P}$ is constructed by smooth vector fields of type* (0,1) *and* rank $\mathcal{P} = m$ *on U. The condition* ($*$) *on the system $\mathcal{P}$ in the previous theorem is necessary and sufficient for the existence of new local complex coordinates $\xi = (\xi_1, \cdots, \xi_n)$ such that $\mathcal{P}$ is equivalent to the system of vector fields generated by*

$$\partial/\partial\bar{\xi}_\lambda\,, \quad \lambda = 1, 2, \cdots, m\,.$$

In this case the transformation $z \rightarrow \xi$ can be chosen so as to be holomorphic.

Proof. It is sufficient to show the second half of the theorem. Through each point z^0 in U there exists locally a real $2m$-dimensional submanifold σ of U^0 $(=U(z^0))$, a neighborhood of z^0 in U, because the system of vector fields $\mathcal{R}$ generated by

$$\operatorname{Re} P^k = \frac{1}{2}(P^k + \bar{P}^k)\,, \quad \operatorname{Im} P^k = \frac{1}{2i}(P^k - \bar{P}^k)\,,$$

$k = 1, \cdots, m$, is the system of real vector fields and satisfies the conditions of the (real) Frobenius theorem. Each element of $\mathcal{R}$ is tangent to σ at every point of σ. The real rank of $\mathcal{R}$ on U^0 is $2m$ and the complex one of $\mathcal{P}$ is m. Therefore we can see that σ is an m-dimensional complex submanifold of U^0 (see [18]). Let $\mathcal{P}^*$ be the system of smooth (0, 1)-forms dual to $\mathcal{P}$ at each point in U^0 such that $\mathcal{P}^*$ is generated by $P^{*1}, \cdots, P^{*(n-m)}$ over $C^\infty(U^0)$. The system $\mathcal{P}^*$ is equivalent to the one of smooth (0, 1)-forms generated by $d\bar{h}_{m+j}$, $j = 1, 2, \cdots, n-m$ over

$C^\infty(U^0)$, where h_{m+j} are holomorphic functions and functionally independent in U^0. If the $(n-m)\times(n-m)$ matrix $(\partial h_{m+j}/\partial z_k)$, $1\leqq j\leqq n-m$, $1\leqq k\leqq n-m$, is non-singular on U^0, we may consider the biholomorphic mapping on U^0 (restricted further if necessary)

$$\begin{cases}\xi_j = h_{m+j}\,, & j=1,2,\cdots,n-m\,,\\ \xi_{n-m+k} = z_{n-m+k}\,, & k=1,2,\cdots,m\,,\end{cases}$$

then we obtain the desired transformation. Q.E.D.

Let us consider the following $\boldsymbol{R}$-semi-linear but not $\boldsymbol{C}$-semi-linear system of first order partial differential equations:

$$(1.1)\qquad \bar\partial_j u + A(z)\,\bar\partial_j \bar u = B_j(z, u)\,.$$

Here A is an $m\times m$ matrix entries of which are smooth on an open subset of G in $\boldsymbol{C}^n$, and B_j are smooth m-column vectors on an open subset $G\times U$ of $\boldsymbol{C}^n\times\boldsymbol{C}^m$, and u is unknown m-column vector.

We shall assume that for every point (z,u) in $G\times U$

$$(1.2)\qquad \det(E-A\bar A)\neq 0\,.$$

Now, by thinking of the transformation $u=w-A\bar w$, we can reduce Eq (1.1) to the $\boldsymbol{C}$-semi-linear system of equations of the following type

$$(1.3)\qquad \bar\partial_j w = f_j(z, w)\,.$$

Thus we verify that the problem of solving Eq (1.1) is reduced to the one of solving Eq (1.3). The present author treated the special type of Eq (1.1) ([13]). Eq (1.3) is completely integrable in $G\times W$ (that is, for each $(z^0, w^0)\in G\times W$ there is a local solution whose graph passes through (z^0, w^0)), if and only if: in $G\times W$

$$(1.4)\qquad \bar\partial_\mu f^j_\lambda - \bar\partial_\lambda f^j_\mu + \textstyle\sum_{k=1}^m (\partial'_k f^j_\lambda) f^k_\mu - (\partial'_k f^j_\mu) f^k_\lambda = 0\,,\quad 1\leqq\mu<\lambda\leqq n\,,$$

$$(1.5)\qquad \bar\partial'_k f^j_\lambda = 0\,,\quad 1\leqq j, k\leqq m;\ 1\leqq\lambda\leqq n\,,$$

where $\partial'_s=\partial/\partial w_s$, $\bar\partial'_s=\partial/\partial\bar w_s$, and f^j_λ are j-th component of vector f_λ (see [20]).

It should be observed that the condition (1.5) means all f^j_λ are holomorphic in $w_1, w_2, \cdots, w_m$. If Eq (1.3) is not completely integrable on $G\times W$, when does there exist a (local) solution? And assuming the existence of global solutions to Eq (1.3), what sort of properties do they possess? Let us consider the simple case where Eq (1.3) with $m=1$ is an $\boldsymbol{R}$-linear system defined on an open subset G in $\boldsymbol{C}^n$ $(n\geqq 1)$, that is,

$$(1.6)\qquad \bar\partial_j w = a_j(z)\,w + b_j(z)\,\bar w + c_j(z)\,.$$

Then, we verify immediately that conditions (1.4), (1.5) are equivalent to the following: for $n \geqq 2$

$$\begin{cases} \bar{\partial}_k a_j - \bar{\partial}_j a_k = 0, \quad b_j = 0, \\ \bar{\partial}_k c_j - \bar{\partial}_j c_k + a_j c_k - a_k c_j = 0, \end{cases} \tag{1.7}$$

Here if G is pseudoconvex and condition (1.7) holds in G, then we can change Eq (1.6) into the inhomogeneous Cauchy-Riemann equation ([9]). Therefore it follows easily that the general solutions of Eq (1.6) ($\boldsymbol{C}$-linear) are given by the form

$$w(z) = \phi(z) \exp(\omega) + w_0(z), \tag{1.8}$$

where ϕ is holomorphic on G, ω satisfies $\bar{\partial}\omega = \boldsymbol{a} (= \sum a_j d\bar{z}_j)$ and w_0 is a particular solution of Eq (1.6). From (1.8) we are able to recognize that, in the $\boldsymbol{C}$-linear case, the solutions preserve many properties of analytic functions for one or several complex variables.

Let G be an open set in $\boldsymbol{C}^n$ and let $C^\infty(G)$ denote the space of smooth complex-valued functions on G. We denote by $C^\infty_{(p,q)}(G)$ the space of infinitely differentiable differential forms of type (p, q) on G, and in what follows we shall call each element of $C^\infty_{(p,q)}(G)$ a smooth (p,q)-differential form on G. The operation of conjugation maps $C^\infty_{(p,q)}(G)$ to $C^\infty_{(q,p)}(G)$, the form $\tau\, dz_J \wedge d\bar{z}_K$ being taken to $\bar{\tau}\, d\bar{z}_J \wedge dz_K$, where the coefficient τ is smooth, $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p}$, $d\bar{z}_K = d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}$ and where the following notations are used: $J = (j_1, \cdots, j_p)$ and $K = (k_1, \cdots, k_q)$ are multi-indices of integers $1 \leqq j_\lambda, j_\mu \leqq n$. We denote by ∂ and $\bar{\partial}$ the exterior differentiations

$$\sum \partial_j dz_j, \quad \text{and} \quad \sum \bar{\partial}_j d\bar{z}_j, \quad \text{respectively.}$$

Let us now consider the mapping ν of $C^\infty(G)$ to $C^\infty_{(0,1)}(G)$:

$$w \to \bar{\partial} w - f(z, w)$$

for $f(z, w)$ in $C^\infty_{(0,1)}(G \times \boldsymbol{C})$. Then we see the inverse images of zeroes are the set of all solutions to Eq (1.3) with $m=1$. Since Eq (1.3) is in general over-determined system, the compatibility conditions on f are necessary for the existences of solutions. To the end of finding those conditions we shall avail ourselves of the calculus of differential forms on $\boldsymbol{C}^n$. If $g(z, w)$ is a smooth (p, q)-form with respect to z on $G \times \boldsymbol{C}^m$, the operation ∂_z or $\bar{\partial}_z$ on $g(z, w)$ stands for the exterior differentiation with respect to z (when thinking of w as the independent variables). When w depends smoothly on z, then the operation ∂ or $\bar{\partial}$ on $g(z, w)$ can be written by the following formulas:

$\delta g(z, w(z)) = \delta_z g + (-1)^{p+q} \sum_{\lambda=1}^m (g_w \wedge \delta w_\lambda + g_{\bar{w}} \wedge \delta \bar{w}_\lambda)$, where δ denotes ∂ or $\bar{\partial}$, and where g_w and $g_{\bar{w}}$ denote the following (p, q)-forms:

$$g_u = \sum' \frac{\partial}{\partial u} g_{IJ}\, dz_I \wedge d\bar{z}_J\,.$$

where $\sum'$ means that the summation is performed only over strictly increasing multi-indices and $u=w_\lambda$, $\bar{w}_\lambda$, $\lambda=1, 2, \cdots, m$.

We shall be able to define the formal exterior product of (p, q)-form α and ∂ (or $\bar{\partial}$), i.e. $\alpha\wedge\partial$ (or $\alpha\wedge\bar{\partial}$), by $\alpha\wedge\partial w$ (or $\alpha\wedge\bar{\partial}w$) for any smooth function w on G. Thus we are in a position to state the following identification of the formal exterior product A with the system of vector fields B on G.

A	B
formal exterior product	system of vector fields on G
$\boldsymbol{b} = \sum b_j\, dz_j$	$1\leqq j, k, s, t\leqq n;\, j<k,\, s<t$
$\partial\,(\bar{\partial})$	$\{\partial_j\}\quad(\{\bar{\partial}_j\})$
$\boldsymbol{b}\wedge\partial\,(\bar{\boldsymbol{b}}\wedge\bar{\partial})$	$\{X_{jk}\},\, X_{jk} = b_j\,\partial_k - b_k\,\partial_j$ $(\{\bar{X}_{jk}\},\, \bar{X}_{jk} = \bar{b}_j\,\bar{\partial}_k - \bar{b}_k\,\bar{\partial}_j)$
$\bar{\partial}\boldsymbol{b}\wedge\bar{\boldsymbol{b}}\wedge\partial$	$\{\bar{X}_{st}(b_j)\,\partial_k - \bar{X}_{st}(b_k)\,\partial_j\}$
$[\boldsymbol{b}\wedge\partial, \bar{\boldsymbol{b}}\wedge\bar{\partial}]$	$\{[X_{jk}, \bar{X}_{st}]\}$

Here $[\boldsymbol{b}\wedge\partial, \bar{\boldsymbol{b}}\wedge\bar{\partial}]$ means $\boldsymbol{b}\wedge\partial\bar{\boldsymbol{b}}\wedge\bar{\partial}-\bar{\boldsymbol{b}}\wedge\bar{\partial}\boldsymbol{b}\wedge\partial$, and $[\ ,\ ]$ the commutator.

2. Properties of the solutions I: the zero sets

In the previous section we considered the complete integrability conditions of Eq (1.3). However, the pseudoanalytic functions introduced by Bers [3] and Vekua functions ([21]) are surely the functions satisfying Eq (1.6) which is not always completely integrable, that is, b_j does not always identically vanish ($j=1$ only, i.e. $n=1$).

We shall deal with the following equation

$$\text{(2.1)}\qquad \begin{cases} \bar{\partial}w = f(z, w)\,, \\ f(z, 0) = 0\,, \quad \text{everywhere in } G. \end{cases}$$

Eq (2.1) is said to be *quasi-homogeneous* in $G\times\boldsymbol{C}$.

Let w be a nontrivial smooth solution of Eq (2.1) defined on a domain $\Omega\subset G$. Unless stated otherwise, the solutions of differential equations dealt with from now on means smooth ones.

We define the set $N=\{z\in\Omega: w(z)=0\}$, and then N is relatively closed in Ω. We have the

Theorem 2.1 (Identity theorem). *If N has an interior point, $w(z)$ vanishes identically on Ω.*

Proof. Let a point a belong to the interior of N. Then we have an open ball $B=B(a;r)$ (about a with radius $r>0)\subset N$. We fix an arbitrarily chosen complex line $\Lambda=\Lambda(c)$ through a: for a unit vector $c\in\boldsymbol{C}^n$

$$\Lambda = \{z = a+\lambda c; \lambda\in\boldsymbol{C}\} .$$

Define $\Delta(c)=\{\lambda\in\boldsymbol{C}; z=a+\lambda c\in\Omega\cap\Lambda\}$. If we put $v(\lambda)=w(a+\lambda c)$, Eq. (2.1) turns out to be an equation of the following type with respect to $\lambda\in\Delta(c)$

$$(2.2) \qquad (\partial/\partial\bar{\lambda})\, v = h(\lambda, v)\,,$$

where $h(\lambda, 0)=0$ everywhere in $\Delta(c)$. Since $w=0$ on B, by applying the identity theorem of Vekua functions of one variable ([4], [21]), the v vanishes on the connected component $\Gamma(c)$ of $\Delta(c)$ containing the origin in the λ-plane. This fact is valid for every c. For any point $b\in\Omega$ we can join it with a in Ω by a chain of balls $B_1, \cdots, B_p$ such that $B_{k-1}\cap B_k\neq 0$, $1\leqq k\leqq p(B=B_0)$. Let point b belong to B_p. From $B_1\subset\{z\in\boldsymbol{C}^n; z=a+\lambda c, \lambda\in\Gamma(c), c\in\boldsymbol{C}^n, |c|=1\}$, we see $w=0$ everywhere in B_1. By induction we obtain $w=0$ everywhere in B_p and hence $w(b)=0$. Q.E.D

Proposition 2.1. *The set $\Omega\backslash N$ is a domain.*

Proof. It suffices to prove that $\Omega\backslash N$ is connected. Take an open ball B in Ω. If $B\backslash N$ is proven to be connected, then the rest of the proof follows the same pattern as the proof of the above theorem.

For arbitrary points a, b in $B\backslash N$ we fix a complex line Λ joining them: $z=a+\lambda c$, $c=(b-a)/|b-a|$. Let $\sigma=B\cap\Lambda\cap N$. If σ is nonvoid, by considering Eq (2.2) on $\Delta(c)$, it is discrete (possibly after taking a suitable b' for b in a small open ball with center b contained in $B\backslash N$) and hence a and b can be joined with a path in $B\cap\Lambda\backslash N$. Q.E.D.

We shall prove later (see Theorem 4.1 in Section 4) that under certain conditions the set N is an analytic set of Ω.

3. Compatibility conditions

In this section we shall deal exclusively with Eq (1.3) in one unknown function. We rewrite it.

$$(3.1) \qquad \bar{\partial}w = f(z, w)\,,$$

where $f\in C^\infty_{(0,1)}(G\times\boldsymbol{C})$ and $f = \sum f_j\, d\bar{z}_j$.

As noted in Section 1, when Eq (3.1) is completely integrable $G\times\boldsymbol{C}$, roughly speaking, we can reduce the problem to solve it to the $\bar{\partial}$-problem. Therefore we shall treat the problem of solving Eq (3.1) under the following condition

$$(H_1) \qquad \partial f/\partial\bar{w}\neq 0 \qquad \text{for all} \quad (z, w)\in G\times\boldsymbol{C}\,.$$

From now on we assume that there exists a nondegenerate smooth solution $w(z)$ defined on a domain $\Omega \Subset G$.

We have, by operating $\bar{\partial}$ on Eq (3.1), an additional system of equations to which the $w(z)$ is subject:

$$[\bar{f}_w] \wedge \partial w = [\partial_z \bar{f} + \bar{f} \wedge \bar{f}_{\bar{w}}] , \tag{3.2}$$

where [ ] means the substitution of solution $w(z)$ for w. In other words, [*] means the restriction of * to the graph Γ of $w(z)$ on Ω. Consider furthermore the exterior product of $[f_{\bar{w}}]$ and (3.1). Then we have

$$[f_{\bar{w}}] \wedge \bar{\partial} w = [-f \wedge f_{\bar{w}}] . \tag{3.3}$$

It follows from the assumption (H_1) that the rank of $[\bar{f}_w] \wedge \partial$ in (3.2) is $n-1$ at each point in Ω, which obviously does not depend on solution $w(z)$.

We are given another additional system of equations which we shall need. Hereafter for economy of writing we shall omit [ ] or use the phrase "on Γ". Consider

$$[\bar{\partial}, \bar{f}_w \wedge \partial]\, w = \bar{\partial}(\bar{f}_w \wedge \partial w) - \bar{f}_w \wedge \partial \bar{\partial} w . \tag{3.4}$$

The right-hand side of (3.4) equals, by (3.2) and (3.1),

$$\bar{\partial}(\partial_z \bar{f} + \bar{f} \wedge \bar{f}_{\bar{w}}) - \bar{f}_w \wedge \partial f . \tag{3.5}$$

On the other hand, the right-hand side of (3.4) equals

$$\bar{\partial} \bar{f}_w \wedge \partial w . \tag{3.6}$$

We have, by making ue of (3.1) and (3.2), (3.5) is equivalent to

$$(\partial_z \bar{f}_w + \bar{f} \wedge \bar{f}_{\bar{w}\bar{w}}) \wedge \bar{\partial} \overline{w} + \alpha(z, w(z)) \quad \text{on} \quad \Omega ,$$

where $\alpha(z, w)$ is the smooth (2, 1)-form in z on $G \times \boldsymbol{C}$ determined by the form f only (and hence $\alpha(z, w(z))$ has no derivatives of $w(z)$). In this section, by the symbol α we shall denote smooth differential forms in z of various types (defined on $G \times \boldsymbol{C}$) which depend on the form f only. Therefore we obtain

$$\bar{\partial} \bar{f}_w \wedge \partial w - (\partial_z \bar{f}_{\bar{w}} + \bar{f} \wedge \bar{f}_{\bar{w}\bar{w}}) \wedge \bar{\partial} \overline{w} = \alpha . \tag{3.7}$$

REMARK 3.1. We have $\bar{\partial} \bar{f}_w = \bar{\partial}_z \bar{f}_w + f \wedge \bar{f}_{ww} - \bar{f}_{w\bar{w}} \wedge \bar{\partial} \overline{w}$ on Γ and hence $\bar{\partial} \bar{f}_w$ contains the derivatives of $w(z)$.

Let us consider the commutator

$$[\bar{f}_w \wedge \partial, f_{\bar{w}} \wedge \bar{\partial}] = \bar{f}_w \wedge \partial f_{\bar{w}} \wedge \bar{\partial} - f_{\bar{w}} \wedge \bar{\partial} \bar{f}_w \wedge \partial . \tag{3.8}$$

The system of vector fields of right-hand side of (3.8) have no derivatives of

$w(z)$. Ih fact we have

$$
\begin{aligned}
f_{\bar{w}}\wedge\bar{\partial}\bar{f}_{w} &= f_{\bar{w}}\wedge(\bar{\partial}_{z}\bar{f}_{w}-\bar{f}_{ww}\wedge f)+\bar{f}_{w\bar{w}}\wedge(\bar{\partial}_{z}f+f\wedge f_{w}) \\
&\qquad \text{(by the complex conjugation of (3.2))} \\
&= \alpha\,.
\end{aligned}
\tag{3.9}
$$

On the other hand, from (3.2) and (3.3), we have

$$
\begin{aligned}
&[\bar{f}_{w}\wedge\partial, f_{\bar{w}}\wedge\bar{\partial}]\, w = \\
&= \bar{f}_{w}\wedge\partial(-f\wedge f_{\bar{w}})-f_{\bar{w}}\wedge\bar{\partial}(\partial_{z}\bar{f}+\bar{f}\wedge\bar{f}_{\bar{w}}) \\
&= \alpha\,.
\end{aligned}
\tag{3.10}
$$

From (3.8), (3.9) nad (3.10) we obtain an additional equation which $w(z)$ must satisfy:

$$\bar{f}_{w}\wedge\partial f_{\bar{w}}\wedge\bar{\partial}w-f_{\bar{w}}\wedge\bar{\partial}\bar{f}_{w}\wedge\partial w = \alpha\,. \tag{3.11}$$

From (3.1) and (3.11) it follows

$$f_{\bar{w}}\wedge\bar{\partial}\bar{f}_{w}\wedge\partial w = \alpha\,. \tag{3.12}$$

Let us think of the set $\mathcal{S}^*$ of all the additional equations to which $w(z)$ is subject, and the set $[\mathcal{S}]$ of all the vector fields of type (1,0) which correspond to each equation of type (1,0), i.e. $\sum \alpha_j\, \partial_j\, w=\alpha$, belonging to $\mathcal{S}^*$. We denote by $T_z^{\boldsymbol{C}}$ the complex tangent space of Ω at z. Then the $\boldsymbol{C}$-vector space spanned by $[\mathcal{S}]$ at $z\in\Omega$ is a subspace of $T_z^{\boldsymbol{C}}$. Denoting this subspace by $[\mathcal{S}]\,(z)$, we define the *rank* of $[\mathcal{S}]$ at point $z\in\Omega$ by $\dim[\mathcal{S}](z)$. Then we see $\operatorname{rank}([\mathcal{S}])\geqq n-1$ at each $z\in\Omega$, because (H_1) shows the rank of $\bar{f}_w\wedge\partial$ (which belongs to $[\mathcal{S}]$) is $n-1$ on Ω. Assume $\dim[\mathcal{S}]\,(z^0)=n$ for $z^0\in\Omega$. Then we note that, for a neighborhood $U^0(\subset\Omega)$ of z^0, $\operatorname{rank}([\mathcal{S}])=n$ at every point of U^0. On the contrary, if $\operatorname{rank}([\mathcal{S}])=n-1$ at z^0, it does not always follow that there is a neighborhood of z^0 on which the rank of $[\mathcal{S}]$ is $n-1$.

Let us assume that $\operatorname{rank}([\mathcal{S}])$ is constant near each point of Ω. Then we needs to discuss the following two cases.

Case 1. $\operatorname{rank}([\mathcal{S}])=n$. For a suitable neighborhood U of each point of Ω, $[\mathcal{S}]$ is generated by $\partial_1, \cdots, \partial_n$ over $C^\infty(U)$ and hence the $w(z)$ satisfies on U the following normal total differential equations

$$
\begin{cases}
\partial w = g(z, w)\,, \\
\bar{\partial} w = f(z, w)\,,
\end{cases}
\tag{3.14}
$$

where $g(z, w)$ is the smooth $(1, 0)$-form in z regarding w as parameter and determined by equations (3.2) and (3.12). More precisely, if, for example, the system of vector fields generated by $\bar{f}_w\wedge\partial, f_{\bar{w}}\wedge\bar{\partial}\bar{f}_w\wedge\partial$ over $C^\infty(U)$ has rank n on U (possibly after contracting it), this system is equivalent to the one generated by $\partial_1, \cdots, \partial_n$ over $C^\infty(U)$. Therefore the $w(z)$ must fulfill Eq (3.14). As a matter of fact, Eq (3.14) can have no solution other than $w(z)$ on Ω unless the zero satisfies Eq (3.14) (in other words, unless Eq (3.1) has the trivial solution). In fact, we fix a point $a\in\Omega$ and for any point $b\in\Omega$ we join them with a smooth curve γ contained in Ω: $z=z(t)$, $0\leqq t\leqq 1$, $z(0)=a$, $z(1)=b$, and have a finite covering $\mathcal{C}$ of γ in such a way that each element of $\mathcal{C}$ has the property stated above. Then $W(t)$, defined by $w(z(t))$, must satisfy the following ordinary differential equation on $\gamma\cap U(U\in\mathcal{C})$:

$$dW/dt = \sum g_j(z(t), W(t))\, dz_j/dt + f_j(z(t), W(t))\, d\bar{z}_j/dt\,.$$

where the initial value at t' such that $z(t')\in\gamma\cap U$ is $W(t')=w(\zeta)$, $\zeta=z(t')$.

Then, by the uniqueness of solution under the Lipschitz condition, $w(b)$ is determined uniquely.

REMARK 3.2. Assume that $g(z, 0)=0$ and $f(z, 0)=0$ everywhere in Ω. Then the system of equations (3.14) has either the trivial solution or the one with no zero on Ω.

Case 2. $\operatorname{rank}([\mathcal{S}])=n-1$. We keep in mind that $\operatorname{rank}([\bar{f}_w\wedge\partial])=n-1$ on Ω, and hence for any point z^0 in Ω there exist a neighborhood U^0 of it and a smooth $(0, 2)$-form ω such that

$$f_{\bar{w}}\wedge\bar{\partial}\bar{f}_w = \omega\wedge\bar{f}_w\,. \tag{3.15}$$

Therefore it is seen from (3.8) that

$$[\bar{f}_w\wedge\partial, f_{\bar{w}}\wedge\bar{\partial}] = \bar{\omega}\wedge f_{\bar{w}}\wedge\bar{\partial}-\omega\wedge\bar{f}_w\wedge\partial\,. \tag{3.16}$$

Next we shall show that the system of vector fields $\bar{f}_w\wedge\partial$ is closed under commutator operation, that is, a Lie algebra over $C^\infty(U^0)$.

Considering the exterior product of $\bar{f}_w$ and (3.2), we have at each point in Ω

$$\bar{f}_w\wedge(\partial_z\bar{f}+\bar{f}\wedge\bar{f}_{\bar{w}}) = 0\,. \tag{3.17}$$

Denoting the smooth $(3, 0)$-form in z on $G\times\boldsymbol{C}$ (regarding w as parameter) $\bar{f}_w\wedge(\partial_z\bar{f}+\bar{f}\wedge\bar{f}_{\bar{w}})$ by $h(z, w)$, we want to prove the following proposition.

Proposition 3.1.

$$h_w = 0\,, \quad h_{\bar{w}} = 0 \quad on \quad \Gamma\,. \tag{3.18}$$

Proof. It suffices to show only that $h_w=0$ on Γ. Assuming $h_w\neq0$ at a point (z^0, w^0) in Γ, we obtain that there are neighborhoods $z^0\in U^0\subset\Omega$ and $w^0\in V^0$ in the w-plane such that $h_w\neq0$ on $U^0\times V^0$. Then we have on $U^0\times V^0$

$$\partial_w h_{i'j'k'}\neq0 \quad \text{for a set of numbers } i', j', k'\,, \tag{3.19}$$

where $h=\sum' h_{ijk}\, dz_i\wedge dz_j\wedge dz_k$ (h_{ijk} is skew-symmetric with respect to any two of (i,j,k). $\sum'$ means that the summation is performed only all over strictly increasing (i,j,k)). From (3.17) it follows that

$$h_{i'j'k'}=0 \quad \text{for every point in} \quad \Gamma\,,$$

and hence, that on Γ

$$\partial h_{i'j'k'}=\partial_z h_{i'j'k'}+(\partial_w h_{i'j'k'})\,\partial w+(\partial_{\bar w} h_{i'j'k'})\,\bar f=0\,. \tag{3.20}$$

Thus we obtain from (3.20) the following equation

$$\partial w=\alpha(z,w)\,,$$

which $w(z)$ satisfies on U^0. This contradicts rank$([S])=n-1$ on Γ. Q.E.D.

Since $h_w=\bar f_w\wedge\partial\bar f_w$ on Γ, we have by virtue of (3.18)

$$\bar f_w\wedge\partial\bar f_w=0 \quad \text{on} \quad \Gamma\,. \tag{3.21}$$

It is readily seen that (3.21) is equivalent to syaing that the vector field system $\bar f_w\wedge\partial$ is closed under commutator operation on Γ. By means of (3.16), (3.21) and Theorem 1.2 we are now in a position to state the

Proposition 3.2. *The system of vector fields on Γ generated by $\bar f_w\wedge\partial$ and $f_{\bar w}\wedge\bar\partial$ is closed under the commutator operation and hence in a neighborhood U^0 of each point z^0 in Ω there exists an $(n-1)$-dimensional complex submanifold σ through z^0 of U^0.*

By (H_1) we may suppose that $[\partial\bar f_1/\partial w)\neq 0$ on U^0. We see from Theorem 1.2 that the system of cotangent vector fields dual to the system of vector fields generated by $\bar f_w\wedge\partial$ is spanned by dh over $C^\infty(U^0)$, where h is holomorphic and $\partial h/\partial z_1\neq 0$ on U^0 (of course h depends on the $w(z)$). Therefore, since the $(1, 0)$-form $[\bar f_w]$ is dual to $[\bar f_w]\wedge\partial$, we obtain

$$[\bar f_w]=b\,dh\,, \tag{3.22}$$

where b is a smooth function defined on U^0 and $\neq 0$ (b also depends on $w(z)$). (3.22) allows us to claim the

Proposition 3.3.

$$\partial_z\bar f_{\bar w}+\bar f\wedge\bar f_{\bar w\bar w}=0 \quad on \quad \Gamma\,. \tag{3.23}$$

Proof. Take any point z^0 of Ω. Then we have a neiborhood U^0, functions h and b as stated above. From (3.22) we have $\bar\partial\bar f_w=\bar\partial b\wedge dh$, and so

$$\bar\partial\bar f_w=(1/b)\,\bar\partial b\wedge\bar f_w \quad \text{on} \quad U^0\,.$$

Then we have by (3.2) $\bar{\partial}\bar{f}_w \wedge \partial w = \alpha$ on U^0. From this and (3.7) we obtain on U^0

$$(\bar{\partial}_z f_w + f \wedge f_{ww}) \wedge \partial w = \alpha .$$

Since $\operatorname{rank}([\mathcal{S}]) = n-1$, we must have on $\Gamma|_{U^0}$

$$\bar{\partial}_z f_w + f \wedge f_{ww} = \beta \wedge \bar{f}_w$$

for a smooth form β of degree 1 defined on U^0. Since it is absurd, such β cannot exist, which indicates (3.23) holds on $\Gamma|_{U^0}$ and hence on Γ. Q.E.D.

REMARK 3.4. It seems likely from (3.23) that we must suppose

(3.24) $\quad \partial_z \bar{f} + \bar{f} \wedge \bar{f}_{\bar{w}}$ is holomorphic in w on $\boldsymbol{C}$ for each z fixed in G.

If so and moreover if f is holomorphic in w on $\boldsymbol{C}$, (3.24) is antiholomorphic in w and hence depends only on z. If f satisfies furthermore that $f(z, 0) = 0$ everywhere in G, we have, by applying the Liouville theorem in the function-theory,

$$\bar{\partial}_z f_w + f \wedge f_{ww} = 0 \quad \text{is equivalent to} \quad \bar{\partial}_z f + f \wedge f_w = 0$$

for any point (z,w) in $G \times \boldsymbol{C}$.

Now we shall be interested in the special holomorphic change of variables Φ: $U^0 \to \boldsymbol{C}^n$ defined by

$$\text{(3.25)} \qquad \begin{cases} \xi_i = h(z) , \\ \xi_j = z_j , \quad j = 2, \cdots, n . \end{cases}$$

where h defines (3.22) on U^0.

Restricting U^0 further if necessary, we may consider Φ to be biholomorphic. We introduce the notation as follows:

$$\begin{aligned} & v = w \circ \Phi^{-1}(\xi) , \quad b' = b \circ \Phi^{-1}(\xi) , \\ & \partial' = \sum \partial'_j \, d\xi_j , \quad \bar{\partial}' = \sum \bar{\partial}' \, d\bar{\xi}_j , \\ & \partial'_j = \partial/\partial \xi_j , \quad \bar{\partial}'_j = \partial/\partial \bar{\xi}_j , \end{aligned}$$

$\partial'_\xi, \bar{\partial}'_\xi$: the exterior differentiations in ξ and $\bar{\xi}$

with v as parameter.

$$f'(\xi, v) = f(\Phi^{-1}(\xi), v) , \quad \alpha'(\xi, v) = \alpha(\Phi^{-1}(\xi), v) .$$

Suppose we are given a table of systems of differential equations (B) into which we transform several systems of additional equations (A) under the Φ discussed above.

	A	B	
(i)	(3.22)	$\bar f'_v = b'\, d\xi_1$ (or $f'_{\bar v} = \bar b'\, d\bar\xi_1$)	
(ii)	(3.1)	$\bar\partial'\, v = f'(\xi, v)$	(3.26)
(iii)	(3.2)	$b'\, d\xi_1 \wedge \partial'\, v = \partial'_\xi \bar f' + \bar f' \wedge \bar f'_{\bar v}$	(3.27)
(iv)	(3.12)	$\bar b'\, d\bar\xi_1 \wedge \bar\partial'(b'\, d\xi_1) \wedge \partial'\, v = \alpha'$	(3.28)
(v)	(3.23)	$\partial'_\xi \bar f'_{\bar v} + \bar f' \wedge \bar f'_{\bar v v} = 0$	(3.29)
		(which is the complex conjugation of $\bar\partial'_\xi f'_v + f' \wedge f'_{\bar v v}$).	

Let us consider the graph $\tilde\gamma$ of $v(\xi)$ on $\Phi(U^0)$. From (i) we get

$$\partial f'_1/\partial\bar v = \bar b', \quad f'_{\bar v} \wedge d\bar\xi_1 = 0 \quad \text{on} \quad \tilde\gamma. \tag{3.30}$$

We may express (3.26) in the following form, to which we refer later on.

$$\bar\partial'_1\, v = f'_1(\xi, v), \quad \bar\partial'\, v \wedge d\bar\xi_1 = f' \wedge d\bar\xi_1 \quad \text{on} \quad \tilde\gamma. \tag{3.31}$$

From (3.27) we have on $\tilde\gamma$

$$b'\, \partial'_j\, v = \partial'_1 \bar f'_j - \partial'_j \bar f'_1 + \bar f'_1 (\bar f'_j)_{\bar v} - \bar f'_j (\bar f'_1)_{\bar v}, \quad 2 \leqq j \leqq n, \tag{3.32}$$

$$(\partial'_\xi \bar f' + \bar f' \wedge \bar f'_{\bar v}) \wedge d\xi_1 = 0. \tag{3.33}$$

4. Properties of the solutions II: The representation theorem

In the previous section we have established a series of compatibility conditions in two cases where a certain system of vector fields has rank $n-1$ or n near each point under the existence of solutions to Eq (3.1), and knew the coefficients of such vector fields in general depend on the solutions. This is the reason why we must impose several hypotheses for existence of solutions of Eq (3.1). We shall describe such hypotheses in the following section. In this section, we will write some properties of solutions of Eq (3.1) by reducing it to an $\boldsymbol{R}$-linear equation and availing ourselves of some results obtained in the previous section. We shall treat the case that Eq (3.1) is quasi-homogeneous:

$$\begin{cases} \bar\partial w = f(z, w), \\ f(z, 0) = 0 \quad \text{everywhere in } G. \end{cases} \tag{4.1}$$

REMARK 4.1. Let us assume that $f(z, 0)$ does not vanish on G. If there exists a smooth solution w^0 of Eq (3.1) defined on a domain $\Omega \subset G$, we have the system of equations

$$\bar\partial w = f(z, w + w^0) - f(z, w^0). \tag{4.1$'$}$$

Then we see Eq (4.1)$'$ is quasi-homogeneous on $\Omega \times \boldsymbol{C}$. Therefore if Eq (3.1) has two smooth different solutions defined on Ω, it can be found that solutions of Eq (3.1) also have the same properties as solutions of Eq (4.1).

From now on we shall discuss only nontrivial solutions defined on $\Omega \Subset G$, and follow the notation in Sections 2 and 3. By $\mathcal{F}(\Omega)$ we denote the set of all the solutions of Eq (4.1) on Ω. By $\mathcal{F}_0(\Omega)$ we denote $\mathcal{F}(\Omega)-\{0\}$, where 0 indicates the trivial solution.

We note (regarding w as parameter) that the (0,1)-form $f(z, w)$ on $G\times \boldsymbol{C}$ to Eq (4.1) may be expressed in

$$(4.2) \qquad f(z, w) = w\, F_1(z, w)+\bar{w}\, F_2(z, w)\,,$$

where $F_\lambda \in C^\infty_{(0,1)}(G\times \boldsymbol{C})$ $(\lambda=1, 2)$ is the form in z with parameter w.

In this section we *suppose* $F_2 \neq 0$ on $G\times \boldsymbol{C}$.

Denoting by Γ the graph of the $w(z)$ on Ω, we define $\boldsymbol{a}$ and $\boldsymbol{b}$ by $\bar{F}_1|_\Gamma$ and $\bar{F}_2|_\Gamma$, respectively. We have thus the following $\boldsymbol{R}$-linear and homogeneous differential equation

$$(4.3) \qquad \bar{\partial} w = \bar{\boldsymbol{a}} w+\bar{\boldsymbol{b}}\bar{w}\,.$$

Under the *assumption* that $\operatorname{rank}(\mathcal{S}|_\Gamma)$ is constant *near* each point of Ω, repeating the argument of the same pattern as in Section 3, we arrive at the two cases: $\operatorname{rank}(\mathcal{S}|_\Gamma)=n$ or $n-1$ near each point of Ω. Let us remark that $\operatorname{rank}(\mathcal{S}|_\Gamma)=n$ at a point implies $\operatorname{rank}(\mathcal{S}|_\Gamma)=n$ near that point.

We shall see a simple propery in the first case $\operatorname{rank}(\mathcal{S}|_\Gamma)=n$.

Proposition 4.1. *Assume* $w\in \mathcal{F}_0(\Omega)$ *and* $\operatorname{rank}(\mathcal{S}|_\Gamma)=n$ *on* Ω. *Then* $w(z)$ *has no zero in* Ω.

Proof. Eq (4.3) is $\boldsymbol{R}$-linear and homogeneous, and so is Eq (3.14). By Remark 3.2 we obtain the result. Q.E.D.

As before we write the zeroes of $w\in \mathcal{F}_0(\Omega)$ by N.

Proposition 4.2. *Assume* $N\neq 0$. *Then for any* $z\in N$ *there exist a neighborhood* U *of* z *such that on* U

$$(4.4) \qquad \begin{cases} \partial \boldsymbol{a} = 0\,, \\ \boldsymbol{b}\wedge d\boldsymbol{b} = 0\,. \end{cases}$$

Proof. Fix z^0 of N. Then, by assumption for $\operatorname{rank}(\mathcal{S}|_\Gamma)$ and Proposition 4.1, we can choose a neighborhood U^0 of z^0 such that $\operatorname{rank}(\mathcal{S}|_\Gamma)=n-1$ there. If we repeat the same discussion for Eq (4.3) as in the case $\operatorname{rank}(\mathcal{S}|_\Gamma)=n-1$ in Section 3, then we get (4.4), owing to Propositions 3.2 and 3.3. Q.E.D.

If $w\in \mathcal{F}_0(\Omega)$ and $N\neq 0$, we can see from Proposition 4.2 that for a neighborhood U^0 of each point z^0 of N we have a biholomorphic change of variables Φ

onU^0 such as (3.25). Choosing such a U^0, we fix U^0. Then Φ transforms Eq (4.3) into

$$\bar{\partial}' v = \bar{\boldsymbol{a}}' v + b' \bar{v} d\xi_1 . \tag{4.4.1}$$

Now we can take a polydisc $D = D_1^0 \times D'^0 \subset \Phi(U^0)$ such that $\Phi(U^0 \cap N) \cap D \neq 0$, where D_1^0 is an open disc with the center $\xi_1^0 = h(z^0)$ in the ξ_1-complex plane and D'^0 is a polydisc in $\boldsymbol{C}^{n-1}$ with the center $\xi'^0 = z'^0$, $\xi' = (\xi_2, \cdots, \xi_n)$, $z' = (z_2, \cdots, z_n)$ (see (3.25)). From the fact that $\partial' \boldsymbol{a}' = 0$ on $\Phi(U^0)$, we have a smooth function ω on $\Phi(U^0)$ satisfying $\bar{\partial}'\omega = \bar{\boldsymbol{a}}'$(restricting U^0 if necessary).

Then, by $v = v_1 \exp \omega$, we transform (4.3.1) into

$$\bar{\partial}' v_1 = \bar{v}_1 \bar{c}\, d\bar{\xi}_1 , \tag{4.4.2}$$

where $c = b' \circ \Phi^{-1}(\xi) \exp(\bar{\omega} - \omega)$.

Then we are in a position to state the following lemma on a neighborhood of ξ^0 $(=\Phi(z^0),\ z^0 \in N)$.

Lemma 4.1. *$v(\xi) = \psi(\xi_1)\, W(\xi)$, where $\psi(\xi_1)$ is holomorphic in D_1^0 and $W(\xi)$ is smooth and bounded away from zero on D.*

Proof. (4.4.2) shows v_1 is holomorphic on D'^0 for each fixed $\xi_1 \in D_1^0$ and is Vekua function in $\xi_1 \in D_1^0$ for each fixed $\xi' \in D'^0$. Hence we can put $c = \phi/\bar{\Psi}$, where ϕ, Ψ are smooth on D and holomorphic on D'^0 for each fixed $\xi_1 \in D_1^0$. Define $\rho = (1/2) \log(\phi\, \Psi)$, where log means a branch. By considering $v_2 = v_1 \exp(-\rho)$, Eq (4.4.2) is transformed into

$$\bar{\partial} v_2 = \{-(\bar{\partial}_1 \rho)\, v_2 + |c|\, \bar{v}_2\}\, d\bar{\xi}_1 \tag{4.4.3}$$

where we omit " $'$ " for brevity.

We want to claim that $|c|$ and $\bar{\partial}_1 \rho$ depends only on ξ_1. To do this, operating $\bar{\partial}$ on the both side of (4.4.3), we have

$$\partial v_2 \wedge d\xi_1 = -v_2 \partial \log|c| \wedge d\xi_1 . \tag{4.4.4}$$

By operating ∂ on (4.4.3) we have, noting $\bar{\partial} v_2 \wedge d\bar{\xi}_1 = 0$ on D,

$$\partial\bar{\partial} v_2 \wedge d\xi_1 = -\bar{v}_2 \partial|c| \wedge d\xi_1 \wedge d\bar{\xi}_1 + v_2 \beta , \tag{4.4.5}$$

where β denotes the smooth (2,1)-form on D.

From (4.4.4) and (4.4.5) we have

$$v_2 \bar{\partial}|c| \wedge d\xi_1 \wedge d\bar{\xi}_1 = \bar{v}_2 \gamma , \tag{4.4.6}$$

where γ denotes the smooth (1,2)-form on D. Let us assume $\bar{\partial}|c| \wedge d\xi_1 \neq 0$ on a neighborhood $\Delta \subset D$. Then, for a number $j\,(2 \leqq j \leqq n)$, $\bar{\partial}_j |c| \neq 0$ on Δ. From (4.4.6),

$$(\bar\partial_j |c|)\, v_2 = \gamma_j \bar v_2\,, \tag{4.4.7}$$

where γ_j denotes the coefficient of $d\xi_1 \wedge d\bar\xi_1 \wedge d\bar\xi_j$ in γ. From (4.4.7) and Eq (4.3.3), we obtain on Δ

$$\partial v_2 = v_2 \delta_1 + \bar v_2 \delta_2\,, \tag{4.4.8}$$

where δ_1, δ_2 are the smooth (1, 0)-form on Δ. The equations (4.4.3), (4.4.8) are normal total differential equations. Hence, noting $v_2(\xi^0)=0$ and the nontriviality of v_2, we see from Theorem 2.1 $\bar\partial |c| \wedge d\bar\xi_1 = 0$ on D. Similarly we have $\partial |c| \wedge d\xi_1 = 0$ on D.

From (4.4.3), we have

$$\bar\partial_1 v_2 + (\bar\partial_1 \rho)\, v_2 = |c| \bar v_2\,. \tag{4.4.9}$$

Since v_2 and ρ are holomorphic on D'^0 for each fixed $\xi_1 \in D_1^0$ and $|c|$ depends only on ξ_1, we can see at once v_2 depends only on ξ_1 and so does $\bar\partial_1 \rho$.

Thus Eq (4.4.9) turns out the Vekua equation of one variable. From the representation theorem for Vekua equation of one variable ([3], [4], [21]) we have $v_2 = \psi(\xi_1)\, B(\xi_1)$, where ψ is holomorphic in D_1^0, and B smooth and bounded away from zero in D_1^0. And hence $v = \psi\, B \exp(\rho + \omega)$, which completes the proof.

Q.E.D.

Theorem 4.1. *Assume $w \in \mathscr{F}_0(\Omega)$ and $N \neq 0$. Then N is an analytic set of Ω.*

Proof. We note N is closed in Ω. Fix $z^0 \in N$. Then, using the same notation as in the proof of Lemma 4.1, we have from this lemma

$$v(\xi) = \psi(\xi_1)\, W(\xi)\,.$$

Therefore $v(\xi)=0$ is equivalent to $\psi(\xi_1)=0$. Thus, by shrinking U^0 further if necessary, it follows that the set of zeroes of ψ is finite and so

$$N \cap U^0 = \cup \{z \in U^0 \colon h(z) = c \quad \text{such that} \quad \psi(c) = 0\}\,,$$

where the union is performed over the set of zeroes of ψ. Q.E.D.

Let Ω be a pseudoconvex domain in G. Then it is well-known that there exists a smooth function v defined on Ω in such a way that $\bar\partial v = \bar{\boldsymbol{a}}$ on Ω ([9], [10], [11]). If we transform Eq (4.3) by $w = W \exp(v)$, we have

$$\bar\partial W = \bar W \exp(\bar v - v)\, \bar{\boldsymbol{b}}\,. \tag{4.5}$$

Proposition 4.3. *W is 1-holomorphic on Ω* ([1]), *that is,*

$$\bar\partial W \wedge \partial\bar\partial W = 0 \quad on \quad \Omega\,. \tag{4.6}$$

Proof. We can obtain the result by a direct computation. Q.E.D.

Since the second part of (4.4) is invariant under scalar multiplication, we can locally change Eq (4.4), by the transformation Φ (which is derived under the assumption $b_1 \neq 0$), and change Eq (4.5) into the system of equations of type (3.30), that is,

$$\begin{cases} \bar{\partial}_1' W' = c\,\bar{W}'\,, \\ \bar{\partial}_j' W' = 0\,, \quad j = 2, 3, \cdots, n\,. \end{cases} \tag{4.7}$$

From (4.7) we know that W' depends holomorphically on $\xi' = (\xi_2, \cdots, \xi_n)$ (see [1] and [2] as for the q-holomorphic functions $(0 \leqq q < n)$).

For any (0, 1)-form $\theta = \sum \theta_j\, d\bar{z}_j$, $|\theta|$ denotes $\sum |\theta_j|$.

Proposition 4.4. *Assume that $|f_w| \leqq L(z)$ on $G \times \boldsymbol{C}$ for a smooth $L(z)$ on G. Let $\Omega \Subset G$ be a strictly pseudoconvex domain with C^2 boundary. And if $w \in \mathcal{F}_0$ is continuous on $\bar{\Omega}$, then Maximum Principle is valid for $w(z)$ in the following sense:*

$$\mathrm{Max}_{z\in\bar{\Omega}} |w(z)| \leqq K\, \mathrm{Max}_{z\in\partial\Omega} |w(z)|\,,$$

where K is a positive constant $(\geqq 1)$ depending only on Ω.

Proof. Note that $\boldsymbol{a} = \bar{F}_1(z, w(z))$, where

$$F_1(z, w) = \int_0^1 f_w(z, wt)\, dt\,.$$

Hence, $|\boldsymbol{a}| \leqq L' = \sup_{z\in\bar{\Omega}} L(z) < \infty$ on Ω. Since $\boldsymbol{a}$ satisfies $\partial\, \boldsymbol{a} = 0$ on Ω by Proposition 4.2, there exists a smooth function v continuous on $\bar{\Omega}$ satisfying $\bar{\partial} v = \bar{\boldsymbol{a}}$ on Ω and $\mathrm{Max}_{z\in\bar{\Omega}} |v(z)| \leqq K' \sup_{z\in\bar{\Omega}} |\boldsymbol{a}(z)| \leqq L'$ (see [8], [10]). The function W defined by $w \exp(-v)$ is, by Proposition 4.3, 1-holomorphic and so satisfies

$$\mathrm{Max}_{z\in\bar{\Omega}}\ |W(z)| = \mathrm{Max}_{z\in\partial\Omega}\ |W(z)|\,.$$

Thus we obtain the result. Q.E.D.

REMARK 4.2. If f_w is holomorphic in w on $G \times \boldsymbol{C}$, the condition $|f_w| \leqq L(z)$ implies on account of Liouville theorem that f_w never includes w. When f is $\boldsymbol{R}$-linear with respect to w, the condition is fulfilled.

Given two functions σ and τ on a domain $\Omega \subset \boldsymbol{C}^n$. We say that σ and τ are similar on Ω (see [4], [6]) if, for a bounded function ρ continuous on $\bar{\Omega}$ and bounded away from zero, $\sigma = \tau\, \rho$ on Ω.

From now on, we shall use the same notation and terminology as in Lemma 4.1 and Theorem 4.1. We state the

Lemma 4.2. *Let $w \in \mathcal{F}_0(\Omega)$. Then $\partial_j w/w$ is locally summable on Ω for all*

j, $1 \leqq j \leqq n$.

Proof. One may assume $N \neq 0$ and it suffices to see that $\partial_j w/w$ is summable on a neighborhood U^0 of $z^0 \in N$. From Lemma 4.1 the summability of $\partial_j w/w$ on U^0 is reduced to that of $(d\psi/d\xi_1)/\psi$ on $\Phi(U^0)$ under Φ on U^0. One may consider $\Phi(U^0) = D(=D_1^0 \times D'^0)$. $(d\psi/d\xi_1)/\psi$ has the singularity of order one at each point of $h(U^0 \cap N)$. We remark $h(U^0 \cap N)$ is a finite set (see the proof of Theorem 4.1). This shows that $(d\psi/d\xi_1)/\psi$ is summable on D. Q.E.D.

We are now in a position to state the representation theorem for $\mathcal{F}(\Omega)$.

Theorem 4.2. *Let $\Omega \Subset G$ be a strictly pseudoconvex domain with C^4 boundary. For bounded $w \in \mathcal{F}(\Omega)$ there exist a holomorphic function τ on Ω and a Hölder continuous function ρ on $\bar{\Omega}$ such that w is similar to τ on Ω.*

Proof. It suffices to treat $w \in \mathcal{F}_0(\Omega)$. Fix $w \in \mathcal{F}_0(\Omega)$. Let N be the set of zeroes of w. We define the (0, 1)-form $g(z)$ on by

$$g(z) = \begin{cases} f(z, w(z))/w(z) & \text{on} \quad \Omega \backslash N, \\ 0 & \text{on} \quad N. \end{cases} \tag{4.8}$$

Put $L_1 = \sup_{z \in \Omega} |w(z)|$. We have $|f| \leqq L_2 |w|$ on Ω, where $L_2 = \mathrm{Max}_{(z,\zeta) \in \bar{\Omega} \times \bar{\Delta}} \{|F_1(z, \zeta)| + |F_2(z, \zeta)|\}$, $\Delta = \{\zeta \in \boldsymbol{C};\ |\zeta| < L_1\}$. Hence, $g(z)$ is summable on Ω. We want to prove $\bar{\partial} g = 0$ in the sense of distributions. To do this, it suffices to show $\bar{\partial}_j g$ is summable on a neighborhood U^0 of any point z^0 of N. Fix $z^0 \in N$ and U^0. From (4.8), at each point of $U^0 \backslash N$

$$\begin{aligned} \bar{\partial}_j g = (1/w)\ \{\bar{\partial}_j f + (\bar{\partial}_j w)\, \partial_w f + (\bar{\partial}_j \bar{w})\, \partial_{\bar{w}} f\} \\ -(1/w^2)\, (\bar{\partial}_j w) f\,. \end{aligned} \tag{4.9}$$

From (4.2) $\bar{\partial}_j f(z, 0) = 0$ on G and hence

$$\bar{\partial}_j f(z, w(z))/w(z)$$

is summable on Ω. Thus, Lemma 4.2 shows $\bar{\partial}_j g$ is summable on U^0.

We want to prove that, for all j, $\bar{\partial}_j g$ exists on Ω in the sense of distributions. However it suffices to prove the existence on U^0. One may consider U^0 to be a polydisc $D = D_1 \times \cdots \times D_n$. For a fixed $z'_j \in D'_j = D_1 \times \cdots \times D_{j-1} \times D_{j+1} \times \cdots \times D_n$, $1 \leqq j \leqq n$, by $D_j(z'_j)$ we denote $\{z_1\} \times \cdots \times \{z_{j-1}\} \times D_j \times \{z_{j+1}\} \times \cdots \times \{z_n\}$. Let ϕ be a smooth function with compact support $\subset D$. Fix $z'_j \in D'_j$ such that $\int_{D_j(z'_j)} g \bar{\partial}_j \phi \, dV_j$ exists, where dV_j is two dimensional volume element. For simplicity, we write δ for $D_j(z'_j)$. By the proof of Proposition 2.1, $N \cap \delta$ is finite set $\{a_1, \cdots, a_l\}$. Let us think of the closed disc Δ_k with center a_k and radius ε such that $\Delta_k \Subset \delta$ and $\Delta_p \cap \Delta_q = 0$ for $p \neq q$, $1 \leqq k, p, q \leqq l$. Let $\Delta(\varepsilon)$ be the set deleted all Δ_k from δ and let $\partial\Delta(\varepsilon)$ be the boundary of $\Delta(\varepsilon)$ contained in δ

$$\begin{aligned}\int_\delta g\bar{\partial}_j\phi\, dV_j &= \lim_{\varepsilon\to 0}\int_{\Delta(\varepsilon)} g\bar{\partial}_j\phi\, dV_j \\ &= \lim_{\varepsilon\to 0}\left\{\frac{1}{2i}\int_{\partial\Delta(\varepsilon)} g\phi\, dZ_j - \int_{\Delta(\varepsilon)} (\bar{\partial}_j g)\phi\, dV_j\right\} \\ &= -\int_\delta (\bar{\partial}_j g)\phi\, dV_j\,,\end{aligned}$$

where we make use of the boundedness of g and the summability of $\bar{\partial}_j g$ on D.

Thus we obtain by the Fubini theorem

$$\begin{aligned}\int_D g\bar{\partial}_j\phi\, dV &= \int_{D_j'} dV_j' \int_{D_j(z_j')} g\bar{\partial}_j\phi\, dV_j \\ &= -\int_{D_j'} dV_j' \int_{D_j(z_j')} (\bar{\partial}_j g)\phi\, dV_j \\ &= -\int_D (\bar{\partial}_j g)\phi\, dV\,,\end{aligned}$$

where dV_j' and dV mean $(2n-2)$- and $2n$-dimensional volume elements respectively. Hence, we see $\bar{\partial}g=0$ on Ω in the sense of distributions and the (0, 1)-form g is bounded on Ω. By applying Kerzman's theorem ([10]), we have an α-Hölder continuous function u on $\bar{\Omega}$ $(\alpha<1/2)$ such that $\bar{\partial}u=g$. Putting $\phi=w\exp(-u)$, we can see ϕ is holomorphic on Ω, because w and u are continuous on Ω. Q.E.D.

5. Integrability conditions and local solvability

In Section 3 we have found several compatibility conditions under Assumption (H_1) and $\operatorname{rank}([\mathcal{S}])=n-1$. These conditions are the additional equations other than Eq (3.1) which we must treat to solve (3.1). By a suitable change of variables we change Eq (3.1) and the additional equations into the semi-linear $\bar{\partial}$-equation in one variable (see the first equation of (3.31)) and the normal total differential equation in the remainder of variables (see the second equation of (3.31) and (3.32)). From those we wish to find the integrability conditions of Eq (3.1).

We suppose

(5.1) $$f_{\bar{w}} \text{ does not depend on } w \text{ on } G\times \boldsymbol{C}\,.$$

Hypothesis (5.1) indicates that the given (0,1)-form f is of the following type,

(5.2) $$f(z, w) = g(z, w)+\bar{w}\bar{\boldsymbol{b}}\,,$$

where (i) $g(z, w)$ is a $(0, 1)$-form with respect to $z\in G$, smooth in $(z, w)\in G\times \boldsymbol{C}$ and holomorphic in w and (ii) $\boldsymbol{b}$ is a smooth $(1, 0)$-form in G and does not depend on w.

Hypothesis (H_1) shall mean, therefore, that

$$(H_1) \qquad \boldsymbol{b}\neq 0 \quad \text{on} \quad G.$$

We shall next think of the assumptions on $\boldsymbol{b}$ (see Proposition 3.2):

(H_2) The system of vector fields $\tilde{\mathscr{P}}$ generated by $\boldsymbol{b}\wedge\partial$ and $\bar{\boldsymbol{b}}\wedge\bar{\partial}$ over $C^\infty(G)$ is a Lie algebra over $C^\infty(G)$, that is, denoting by $\mathscr{P}$ the system of vector fields spanned by $\boldsymbol{b}\wedge\partial$ over $C^\infty(G)$, $\mathscr{P}$ and $\tilde{\mathscr{P}}$ satisfy the condition $(*)$ in Theorem 1.1.

From (3.17) and (3.23) we have the hypotheses:

$$(H_3) \qquad f_{\bar{w}}\wedge(\bar{\partial}_z f+f\wedge f_w)=0 \quad \text{everywhere in} \quad G\times\boldsymbol{C}\,.$$

$$(H_4) \qquad \bar{\partial}_z f_w+f\wedge f_{ww}=0 \quad \text{everywhere in} \quad G\times\boldsymbol{C}\,.$$

From now on, for brevity we assume that

$$(5.3) \qquad f(z, 0)=0 \quad \text{everywhere in} \quad G\,.$$

From (5.2) the hypotheses (H_3) and (H_4) can be written in the form, respectively: for all (z, w) in $G\times\boldsymbol{C}$.

$$(5.4) \qquad \bar{\boldsymbol{b}}\wedge\{\bar{\partial}_z g+g\wedge g_w+(\bar{\partial}\bar{\boldsymbol{b}}+\bar{\boldsymbol{b}}\wedge g_w)\,\bar{w}\}=0\,,$$

$$(5.5) \qquad \bar{\partial}_z g_w+g\wedge g_{ww}+\bar{w}\bar{\boldsymbol{b}}\wedge g_{ww}=0\,.$$

Noting that g is holomorphic in w, we obtain from (5.4) and (5.5) that at every $(z, w)\in G\times\boldsymbol{C}$

$$(5.6) \qquad \boldsymbol{b}\wedge\partial\boldsymbol{b}=0\,,$$

$$(5.7) \qquad \bar{\boldsymbol{b}}\wedge(\bar{\partial}_z g+g\wedge g_w)=0\,,$$

$$(5.8) \qquad \bar{\boldsymbol{b}}\wedge g_{ww}=0\,,$$

$$(5.9) \qquad \bar{\partial}_z g_w+g\wedge g_{ww}=0\,.$$

Since (5.6) follows Hypothesis (H_2), we see (5.6) becomes useless. The form g has the property $g(z, 0)=0$ in G, and so if we recall Remark 3.4 and replace f in the remark with g in (5.9)., it follows from (5.9) that

$$(5.10) \qquad \bar{\partial}_z g+g\wedge g_w=0 \quad \text{everywhere in} \quad G\times\boldsymbol{C}\,.$$

From this fact, (5.7) and (5.9) are replaced by (5.10) only.

It is verified from (5.8) that there exists a smooth function $\lambda(z, w)$ in $G\times \boldsymbol{C}$ and holomorphic in w such that $\lambda(z, 0)=0$ everywhere in G and

$$g_{ww} = \lambda\bar{\boldsymbol{b}}\,. \tag{5.11}$$

Hence g has the expression of the following type

$$g(z, w) = \lambda(z, w)\,\bar{\boldsymbol{b}}+w\bar{\boldsymbol{a}}\,, \tag{5.12}$$

where $\boldsymbol{a}$ is a smooth (1, 0)-form in G whcih contains no w.

Substituting (5.12) into (5.10), we obtain

$$w\bar{\partial}\bar{\boldsymbol{a}}+(\bar{\partial}_z\lambda)\wedge\bar{\boldsymbol{b}}+\lambda\bar{\partial}\bar{\boldsymbol{b}}+(w\lambda_w-\lambda)\,\bar{\boldsymbol{a}}\wedge\bar{\boldsymbol{b}} = 0 \quad \text{everywhere in} \quad G\times\boldsymbol{C}\,. \tag{5.13}$$

In order to get the integrability conditions of Eq (3.1) we desire precise relations among λ, $\boldsymbol{a}$ and $\boldsymbol{b}$. Since λ is smooth and holomorphic in w for each fixed $z\in G$, λ has a power series in w with coefficients in $C^\infty(G)$,

$$\lambda(z, w) = \textstyle\sum_{j=1}^\infty \lambda_j(z)\,w^j\,, \tag{5.14}$$

where the radius of convergence is ∞ for any z in G. Therefore we have, for any $z\in G$

$$\bar{\partial}_z\lambda = \textstyle\sum_{j=1}^\infty (\bar{\partial}\lambda_j)\,w^j \quad ((0, 1)\text{-form})\,, \tag{5.15}$$

$$w\lambda_w-\lambda = \textstyle\sum_{j=1}^\infty (j-1)\,\lambda_j(z)\,w^j\,. \tag{5.16}$$

Substituting (5.14), (5.15) and (5.16) into (5.13), we obtain

$$\{\bar{\partial}\bar{\boldsymbol{a}}+\bar{\partial}\lambda_1\wedge\bar{\boldsymbol{b}}+\lambda_1\bar{\partial}\bar{\boldsymbol{b}}\}\,w + \textstyle\sum_{j=2}^\infty \bar{\partial}\lambda_j\wedge\bar{\boldsymbol{b}}+\lambda_j\bar{\partial}\bar{\boldsymbol{b}}+(j-1)\,\lambda_j\,\bar{\boldsymbol{a}}\wedge\bar{\boldsymbol{b}}\}\,w^j = 0\,.$$

From this we obtain on G

$$\boldsymbol{b}\wedge\partial\bar{\lambda}_1-\bar{\lambda}_1\,\partial\boldsymbol{b}-\partial\boldsymbol{a} = 0\,, \tag{5.17}$$

$$\boldsymbol{b}\wedge\partial\bar{\lambda}_j-\{\partial\boldsymbol{b}+(j-1)\,\boldsymbol{a}\wedge\boldsymbol{b}\}\,\bar{\lambda}_j = 0\,, \quad j = 2, 3, \cdots \tag{5.18}$$

We want to find the integrability conditions on the system of equations discussed. To this end, a change of coordinates Φ determined by (H_1) and (H_2) (see below) plays an essential role (see (3.25)). We will use Φ under the same situation as in the case of (3.25) except that Φ is independent of any solution. We intend to use the same notation as used in Section 3.

We substitute (5.12) into (5.2) and obtain

$$\bar{\partial}w = w\bar{\boldsymbol{a}}+\{\lambda(z, w)+\bar{w}\}\,\bar{\boldsymbol{b}}\,. \tag{5.19}$$

Substituting the right-hand side of (5.19) into (3.2), we get by (5.13)

$$\boldsymbol{b}\wedge\partial w = w(-\boldsymbol{a}\wedge\boldsymbol{b}+\partial\boldsymbol{b})\,. \tag{5.20}$$

From (H_1) and (H_2), and by virtue of Theorem 1.2 we can state the following proposition.

Proposition 5.1. *(H_2), together with (H_1), leads to*

$$\boldsymbol{b}\wedge d\boldsymbol{b} = 0 \quad \textit{on} \quad G\,. \tag{5.21}$$

Proof. Note that $\operatorname{rank}(\boldsymbol{b}\wedge\partial)=n-1$. The system of vector fields $\{X_{jk}\}$ (see Section 1) is a Lie algebra over $C^\infty(G)$. Since it suffices to show (5.21) locally, we fix a neighborhood U^0 such that $b_1\neq 0$ there. The system $\{X_{jk}, \bar{X}_{j'k'}\}$ (see Section 1) is a Lie algebra over $C^\infty(U^0)$, and hence, owing to Theorem 1.2, we have functions h holomorphic on U^0 (possibly after shrinking U) and b smooth such that

$$\boldsymbol{b} = b\; dh\,, \tag{5.22}$$

where $b\neq 0$ and $\dfrac{\partial h}{\partial z_1}\neq 0$ on U^0. Q.E.D.

From (5.17) and (5.21) we have at once

$$\boldsymbol{b}\wedge\partial\boldsymbol{a} = 0 \quad \text{on} \quad G\,. \tag{5.23}$$

We shall fix U^0 mentioned in the proof of Proposition 5.1 once and for all, and find the integrability conditions on U^0 for the system of equations (5.19) and (5.20). Using the h mentioned above, we consider the same change of coordinates Φ on U^0 as (3.25). We follow the notation in Section 3.

By Φ (5.19) and (5.20) are transformed into the following equations, respectively (see also (3.26), (3.27)). Let us note the differential forms are invariant under holomorphic mappings.

Hereafter we shall also use the notation in Section 4. One may regard $\Phi(U^0)$ as the polydisc $D=D_1^0\times D'^0$.

$$\begin{cases}\bar\partial' v = v\bar{\boldsymbol{a}}'+\{\lambda'(\xi, v)+\bar{v}\}\bar{b}'\; d\bar\xi_1'\,,\\ b'\partial' v\wedge d\xi_1 = v(b'\boldsymbol{a}'-\partial' b')\wedge d\xi_1 \quad \text{on} \quad \Phi(U^0)\times\boldsymbol{C}\,.\end{cases} \tag{5.24}$$

Let us rewrite them as follows:

$$\bar\partial_1' v = \bar{a}_1' v+\bar{b}'(\lambda'+\bar{v})\,, \tag{5.25.1}$$

$$\bar\partial' v\wedge d\bar\xi_1 = v\bar{\boldsymbol{a}}'\wedge d\bar\xi_1\,, \tag{5.25.2}$$

$$\partial' v\wedge d\xi_1 = v(\boldsymbol{a}'-\partial'\log b')\wedge d\xi_1\,, \tag{5.25.3}$$

where log denotes a branch, ∂', $\bar{\partial}'$ denote $\partial'=\sum \partial/\partial\xi_j\, d\xi_j$, $\bar{\partial}'=\sum \partial/\partial\bar{\xi}_j\, d\bar{\xi}_j$ respectively and the forms transformed by Φ are expressed by superscript " ' ". However, for simplicity, hereafter we shall omit it.

Similarly we transform (5.17), (5.18) and (5.23) by Φ, respectively, and modify those which are transformed into.

$$b(\partial\bar{\lambda}_1+\bar{\lambda}_1\partial \log b)\wedge d\xi_1 = -\partial \boldsymbol{a}\,, \tag{5.26.1}$$

$$\{\partial\bar{\lambda}_j+\bar{\lambda}_j(\partial \log b+(j-1)\,\boldsymbol{a})\}\wedge d\xi_1 = 0\,, \quad j=2, 3, \cdots. \tag{5.26.2}$$

$$\partial\boldsymbol{a}\wedge d\xi_1 = 0\,. \tag{5.26.3}$$

We see that the system of equations (5.25.2) and (5.25.3) is a normal total differential equations in $\xi'\in D'^0$ for each fixed $\xi_1\in D_1^0$. Thus we can assert that the complete integrability conditions of this system of equations on $D'^0\times \boldsymbol{C}$, for fixed $\xi_1\in D_1^0$, fulfill (5.26.3) and

$$(\partial\bar{\boldsymbol{a}}+\bar{\partial}\boldsymbol{a}+\partial\bar{\partial}\log b)\wedge d\xi_1\wedge d\bar{\xi}_1 = 0\,. \tag{5.27}$$

Next, we are going to seek the compatibility conditions of the system of (5.25.1), (5.25.2) and (5.25.3). (This system is the normal form). One will find such conditions to contain the conditions (5.26.3) and (5.27).

From (5.24) and (5.25.3), after a lengthy computation we obtain

$$\begin{aligned}
&(\partial\bar{\boldsymbol{a}}+\bar{\partial}\boldsymbol{a}+\partial\bar{\partial}\log b)\, v\wedge d\xi_1\\
&\quad-(\lambda\partial\bar{b}+\bar{b}\partial_\xi\,\lambda+\bar{b}\lambda\partial\log b-\bar{b}\lambda\boldsymbol{a})\wedge d\xi_1\wedge d\bar{\xi}_1\\
&\quad-\bar{b}v\lambda_v(\boldsymbol{a}-\partial\log b)\wedge d\xi_1\wedge d\bar{\xi}_1\\
&\quad-(\partial\bar{b}+\bar{b}\partial\log b)\,\bar{v}d\xi_1\wedge d\bar{\xi}_1\\
&=0\,.
\end{aligned} \tag{5.28}$$

We require that (5.28) holds for all (ξ, v) in $D\times\boldsymbol{C}$. Then we have, by differentiating with respect to $\bar{v}$,

$$\partial\bar{b}+\bar{b}\partial\log b = 0 \quad \text{on} \quad D$$

and hence

$$\partial \log |b| \wedge d\xi_1\wedge d\bar{\xi}_1 = 0 \quad \text{on} \quad D\,,$$

which is equivalent to

$$|b| \quad \text{depends only on} \quad \xi_1\in D_1^0\,. \tag{5.29}$$

Let us substitute (5.14) into (5.28) with (5.29). Then we have, after lengthy computation, for all $(\xi, v)\in D\times\boldsymbol{C}$

$$
\begin{aligned}
&(\partial\bar{\boldsymbol{a}}+\bar{\partial}\boldsymbol{a}+\partial\bar{\partial}\log b)\, v\wedge d\xi_1 \\
&\quad -\bar{b}(\partial\lambda_1+\lambda_1\partial\log\bar{b})\, v\wedge d\xi_1\wedge d\bar{\xi}_1 \\
&\quad -\textstyle\sum_{j=2}^{\infty}\bar{b}[\partial\lambda_j+\lambda_j\{(j-1)\,\boldsymbol{a}-j\partial\log b\}]\, v^j\wedge d\xi_1\wedge d\bar{\xi}_1 \\
&=0\,,
\end{aligned}
$$

where the third term of the left-hand side is modified by using (5.29).

We require thus the following conditions.

$$
\begin{aligned}
(5.30.1)\qquad &\bar{b}(\partial\lambda_1+\lambda_1\partial\log\bar{b})\wedge d\xi_1\wedge d\bar{\xi}_1 \\
&\quad =(\partial\bar{\boldsymbol{a}}+\bar{\partial}\boldsymbol{a}+\partial\bar{\partial}\log b)\wedge d\xi_1\,,
\end{aligned}
$$

$$
(5.30.2)\qquad [\partial\lambda_j+\lambda_j\{(j-1)\,\boldsymbol{a}-j\partial\log b\}]\wedge d\xi_1=0 \qquad j=2,3,\cdots.
$$

It is natural to raise a question whether there is a function $\lambda(\xi, v)$ satisfying the conditions (5.26.1), (5.26.2), (5.30.1) and (5.30.2) or not.

First we study the probelm of solving the system of equations (5.26.1) and (5.30.1). We put $\Lambda=\bar{b}\lambda_1$ and then have

Proposition 5.2. *The system of equations* (5.26.1), (5.30.1) *is locally solvable under the conditions* (5.26.3), (5.27).

Proof. (5.26.1) and (5.30.1) are changed into the following, respectively.

$$
(5.31)\qquad \partial\bar{\Lambda}\wedge d\xi^1=-\partial\boldsymbol{a}\,,
$$

$$
(5.32)\qquad \partial\Lambda\wedge d\xi_1\wedge d\bar{\xi}_1=(\partial\bar{\boldsymbol{a}}+\bar{\partial}\boldsymbol{a}+\partial\bar{\partial}\log b)\wedge d\xi_1\,.
$$

Note that the system of equations (5.31) and (5.32) is the normal total differential equation in Λ of variables $\xi'\in D'^0$ for each fixed $\xi_1\in D_1^0$. Since this system is completely integrable on $D'^0\times\boldsymbol{C}$ for each $\xi_1\in D_1^0$, there is a smooth solution Λ on D'^0 which depends smoothly on $\xi_1\in D_1^0$. Q.E.D.

Proposition 5.3. *Under the conditions* (5.26.3), (5.27) *and* (5.29) *the system of equations* (5.26.2) *and* (5.30.2) *is locally solvable.*

Proof. We can consider $\Lambda_{j,0}$ on $D(j=2,3,\cdots)$ such that

$$
(5.33)\qquad \bar{\partial}\Lambda_{j,0}\wedge d\xi_1=\overline{\{\partial\log b+(j-1)\,\bar{\boldsymbol{a}}\}}\wedge d\xi_1\,,
$$

because (5.26.3) holds on D. Putting $\lambda_j=\Lambda\exp(-\Lambda_{j,0})$, we have

$$
(5.34)\qquad \bar{\partial}\Lambda\wedge d\xi_1=0\,.
$$

On the other hand, substituting $\lambda_j=\Lambda\exp(-\Lambda_{j,0})$ into (5.30.2), we get

$$
(5.35)\qquad \partial\Lambda+\Lambda\{-\partial\Lambda_{j,0}+(j-1)\,\boldsymbol{a}-j\,\partial\log b\}\wedge d\xi_1\wedge d\bar{\xi}_1=0\,.
$$

By (5.34) and (5.35), we obtain

$$\begin{aligned}(5.36)\qquad &\{\partial\bar{\partial}\log|b|^2+\\ &+(j-1)(\partial\bar{\partial}\log b+\partial\bar{\boldsymbol{a}}+\bar{\partial}\boldsymbol{a})\}\wedge d\xi_1\wedge d\bar{\xi}_1=0\,.\end{aligned}$$

From (5.30), we have (5.27), which shows that the normal total differential equation (with respect to $\xi'\in D'^0$) composed of equations (5.34) and (5.35) is completely integrable on $D\times\boldsymbol{C}$, when we regard $\xi_1\in D_1^0$ as parameter of this equation. Q.E.D.

Thus we are able to choose $\lambda(\xi, v)$ given by

$$(5.37)\qquad \lambda(\xi, v)=c_1\lambda_1(\xi)\,v+\sum_{j=2}^{N} c_j\,\lambda_j(\xi)\,v^j$$

for any positive integer $N\geqq 2$, any $c_1\in\boldsymbol{R}$ and $c_j\in\boldsymbol{C}$, $j=2, \cdots, N$.

EXAMPLE 1. Let $f(z, w)=w\bar{\boldsymbol{a}}+(w^2+\bar{w})\,\bar{\boldsymbol{b}}$ on $\boldsymbol{C}^3$, where

$$\begin{aligned}\boldsymbol{a}&=-\partial\log(1+|z_1+z_2^2|^2)+\\ &\quad+(\sin|z_1+z_2^2|^2)\,(dz_1+2z_2\,dz_2)\,,\\ \boldsymbol{b}&=(1+|z_1+z_2^2|^2)\,(dz_1+2z_2\,dz_2)\,,\end{aligned}$$

and log denotes the principal branch.

Then we have that $\boldsymbol{a}\wedge\boldsymbol{b}=0$, $\partial\boldsymbol{a}=0$ and $\partial\boldsymbol{b}=0$. By (5.20) we have $\boldsymbol{b}\wedge\partial w=0$, while $\bar{\boldsymbol{b}}\wedge\bar{\partial}w=0$. If we consider Φ: $\xi_1=z_1+z_2^2$, $\xi_2=z_2$ on a small neighborhood of the origin, we see $v=w\circ\Phi^{-1}(\xi)$ depends only on ξ_1 and have

$$\begin{aligned}\bar{\partial}_1 v&=\bar{a}_1\,v+\bar{b}(v^2+\bar{v})\,,\\ a_1&=-\partial_1\log(1+|\xi_1|^2)+\sin|\xi_1|^2\,,\\ b&=1+|\xi_1|^2\,.\end{aligned}$$

REMARK 5.1. If $\lambda(z, w)$ or $\lambda_j(z, w)$ $(j=2, 3, \cdots)$ vanishes on G, the system considered is $\boldsymbol{R}$-linear, that is, the type of $f(z, w)=w\bar{\boldsymbol{a}}+\bar{w}\bar{\boldsymbol{b}}$. Assume that λ_j is nonzero constant for a number $j\geqq 2$. Then, it follows from (5.26.2) and (5.30.2) that $\partial\log b\wedge d\xi_1=0$ and $\boldsymbol{a}\wedge d\xi_1=0$. Therefore we must have

$$\partial\boldsymbol{b}=0\,,\quad \boldsymbol{a}\wedge\boldsymbol{b}=0\quad\text{on}\quad G\,.$$

Hence, from (5.19) we have $\bar{\boldsymbol{b}}\wedge\bar{\partial}w=0$ and from (5.20) $\boldsymbol{b}\wedge\partial w=0$, from which we obtain the conclusion that Eq (5.20) can be locally reduced to the equation of one variable.

EXAMPLE 2. $\lambda(w)=c_1w+cw^2\sum_{j=0}^{N}\dfrac{w^j}{(j+1)(j+2)!}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ being the same ones as in Example 1. Then

$$\bar{\partial}_1 v=\bar{a}_1 v+\bar{b}(\lambda(v)+\bar{v})\,.$$

6. The existence theorem and local similarity principle

In this section we shall discuss the local existence of solutions to the system of equations (5.19) defined on G. We will use the notations in the previous section.

We are given the following system of equations on a domain $G\subset \boldsymbol{C}^n$

$$\bar{\partial} w = w\bar{\boldsymbol{a}}+\{\lambda(z, w)+\bar{w}\}\bar{\boldsymbol{b}}, \tag{6.1}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are smooth (1,0)-forms denfied on G, and where $\lambda(z, w)$ is a smooth function $G\times \boldsymbol{C}$.

Since we handle with the problem of solving Eq (6.1) locally, we consider a neighborhood U^0 of z^0 such that

(i) $b_1\neq 0$ on U^0,

(ii) there is a biholomorphic mapping Φ: $U^0\rightarrow\Phi(U^0)\Subset \boldsymbol{C}^n$ such as (3.25) ((6.2.3) below assures this property). Furthermore the assumptions on λ, $\boldsymbol{a}$ and $\boldsymbol{b}$ are as follows:

(6.2.1) λ is an entire function in w for each $z\in G$ and $\lambda(z, 0)=0$ everywhere in G.

For every point in G the followings are valid.

$$\partial\boldsymbol{a}\wedge\boldsymbol{b}=0, \tag{6.2.2}$$

$$\boldsymbol{b}\wedge d\boldsymbol{b}=0, \tag{6.2.3}$$

$$d(\boldsymbol{b}\wedge\bar{\boldsymbol{b}})=0, \tag{6.2.4}$$

$$\begin{cases}\partial\lambda_1\wedge\boldsymbol{b}\wedge\bar{\boldsymbol{b}}-\lambda_1\boldsymbol{b}\wedge\partial\bar{\boldsymbol{b}}=(\partial\bar{\boldsymbol{a}}+\bar{\partial}\boldsymbol{a}+\partial\bar{\partial}\log b)\wedge\boldsymbol{b}, \\ \partial\bar{\lambda}_1\wedge\boldsymbol{b}+\bar{\lambda}_1\partial\boldsymbol{b}=-\partial\boldsymbol{a}, \end{cases} \tag{6.2.5}$$

where b is a smooth function defined on U^0 and depends on $\boldsymbol{b}$ (see the proof below),

$$\begin{cases}\partial\lambda_j\wedge\boldsymbol{b}+\lambda_j\{(j-1)\,\boldsymbol{a}\wedge\boldsymbol{b}-j\,\partial\boldsymbol{b}\}=0, \\ \partial\bar{\lambda}_j\wedge\boldsymbol{b}+\bar{\lambda}_j\{(j-1)\,\boldsymbol{a}\wedge\boldsymbol{b}+\partial\boldsymbol{b}\}=0. \end{cases} \tag{6.2.6}$$

We are now in a position to state the local existence:

Theorem 6.1. *Under the assumptions mentioned above,* Eq (6.1) *has a smooth solution locally.*

Proof. We fix U^0. However we note that in this proof we restrict U^0 further if necessary. We have seen that Eq (6.1) is solvable on U^0 if and only if the following system of equations is solvable on U^0:

(6.1) (= (5.19)), (5.20) and

$$\boldsymbol{b}\wedge\bar{\partial}w = -w\,\bar{\boldsymbol{a}}\wedge\bar{\boldsymbol{b}}\,. \tag{6.3}$$

Assumption (6.2.3) is equivalent to

$$\boldsymbol{b}\wedge\partial\boldsymbol{b} = 0\,,\quad \boldsymbol{b}\wedge\bar{\partial}\boldsymbol{b} = 0 \quad \text{on} \quad G\,, \tag{6.4}$$

Then (6.4) shows the $C^\infty(U^0)$-module generated by the system of vector fields $\boldsymbol{b}\wedge\partial$ and $\boldsymbol{b}\wedge\bar{\partial}$ is a Lie algebra. Hence, by Theorem 2.1, we have the holomorphic change of variables on U^0:

$$\Phi:\begin{cases}\xi_1 = h(z)\,,\\ \xi_j = z_j\,,\quad 2\leqq j\leqq n\,.\end{cases} \tag{6.4)$'$ (}$$

Now, we consider the same polydisc D as in the proof of Lemma 4.1 except that the center of D is on N.

Operating the change of variables Φ on (6.1), (5.20) and (6.3) (see also (3.22) $\sim$(3.28)), we have (5.24) or (5.25.1)$\sim$(5.25.3), that is (for economy of writing we shall omit the superscript " $'$ " similarly as in the previous section),

$$\bar{\partial}_1 v = \boldsymbol{a}_1 v + \bar{\boldsymbol{b}}(\lambda+\bar{v})\,, \tag{6.5.1}$$

$$\bar{\partial}v\wedge d\xi_1 = v\,\bar{\boldsymbol{a}}\wedge d\xi_1\,, \tag{6.5.2}$$

$$\partial v\wedge d\xi_1 = v(\boldsymbol{a}-\partial\log b)\wedge d\xi_1\,, \tag{6.5.3}$$

where log denotes a branch.

Assumptions (6.2.2), (6.2.4), (6.2.5), and (6.2.6) show that the normal total system of equations (6.5.2), (6.5.3) is completely integrable on $D'^0\times\boldsymbol{C}$ with respect to the variable $\xi'\in D'^0$, where $\xi_1\in D_1^0$ is regarded as parameter. Therefore we can consider the solution v of this total system (with the initial condition: $v(\xi_1, \xi'^0)=v^0$, $(\xi_1, \xi'^0, v^0)\in D\times\boldsymbol{C}$ for each fixed $\xi_1\in D_1^0$) which depends smoothly on $\xi_1\in D_1^0$. If we can choose v so that it fulfills furthermore Eq (6.5.1) for each fixed $\xi'\in D'^0$, we complete the proof. Owing to (6.2.2), there exists a smooth function A on D such that

$$\bar{\partial}A\wedge d\xi_1 = \bar{\boldsymbol{a}}\wedge d\xi_1\,, \tag{6.6}$$

for any fixed $\xi_1\in D_1^0$. By the change of dependent variable

$$v = W\exp A$$

we turn the system of equations (6.5.1), (6.5.2) and (6.5.3) into

$$\begin{aligned}\bar{\partial}_1 W = (-\bar{\partial}_1 A+\boldsymbol{a}_1)\,W+\bar{\boldsymbol{b}}\,\lambda(\xi, W\exp A)\exp(-A)\\ +\bar{\boldsymbol{b}}\exp(\bar{A}-A)\,\bar{W}\,,\end{aligned} \tag{6.7.1}$$

$$\bar{\partial}W\wedge d\xi_1 = 0\,, \tag{6.7.2}$$

(6.7.3) $$\partial W\wedge d\xi_1=(-\partial A+\boldsymbol{a}-\partial\log b)\,W\wedge d\xi_1\,.$$

Furthermore we can express the W in the following form

(6.8) $$W=Y\exp B\,,$$

provided B satisfies in D

(6.9.1) $$\partial B\wedge d\xi_1=(-\partial A+\boldsymbol{a}-\partial\log b)\wedge d\xi_1\,,$$

(6.9.2) $$\bar\partial B\wedge d\bar\xi_1=0\,.$$

Such a B exists. In fact, from assumptions (6.2.3) and the first part of (6.2.5), we can derive

$$(\partial\bar{\boldsymbol{a}}+\bar\partial\boldsymbol{a}+\partial\bar\partial\log b)\wedge\boldsymbol{b}\wedge\bar{\boldsymbol{b}}=0\quad\text{on}\quad U^0\,.$$

From this, changing z into ξ under Φ, we have

$$(\partial\bar{\boldsymbol{a}}+\bar\partial\boldsymbol{a}+\partial\bar\partial\log b)\wedge d\xi_1\wedge d\bar\xi_1=0\quad\text{on}\quad D\,.$$

Thus we obtain from (6.7.1) and (6.8)

(6.10) $$\bar\partial_1 Y=\alpha\,Y+\beta+\gamma\,\bar Y\,,$$

where
$$\begin{aligned}\alpha&=-\bar\partial_1(A+B)+\bar{a}_1\,,\\ \beta&=\bar b\lambda(\xi,\,Y\exp(A+B))\exp(-A-B)\,,\\ \gamma&=\bar b\exp(\bar A+\bar B-A-B)\,.\end{aligned}$$

We note that

$$\beta=\bar b\textstyle\sum_{j=1}^{\infty}\lambda_j(\xi)\exp\{(j-1)(A+B)\}\;Y^j\,,$$

We consider the right-hand side of Eq (6.10). We define Δ, Δ^j:

$$\begin{aligned}&\Delta:=\text{the coefficient of }Y=-\bar\partial_1(A+B)+\bar a_1+\bar b\lambda_1\,,\\ &\Delta^j:=\text{the coefficient of }Y^j=\bar b\lambda_j\exp\{(j-1)(A+B)\}\,,\,j\geqq 2\end{aligned}$$

We wish to prove that Δ, Δ^j and γ depend only on ξ_1. Operating the change of variables Φ on (6.2.5) and (6.2.6), we have (5.30.1), (5.26.1), (5.30.2) and (5.26.2) respectively. We see:

$$\partial_j\Delta=-\bar\partial_1 a_j+\partial_j\bar a_1+\bar\partial_1\partial_j\log b+(\partial_j\bar b)\lambda_1+\bar b\,\partial_j\lambda_1\,.$$

Note that (5.31.1) is equivalent to

(6.11) $$\begin{cases}\bar b\,\partial_j\lambda_1+(\partial_j\bar b)\lambda_1=\bar\partial_1 a_j-\partial_j\bar a_1-\partial_j\bar\partial_1\log b\,,\\ \partial_j\bar a_k-\bar\partial_k a_j+\partial_j\bar\partial_k\log b=0\,,\quad 2\leqq j,\,k\leqq n\,.\end{cases}$$

Hence we obtain

$$\partial_j\Delta = 0\,, \quad 2\leqq j\leqq n\,.$$

Similarly it is obtained

$$\bar{\partial}_j\Delta = -\bar{\partial}_j\,\bar{\partial}_1(A+B)+\bar{\partial}_j\,\bar{a}_1+(\bar{\partial}_j\,\bar{b})\,\lambda_1+\bar{b}\,\bar{\partial}_j\,\lambda_1\,.$$

However, from (6.6) and (6.9.2) we have $\bar{\partial}_j\,A+\bar{\partial}_j\,B=\bar{a}_j$, $2\leqq j\leqq n$. And we note that (5.27.1) is equivalent to

$$\text{(6.12)}\qquad \begin{cases} b\,\partial_j\,\bar{\lambda}_1+(\partial_j\,b)\,\bar{\lambda}_1 = \partial_1\,a_j-\partial_j\,a_1\,, & 2\leqq j\leqq n\,,\\ \partial_j\,a_k-\partial_k\,a_j = 0\,, & 2\leqq j<k\leqq n\,. \end{cases}$$

Therefore we obtain $\bar{\partial}_j\Delta=0$, $2\leqq j\leqq n$. Thus we see that Δ depends only on ξ_1.

(5.26.2) and (5.30.2) are, respectively, equivalent to

$$\text{(6.13)}\qquad \partial_j\,\bar{\lambda}_p+\{(p-1)\,a_j+\partial_j\log b\}\,\bar{\lambda}_p = 0\,,$$

$$\text{(6.14)}\qquad \partial_j\,\lambda_p+\{(p-1)\,a_j-p\partial_j\log b\}\,\lambda_p = 0\,,$$
$$2\leqq j\leqq n;\ p = 2, 3, \cdots\,.$$

And hence we have, by (6.2.4) and (6.14),

$$\begin{aligned} \partial_j\Delta^p &= \bar{b}\,\{(\partial_j\log\bar{b})\,\lambda_p+\partial_j\,\lambda_p \\ &\quad +(p-1)\,(a_j-\partial_j\log b)\,\lambda_p\}\,\exp\,\{(p-1)\,(A+B)\} \\ &= \bar{b}\,\{(\partial_j\log|b|^2)\,\lambda_p+\partial_j\,\lambda_p \\ &\quad +((p-1)\,a_j-p\,\partial_j\log b)\,\lambda_p\}\,\exp\,\{(p-1)\,(A+B)\} \\ &= 0\,. \end{aligned}$$

In a similar way we obtain

$$\bar{\partial}_j\Delta^p = 0,\ \partial_j\,\gamma = \bar{\partial}_j\,\gamma = 0\,, \quad 2\leqq j\leqq n;\ p = 2, 3, \cdots\,.$$

Thus, Y, defined by (6.8), depends only on ξ_1, because B satisfies (6.9.1) and (6.9.2) on D'^0 for each fixed $\xi_1\in D_1^0$. We keep in mind that the v satisfying (6.5.2) and (6.5.3) on D'^0 for each fixed $\xi_1\in D_1^0$ depends smoothly on $\xi_1\in D_1^0$.

We shall now show the existence of Y satisfying Eq (6.10). Denoting by κ the right-hand side of Eq (6.10), we have the following integral equation equivalent to Eq (6.10)

$$\text{(6.15)}\qquad Y(\xi_1) = \tau(\xi_1)-\frac{1}{\pi}\int_{D_1^0}\frac{\kappa(\zeta,\,Y(\zeta))}{\zeta-\xi_1}\,dV\,,$$

where τ is holomorphic on D_1^0, and dV the 2-dimensional volume element.

We consider the Banahah space $\boldsymbol{C}(\bar{D}_1^0)$ equipped with the maximum norm $||\,.\,||$, denoted by $\boldsymbol{C}^0$ for brevity and the operator T of $\boldsymbol{C}^0$ into itself, defined by

$$TZ(\xi_1) = -\frac{1}{\pi}\int_{D_1^0} \frac{\kappa(\zeta, Z(\zeta))}{\zeta-\xi_1} dV .$$

Let τ be holomorphic in D_1^0 and continuous on $\bar{D}_1^0$. Then denoting c sup $\{|\tau(\xi_1)|; \xi_1 \in \bar{D}_1\}$, we define $\boldsymbol{\Lambda}$ to be the family $\{Z \in \boldsymbol{C}^0; ||Z|| \leqq 2c\}$. And further, by $\bar{\Delta}$ we denote the closed disc $\{\eta \in \boldsymbol{C}; |\eta| \leqq 2c\}$. The right-hand side of (6.10) can be written of the form

$$\kappa(\zeta, Y) = \mu(\zeta, Y) + \gamma(\zeta)\bar{Y},$$

where $\mu(\zeta, Y) = \alpha Y + \beta$. By assumption (6.2.1) there is a function $\tilde{\mu}$ defined in $\bar{D}_1^0 \times \bar{\Delta} \times \bar{\Delta}$ which is smooth in all variables, holomorphic in $(\eta_1, \eta_2) \in \bar{\Delta} \times \bar{\Delta}$ for each fixed $\zeta \in \bar{D}_1^0$, and

$$\kappa(\zeta, \eta_2) - \kappa(\zeta, \eta_1) = (\eta_2 - \eta_1)\tilde{\mu}(\zeta, \eta_1, \eta_2) + \gamma(\zeta)(\bar{\eta}_2 - \bar{\eta}_1).$$

Setting $L = \sup \{|\tilde{\mu}| + |\gamma|; (\zeta, \eta_1, \eta_2) \in \bar{D}_1^0 \times \bar{\Delta} \times \bar{\Delta}\}$, we have the Lipschitz condition

$$(6.16) \qquad |\kappa(\zeta, \eta_2) - \kappa(\zeta, \eta_1)| \leqq L|\eta_2 - \eta_1|, \quad \text{for} \quad \eta_1, \eta_2 \in \bar{\Delta}.$$

Thus, from (6.16) we obtain, noting $\kappa(\zeta, 0) = 0$ everywhere in $\bar{D}_1^0$,

$$\begin{aligned} ||TZ|| &\leqq L||Z|| \sup \{\frac{1}{\pi}\int_{D_1^0} \frac{1}{|\zeta-\xi_1|} dV_\zeta; \xi_1 \in \bar{D}_1^0\} \\ &\leqq 2L\varepsilon||Z||, \end{aligned}$$

where ε denotes the radius of D_1^0.

Therefore, we have

$$(6.17) \qquad ||TZ|| \leqq (1/2)||Z|| \qquad \text{for} \quad \varepsilon < 1/(4L).$$

Now we wish to solve Eq (6.15) in $\bar{D}_1^0$ with radius $\varepsilon < 1/(4L)$. In this case τ is holomorphic in D_1^0 and continuous on $\bar{D}_1^0$. We consider the operator $\tilde{T}$ defined in $\boldsymbol{\Lambda}$ to be $\tilde{T}Z = \tau + TZ$, and then in virtue of (6.17) the operator $\tilde{T}$ maps $\boldsymbol{\Lambda}$ into itself. Moreover, from (6.16) we obtain

$$(6.18) \qquad ||\tilde{T}Z_2 - \tilde{T}Z_1|| \leqq (1/2)||Z_2 - Z_1|| \qquad \text{for} \quad Z_1, Z_2 \in \boldsymbol{\Lambda},$$

which shows the operator $\tilde{T}$ is contracting. By a well-known argument the functions $Z^{(p)}$ defining by iteration

$$Z^{(p+1)} = \tilde{T}Z^{(p)}, \quad Z^{(1)} = 0, \quad p = 1, 2, \cdots$$

converge in $\boldsymbol{C}^0$ to a function $Y \in \boldsymbol{\Lambda}$ which satisfies (6.15) (see [6]). Q.E.D.

Taking any $z^0 \in \Omega \Subset G$, we fix it and consider a suitable neighborhood U^0 of z^0, on which the above Φ is considered. Assume Eq (6.1) satisfies the assumptions in Theorem 6.1.

Theorem 6.2. *For every bounded holomorphic function v in $h(U^0)$ there is a function w on U^0 which fulfills* Eq (6.1) *and is similar to v.*

Proof. We use the same notation as in the proof of Theorem 6.1. And let the center of polydisc D be the point $\Phi(z^0)$. We define the function $\mu_0(\zeta, Y)$ in $\bar{D}_1^0 \times \boldsymbol{C}$ as follows.

$$\mu_0(\zeta, Y) = \begin{cases} \dfrac{1}{\nu(\zeta)} \mu(\zeta, \nu Y) + \gamma(\zeta) \dfrac{\bar{\nu}(\zeta)}{\nu(\zeta)} \bar{Y}, & \text{for } \nu \neq 0, \\ 0 & \text{for } \nu = 0, \ \zeta \in D_1^0 \end{cases}$$

Let us consider the following integral equation introduced by Vekua [21] for $\boldsymbol{R}$-linear equation.

$$\begin{aligned} v_0(\xi_1) + \frac{1}{\pi} \int_{D_1^0} \frac{\mu_0(\zeta, v_0(\zeta))}{\zeta - \xi_1} dV \\ - \frac{1}{\pi} \int_{D_1^0} \frac{\mu_0(\zeta, v_0(\zeta))}{\zeta - t^0} dV = 1, \end{aligned} \tag{6.19}$$

where $t^0 (= h(z^0))$ is the center of D_1^0.

We define the operator T of $\boldsymbol{C}^0$ into itself to be

$$\begin{aligned} TZ = -\frac{1}{\pi} \int_{D_1^0} \frac{\mu_0(\zeta, Z(\zeta))}{\zeta - \xi_1} dV \\ + \frac{1}{\pi} \int_{D_1^0} \frac{\mu_0(\zeta, Z(\zeta))}{\zeta - t^0} dV. \end{aligned}$$

Let $\boldsymbol{\Lambda}$ denote the family $\{Z \in \boldsymbol{C}^0; \|Z\| \leqq 2\}$ which is the same one as considered in the proof of Theorem 6.1 except that holomorphic function τ is 1. Then in the same way as we obtained the existence of unique solution in Theorem 6.1, one can obtain the unique solution v_0 on $\bar{D}_1^0$ of Eq (6.19). We may think of $v_0 \neq 0$ on $\bar{D}_1^0$, because $v_0(t) = 1$ and one can restrict D_1^0 further if necessary. Now, Putting $v = \tau v_0$, from (6.19) we can easily show v satisfies Eq (6.10) on D_1^0. Thus the function $w(z)$, given by $(v \circ h)(z)$, is the desired one. This completes the proof (see [21]). Q.E.D.

Consider a family $\mathscr{F}$ of smooth functions on Ω. According to Bers [4] we define Similarity Principle. We say that $\mathscr{F}$ fulfills Similarity Principle on the domain, if the following condition holds: for each $\sigma \in \mathscr{F}$ there exists a holomorphic function τ on Ω such that σ is similar to τ on Ω with respect to a bounded Hölder continuous function ρ on $\bar{\Omega}$ and bounded away from zero. and vice versa.

From Lemma 4.1 and Theorem 6.2 it follows that the local Similarity Principle is valid for the family $\mathcal{F}(\Omega)$ of all the solutions in Ω of Eq (6.1) with the assumptions in Theorem 6.1. On the contrary the family of all Vekua functions of one variable on a domain (in the complex plane) is subject to Similarity Principle on the domain. It is an open question whether $\mathcal{F}(\Omega)$ obeys this principle on an arbitrary domain in $\boldsymbol{C}^n$.

Proposition 6.1. *Under the same assumption as in Theorem* 6.1, *any uniformly bounded subfamily $\mathcal{F}'(\Omega)$ of $\mathcal{F}(\Omega)$ is compact in the topology of uniform convergence on compact sets.*

Proof. Take a point $z^0 \in \Omega$ and fix it. We use again the same notation as in the proof of Theorem 6.1. By the hypothesis on $\mathcal{F}'(\Omega)$, there is a constant $M>0$ such that the inequality $|w| \leqq L$ holds for any $w \in \mathcal{F}'(\Omega)$. Calling $v= w \circ \Phi^{-1}(\xi)$ and $Y=v \exp(-A-B)$, we have $|v|<M$ on D for any v such that $w|_{U^0} \in \mathcal{F}'(U^0)$ and $|Y(\xi_1)|<M'$ on D_1^0 (M' depends on λ, $\boldsymbol{a}$, $\boldsymbol{b}$, M and D_1^0). Noting that we must replace the constant c (in the proof of Theorem 6.1) with M', it follows from (6.15) and (6.17) that $|\tau|<2M'$ for any $\tau \in A_{\mathcal{F}'}$ ($A_{\mathcal{F}'}$ is the family of those holomorphic functions in D_1^0 which correspond to elements of $\mathcal{F}'(U^0)$ under the formula (6.15) and $Y=w \circ \Phi^{-1}(\xi) \exp(A-B)$). Therefore any sequence $\{\tau_k\} \subset A_{\mathcal{F}'}$ has a subsequence which converges uniformly on D_1^0 to a holomorphic function τ_0. Without loss of generality we may consider the subsequence to be the $\{\tau_k\}$. On the other hand, we have

$$Y_k(\xi_1) = \tau_k(\xi_1)+(TY_k)(\xi_1) \qquad \text{for each} \quad \tau_k\,.$$

From (6.16) we obtain $\|TY_q-TY_p\| \leqq (1/2)\|Y_q-Y_p\|$ and then

$$\|Y_q-Y_p\| \leqq 2\|\tau_q-\tau_p\|\,.$$

Thus there is a continuous function Y_0 in D_1^0 and

$$Y_0(\xi_1) = \tau_0(\xi_1)+(TY_0)(\xi_1)\,.$$

Therefore $w_k=v_k \circ \Phi(z)$ converges to $w_0=v_0 \circ \Phi(z)$ uniformly on U^0, where $v_0= Y_0 \exp(A+B)$.

If $\{U_p\}$ ($p=1, 2, \cdots$) is a sequence of open sets covering Ω such that, for each p, any sequence of elements of $\mathcal{F}'(\Omega)$ has a subsequence converging uniformly on U_p, it follows by the well-known diagonal method that any sequence $\{w_q\}$, $w_q \in \mathcal{F}'(\Omega)$, has a subsequence $\{w_{q_\sigma}\}$ converging uniformly on U_p for each p. It is clear this subsequence converges uniformly on compact subsets of Ω.

Q.E.D.

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