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SOME HIGHER DIMENSIONAL KNOTS

Dedicated to Professor Itiro Tamura on his 60 th birthday

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Introduction

In this paper, we construct some classes of higher dimensional knots, and investigate geometrical and algebraic properties of the knots.

For classical knots, many concrete examples are known and studied. On the other hand, for higher dimensional cases, not so many examples, or constructions, are known. One of reasons for this difference is seemed to be derived from the existence of unknotting operations for classical knots to change into the unknot, that is, any 1-knot is changed to the unknot by exchanging the crossings suitably. In [5], F. Hosokawa and A. Kawauchi study an unknotting operation for 2-knots, and recently Kawauchi argues this from more general points of view in [11]. The author does not know whether there exist simple unknotting operations for any n-knots.

In § 1, we first give modifications which change some knots to the unknot, and we call such knots to be of type p, then we prove that *any 2-knot is of type* 2 in Theorem 1.2. Thus our defining 'unknotting operation' is valid for any 2-knots. We have a relationship between n-knots of type p and some disk pairs in Theorem 1.10, and this is very useful in the later sections.

In § 2, we show that *an n-knot of type p is also of type* $(n-p+1)$, and this is a geometrical description of the algebraic duality.

In § 3, we first generalize the notions of semi-unknotted manifolds and ribbon maps [24]. Then we argue relationships between bounding manifolds of knots and immersed disks.

In § 4, we discuss knots of type p and the bounding manifolds. Combining results in §§ 3 and 4, we can conclude that *any 2-knot is the boundary of an immersed disk with only double points singularities* (Corollary 4.2.2).

In §§ 5 and 6, we calculate the Alexander modules of knots, and introduce some algebras to obtain an exact sequence containing a semi-group of knots.

0. Preliminaries

Throughout the paper, we shall work in the piecewise linear category, and we shall assume that all submanifolds in a manifold are locally flat. The results in the paper remain valid in smooth category.

By an *n-knot* K^n, we mean that K^n is an embedded n-sphere in an $(n+2)$-sphere S^{n+2}, and we may also denote an n-knot by (S^{n+2}, K^n). We define that a submanifold W in a manifold M is *proper* if $W \cap \partial M = \partial W$. We say that a proper n-disk D^n in M is *unknotted* if there exists an n-disk D' in ∂M such that $\partial D = \partial D'$ and $D \cup D'$ bounds an $(n+1)$-disk in M.

Let $\{f_i\}$ be a family of disjoint embedding of $S^p \times D^q$ in a manifold M, and $A_i = f_i(S^p \times D^q)$. Suppose that $A_i \subset \operatorname{int} M$ or A_i is proper in M for each i. Then we say that $\{f_i\}$, or $\{A_i\}$, is *trivial* in M, if there exists a family of disjoint embeddings $\{\bar{f}_i\}$ of $B^{p+1} \times D^q$ in M such that

(1) $\bar{f}_i | \partial B^{p+1} \times D^q = f_i$ for each i, where we identity ∂B^{p+1} with S^p,

(2) $\bar{f}_i(B^{p+1} \times D^q) \subset \operatorname{int} M$ if $A_i \subset \operatorname{int} M$, and $\bar{f}_i^{-1}(\partial M) = B^{p+1} \times \partial D^q$ if A_i is proper in M.

For two manifold pairs (X_1, Y_1) and (X_2, Y_2), we define that (X_1, Y_1) and (X_2, Y_2) are *equivalent* if there exists an orientation preserving homeomorphism from X_1 onto X_2 which induces an orientation preserving homeomorphism from Y_1 onto Y_2.

Let f be a map from a subspace X_0 of X to Y, then $X \underset{f}{\cup} Y$ is defined to be a space obtained from the disjoint union of X and Y by identifying x and $f(x)$ for $x \in X_0$.

Let h^r be an r-handle on an m-manifold M, then we may identify h^r with an embedding $h^r: B^r \times D^{m-r} \to M \cup h^r$. For an r-handle h^r on M, $h^r | \partial B^r \times D^{m-r}$ is said to be the *attaching map* of h^r, and denoted by $\alpha(h^r)$. By $a(h^r)$, we denote an *attaching sphere* of h^r, i.e., $h^r(\partial B^r \times *)$ for $* \in D^{m-r}$. For r-handles $h_1^r, \cdots, h_\nu^r$ on M, by $M \cup \bigcup h_i^r$ we denote the manifold $M \cup \bigcup \{h_i^r | 1 \leqq i \leqq \nu\}$, if any confusion does not occur.

We say that $\{f_t\}$ is an *ambient isotopy* on M if there exists an level preserving homeomorphism $F: M \times I \to M \times I$ such that $f_t(x) = F(x, t)$ for $(x, t) \in M \times I$, and that f_0 is the identity map on M. For two maps g, h from X to M, we define that g and h are *ambient isotopic* if there exists an ambient isotopy $\{f_t\}$ on M such that $f_1 \circ g = h$.

For subspaces P and Q in X, we denote the *regular neighbourhood of P in X mod Q* by $N(P; X \bmod Q)$, and by $N(P; X)$ we denote the *regular neighbourhood of P in X* (see [9]).

For other terminologies, we refer the readers to [8], [19], [22] and [25].

1. Knots and disk pairs of type p

DEFINITION 1.1. For a manifold M, a family $\{h_i^q\}$ of q-handles on M is *trivial* if the family of attaching maps $\{\alpha(h_i^q)\}$ of q-handles is trivial in ∂M.

Let p be an integer with $1 \leqq p \leqq n$, then an n-knot K^n in $S^{n+2} = \partial D^{n+3}$ is of *type* p if there exist trivial $(n-p+2)$-handles $\{h_i^{n-p+2}\}$ on D^{n+3} such that

(1) $K^n \cap h_i^{n-p+2} = \emptyset$ for each i,

(2) K^n bounds an $(n+1)$-disk in $\partial(D^{n+3} \cup \bigcup h_i^{n-p+2})$, that is, K^n is unknotted in a manifold obtained from S^{n+2} by 'trivial surgery' along a trivial link consisting of $(n-p+1)$-spheres in S^{n+2}.

By $\mathcal{K}_n(p)$, we denote the set of equivalence classes of n-knots of type p. Then $\mathcal{K}_n(p)$ naturally forms a commutative semi-group under the knot sum.

1.1.1. Examples (1) In general, a ribbon n-knot K^n is obtained from a trivial link of n-spheres $S_0^n, S_1^n, \cdots, S_m^n$ in S^{n+2} by connecting with m bands (for example, see [1]). Choose m n-spheres Σ_i $(i=1, \cdots, m)$ very near to S_i^n which is parallel to S_i^n. Let $\{h_i^{n+1}\}$ be trivial handles on D^{n+3} with attaching spheres $\{\Sigma_i\}$ such that $h_i^{n+1} \cap K^n = \emptyset$. Then K^n is unknotted in $\partial(D^{n+3} \cup \bigcup h_i^{n+1})$. (See Fig. 1.1. for $m=1$) This shows that every ribbon n-knot is of type 1.

(2) We choose a simple closed curve γ as in Fig. 1.1. Then γ is unknotted in S^{n+2}, and the band is deformed to be 'straight' in the manifold obtained from S^{n+2} by performing trivial surgery along γ. Hence this ribbon n-knot is of type n.

In the later, we will describe more general forms for ribbon knots.

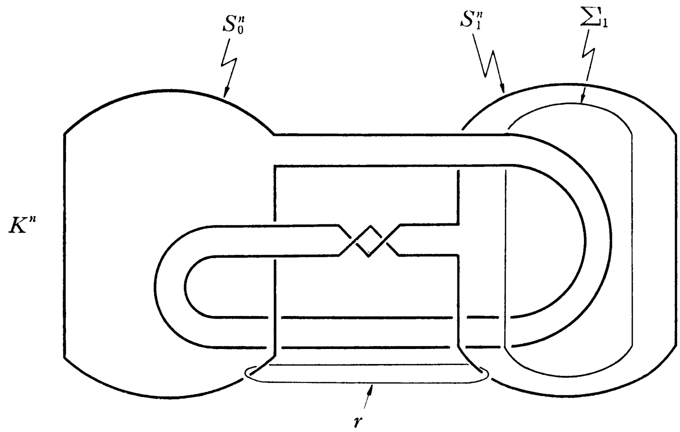

Fig. 1.1.

(3) Any *cable knots* of an n-knots [10] of type p are also of type p.

Theorem 1.2. *Any* 2-*knot is of type* 2.

Proof. Let K^2 be a 2-knot in S^4, then we can choose a presentation of the knot group

$$\pi_1(S^4 - K^2) = \langle x_0, x_1, \cdots, x_m \mid \{\text{relations}\} \rangle$$

such that each x_i is represented by a meridianal loop. Let $\{c_i\}$ be disjoint simple closed curves in S^4-K^2 such that c_i with a path connecting to the base point represents $x_0x_i^{-1}$ in $\pi_1(S^4-K^2)$ for each i. Let $\partial D^5=S^4$ and $\{h_i^2\}$ trivial 2-handles on D^5 with attaching spheres $\{c_i\}$ such that h_i^2 is disjoint from K^2 for each i. Then it holds that

(1) $\mathrm{lk}(K^2, c_i)=0$ for all i, where $\mathrm{lk}(\ ,\ )$ is a linking number,

(2) let $V=D^5\cup\bigcup h_i^2$, and $M=\partial V$, then M is simply connected and $\pi_1(M-K^2)\cong Z$.

It follows from the above condition (1) that K^2 is homotopic to zero in M. Thus we can apply T. Matumoto's result [16] to assert that K^2 is unknotted in $M\underset{n}{\#}(\# S^2\times S^2)$ for some n. Hence this implies that K^2 is of type 2.

We will investigate properties for handle decompositions with the following condition:

1.3. Let $\{h_i^p\}$ be p-handles on an m-disk D_0^m, and $\{h_i^{p+1}\}$ $(p+1)$-handles on $V=D_0^m\cup\bigcup h_i^p$ such that

(1.3.1) h_i^p and h_i^{p+1} are complementary handles for each i, i.e., an attaching sphere $a(h_i^{p+1})$ of h_i^{p+1} intersects a belt sphere of h_i^p in one point,

(1.3.2) $\{h_i^p\cup h_i^{p+1}\}$ are disjoint m-disks.

We remark that $V\cup\bigcup h_i^{p+1}$ is an m-disk, say D^m.

The following is trivial by the definition, and we omit the proof:

Proposition 1.4. *Let $D_0^m\cup\bigcup h_i^p\cup\bigcup h_i^{p+1}$ satisfy* (1.3.1) *and* (1.3.2), *then $\{h_i^p\}$ are trivial p-handles on D_0^m.*

Next we will show an analogy with Proposition 1.4 for $(p+1)$-handles:

Theorem 1.5. *Let $V\cup\bigcup h_{i0}^{p+1}$ and $V\cup\bigcup h_{i1}^{p+1}$ be handle decompositions of D^m, both of which satisfies* (1.3.1) *and* (1.3.2). *Let α_{i0} (resp. α_{i1}) be the attaching map of h_{i0}^{p+1} (resp. h_{i1}^{p+1}). Let $q=m-p-1$. Then there exist an orientation preserving homeomorphism $f\colon V\to V$ and a homeomorphism $g_i\colon B^{p+1}\times D^q\to B^{p+1}\times D^q$ such that $f\circ\alpha_{i0}=\alpha_{i1}\circ(g_i|\partial B^{p+1}\times D^q)$ for each i, and that $f_*\colon \pi_p(V)\to\pi_p(V)$ is the identity map.*

Proof. From the conditions (1.3.1) and (1.3.2), we can first assume that

$$h_{i0}^{p+1}(B_+^p\times D^q)=h_i^p(B^p\times D_+^q),\quad\text{and}$$
$$h_{i1}^{p+1}(B_+^p\times D^q)=h_i^p(B^p\times D_+^q),$$

where $B_+^p\subset\partial B^{p+1}$ and $D_+^q\subset\partial D^{q+1}$ are regarded as inclusions of hemispheres, and $B_+^p\times D^q$ is naturally identified with $B^p\times D_+^q$. For proving the theorem, it suffices to consider the case $h_{i0}^{p+1}|B_+^p\times D^q=h_i^p|B^p\times D_+^q$ for each i. Then $\psi_i=(h_i^p)^{-1}\circ(h_{i1}^{p+1}|B_+^p\times D^q)$ is a homeomorphism from $B_+^p\times D^q$ onto itself. As $B^{p+1}\times D^q$ is homeomorphic to $(B^p_+ \times D^q) \times I$, it is easily seen that ψ_i can be extended to a homeomorphism from $B^{p+1} \times D^q$ onto itself, thus we can find a homeomorphism g_i: $B^{p+1} \times D^q \to B^{p+1} \times D^q$ such that $\psi_i \circ (g_i | B^p_+ \times D^q)$ is the identity map, hence it holds that $h^{p+1}_{i1} \circ (g_i | B^p_+ \times D^q) = h^p_i | B^p \times D^q_+$. We remark that g_i is not necessarily orientation preserving. Let $B^p_- = \mathrm{cl}(\partial B^{p+1} - B^p_+)$. By our identification, $\partial B^p \times D^{q+1} \cup B^p_- \times D^q$ can be identified with an $(m-1)$-disk, say Δ^{m-1}. We can define an embedding ϕ_{i0} of Δ^{m-1} in ∂D^m_0 by $\phi_{i0} = (h^p_i | \partial B^p \times D^{q+1}) \cup (h^{p+1}_{i0} | B^p_- \times D^q)$. Similarly, we can define ϕ_{i1} by $\phi_{i1} = (h^p_i | \partial B^p \times D^{q+1}) \cup (h^{p+1}_{i1} \circ (g_i | B^p_- \times D^q))$. Give an orientation to Δ^{m-1} such that $\phi_{10}(\Delta^{m-1})$ and ∂D^m_0 have a compatible orientation, then $\phi_{i1}(\Delta^{m-1})$ has an orientation compatible with that of ∂D^m_0 from our construction. Then by well known fact, it is easily shown that there exists an orientation preserving homeomorphism f' of D^m_0 onto itself such that $f' \circ \phi_{i0} = \phi_{i1}$ for each i. As $f' \circ (h^p_i | \partial B^p \times D^{q+1}) = h^p_i | \partial B^p \times D^{q+1}$, f' is extended to an orientation preserving homeomorphism f of $V = D^{n+3}_0 \cup \bigcup h^p_i$ onto itself, which is easily seen to be a required one.

DEFINITION 1.6. A handle decomposition $D^m_0 \cup \bigcup h^p_i \cup \bigcup h^{p+1}_i$ of D^m satisfying (1.3.1) and (1.3.2) is uniquely determined in the sense of Theorem 1.5. We say that this handle decomposition is *trivial*, or a *trivial handle decomposition* of D^{n+3}, and that $\{a(h^{p+1}_i)\}$ are *standard p-spheres* on V.

For $(p+1)$-handles $\{h'_i\}$ on the above V, we say that $\{h'_i\}$ are *geometrical cancelling handles* for $\{h^p_i\}$ if there exists an ambient isotopy $\{f_t\}: V \to V$ such that $\{f_1(a(h^p_i))\}$ are standard p-spheres on V.

DEFINITION 1.7. Let $D^{n+3}_0 \cup \bigcup h^p_i \cup \bigcup h^{p+1}_i$ be a handle decomposition of D^{n+3}, and Δ a proper, unknotted $(n+1)$-disk in $V = D^m_0 \cup \bigcup h^p_i$. Assume the following are satisfied:

(1) $h^{p+1}_i \cap \partial\Delta = \emptyset$ for any i, and

(2) $\{h^{p+1}_i\}$ are geometrically cancelling handles for $\{h^p_i\}$.

In general, Δ is knotted in D^{n+3}. Regarding Δ as a proper disk in D^{n+3}, we denote β instead of Δ. Let α_i be the attaching map of h^{p+1}_i on V, and $\mathcal{D}$ a triple $(V, \{\alpha_i\}, \Delta)$. Then we say that $\mathcal{D}$ is a *p-decomposition* of (D^{n+3}, β), and we call that (D^{n+3}, β) is a *disk pair of type p*, and we say that the *dimension* of a disk pair (D^{n+3}, β) is $n+1$. By $\mathcal{B}_n(p)$, we denote the set of equivalence classes of n-dimensional disk pairs of type p. Then $\mathcal{B}_n(p)$ naturally forms a commutative semigroup under the usual boundary connected sum for pairs.

DEFINITION 1.8. Let $\mathcal{D} = (V, \{\alpha(h^{p+1}_i)\}, \Delta)$ be a p-decomposition of a disk pair (D^{n+3}, β). In the case of $p=1$, if $\mathrm{lk}(\partial\Delta, a(h^2_i)) = 0$ for each i, then we say that $\mathcal{D}$ is *good*. Let $\bar{\Delta}$ be an $(n+1)$-disk in ∂V such that $\partial\bar{\Delta} = \partial\Delta$. In the case of $p \geqq 2$, if we can choose $\bar{\Delta}$ such that each $a(h^{p+1}_i)$ intersects $\bar{\Delta}$ transversally and that the intersection of $a(h^{p+1}_i)$ and $\bar{\Delta}$ consists of finitely many $(p-1)$-spheres, then we say that $\mathcal{D}$ is *good*.

REMARK 1.8.1. (1) For a p-decomposition $\mathcal{D}=(V, \{\alpha_i\}), \Delta)$, it trivially holds that $\mathrm{lk}(\partial\Delta, a(h_i^{p+1}))=0$ if $p\geqq 2$.

(2) Under the notation of Definition 1.8, let $\mathcal{D}$ be a good 1-decomposition. By moving $\bar{\Delta}$ in general position with respect to $\{a(h_i^2)\}$, $\bar{\Delta}\cap a(h_i^2)$ consists of even many points, half of which have the sign $+$, and the other half have the sign $-$, where a point has the sign $+$ if and only if $a(h_i^2)$ intersects $\bar{\Delta}$ with the intersection number $+1$ in a neighbourhood of the point.

Lemma 1.9. *Assume that (D^{n+3}, β) has a p-decomposition, and $p=1$, 2 or $3\leqq p\leqq n-1$. Then we can find a good p-decomposition of (D^{n+3}, β).*

Proof. Let $\mathcal{D}=(V, \{\alpha_i\}, \Delta)$ be a p-decomposition of (D^{n+3}, β), and $\bar{\Delta}$ an $(n+1)$-disk in ∂V such that $\partial\Delta=\partial\bar{\Delta}$.

In the case of $p=2$, by moving $\bar{\Delta}$ rel $\partial\bar{\Delta}$ in general position with respect to $\{a(h_i^{p+1})\}$, $\bar{\Delta}\cap a(h_i^{p+1})$ consists of compact, closed 1-manifolds, hence finitely many 1-spheres.

In the case of $3\leqq p\leqq n-1$, we can prove Lemma 1.9 by the same argument as D. Hacon's result (Theorem 2.4 in §2 of [3]), except the algebraic duality in (p. 442, line 14 in [3]). But the duality is easily checked to be valid if $1\leqq i\leqq p-2$ under the notation in [3], and this restriction does not affect any argument in the proof.

In the case of $p=1$, the assertion is proved in Lemma 2.2 of author's previous paper [15].

We now state the main theorem in this section:

Theorem 1.10. *Let K^n be an n-knot in S^{n+2}, and p an integer with $1\leqq p\leqq n$. Then K^n is of type p if and only if there exists a disk pair (D^{n+3}, β) of type p such that $(S^{n+2}, K^n)=\partial(D^{n+3}, \beta)$.*

Proof. First we will prove the necessity.

Suppose K^n is an n-knot of type p in $S^{n+2}=\partial D_1^{n+3}$, and $q=n-p+2$, then there exist trivial q-handles $\{h_i^q\}$ on D_1^{n+3} such that

(1) $K^n\cap h_i^p=\emptyset$ for each i,

(2) K^n bounds an $(n+1)$-disk $\bar{\Delta}$ in $\partial(D_1^{n+3}\cup\bigcup h_i^q)$.

By the triviality of $\{h_i^q\}$ and the contractness of $\bar{\Delta}$, there exist geometrically cancelling handles $\{h_j^{q+1}\}$ for $\{h_j^q\}$ on $D_1^{n+3}\cup\bigcup h_i^q$ such that $h_j^{q+1}\cap\bar{\Delta}=\emptyset$ for all j. Then $D_1^{n+3}\cup\bigcup h_i^q\cup\bigcup h_i^{q+1}$ is an $(n+3)$-disk, say B_1. Therefore there exists an $(n+3)$-handle D_0^{n+3} on B_1 to obtain a handle decomposition of S^{n+3}. Turning this handle decomposition of S^{n+3} upside down, S^{n+3} consists of one 0-handle D_0^{n+3}, p-handles $\{\bar{h}_i{}^p\}$, $(p+1)$-handles $\{\bar{h}_i{}^{p+1}\}$ and one $(n+3)$-handle D_1^{n+3}, where $\bar{h}_i{}^p=h_i^{q+1}$ and $\bar{h}_i{}^{p+1}h=h_i^q$ as setwise. Then $D_0^{n+3}\cup\bigcup\bar{h}_i{}^p\cup\bigcup\bar{h}_i{}^{p+1}$ gives a trivial handle decomposition of an $(n+1)$-disk, say D^{n+3}. Let $V=D_0^{n+3}\cup\bigcup\bar{h}_i{}^p$. By our construction, it holds that

(1) $\bar{\Delta}\subset\partial V$ and $(S^{n+2}, K^n)=(\partial D^{n+3}, \partial\bar{\Delta})$,

(2) $\bar{\Delta}\cap\bar{h}_i^p=\emptyset$ and $\partial\bar{\Delta}\cap\bar{h}_i^{p+1}=\emptyset$ for each i.

We remark that $\bar{\Delta}\cap\bar{h}_j^{p+1}\neq\emptyset$ in general. From the above (1), we obtain a proper unknotted $(n+1)$-disk Δ in V such that $\partial\Delta=\partial\bar{\Delta}$. Then $(V, \{\alpha(\bar{h}_i^{p+1})\}, \Delta)$ is a p-decomposition of a required disk pair of type p.

Next we will prove the sufficiency, and the proof is similar to that for the necessity.

Let $(V, \{\alpha_i\}, \Delta)$ be a p-decomposition of (D^{n+3}, β) with $(\partial D^{n+3}, \beta)=(S^{n+2}, K^n)$, and D_1^{n+3} be an $(n+3)$-handle on D^{n+3} to obtain a handle decomposition $V\cup\bigcup h_i^{p+1}\cup D_1^{n+3}$ of S^{n+3}. By our assumptions, we can regard $K^n\subset\bigcup\partial D_1^{n+3}$. Identifying h_i^{p+1} with a q-handle $\bar{h}_i^q$ on D_1^{n+3}, it is easy to see that $\{\bar{h}_i^q\}$ are trivial handles on D_1^{n+3}, and it holds $K^n\cap\bar{h}_i^q=\emptyset$ for each i, since $\{h_i^{p+1}\}$ are geometrically cancelling for $\{h_i^p\}$ on V. From the unknottedness of Δ in V, we can choose an $(n+1)$-disk $\bar{\Delta}$ in $\partial(D_1^{n+3}\cup\bigcup\bar{h}_i^q)$ with $\partial\bar{\Delta}=K^n$. Therefore K^n is of type p.

The notion of a disk pair of type p is derived from [4], [6] ,[15] and [21]. A. Omae [18] argues a relationship between ribbon 2-knots and disk pairs obtained in [6] and [21]. In [1], we prove that K^n is a ribbon knot if and only if there exists a ribbon disk pair (D^{n+3}, β) such that $(S^{n+2}, K^n)=\partial(D^{n+3}, \beta)$. We note that the notions of ribbon disk pairs and disk pairs of type 1 coincide. Combining this result and Theorem 1.10, we have the following:

Corollary 1.10.1. *Let K^n be an n-knot, then K^n is a ribbon n-knot if and only if K^n is of type* 1.

Combining Lemma 1.9 and Theorem 1.10, we have the following:

Corollary 1.10.2. *Let $1\leqq p\leqq n$, K^n an n-knot, and (D^{n+3}, β) a disk pair with $\partial(D^{n+3}, \beta)=(S^{n+2}, K^n)$. Consider the following assertions* (1)–(3):

(1) *K^n is an n-knot of type p,*

(2) *(D^{n+3}, β) is a disk pair of type p,*

(3) *(D^{n+3}, β) is a disk pair of type p with a good p-decomposition.*

Then $(1)\Leftrightarrow(2)\underset{(*)}{\Leftarrow}(3)$. *The converse of* $(*)$ *holds if* $p=1$, 2 *or* $3\leqq p\leqq n-1$.

2. A relationship between n-knots of distinct types

In this section, we investigate a relationship between knots of different types, which is a geometrical realization of the algebraic duality.

The following is well-known, but for the completeness we will give a proof.

Lemma 2.1. *Let M be a $(2m+2)$-manifold, p an integer with $1\leqq p\leqq m$, and L_0 and L_1 links consisting of p-spheres in* int M. *If L_0 and L_1 are homotopic in M, then L_0 and L_1 are ambient isotopic in M.*

Proof. Let Σ be a disjoint union of μ p-spheres $S_1^p, \cdots, S_\mu^p$, and $F\colon \Sigma\times I\to M$ be a homotopy such that $F(\Sigma\times 0)=L_0$ and $F(\Sigma\times 1)=L_1$. Let $S(F)$ be the set of *singular points* of F, i.e.,

$$S(F) = \mathrm{cl}\{x\in\Sigma\times I \mid \# F^{-1}F(x)\geqq 2\}.$$

If $1\leqq p<m$, then $S(F)=\emptyset$ by moving F in general position, thus F is an isotopy. By [7], we can extend F to an ambient isotopy of M.

Suppose $p=m$, then we can assume $S(F)$ consists of finitely many points by general position argument. Let $S(F)=\{x_1, \cdots, y_1, \cdots\}$ such that $F(x_i)=F(y_i)$ for each i. Then there exists a simple arc γ_j in $S_j^p\times I$, for each j, such that

(1) one end point of γ_j is in int $S_j^p\times I$, the other end point is in $S_j^p\times 0$, and int $\gamma_j\subset$ int $S_j^p\times I$,

(2) γ_j passes all singular points x_j's in $S_j^p\times I$, but γ_j does not pass any singular points y_j's in $S_j^p\times I$.

Let B_j be a regular neighbourhood of γ_j in $S_j^p\times I$ mod $S_j^p\times 0$, $X=\bigcup_j B_j\cup N(\Sigma\times 0;\ \Sigma\times I)$, and $X'=\mathrm{cl}(\Sigma\times I-X)$, where we choose $N(\Sigma\times 0;\ \Sigma\times I)$ which does not contain any singular points $\{y_j\}$. Let $\partial X=L'\cup(\Sigma\times 0)$, then $\partial X'=L'\cup(\Sigma\times 1)$. Therefore $F|X$ is an embedding, and X is homeomorphic to $\Sigma\times I$, thus $f(\Sigma\times 0)$ is ambient isotopic to $f(L')$ in M. In the same way, by the embedding $f|X'$, $f(L')$ is ambient isotopic to $f(\Sigma\times 1)$ in M. Therefore L_0 and L_1 are ambient isotopic in M.

Theorem 2.2. *If $2\leqq 2p\leqq n$, then it holds that $\mathcal{K}_n(p)\subset\mathcal{K}_n(n-p+1)$, i.e., an n-knot of type p is also of thpe $(n-p+1)$.*

Proof. Let K^n be an n-knot of type p in $S^{n+2}=\partial D^{n+3}$, $\mathcal{D}=(V, \{\alpha_i\}, \Delta)$ a p-decomposition of (D^{n+3}, β) such that $\partial(D^{n+3}, \beta)=(S^{n+2}, K^n)$, and $V=D_0^{n+3}\cup\bigcup h_i^p$. From Lemma 1.9, we can assume that $\mathcal{D}$ is good in the case of $p=1$. Let $\bar\Delta$ be an $(n+1)$-disk in ∂V such that $\partial\bar\Delta=\partial\Delta$. By the contractness of $\bar\Delta$, there exist geometrically cancelling handles $\{\tilde h_i^{p+1}\}$ for $\{h_i^p\}$ on V such that $\tilde h_i^{p+1}\cap\bar\Delta=\emptyset$ for all i, hence it holds that $\partial\Delta\cap\tilde h_i^{p+1}=\emptyset$ for all i. We can choose $\{\tilde h_i^{p+1}\}$ so that $\{\alpha(\tilde h_i^{p+1})\}$ and $\{\alpha(h_i^{p+1})\}$ are ambient isotopic on ∂V. Let $\tilde a_i=a(\tilde h_i^{p+1})$. By the general position argument and $2p\leqq n$, we may assume that

$$\dim(\tilde a_i\cap a(h_j^{p+1}))\leqq 2p-(n+2)<0\,,$$

i.e., $\tilde a_i\cap a(h_j^{p+1})=\emptyset$ for all i and j, thus we can assume that $\tilde h_i^{p+1}\cap h_j^{p+1}=\emptyset$ for all i and j. Hence $\bigcup\tilde a_i$ is regarded as a link consisting of p-spheres in $\partial D^{n+3}-K^n$. From our choice, $\bigcup\tilde a_i$ is a trivial link in ∂D^{n+3}, hence $\{\tilde h_i^{p+1}\}$ are trivial handles on D^{n+3}.

For proving K^n to be of type $(n-p+1)$, it suffices to show that K^n is unknotted in $\partial(D^{n+3}\cup\bigcup\tilde h_i^{p+1})$. By our construction, $D_0^{n+3}\cup\bigcup h_i^p\cup\bigcup\tilde h_i^{p+1}$ is an

$(n+3)$-disk, say D_1^{n+3}. Then we can regard K^n as an n-knot in ∂D_1^{n+3}. We remark that K^n is unknotted in ∂D_1^{n+3}, since $\bar{\Delta} \cap \tilde{h}_i^{p+1} = \emptyset$ for all i. If $p=1$, then by the goodness of p-decomposition, we have $\mathrm{lk}(K^n, a(h_i^{p+1}))=0$ in $\partial(D_0^{n+3} \cup \bigcup h_i^p)$ for each i, which holds if $p>1$, by Remark 1.8.1. Hence $a(h_i^{p+1})$ is null homotopic in $\partial D_1^{n+3} - K^n$, since K^n is unknotted in ∂D_1^{n+3}. By Lemma 2.1, $\{a(h_i^{p+1})\}$ is a trivial link in $\partial D_1^{n+3} - K^n$. Therefore $K^n \cup \bigcup \{a(h_i^{p+1})\}$ is a trivial link in ∂D_1^{n+3}. Hence K^n is unknotted in $\partial(D_1^{n+3} \cup \bigcup h_i^{p+1})$. From the disjointness of $\{h_i^{p+1}\}$ and $\{\tilde{h}_i^{p+1}\}$ on $D_0^{n+3} \cup \bigcup h_i^p$ it follows that $D_1^{n+3} \cup \bigcup h_i^{p+1} = D^{n+3} \cup \bigcup \tilde{h}_i^{p+1}$. Thus K^n is unknotted in $\partial(D^{n+3} \cup \bigcup \tilde{h}_i^{p+1})$. This completes the proof of Theorem 2.2.

From Corollary 1.10.1 and the existence of non-ribbon 2-knots, the inverse inclusion in Theorem 2.2 does not hold in general.

3. Bounding manifolds and immersed disks for knots

First, we will give some definitions used in this and the later sections.

DEFINITION 3.1. Let $f_i\colon \partial B^p \times D^{m-p+1} \to \mathrm{int}\, W$ be disjoint embeddings, where W is anorientable m-manifold and $1 \leqq p \leqq m$. Let $q=m-p+1$. Assume that the orientation of $f_i(\partial B^p \times D^q)$ induced from $B^p \times D^q$ is compatible with that of W. We denote $f_i | \partial B^p \times \partial D^q$ by $f_i | \partial$, then $f_i | \partial$ is regarded as an embedding of $(\partial B^p \times \partial D^q)_i$ in int W. Then we define $\chi(W;\ \{f_i\})$ as the manifold

$$(W - \bigcup_i f_i(\partial B^p \times D^q)) \underset{\{f_i|\partial\}}{\cup} \bigcup_i (B^p \times \partial D^q)_i\,,$$

which is said to be *obtained from W by performing surgeries,* whose *index* is p.

If W is embedded in a manifold M, and each f_i can be extended to an embedding $\bar{f}_i\colon B^p \times D^q \to M$ such that $\bar{f}_i(B^p \times D^q) \cap W = f_i(\partial B^p \times D^q)$, then we say $\chi(M;\ \{f_i\})$ is *obtained from W by performing ambient surgeries.* In the case of $\dim M = m+1$, let $D_i = \bar{f}_i(B^p \times *)$ for $* \in \mathrm{int}\, D^q$, then $\bar{f}_i(B^p \times D^q)$ can be regarded as a regular neighbourhood of D_i in M mod W. In this case, we also denote $\chi(M;\ \{D_i\})$ instead of $\chi(M; \{f_i\})$, if any confusion does not occur.

DEFINITION 3.2. Let W be an orientable $(n+1)$-manifold in S^{n+2} such that ∂W is homeomorphic to an n-sphere. Then we say that W is *semi-unknotted of type p* if there exist disjoint embeddings $\nu_i\colon S^{n-p+1} \times D^p \to \mathrm{int}\, W$ such that

(1) $\chi(W;\ \{\nu_i\})$ is an $(n+1)$-disk, and

(2) $\{j \circ \nu_i\}$ is trivial in S^{n+2}, where $j\colon W \to S^{n+2}$ is the inclusion map.

We say that $\{\nu_i\}$, or simply $\{\nu_i(S^{n-p+1} \times *)\}$ for $* \in \mathrm{int}\, D^p$, is a *trivial system* of W.

DEFINITION 3.3. (i) A map $\rho\colon D^{n+1} \to S^{n+2}$ is a *pseudo-ribbon map* for an n-knot K^n if the following are satisfied:

(1) $\rho | \partial D^{n+1}$ is an embedding, and $\rho(\partial D^{n+1})=K^n$,

(2) for each $x \in D^{n+1}$, $\sharp \rho^{-1}\rho(x) \leqq 2$ and there exists neighbourhood V of x in D^{n+1} such that $\rho | V$ is an embedding,

(3) for x_1, x_2 with $\rho(x_1)=\rho(x_2)$ and $x_1 \neq x_2$, there exist neighbourhoods V_i of x_i ($i=1, 2$) such that $\rho | V_1$ and $\rho | V_2$ are embeddings and $\rho(V_1)$ intersects $\rho(V_2)$ transversally.

(ii) A pseudo-ribbon map ρ: $D^{n+1} \to S^{n+2}$ is of *type p* if the following are satisfied:

each component of $\{x | \sharp \rho^{-1}\rho(x) \geqq 2\}$ is homeomorphic to $S^{p-1} \times D^{n-p+1}$, say T_1^+, T_1^-, T_2^+, T_1^+, $\cdots$, which satisfy that

(1) $\rho(T_i^+)=\rho(T_i^-)$ for each i,

(2) T_i^+ is proper in D^{n+1}, and $T_i^- \subset \operatorname{int} D^{n+1}$,

(3) $\bigcup_i T_i^+$ is trivial in $D^{n+1} - \bigcup_j T_j^-$.

Definitions 3.2 and 3.3 are generalizations of [2], [15] and [24]. In fact, a semi-unknotted manifold in the sense of T. Yanagawa's [24] is semi-unknotted of type 1 in our definition.

It is easily seen that any embedded $\sharp_m S^1 \times S^n - \operatorname{int} B^{n+1}$ in S^{n+2} are semi-unknotted of type n if $n>1$. In Theorem 3 of [14], the author constructs infinitely many n-knots bounding semi-unknotted $(n+1)$-manifolds of type p, and also of type q, where $2 \leqq p \leqq q$ and $q=n-p+1$.

First we will argue relationships between bounding manifolds and pseudo-ribbon maps for knots.

Theorem 3.4. *Let ρ be a pseudo-ribbon map of type p for an n-knot K^n, then there exists a semi-unknotteu $(n+1)$-manifold of type p bounding K^n in S^{n+2}.*

Proof. We use the notation in Definitions 3.2 and 3.3. Let $q=n-p+1$. By the triviality of $\bigcup T_i^+$, there exist disjoint embeddings ϕ_i: $B^p \times D^q \to D^{n+1} - \bigcup T_j^-$ such that $\phi_i(\partial B^p \times D^q)=T_i^+$ and $\phi_i^{-1}(\partial D^{n+1})=B^p \times \partial D^q$ for each i. Then $\rho \circ \phi_i$ are disjoint embeddings of $B^p \times D^q$ in S^{n+2}. There exist disjoint embeddings $\bar{\phi}_i$: $B^p \times D^{q+1} \to S^{n+2}$ such that

(1) $\bar{\phi}_i | B^p \times D^q = \rho \circ \phi_i$, where $D^q \subset \operatorname{int} D^{q+1}$,

(2) there exists a regular neighbourhood V_i of T_i^- in D^{n+1} such that $\rho(V_i)$ is the intersection of $\bar{\phi}_i(B^p \times D^{q+1})$ and $\rho(D^{n+1} - \phi_i(B^p \times D^q))$,

thus $\bar{\phi}_i(B^p \times D^{q+1})$ is a regular neighbourhood of $\rho \circ \phi_i(B^p \times D^q)$ in S^{n+2} mod $\rho(D^{n+1} - \phi_i(B^p \times D^q))$. Let $W = \rho(D^{n+1} - \bigcup V_i) \cup \bigcup \bar{\phi}_i(B^p \times \partial D^{q+1})$, then W is an orientable $(n+1)$-manifold with $\partial W = \rho(\partial D^{n+1})$. From our construction, it follows that $\{(\bar{\phi}_i | B^p \times \partial D^{q+1})\}_i$ is a trivial system for W to be semi-unknotted of type p.

Lemma 3.5. *Let W be a semi-unknotted $(n+1)$-manifold of type p with trivial system $\{\nu_i\}$, and $L = \bigcup \nu_i(S^{n-p+1} \times *)$ for $* \in \operatorname{int} D^p$. Then there exists a* *link* L_0 *consisting of* $(p-1)$*-spheres in* $\operatorname{int} D^{n+1}$ *such that* $\operatorname{cl}(D^{n+1}-N(L_0; D^{n+1}))$ *is homeomorphic to* $\operatorname{cl}(W-N(L; W))$.

Proof. From the definition, we can put

$$\chi(W; \{\nu_i\}) = \operatorname{cl}(W-N(L; W)) \underset{\{\nu_i|\partial\}}{\cup} \bigcup_i (B^{n-p+1}\times \partial D^p)_i ,$$

and there exists a homeomorphism $f\colon \chi(W; \{\nu_i\}) \to D^{n+1}$. Then $L_0 = f(\bigcup_i (*\times \partial D^p)_i)$ is a desired link for $*\in \operatorname{int} B^{n-p+1}$.

The following is well-known (for example, see Theorem 8.2, page 246 in J.F.P. Hudson's Book [8], but we remark that in his book [8] the assertion is misprinted,):

Handle structure Theorem. *Let* $(W; \partial_- W, \partial_+ W)$ *be a cobordism, and* $(W, \partial_- W)$ *be r-connected for* $r\leqq n-3$ *and* $n\geqq 5$. *Then W has a handle decomposition on* $\partial_- W$ *consisting of handles of index greater than or equal to* $r+1$.

Lemma 3.6. *Let L be a link consisting of p-spheres in* $\operatorname{int} D^n$ *for* $1\leqq p\leqq n-2$. *Then* D^n *has a handle decomposition on* $N(L; D^n)$ *consisting of handles with index less than or equal to* $p+1$.

Proof. Let $W=\operatorname{cl}(D^n-N(L; D^n))$, then it suffices to show our assertion that W has a handle decomposition on $\partial N(L; D^n)$ consisting of handles with index less than or equal to $p+1$.

In the case of $p=n-2$, the assertion is trivial. If $p=1$ and $n\geqq 4$, then L is a trivial link. Hence the assertion is easily seen to hold. Thus we assume $2\leqq p\leqq n-3$. Let $\partial_- W=\partial D^n$ and $\partial_+ W=\partial N(L; D^n)$, then $(W; \partial_- W, \partial_+ W)$ is a connected cobordism. From the dimensional assumption, it follows that W and $(W, \partial_- W)$ is simply connected. It is easily checked that $H_i(W, \partial_- W)=0$ for $1\leqq i\leqq n-p-2$. Therefore $(W, \partial_- W)$ is $(n-p-2)$-connected. From Handle Structure Theorem, W has a handle decomposition on $\partial_- W$ consisting of handles of index greater than or equla to $n-p-1$. Turning this handle decomposition upside-down, we obtain a desired one.

Combining Lemmas 3.5 and 3.6, we directly have the following:

Lemma 3.7. *Let W be a semi-unknotted* $(n+1)$*-manifold of type p with a trivial system* $\{\nu_i\}$, *and* $L=\bigcup \nu_i(S^{n-p+1}\times *)$, $*\in \operatorname{int} D^p$. *Then W has a handle decomposition on* $N(L; W)$ *consisting of handles of index less than or equal to p.*

We will show the converse of Theorem 3.4. For simplifying the proof, we first give the following notation:

3.8. We put

$$J=[-1,1],$$
$$\Delta=[-1/2,1/2] \quad \text{and}$$
$$X=J^q\times J^p,$$

where $q=n-p+2$. We put

$$T=(J^q\times\partial\Delta^p)\cap(\Delta^{q-1}\times J^p)=\Delta^{q-1}\times\partial\Delta^p,$$

then T is homeomorphic to $S^{p-1}\times D^{n-p+1}$. Regarding $T\subset J^q\times\partial\Delta^p$, we denote T^- instead of T, and we denote T by T^+ if we regard $T\subset\Delta^{q-1}\times J^p$. Then

$$T^-\subset\operatorname{int}J^q\times\partial\Delta^p,$$
$$T^+ \text{ is proper in } \Delta^{q-1}\times J^p \quad \text{and}$$
$$T^+ \text{ is unknotted in } \Delta^{q-1}\times J^p.$$

The following is the converse of theorem 3.4:

Theorem 3.9. *Let W be a semi-unknotted $(n+1)$-manifold of type p in S^{n+2}, then there exists a pseudo-ribbon map of type p for an n-knot ∂W,*

Proof. Let $\{\nu_i\}$ be a trivial system of W, and $q=n-p+2$. Let $j\colon W\to S^{n+2}$ be the inclusion map. From the triviality of $\{j\circ\nu_i\}$, there exist disjoint embeddings $\phi_i\colon B^q\times D^p\to S^{n+2}$ such that $\phi_i|\partial B^q\times D^p=j\circ\nu_i$ for each i. First we may assume that $\phi_i(B^q\times *)$ intersects W transversally for $*\in\operatorname{int}D^p$, and that $\phi_i(B^q\times D^p)\cap W$ is homeomorphic to $(\phi_i(B^q\times *)\cap W)\times D^p$. Let $W_0=\operatorname{cl}(W-\bigcup\nu_i(\partial B^q\times D^p))$, then $W_0\underset{\{\nu_i|\partial\}}{\cup}\bigcup(B^q\times\partial D^p)_i$ is an $(n+1)$-disk, say D^{n+1}, where we regard $\nu_i|\partial=\nu_i|S^{q+1}\times\partial D^p$ as an embedding of $(\partial B^q\times\partial D^p)_i$ in W. We define a map $\rho\colon D^{n+1}\to S^{n+2}$ by the following:

$$\rho(x)=\begin{cases} j(x) & \text{if } x\in W_0,\\ \phi_i(x) & \text{if } x\in(B^q\times\partial D^p)_i,\end{cases}$$

where we regard ϕ_i as an embedding from $(B^q\times D^p)_i$ in S^{n+2}. Then ρ is a pseudo-ribbon map for $K^n=\partial W$. Let $\partial_+W=\partial W_0-K^n$, then from Lemma 3.7 it follows that

$$W_0=\partial_+W\times I\cup\{\text{handles of index}\leqq p\}.$$

Let $B_i^q=\phi_i(B^q\times *)$ for $*\in\operatorname{int}D^p$. For $1\leqq j\leqq p-1$, it holds that $B_i^q\cap(\text{cores of } j\text{-handles})=\emptyset$ by the general position argument, i.e., we can move W_0 ambient isotopically keeping $\partial_+W\times I$ fixed so that $B_i^q\cap(j\text{-handles})=\emptyset$ for each i. Thus we can assume that $\phi_i(B^q\times D^p)\cap(j\text{-handles})=0$ for each i. Then $\Sigma=B_i^q\cap(\text{cores of } p\text{-handles})$ consists of finitely many points. Let $z\in\Sigma$, then there exists a regular neighbourhood V of z in S^{n+2} and a homeomorphism $f_z\colon V\to X$, where X is defined in 3.8, such that

(1) $f_z(z)=(0, 0, \cdots, 0)$,

(2) $f_z(\phi_i(B^q\times D^p)\cap V)=J^q\times J^p$, and $f_z((p\text{-handle})\cap V)=\Delta^{q-1}\times J^p$.

Let $T_z^+=(f_z\circ\rho)^{-1}(T^+)$ and $T_z^-=(f_z\circ\rho^{-1})(T^-)$, and repeat this construction for each $z\in\Sigma$. Then $\{x\,|\,\#\rho^{-1}\rho(x)\geqq 2\}=\{T_z^+, T_z^-\,|\,z\in\Sigma\}$. From the argument in 3.8 it is easily concludeed that Definition 3.3-(ii) is satisfied. Hence ρ is a pseudo-ribbon map of type p for K^n. This completes the proof of Theorem 3.9.

4. Knots of type p and bounding manifolds

In this section, we will study a bounding manifold for a knot of type p. We first prove the following:

Theorem 4.1. *Let $1\leqq p\leqq n$. If an n-knot bounds a semi-unknotted $(n+1)$-manifold of type p in S^{n+2}, then the n-knot is of type p.*

Proof. Let W be a semi-unknotted $(n+1)$-manifold of type p in S^{n+1} with $\partial W=K^n$, and $q=n-p+2$. We use the notation in Definition 3.2.

There exist disjoint embeddings $\bar{\nu}_i$: $S^{q-1}\times D^{p+1}\to S^{n+2}$ such that

(1) $\bar{\nu}_i\,|\,S^{q-1}\times D^p=j\circ\nu_i$, where D^{p+1} is identified with $D^p\times[-1, 1]$ and $D^p=D^p\times 0$.

(2) $\bar{\nu}_i(S^{q-1}\times D^{p+1})\cap W=j\circ\nu_i(S^{q-1}\times D^p)$,

(3) $\bar{\nu}_i$ gives a trivial framing for $\nu_i(S^{q-1}\times *)$ for $*\in\operatorname{int} D^p$.

Identifying S^{n+2} with ∂D^{n+3}, add q-handles $\{h_i^q\}$ on D^{n+3} with attaching maps $\bar{\nu}_i$. Then it is easily seen that $K^n\cap h_i^q=\emptyset$ for each i, and that surgeries for W using $\{\nu_i\}$ are realized as ambient surgeries in $\partial(D^{n+3}\cup\bigcup h_i^q)$. Hence K^n is unknotted in $\partial(D^{n+3}\cup\bigcup h_i^q)$. Therefore K^n is of type p.

Next we will consider the converse of the above:

Theorem 4.2. *Assume $p=1$, 2 or $3\leqq p\leqq n-1$. Let K^n be an n-knot of type p in S^{n+2}. Then K^n bounsd a semi-unknotted $(n+1)$-manifold of type p in S^{n+2}.*

Proof. By Lemma 1.9 and Theorem 1.10, there exists a good p-decomposition $\mathcal{D}=(V, \{\alpha_i\}, \Delta)$ of (D^{n+3}, β) such that $(S^{n+2}, K^n)=\partial(D^{n+3}, \beta)$, where $V=D_0^{n+3}\cup\bigcup h_i^p$. We can choose an $(n+1)$-disk $\bar{\Delta}$ in ∂V such that $\bar{\Delta}\cap\bigcup a(h_i^{p+1})$ consists of finitely many $(p-1)$-spheres, because of the goodness of $\mathcal{D}$. We remark that 0-spheres mean to be even many points mentioned in Remark 1.8.1. Let $\bar{\Delta}\cap a(h_p^{i+1})=\{C_{ij}\,|\,1\leqq j\leqq\mu_i\}$, where C_{ij} is a $(p-1)$-sphere, and C_{ij} consists of two points with different signs if $p=1$, as stated in Remark 1.8.1. There exist p-disks d_{ij}, $1\leqq j\leqq\mu_i$, in $a(h_i^{p+1})$ such that $\partial d_{ij}=C_{ij}$ for $j=1, \cdots, \mu_i$, and $C_{ij}\cap d_{ik}=\emptyset$ if $1\leqq k<j\leqq\mu_i$, i.e., $d_{i1}, \cdots, d_{i\mu_i}$ are innermostly arranged in this order. Note that, if $p=1$, the intersection number of $N(d_{ij};\, a(h_i^{p+1}))$ and $\bar{\Delta}$ is 0 for all i, j. Hence we can perform an ambient surgery for $\bar{\Delta}$ by using d_{11} to

obtain W_1. Then d_{12} determines an ambient surgery for W_1 to obtain W_2. Repeating this procedure over all d_{ij}, we finally obtain W_μ in ∂V, $\mu=\mu_1+\cdots+\mu_r$, such that $\partial W_\mu=\partial\bar{\Delta}$ and $W_\mu\cap\bigcup a(h_k^{p+1})=\emptyset$, i.e., there exist embeddings $f_{ij}\colon B^p\times D^q\to\partial V$, where $q=n-p+2$, such that

(1) $f_{ij}(B^p\times D^q)\cap\bigcup a(h_k^p)=f_{ij}(B^p\times *)=d_{ij}$, where $*\in\operatorname{int}D^q$,

(3) $f_{ij}(B^p\times D^q)\cap f_{im}(B^p\times\partial D^q)=\emptyset$ if $j<m$,
$f_{ij}(B^p\times D^q)\cap f_{st}(B^p\times D^q)=\emptyset$ if $i\neq s$,

(3) $W_0=\Delta^{n+1}$, $W_1=\chi(W_0;\, d_{11})$, $W=\chi(W_1;\, d_{12})$, $\cdots$, $W_\mu=\chi(W_{\mu-1};\, d_{r\mu_r})$,

where r is the number of p-handles of $D_0^{n+3}\cup\bigcup h_i^p$. (See Fig. 4.1.)

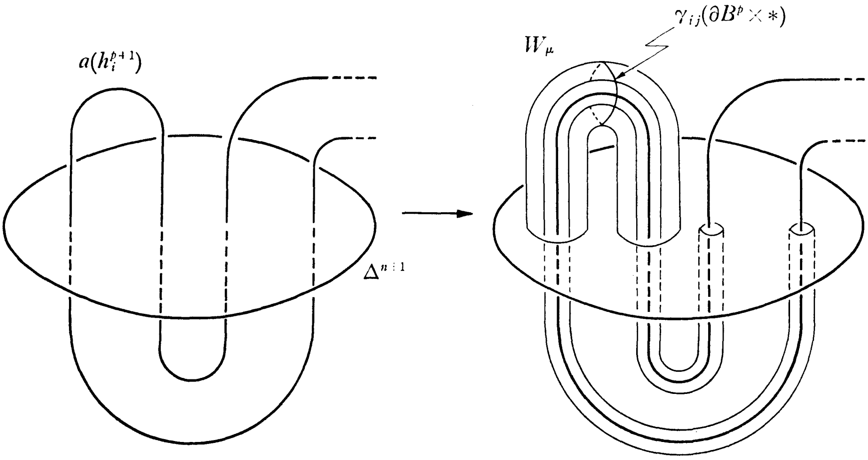

Fig. 4.1.

Hence W_μ is a bounding manifold of K^n in S^{n+2}. Let $\nu_{ij}=f_{ij}|\partial B^p\times D^q$. For proving Theorem 4.2 it suffices to show that $\{\nu_{ij}\}$ is a trivial system of $W=W_\mu$. From our construction, $\{\nu_{ij}\}$ are disjoint embeddings into $\operatorname{int}W_\mu$, and $\chi(W_\mu;\,\{\nu_{ij}\})$ is an $(n+1)$-disk. From the definition of p-decompositions, there exist a homeomorphism $g\colon V\to V$, an q-disk E^q in ∂D^{q+1}, and disjoint p-disks $B_1^p,\cdots,B_\mu^p$ in $\operatorname{int}B^p$ such that

(1) $h_i^p(B^p\times *)\subset g(a(h_i^{p+1}))$, for $*\in\operatorname{int}E^q$,

(2) $g\circ(f_{ij}|B^p\times\partial D^q)=h_i^p|B_j^p\times\partial E^q$, where we naturally identify D^q with E^q, and B^p with B_j^p. (See Fig. 4.2)

Let $E'=\operatorname{cl}(\partial D^{q+1}-E^q)$, and $f'_{ij}=g^{-1}\circ(h_i^p|B_j^p\times E')$. Then $f'_{ij}\colon B_j^p\times E'\to\partial V-\bigcup a(h_k^{p+1})$ are disjoint embeddings such that $f'_{ij}|\partial B_j^p\times\partial E'=\nu_{ij}|\partial B^p\times\partial D^q$. This implies that $\{\nu_{ij}\}$ is a trivial system of W. This completes the proof of Theorem 4.2.

Combining Theorems 3.4, 3.9, 4.1 and 4.2, we have

Corollary 4.2.1. *If* $p=1$, 2 *or* $3\leqq p\leqq n-1$, *then the following* (1)–(3) *are equivalent*:

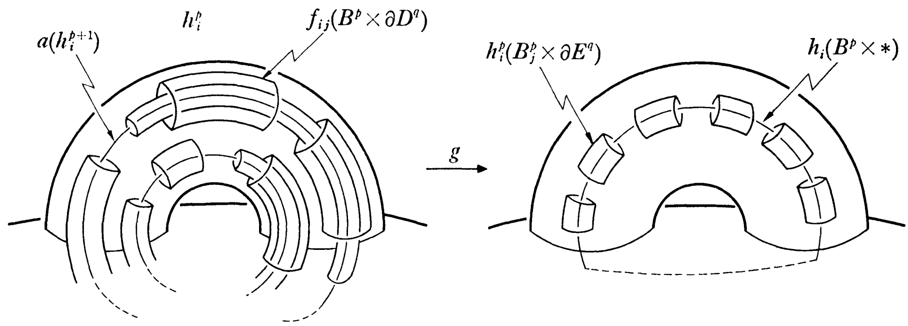

Fig. 4.2.

(1) *K^n is an n-knot of type p.*

(2) *There exists a pseudo-ribbon map of type p for K^n.*

(3) *There exists a semi-unknotted manifold of type p bounding K^n.*

If $1 \leqq p \leqq n$, then (2) *and* (3) *are equivalent, and* (3) *implies* (1).

For any 1-knot K^1, it is well known that K^1 bounds an immersed disk with clasp singularities. As an analogy for 2-knots, Theorems 1.3 and 4.2 imply the following:

Corollary 4.2.2. *For any 2-knot, there exists a pseudo-ribbon map for the 2-knot.*

In [24], T. Yanagawa gives a characterization of ribbon knots by means of ribbon maps, we here characterize ribbon knots by a weaker condition than Yanagawa's. We first define that:

A pseudo-ribbon map ρ: $D^{n+1} \to S^{n+2}$ is *separable* if there exist disjoint $(n+1)$-disks Δ_i in int D^{n+1} such that

$$\rho | (D^{n+1} - \bigcup \Delta_i) \text{ and } \rho | \bigcup \Delta_i \text{ are embeddings.}$$

Theorem 4.3. *An n-knot is a ribbon knot if and only if there exists a separable pseudo-ribbon map for the knot.*

Proof. First we will show the sufficiency. Let ρ be a separable pseudo-ribbon map for K^n, and Δ_i be disjoint $(n+1)$-disks in int D^{n+1} satisfying the condition in the above definition.

Let $\mathfrak{A}_i = \partial \Delta_i$ for each i, then we can add trivial $(n+1)$-handles h_i^{n+1} on D^{n+3} with attaching spheres $\mathfrak{A}_i$ such that $h_i^{n+1} \cap \rho(N(\Delta_i; D^{n+1})) = \rho(N(\mathfrak{A}_i; D^{n+1}))$, where S^{n+2} is identified with ∂D^{n+3}. We can easily deform $\rho(D^{n+1})$ to get an $(n+1)$-disk in $\partial(D^{n+3} \cup \bigcup h_i^{n+1})$ bounding K^n, thus K^n is an n-knot of type 1, hence ribbon n-knot by Corollary 1.10.1.

Conversely let K^n be a ribbon n-knot, hence an n-knot of type 1 by Corollary

1.10.1. By Corollary 4.2.1, we can choose a pseudo-ribbon map ρ of type 1 for K^n. By the definitions, it is easily seen that ρ is separable. This completes the proof of Theorem 4.3.

5. Alexander modules of knots

In this and the next secions, we may use the following notation 5.1 without any specifications:

Notation 5.1. Let K^n be an n-knot of type p. By Theorem 1.10, we can find a disk pair (D^{n+3}, β) with a p-decomposition $\mathcal{D}=(V, \{\alpha_i\}, \Delta)$ such that $(S^{n+2}, K^n)=\partial(D^{n+3}, \beta)$, where α_i is the attaching map of h_i^{p+1}. Let ν be the number of $(p+1)$-handles $\{h_i^{p+1}\}$ on $V=D_0^{n+3}\cup\bigcup h_i^p$. By Lemma 1.9, we can assume $\mathcal{D}$ is good if $p=1$. Let W be an exterior of β in D^{n+3}, i.e., $W=\mathrm{cl}(D^{n+3}-N(\beta; D^{n+3}))$, and X an exterior of K^n in S^{n+2}. We put $q=n-p+2$. Let $\tilde{W}$ be the infinite cyclic covering space of W associated with the Hurewicz homomorphism $\pi_1(W)\to H_1(W)$, then the covering transformation group of $\tilde{W}$ is isomorphic to Z, and we choose a generator t. Let $\Lambda=Z[t, t^{-1}]$. We now introduce some notation:

V_0 := $\mathrm{cl}(V-N(\Delta; V))$,
$\tilde{V}_0$: the lift of V_0 in $\tilde{W}$,
$\{\xi_i\}$: standard p-spheres in ∂V such that $\partial\Delta$ is unknotted in $\partial V-\bigcup\xi_i$,
T := $\mathrm{cl}(\partial N(\beta; D^{n+3})\cap\mathrm{int}\, D^{n+3})$,
$\tilde{T}$: the lift of T in $\tilde{W}$,
$\tilde{X}$:= $\mathrm{cl}(\partial\tilde{W}-\tilde{T})$,
$\tilde{X}_0$:= $\mathrm{cl}(\partial\tilde{V}_0-\tilde{T})$,
$\tilde{X}_1$:= $\mathrm{cl}(\tilde{X}_0-\cup\tilde{\alpha}_{ik}(\partial B^{p+1}\times D^q))$,

where we denote lifts of h_i^p, α_i, h_i^{p+1}, ξ_i in W by $\tilde{h}_{ik}^p$, $\tilde{\alpha}_{ik}$, $\tilde{h}_{ik}^{p+1}$, $\tilde{\xi}_{ik}$ respectively such that $\tilde{\alpha}_{ik}=\alpha(\tilde{h}_{ik}^{p+1})$. Let $\tilde{a}_{ik}=\tilde{\alpha}_{ik}(\partial B^{p+1}\times *)$. We choose indices of $\tilde{\alpha}_{ik}$, $\tilde{\xi}_{ik}$ etc. such that $t\tilde{\alpha}_{ik}=\tilde{\alpha}_{ik+1}$, $t\tilde{\xi}_{ik}=\tilde{\xi}_{ik+1}$ etc. Then $\tilde{X}$ is an infinite cyclic covering space of X associated with the Hurewicz homomorphism $\pi_1(X)\to H_1(X)$. We consider all homology groups for subspaces in W as Λ-modules, unless otherwise stated.

Definition 5.2. We can choose $[\tilde{\xi}_{10}], \cdots, [\tilde{\xi}_{\nu 0}]$ as generators of $H_p(\tilde{V}_0)$. Then we can represent

$$[\tilde{a}_{j0}]=\sum_{i=1}^{\nu}\lambda_{ij}(t)\cdot[\tilde{\xi}_{i0}], \qquad \lambda_{ij}(t)\in\Lambda,$$

as an element of Λ-module $H_p(\tilde{V}_0)$. We note that a Laurent polynomial $\lambda_{ij}(t)$ surely exists, from Remark 1.8.1 and goodness of $\mathcal{D}$ if $p=1$. We say that $M(t)=(\lambda_{ij}(t))$ is an *attaching matrix* of $\mathcal{D}$.

We remark that $M(t)$ is a relation matrix of $H_p(\tilde{W})$.

Remark 5.2.1. In the above definition, the choice of an attaching matrix has the following ambiguities:

(1) The choice of the indices and the orientations of ξ_i, and lift $\tilde{\xi}_{ik}$: this affects an attaching matrix as permuting rows, or multiplying some rows by units of Λ.

(2) The choice of the indices and the orientations of a_i, and lift $\tilde{a}_{ik}$: this corresponds to the same modifications of columns for an attaching matrix as the above case (1).

Thus we identify two attaching matrices which differ in the above ambiguities.

In the case of $2 \leqq p \leqq n$, an attaching matrix is easily interpreted as the following, since $H_p(\tilde{V}_0) \cong \pi_p(V_0)$:

5.2.2. Another description for attaching matrices in the case of $2 \leqq p \leqq n$. Choose a base point e in $\partial V - \partial\Delta$. Let ξ_0 be a meridianal loop for $\partial\Delta$ in ∂V. Let γ_i be a path in $\partial V - \partial\Delta$ connecting ξ_i and e for $0 \leqq i \leqq \nu$. Let $\bar{\xi}_0 = \gamma_0^{-1} \cup \xi_0 \cup \gamma_0$, and $\bar{\xi}_i = \xi_i \cup \gamma_i$ if $i \neq 0$. Then $\pi_1(V_0)$ is generated by $t = [\bar{\xi}_0]$, and $\pi_p(V_0)$ is a free $Z\pi_1$-module with basis $[\bar{\xi}_1], \cdots, [\bar{\xi}_\nu]$. As $\pi_1(V_0)$ is isomorphic to Z, we identify $Z\pi_1$ with Λ. Let γ'_i be a path in $\partial V - \partial\Delta$ connecting $a_i = \alpha_i(\partial B^{p+1} \times *)$ and e, for $* \in \operatorname{int} B^q$, and $\bar{a}_i = a_i \cup \gamma'_i$. Then we can represent $[\bar{a}_j] = \sum_{i=1}^{\nu} \lambda_{ij}(t) \cdot [\bar{\xi}_i]$ as an element of $Z\pi_1$-module $\pi_p(V_0)$, where $\lambda_{ij}(t) \in Z\pi_1$, and an attaching matrix of $\mathcal{D}$ is $(\lambda_{ij}(t))$. The ambiguities for the choice of γ_i etc. correspond to Remark 5.2.1.

The above description of attaching matrices is not valid for the case of $p=1$, because $\pi_1(V_0)$ is a free group on $\nu+1$ generators. A similar description is possible in this case, but we omit it.

Remark 5.2.3. Let $M(t)$ be an attaching matrix of a p-decomposition. Then we may assume that $q(1)$ is the identity matrix. Conversely, assume that $M(t)$ is a square matrix on Λ such that $M(1)$ is the identity matrix. By the similar construction to that in [21], we can easily construct a p-decomposition of a disk apir with an attaching matrix $M(t)$.

The following Lemmas 5.3.1–5.3.3 are easily obtained, and we omit the proof:

Lemma 5.3.1.

$$H_r(\tilde{X}_0, \tilde{X}_1) = \begin{cases} \Lambda^\nu & \text{if } \; r=q,\, n+2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

If $r=q$, then generators are

$$[\tilde{\alpha}_{i0}(* \times D^q),\ \tilde{\alpha}_{i0}(* \times \partial D^q)], \quad * \in \partial B^{p+1}, \qquad \text{for} \quad 1 \leqq i \leqq \nu\,.$$

Lemma 5.3.2. *Assume $p \neq q$, then*

$$H_r(\tilde{X}_0) \cong \begin{cases} \Lambda/(t-1) & \text{if } r=0, \\ \Lambda^\nu & \text{if } r=p, q, \\ 0 & \text{otherwise}. \end{cases}$$

If $r=p$, then generators are $[\tilde{\xi}_{10}], \cdots, [\tilde{\xi}_{\nu 0}]$. If $r=q$, then generators are $[\tilde{h}_{i0}^p(* \times \partial D^{q+1})]$, for $* \in \operatorname{int} B^p$ and $1 \leqq i \leqq \nu$.

Lemma 5.3.3.

$$H_r(\tilde{X}, \tilde{X}_1) \cong \begin{cases} \Lambda^\nu & \text{if } r=p+1, n+2, \\ 0 & \text{otherwise}. \end{cases}$$

If $r=p+1$, then generators are $[\tilde{h}_{i0}^{p+1}(B^{p+1} \times *), *\tilde{h}_{i0}^{p+1}(\partial B^{p+1} \times *)]$, for $* \in \partial D^q$, and $1 \leqq i \leqq \nu$.

Lemma 5.4. *If $p \neq q-1$, q, then the connecting homomorphism $\partial_*\colon H_{p+1}(\tilde{X}_1, \tilde{X}) \to H_p(\tilde{X}_1)$ is represented by an attaching matrix $M(t)$ of $\mathcal{D}$.*

Proof. Let $j_*\colon H_p(\tilde{X}_1) \to H_p(\tilde{X}_0)$ be the homomorphism induced by the inclusion map, then j_* is an isomorphism by Lemma 5.3.1. Let $u_i = j_*^{-1}([\tilde{\xi}_{i0}])$, then $\{u_i\}$ are generators of free Λ-module $H_p(\tilde{X}_1)$. It holds that

$$\begin{aligned} \partial_*([\tilde{h}_{k0}^{p+1}(B^{p+1} \times *), \tilde{h}_{k0}^{p+1}(\partial B^{p+1} \times *)]) &= [\tilde{h}_{k0}^{p+1}(\partial B^{p+1} \times *)] \\ &= j_*^{-1}([\tilde{a}_{k0}]) \\ &= \sum_{i=1}^{\nu} \lambda_{ik}(t) \cdot u_i, \end{aligned}$$

where $M(t) = (\lambda_{ik}(t))$. This completes the proof of Lemma 5.4.

Lemma 5.5. *If $p \neq q$, then $i_*\colon H_q(\tilde{X}_0) \to H_q(\tilde{X}_0, \tilde{X}_1)$ is represented by the transposed matrix of $M(t^{-1})$, where i_* is induced by the inclusion map, and $M(t)$ is an attaching matrix of $\mathcal{D}$.*

Proof. Let $\eta_j = [\tilde{h}_{j0}^p(* \times \partial D^{q+1})]$, $* \in \operatorname{int} B^p$, be generators of $H_q(\tilde{X}_0)$, and $\zeta_{i0} = [\tilde{\alpha}_{i0}(* \times D^q), \tilde{\alpha}_{i0}(* \times \partial D^q)]$, $* \in \operatorname{int} \partial B^{p+1}$, be generators of $H_q(\tilde{X}_0, \tilde{X}_1)$. By $I(u, v)$ we denote the algebraic intersection number of u and v in X_0. We remark that $I(t^r \cdot \tilde{\xi}_{s0}, \eta_j) = 1$ if and only if $s=j$ and $r=0$, thus we have

$$\begin{aligned} I(t^m \cdot \tilde{a}_{k0}, \eta_j) &= \sum_{s=1}^{\nu} I(t^m \lambda_{sk}(t) \cdot \tilde{\xi}_{s0}, \eta_j) \\ &= I(t^m \lambda_{jk}(t) \cdot \tilde{\xi}_{j0}, \eta_j) \\ &= \text{the coefficient of } t^m \text{ in } \lambda_{jk}(t^{-1}), \end{aligned}$$

Therefore it holds that

$$i_*(\eta_j) = \sum_{k,m} I(t^m \cdot \tilde{a}_{k0}, \eta_j) t^m \tilde{\zeta}_{k0}$$
$$= \sum_{k=1}^{\nu} \lambda_{jk}(t^{-1}) \cdot \tilde{\zeta}_{k0} \, .$$

This completes the proof of Lemma 5.5, since $H_q(\tilde{X}_0)$ and $H_q(\tilde{X}_0, \tilde{X}_1)$ are free Λ-modules by Lemmas 5.3.1 and 5.3.2.

Lemma 5.6. *Let $M(t)$ be an attaching matrix of a p-decomposition $\mathcal{D}$. Let $\gamma_j(t) \in \Lambda$, and $\boldsymbol{x}(t) = (\gamma_1(t), \cdots, \gamma_\nu(t))^{\mathrm{T}}$, where $(\ )^{\mathrm{T}}$ is the transposed of $(\ )$. If $M(t) \cdot \boldsymbol{x}(t) = \boldsymbol{0}$, then $\boldsymbol{x}(t) = \boldsymbol{0}$, where $\boldsymbol{0}$ is the zero vector.*

Proof. From the definition of attaching matrices, it follows that $M(1)$ is the identity matrix E_ν of degree ν. Then $\boldsymbol{x}(1) = \boldsymbol{0}$. Assume, for some j, $\gamma_j(t) \neq \boldsymbol{0}$, and put $d = \operatorname{Min}_j (\operatorname{Max}\{n \in N \mid \gamma_j(t) \neq 0, (t-1)^n \text{ is a factor of } \gamma_j(t)\})$. Then d is a positive integer. Let $(t-1)^d \boldsymbol{y}(t) = \boldsymbol{x}(t)$, then $\boldsymbol{y}(1) \neq 0$ and $M(t) \cdot \boldsymbol{y}(t) = \boldsymbol{0}$. This contradicts $M(1) = E_\nu$. Therefore $\gamma_j(t) = 0$ for all j, hence $\boldsymbol{x}(t) = \boldsymbol{0}$.

Theorem 5.7. *If $1 \leqq r \leqq n$ and $r \neq p, q-1$, then $H_r(\tilde{X}) \cong 0$. If $2p \leqq n$, then an attaching matrix $M(t)$ of a p-decomposition is a relation matrix of $H_p(\tilde{X})$.*

Proof. Suppose $1 \leqq r \leqq n$. From Lemma 5.3.1 and the Mayer-Vietoris Theorem for $(\tilde{X}_0, \tilde{X}_1)$, we have $H_r(\tilde{X}_1) \cong H_r(\tilde{X}_0)$ if $r \neq q-1, q$. Using Lemma 5.3.3, we have $H_r(\tilde{X}) \cong H_r(\tilde{X}_1)$ if $r \neq p, p+1$. From these results and Lemma 5.3.2, it follows that $H_r(\tilde{X}) \cong 0$ if $r \neq p, p+1, q-1, q$. From Lemmas 5.4, 5.5 and 5.6, it follows that ∂_* is monomorphism if $p \neq q-1, q$, and that so is i_* if $p \neq q$. Using these facts and the well-known duality [17], we have $H_r(\tilde{X}) \cong 0$ if $r \neq q-1, p$. This completes the proof of former assertion. The latter assertion is trivially obtained from the exact sequence:

$$H_{p+1}(\tilde{X}, \tilde{X}_1) \xrightarrow{\partial_*} H_p(\tilde{X}_1) \rightarrow H_p(\tilde{X}) \rightarrow 0 \, ,$$

since $H_{p+1}(\tilde{X}, \tilde{X}_1)$ and $H_p(\tilde{X}_1)$ are free Λ-modules.

Corollary 5.7.1. *Let K^n be an n-knot of type p, and $\tilde{X}$ the infinite cyclic cofering space of the exterior of K^n in S^{n+2}. If $2 \leqq p \leqq n-1$, and $H_p(\tilde{X}) \cong 0$, then K^n is unknotted.*

For a disk pair (D^{n+3}, β) of type p, the same assertion as the above holds.

Proof. From our assumption, it follows that $\pi_1(X) \cong Z$, thus $\tilde{X}$ is the universal covering space. From the duality [17], we have $H_{p-1}(\tilde{X}) \cong 0$. Therefore X is a homotopy circle, and K^n is unknotted by the unknotting theorem of higher dimensional knots [12], [20] and [23].

For the disk pair (D^{n+3}, β), we can prove the assertion by the similar manner to the above.

Corollary 5.7.2. *Let p and p' be distinct integers with $2<2p\leqq n$ and $2<2p'\leqq n$. Assume that K^n is an n-knot of type p, and also of type p', then K^n is unknotted.*

Proof. Suppose $p<p'$, thus $p<p'<n-p'+1<n-p+1$. From our assumption and Theorem 5.7, it follows that $H_p(\tilde{X})\cong 0$ for the infinite cyclic covering space $\tilde{X}$ of the exterior of K^n in S^{n+2}. By Corollary 5.7.1, K^n is unknotted.

Corollary 5.7.3. *For $1\leqq p\leqq n$, $\mathcal{K}_n(p)$ is infinitely generated as a commutative semigroup.*

Proof. Let $\lambda_\mu(t)=\mu t-(\mu-1)$ for $\mu\in Z$, and $M_\mu(t)$ be a 1×1-matrix $(\lambda_\mu(t))$. Then we can find a disk pair (D^{n+3}, β) with a p-decomposition having an attaching matrix $M_\mu(t)$ from Remark 5.2.3. Let $(S^{n+2}, K^n)=\partial(D^{n+3}, \beta)$, then p-dimensional Alexander invariant [13] of K^n is $\lambda_\mu(t)$. It is trivial that the sum of two knots induces the product of Alexander invariants. It is easily seen that there are infinitely many μ such that $\lambda_\mu(t)$ is a prime polynomial. Thus $\mathcal{K}_n(p)$ is infinitely generated.

6. Disk pairs and attaching matrices

In this section, we will use the Notation 5.1 if necessary, without any specifications.

DEFINITION 6.1. Let $\mathrm{Mat}(\Lambda)$ be the set of a square matrices over Λ whose determinant is ± 1 when substituting $t=1$. For $M_1, M_2\in\mathrm{Mat}(\Lambda)$, we define $M_1\oplus M_2$ as the *block sum* of M_1 and M_2, i.e., $M_1\oplus M_2=\begin{bmatrix} M_1 & 0 \\ 0 & M_2\end{bmatrix}$. An *equivalence relation* on $\mathrm{Mat}(\Lambda)$ is defined to be generated by the following operations:

(T_1) Permuting rows or permuting columns.

(T_2) Multiplying a row or a column by a unit of Λ.

(T_3) Adding a multiple of a row (resp. a column) by a unit of Λ to another row (resp. column).

(T_4) Replacing M by $\begin{bmatrix} M & 0 \\ 0 & 1\end{bmatrix}$, or vice versa.

Then we define $\mathcal{M}_1(\Lambda)$ the set of equivalence classes of matrices of $\mathrm{Mat}(\Lambda)$. We remark that $\mathcal{M}_1(\Lambda)$ forms a commutative semigroup with a binary operation naturally induced from $\oplus$.

We define $\mathcal{M}_0(\Lambda)$ the set of equivalence classes of matrices of $\mathrm{Mat}(\Lambda)$, each of which induces an epimorphism from Λ^ν onto itself for some ν.

Lemma 6.2. *Let P be a subspace in a manifold Y, and α_0 and α_1 embeddings of P in a manifold M. Suppose that $\alpha_0(P)=\alpha_1(P)$ and that α_0 and α_1 are* *ambient isotopic in M. Then there exists a homeomorphism $g\colon Y\to Y$ such that*

$$\alpha_0=\alpha_1\circ(g|P).$$

Proof. Let $\{f_t\}$ be an ambient isotopy between α_0 and α_1, hence $f_1\circ\alpha_0=\alpha_1$. Let $M_0=M\underset{\alpha_0}{\cup} Y$. By [7], we can extend $\{f_t\}$ in M_0 so that there exists an ambient isotopy $\{\bar f_t\}: M_0\to M_0$ which satisfies $(\bar f_t|(i(M)))\circ i=i\circ f_t$ for each $t\in I$, where $i\colon M\to M_0$ is the inclusion map. We can find an embedding $\bar\alpha_0\colon Y\to M_0$ such that $\bar\alpha_0|P=\alpha_0$. Let $\bar\alpha_1=\bar f_1\circ\bar\alpha_0$, then $\bar\alpha_1$ is an embedding of Y in M_0. It holds that

$$\bar\alpha_1|P=\bar f_1\circ\bar\alpha_0|P=(\bar f_1|\alpha_0(P))\circ i\circ\alpha_0=i\circ f_1\circ\alpha_0=i\circ\alpha_1.$$

By definition, M_0 is 'separated' into $i(M)$ and $\bar\alpha_0(Y)$ by $\alpha_0(P)$. Thus $\bar\alpha_0(Y)=\bar\alpha_1(Y)$. Hence $g=\bar\alpha_1^{-1}\circ\bar\alpha_0$ is a well-defined homeomorphism from Y onto itself, and this is a required one.

Lemma 6.3. *Let (D^{n+3},β_0) and (D^{n+3},β_1) be disk pairs of type p which have p-decompositions with the same attaching matrix. Assume $2<2p\leqq n$, then the two disk pairs are equivalent.*

Proof. Let $\mathscr{D}_j$ be a p-decomposition of (D^{n+3},β_j), for $j=0, 1$, such that $\mathscr{D}_j$ has an attaching matrix $M(t)$ of degree ν. Without loss of generality, we can assume that $\mathscr{D}_j=(V,\{\alpha_{ij}\},\Delta)$ for $j=0, 1$, where $V=D_n^{0+3}\cup\bigcup h_i^p$. Let ν be the number of p-handles in V.

From Theorem 1.5, there exists an orientation preserving homeomorphism $f\colon V\to V$ and homeomorphisms $g_i\colon B^{p+1}\times D^q\to B^{p+1}\times D^q$ such that $f\circ\alpha_{i0}=\alpha_{i1}\circ(g_i|\partial B^{p+1}\times D^q)$ and $f_*\colon \pi_p(V)\to\pi_p(V)$ is the identity. Let

$$\alpha'_{i0}=\alpha_{i1}\circ(g_i|\partial B^{p+1}\times D^q)^{-1},$$

and $\mathscr{D}'_0=(V,\{\alpha'_{i0}\},\Delta)$. Then $\mathscr{D}'_0$ is a p-decomposition of (D^{n+3},β_0) with an attaching matrix $M(t)$. We now show the following sublemma:

Sublemma. We can find a p-decomposition $\mathscr{D}_3=(V,\{\alpha_{i3}\},\Delta)$ of (D^{n+3},β_0) such that $\alpha_{i3}(\partial B^{p+1}\times D^q)=\alpha_{i1}(\partial B^{p+1}\times D^q)$, and $\{\alpha_{i3}\}$ and $\{\alpha_{i1}\}$ are ambient isotopic in ∂V.

Proof of Sublemma. As $f(\Delta)$ is unknotted in V, there exists an ambient isotopy $\{\phi_t^{(1)}\}: V\to V$ such that $\phi_1^{(1)}f(\Delta)=\Delta$. Put $\alpha_{i2}=\phi_1^{(1)}\circ\alpha'_{i0}$, and $\mathscr{D}_2=(V,\{\alpha_{i2}\},\Delta)$. Then $\mathscr{D}_2$ is a p-decomposition of (D^{n+3},β_0). We first show that an attaching matrix of $\mathscr{D}_2$ is $M(t)$. Let $\vee$ be the wedge product, $C=S^1\vee S_1^p\vee\cdots\vee S_\nu^p$, and $\bar C=B^2\vee S_1^p\vee\cdots\vee S_\nu^p$, then there exist homotopy equivalences $\psi_0\colon V_0\to C$ and $\bar\psi_0\colon V\to\bar C$ such that $\bar\psi_0|V_0=\psi_0$ and $\psi_0(\xi_i)=S_i^p$ for each i. Using the ambient isotopy $\{\phi_t^{(1)}\}$, we can construct an ambient isotopy $\{\hat\phi_t\}$:

$\bar{C}\to\bar{C}$ such that $\hat{\phi}_t\circ\bar{\psi}_0=\bar{\psi}_0\circ\phi_t^{(1)}$ for each $t\in I$. From our choice of f, it follows that $\bar{\psi}_0\circ f(\xi^i)\simeq S_i^p$ in $\bar{C}$. Then $\bar{\psi}_0\phi_1^{(1)}f(\xi_i)\simeq\hat{\phi}_1\bar{\psi}_0 f(\partial_i)\simeq\hat{\phi}_1(S_i^p)\simeq S_i^p$ in $\bar{C}$. Thus we can represent $[\psi_0\phi_1^{(1)}f(\xi_i)]=t^{m_i}\cdot[S_i^p]$ in $\pi_p(C)$ as $Z\pi_1$-module for some m_i. This means that an attaching matrix of $\mathcal{D}_2$ is obtained from $M(t)$ by multiplying some rows by units of $Z\pi_1=\Lambda$. From 5.2.1, we can choose $M(t)$ as an attaching matrix of $\mathcal{D}_2$. From 5.2.2, $\alpha_{i2}(\partial B^{p+1}\times *)$ and $\alpha_{i1}(\partial B^{p+1}\times *)$ are homotopic in $\partial V-\partial\Delta$. By our dimensional assumption and Lemma 2.1, it is easily seen that $\alpha_{i2}(\partial B^{p+1}\times *)$ and $\alpha_{i1}(\partial B^{p+1}\times *)$ are ambient isotopic in $\partial V-\partial\Delta$. Using the uniqueness of regular neighbourhood, we can finally move $\alpha_{i2}(\partial B^{p+1}\times D^q)$ ambient isotopically to $\alpha_{i1}(\partial B^{p+1}\times D^q)$ in $\partial V-\partial\Delta$, i.e., there exists an ambient isotopy $\{\phi_t^{(2)}\}: V\to V$ such that $\phi_t^{(2)}|\Delta$ is the identity map for all $t\in I$, and that $\phi_t^{(2)}\circ\alpha_{i2}(\partial B^{p+1}\times D^q)=\alpha_{i1}(\partial B^{p+1}\times D^q)$. Let $\alpha_{13}=\phi_t^{(2)}\circ\alpha_{i2}$, and $\mathcal{D}_3=(V,\{\alpha_{i3}\},\Delta)$. Then $\mathcal{D}_3$ is a p-decomposition of (D^{n+3},β_0) such that $\alpha_{i3}(\partial B^{p+1}\times D^q)=\alpha_{i1}(\partial B^{p+1}\times D^q)$. Combining ambient isotopies $\{\phi_t^{(1)}\}$ and $\{\phi_t^{(2)}\}$ we have an ambient isotopy $\{\phi_t\}: V\to V$ such that $\phi_1\circ\alpha_{i3}=\alpha_{i1}$. This completes the proof of Sublemma.

By Lemma 6.2, there exist homeomorphisms $g_i': B^{p+1}\times D^q\to B^{p+1}\times D^q$ such that $\alpha_{i3}=\alpha_{i1}\circ(g_1'|\partial B^{p+1}\times D^q)$. Let $j: V\to V$ be the identity map, then we have $j\circ\alpha_{i3}=\alpha_{i1}\circ g_i'|\partial B^{p+1}\times D^q$. This means that j is extended to a homeomorphism

$$\bar{j}: V\underset{\{\alpha_{i3}\}}{\cup}\bigcup(B^{p+1}\times D^q)_i\to V\underset{\{\alpha_{i1}\}}{\cup}\bigcup(B^{p+1}\times D^q)_i$$

such that $\bar{j}\,|\,V=j$. Thus $\bar{j}$ is an orientation preserving homeomorphism from D^{n+3} onto itself such that $\bar{j}(\beta_0)=\beta_1$.

DEFINITION 6.4. In Definition 1.7, if we change the condition (2) by the following condition (2)′, then we say that $(V, \{\alpha_i\}, \Delta)$ is a *weak p-decomposition* of (D^{n+3}, β):

(2)′ $\{h_i^p\}$ are trivial p-handles on D_0^{n+3}.

For a weak p-decomposition $\mathcal{D}$, we can define an *attaching matrix* of $\mathcal{D}$ by the same manner as that in Definition 5.2. We note that the arguments in 5.2.1 and 5.2.1 are valid for weak p-decompositions.

By Proposition 1.4, a p-decomposition is a weak p-decomposition. We remark that an attaching matrix $M(t)$ of a weak p-decomposition satisfies $|\det M(1)|=1$.

Lemma 6.5. *Let $M(t)$ be an attaching matrix of a weak p-decomposition $\mathcal{D}$. If $2<2p\leqq n$ and $M(1)$ is a diagonal matrix such that each diagonal element is ± 1, then $\mathcal{D}$ is a p-decomposition of the disk pair.*

Proof. Let $\mathcal{D}=(V, \{\alpha_i\}, \Delta)$. The assumption on $M(1)$ means that attaching spheres $\{a_i\}$ of $(p+1)$-handles are homotopic to standard spheres on ∂V

by 5.2.2. Hence $\{a_i\}$ are ambient isotopic to standard p-spheres in ∂V by Lemma 2.1, thus $\mathcal{D}$ is a p-decomposition.

Lemma 6.6. *Suppose $2<2p\leqq n$. Let $\mathcal{D}$ be a weak p-decomposition of a disk pair (D^{n+3}, β) with an attaching matrix $M(t)$, and $M'(t)$ be obtained from $M(t)$ by one of the modifications $(T_1)-(T_4)$ in Definition* 6.1. *Then we can find a weak p-decomposition $\mathcal{D}'$ of (D^{n+3}, β) with an attaching matrix $M'(t)$.*

Proof. In the case of (T_1), let $\mathcal{D}'$ be obtained from $\mathcal{D}$ by renumbering p-handles and $(p+1)$-handles, corresponding to the modification for $M(t)$. Then $\mathcal{D}'$ is a weak p-decomposition of (D^{n+3}, β) with an attaching matrix $M'(t)$.

In the case of (T_2), this case corresponds to the prescribed ambiguities for the choice of attaching matrices remarked in 6.2.1. Thus we choose $\mathcal{D}$ as $\mathcal{D}'$.

In the case of (T_3), we can trade a handle of $\mathcal{D}$ corresponding to the modification for $M(t)$, to obtain $\mathcal{D}'$ which is easily seen to be a weak p-decomposition of (D^{n+3}, β) with an attaching matrix $M'(t)$.

Suppose $M'(t)=M(t)\oplus(1)$. Let $h^p_{\nu+1}$ and $h^{p+1}_{\nu+1}$ be a pair of complementary handles on V suh such that the $(n+3)$-disk $h^p_{\nu+1}\cup h^{p+1}_{\nu+1}$ is disjoint from $\partial\Delta$ in ∂V. Let $\alpha_{\nu+1}=\alpha(h^{p+1}_{\nu+1})$, and $\mathcal{D}'=(V\cup h^p_{\nu+1}, \{\alpha_i\}\cup\alpha_{\nu+1}, \Delta)$, then $\mathcal{D}'$ is a weak p-decomposition of (D^{n+3}, β) with an attaching matrix $M'(t)$.

Suppose $M(t)=M'(t)\oplus(1)$. Let $a_i=a(h^{p+1}_i)$ be an attaching sphere for each i. Without loss of generality, we may assume that $[\bar{a}_\nu]=[\xi_\nu]$, under the notation of Definition 5.2.2. Thus the attaching sphere a_ν of h^{p+1}_ν is homotopic to a standard p-sphere ξ_ν in $\partial V-\partial\Delta$ keeping $\{a_i\}_{i<\nu}$ fixed. By Lemma 2.1, a_ν is ambient isotopic to ξ_ν in $\partial V-\partial\Delta$ keeping $\{a_i\}_{i<\nu}$ fixed. Thus $\mathcal{D}'=(V\cup h^{p+1}_\nu, \{\alpha_i\}_{i<\nu}, \Delta)$ gives a weak p-decomposition of (D^{n+3}, β) with an attahcing matrix $M'(t)$. This completes the proof of Lemma 6.6.

Lemma 6.7. *Let $\mathcal{D}_i$ be a p-decomposition of a disk pair (D^{n+3}, β_i) with an attaching matrix $M_i(t)$ for $i=0, 1$. Assume $2<2p\leqq n$ and $M_0(t)$ is equivalent to $M_1(t)$, then (D^{n+3}, β_0) is equivalent to (D^{n+3}, β_1).*

Proof. Without loss of generalitie, we can assume that $M_1(1)$ is the identity matrix, from Remark 5.2.3. By applying Lemma 6.6 repeatedly, we can find a weak p-decomposition $\mathcal{D}'_0$ of (D^{n+3}, β_0) with an attaching matrix $M_1(t)$. By Lemma 6.5, we conclude that $\mathcal{D}'_0$ is a p-decomposition of (D^{n+3}, β). Hence by Lemma 6.3, the given two disk pairs are equivalent.

Theorem 6.8. *If $2<2p\leqq n$, there exists an exact sequence of commutative semigroups:*

$$\mathcal{M}_0(\Lambda)\xrightarrow{i}\mathcal{M}_1(\Lambda)\xrightarrow{j}\mathcal{B}_{n+1}(p)\rightarrow\{1\},$$

where i is the inclusion map, and j is a homomorphism naturally defined by that

$j([M(t)])$ is a disk pair of type p which has a p-decomposition with an attaching matrix $M(t)$.

Proof. By Remark 5.2.3, Lemmas 6.3 and 6.7, j is a well-defined epimorphism. Let $M(t)$ be a square matrix of degree ν which induces an epimorphism from Λ^ν onto itself. We can choose a p-decomposition $\mathcal{D}$ of a disk pair (D^{n+3}, β) with an attaching matrix $M(t)$ by Remark 5.2.3. Let $\tilde{W}$ be the infinite cyclic covering space of an exterior of β in D^{n+3}. From the fact that $M(t)$ is a relation matrix for $H_p(\tilde{W})$, it follows that $H_p(\tilde{W})\cong 0$. By Corollary 5.7.1, β is unknotted in D^{n+3}. Hence $\mathrm{Im}\, i\subset \mathrm{Ker}\, j$. The reverse inclusion is easily proved. This completes the proof of Theorem 6.8.

By the smae argument as in the proof of Theorem 6.8, we have the following.

Corollary 6.8.1. *The same assertion holds in Theorem* 6.8 *for $\mathcal{K}_n(p)$ instead of $\mathcal{B}_n(p)$. That is, if $2<2p\leqq n$,*

$$\mathcal{M}_0(\Lambda) \xrightarrow{i} \mathcal{M}_1(\Lambda) \xrightarrow{j} \mathcal{K}_n(p) \rightarrow \{1\}$$

is an exadct sequence as commutative semigroups, where i is the inclusion map, and j is a homomorphism naturally defined by that $j([M(t)])$ is an n-knot of type p such that there exists a disk pair with a p-decomposition having an attaching matrix $M(t)$.

References

[1] K. Asano, Y. Marumoto and T. Yanagawa: *Ribbon knots and ribbon disks,* Osaka J. Math. **18** (1981), 161–174.

[2] R.H. Fox: *Characterizations of slices and ribbons,* Osaka J. Math. **10** (1973), 69–76.

[3] D. Hacon: *Knotted spheres in tori,* Quart. J. Math. **20** (1969), 431–445.

[4] L.R. Hitt: *Handlebody presentations of knot cobordisms*, Ph. D Thesis, Florida State Univ., 1977.

[5] F. Hosokawa, A. Kawauchi: *Proposal for unknotted surfaces in four-space,* Osaka J. Math. **16** (1979), 233–248.

[6] J.F.P. Hudson and D.W. Sumners: *Knotted ball pairs in unknotted sphere pairs,* J. London Math. Soc. **41** (1966), 717–723.

[7] J.F.P. Hudson: *Extending piecewise linear isotopies,* Proc. London Math. Soc. (3) **16** (1966), 651–668.

[8] J.F.P. Hudson: Piecewise linear topology, Benjamin, New York, 1969.

[9] J.F.P. Hudson and E.C. Zeeman: *On regular neighbourhoods,* Proc. London Math. Soc. (3) **14** (1964), 719–745., ibid. **21** (1970), 513–524.

[10] T. Kanenobu: *Higher-dimensional cable knots and their finite cyclic covering spaces,* Topolgy Appl. **19** (1985), 123–127.

[11] A. Kawauchi: *Knots in the stable 4-space; An overview,* preprint, 1986.
[12] J. Levine: *Unknotting spheres in codimension two,* Topology **4** (1965), 9–16.
[13] J. Levine: *Polynomial invariants of knots of codimension two,* Ann. of Math. **84** (1966), 53–554.
[14] Y. Marumoto: *Higher dimensional ribbon and slice knots,* Math. Sem. Notes Kobe Univ. **7** (1979), 351–362.
[15] Y. Marumoto: *A class of higher dimensional knots,* J. Fac. Ed. Saga Univ. **31** (II), (1984), 177–185.
[16] T. Matumoto: *On a weakly unknotted 2-sphere in a simply connected 4-manifold,* Osaka J. Math. **21** (1984), 489–492.
[17] J.W. Milnor: *Infinite cyclic covers,* Conf. Topology of Manifolds (Michigan State Univ., 1967), Prindle, Weber & Schmidt, Boston, Mass., 1968, 115–133.
[18] A. Omae: *A note on ribbon 2-knots,* Proc. Japan Acad. **47** (1971), 850–853.
[19] C. Rourke and B. Sanderson: Introduction to piecewise-linear topology, Springer-Verlag, 1972.
[20] J.L. Shaneson: *Embeddings of spheres in spheres of codimension two and h-cobordism of* $S^1\times S^1\times S^1$, Bull. Amer. Math. Soc. **74** (1968), 972–974.
[21] D.W. Sumners: *Higher-dimensional slice knots,* Bull. Amer. Math. Soc. **72** (1966), 894–897.
[22] S. Suzuki: *Knotting problems of 2-spheres in 4-spheres,* Math. Sem. Notes Kobe Univ. **4** (1976), 241–371.
[23] C.T.C. Wall: *Unknotting tori in codimension one and spheres in codimension two,* Proc. Cambridge Philos. Soc. **61** (1965), 659–664.
[24] T. Yanagawa: *On ribbon 2-knots, The 3-manifold bounded by the 2-knots,* Osaka J. Math. **6** (1969), 447–464.
[25] E.C. Zeeman: Seminar on combinatorial topology, (mimeographed notes) I.H.-E.S., Paris, 1963.

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