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ON IITAKA SURFACES

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Introduction. Let k be an algebraically closed field of characteristic zero. We consider a pair (V, D) which satisfies the following conditions:

(i) V is a nonsingular, projective and rational surface defined over k and D is a reduced effective divisor on V with simple normal crossings;

(ii) (V, D) is almost minimal;

(iii) $\bar{\kappa}(V-D)=0$ and $\bar{p}_g(V-D):=\dim H^0(V, D+K_V)=1$.

We shall call such a pair (V, D) an Iitaka surface.

A surface of this kind has been studied by Iitaka [4]. Thence comes the naming of Iitaka surface. In Th. 3 [ibid.], he gave an explicit way of writing down possible configurations of the divisor D. However, he did not determine which of these configurations are realizable. To begin with, he did not employ our almost minimal model to classify such surfaces.

Since an almost minimal model in the context of non-complete surfaces is thought as a substitute of a minimal surface in the context of complete surfaces, it would be natural to include the almost minimality in the definition of Iitaka surfaces. Thanks to this definition, we can determine (and classify) all Iitaka surfaces. Our method depends heavily on the theory of peeling in [9], the Mori theory [10] and observations of suitable $\boldsymbol{P}^1$-fibrations and elliptic fibrations.

Our Main Theorem consists of the following two results:

Reduction Theorem. *Let (V, D) be an Iitaka surface. Then the following assertions hold true.*

(1) *There exists a unique decomposition $D=A+N$ with $A>0$ and $N\geqq 0$ such that $A+K_V\sim 0$, N is disjoint from A and the connected components of N consist of (-2) rods and (-2) forks (cf. the terminology below).*

(2) *There exists a birational morphism from V to a minimal rational surface V^*, $u: V\rightarrow V^*$ satisfying the following conditions:*

(i) *V^* is either $\boldsymbol{P}^2$ or a ruled surface $F_m (m\geq 0)$ with a $\boldsymbol{P}^1$-fibration v: $F_m \rightarrow \boldsymbol{P}^1$. Moreover, $A^*:=u_*A$ is a divisor with (at worst) normal crossing singularities and $A^*+K_{V^*}\sim 0$.*

(ii) *Suppose* $V^*=\boldsymbol{P}^2$. *Then* $u_*D=u_*A$.

(iii) *Suppose* $V^*=F_m$. *Let* M *be a minimal section of* V^* *and let* f_i $(1\leqq i\leqq n; n\leqq 4)$ *be all fiber of* v *such that* $f_i\cap A^*$ *consists of one smooth point of* A^*. *There exist a fiber* h_1 *of* v, *a nonsingular rational curve* C_1 *with* $(C_1^2)=2$ *or* 4 *and a nodal rational curve* C_2 *with* $C_2\in|-K_{V^*}|$, *such that* $h_1\neq f_i$ $(1\leqq i\leqq n)$, h_1, C_1 *and* C_2 *are not components of* A^* *and that* $D^*:=u_*D$ *is a part of* $A^*+f_1+\cdots+f_n+M+h_1+C_1+C_2$. *The curves* h_1, C_1 *and* C_2 *are specified in the next condition.*

(iv) *If* h_1, C_1 *or* C_2 *appears in* D^*, *then* A^* *is either an elliptic curve or a nodal curve and* D^* *has one of the following nine configurations, where* $m\leqq 1$ *in Fig.* 7 *and Fig.* 8 *below and* $m=2$ *otherwise and,* A^* *is an elliptic curve in Fig.* 6, *Fig.* 7 *and Fig.* 8.

(v) *If* M *is a component of* D^*, *then* $m\geq 2$.

(vi) *If* $m\geqq 3$, *then* D^* *is given in Lemma* 2.6.

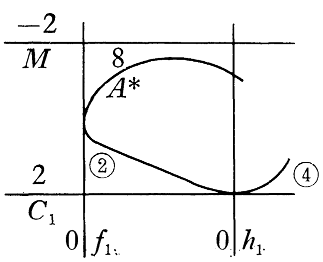


Fig. 1

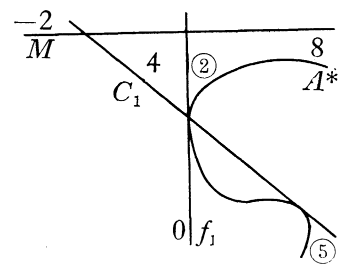


Fig. 2

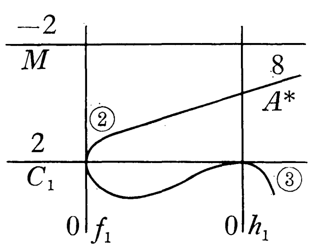


Fig. 3

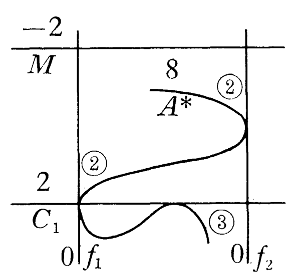


Fig. 4

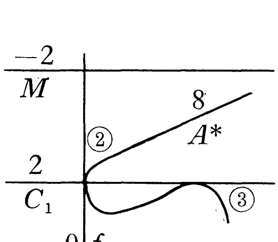


Fig. 5

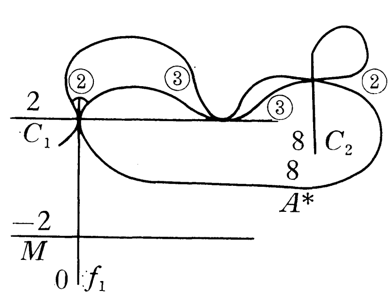


Fig. 6

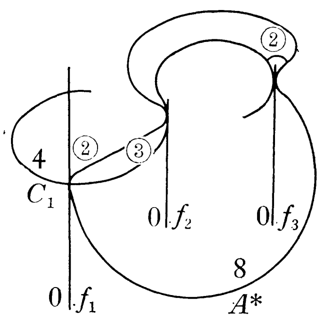


Fig. 7

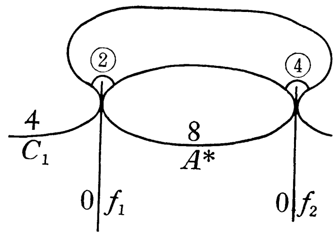


Fig. 8

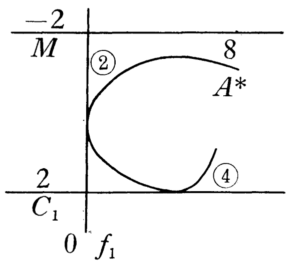


Fig. 9

Existence Theorem. (1) *Let (V, D) be an Iitaka surface with $A \neq 0$. Consider the following operations on D:*

(i) *Let P be a smooth point of A and let w: $V' \to V$ be a sequence of blowing-ups with center at P and its $n(n \geqq 0)$ infinitely near points lying consecutively on the proper transforms of A. Let $R := w^{-1}(P)$—(the last (-1) curve) which is a (-2) rod with n components. Let $A' := w'A$ be the proper transform of A, let $N' := w^*N + R$ and let $D' := A' + N'$.*

(ii) *Let P be a double point of A and let $w: V' \to V$ be the blowing-up with center at P. Let $A' := w^{-1}A$, $N' := w^*N$ and $D' := A' + N'$.*

(iii) *Suppose that there exists a (-1) curve E on V such that any connected component of $E + N$ has either a rod or a fork as its dual graph. Let $P := A \cap E$ and let w: $V' \to V$ be the blowing-up of P. Let $A' := w'A$, $N' := w'E + w^*N$ and $D' := A' + N'$.*

Let (V', D') be a pair obtained from (V, D) by performing finitely many operations of type (i), (ii) or (iii) on D. Then (V', D') is an Iitaka surface.

(2) *Let (V^*, D^*) be a pair as in Reduction Theorem. A minimal resolution of (V^*, D^*) is, by definition, the shortest sequence of blowing-ups u: $V_0 \to V^*$ such that $u^{-1}D^*$ is a divisor with simple normal crossings. Let D_0 be a reduced effective divisor obtained from $u^{-1}D^*$ removing all (-1) curves except for the (-1) curve arising from a possible, unique node of A^*. Then the pair (V_0, D_0) is an Iitaka surface.*

(3) *Every Iitaka surface (V, D) is obtained from an Iitaka surface (V_0, D_0) as considered in the assertion (2) above by repeating the operations considered in*

the assertion (1) *above.*

This paper consists of five sections. In §1, we shall consider under which conditions an Iitaka surface becomes a logarithmic K3-surface. At the begining of §2, we apply the theory of peeling and the Mori theory. By the first theory, we pass from an Iitaka surface (V, D) to a pair $(\bar{V}, \bar{D})$ by contracting BkD, where $\bar{V}$ is a projective normal surface with rational double points. We apply the Mori theory and show that we have only to consider three cases separately. Then we consider an Iitaka surface (V, D) with $\rho(\bar{V}) \geqq 2$; this will cover the first two cases. We treat the third case $\rho(\bar{V})=1$ in §§3 and 4. Finally in §5 we consider complementary cases to complete the proof of Main Theorem.

TERMINOLOGY. For the definitions of $\Omega_V^1(\log D)$ and the logarithmic Kodaira dimension $\bar{\kappa}(V-D)$, we refer to Iitaka [3; Chap. 10 & Chap. 11]. For the definition of an almost minimal surface, we refer to [9; Sect. 1. 11], as well as the relevant definitions like the bark of D, rods, twigs, forks, admissible twigs, rational rods, etc. By a $(-i)$ curve we shall mean a nonsingular rational curve C with $(C^2)=-i$ $(i \geqq 1)$. By a (-2) rod (or (-2) fork, resp.) we shall mean a rod (or fork resp.) whose irreducible components are all (-2) curves. In other words, (-2) rods and (-2) forks have the weighted dual graphs of the minimal resolution of rational double points. A reduced effective divisor with simple normal crossings is abbreviated as an SNC divisor.

NOTATIONS. $\kappa(V)$: the Kodaira dimension of V.

$\bar{\kappa}(X)$: the logarithmic Kodaira dimension of a nonsingular algebraic surface X defined over k.

K_V: the canonical divisor of V.

$\bar{p}_g(V-D) := \dim H^0(V, D+K_V)$.

$\bar{q}(V-D) := \dim H^0(V, \Omega_V^1(\log D))$.

$\rho(V)$: the Picard number of V.

F_m: A minimally ruled rational surface on which there is a minimal section M with $(M^2)=-m$.

In the pictures of the configurations of curves (not the dual graphs), considered in our paper, if an encircled number appears, it means that two curves, between which the number is written, meet each other at a single point with the order of contact indicated by the number.

I would like to thank Professor M. Miyanishi who gave me valuable suggestion during the preparation of the present paper.

1. Logarithmic K3-surfaces

We shall begin with

DEFINITION 1.1. Let (V, D) be a pair of a nonsingular projective surface

V defined over k and a reduced effective divisor D with SNC (simple normal crossings) on V. We call this pair a log $K3$-surface if the following conditions are met:

(i) $\bar{\kappa}(V-D)=0$;

(ii) the log geometric genus $\bar{p}_g(V-D)=1$;

(iii) the log irregularity $\bar{q}(V-D):=\dim H^0(V, \Omega_V^1(\log D))=0$.

We hope to classify log $K3$-surfaces (V, D) by looking into their almost minimal models $(\tilde{V}, \tilde{D})$. But $(\tilde{V}, \tilde{D})$ may not remain being a log $K3$-surface. Indeed, $(\tilde{V}, \tilde{D})$ is an Iitaka surface (cf. [9; Lemma 1.10]), while the condition (iii) above may become false for $(\tilde{V}, \tilde{D})$. However, we have the following

Lemma 1.2. *Let (V, D) be a pair of a nonsingular projective surface V and an SNC divisor D on V. Let $(\tilde{V}, \tilde{D})$ be an almost minimal model of (V, D). Then we have:*

(1) *If $\kappa(V)=0$ and (V, D) is a log $K3$-surface, $(\tilde{V}, \tilde{D})$ is also a log $K3$-surface.*

(2) *Conversely, if $(\tilde{V}, \tilde{D})$ is a log $K3$-surface, then (V, D) is a log $K3$-surface and either $\kappa(V)=-\infty$ or $\kappa(V)=0$.*

Proof. (1) Assume $\kappa(V)=0$. Then there exists an integer $N>0$ such that $|NK_V|\neq\phi$. Let $f\colon V\to\tilde{V}$ be the birational morphism attached to an almost minimal model $(\tilde{V}, \tilde{D})$, where $\tilde{D}=f_*D$. We know that $\bar{q}(\tilde{V}-\tilde{D})=0$ iff $q(\tilde{V})=0$ and irreducible components of $\tilde{D}$ are numerically independent (cf. Iitaka [4; Lemma 2]). We also know that $\bar{p}_g(\tilde{V}-\tilde{D})=\bar{p}_g(V-D)$ and $\bar{\kappa}(\tilde{V}-\tilde{D})=\bar{\kappa}(V-D)$ (cf. [9; Lemma 1.10]).

Now assume that (V, D) is a log $K3$-surface. So, in order to verify the assertion (1) we have only to show that irreducible components of $\tilde{D}$ are numerically independent. By inducting on the number of blowing-ups we have to perform to get (V, D) from $(\tilde{V}, \tilde{D})$, we may assume that f is the contraction of a (-1) curve E on V (which means an exceptional curve of the first kind) such that:

(a) $(D^{\#}+K_V, E)<0$, where $D^{\#}:=D-BkD$;

(b) $BkD+E$ is negative-definite.

If E is a component of D, the assertion (1) is clear. So we assume that E is not a component of D.

Suppose that $\sum_{i=1}^n\alpha_i\tilde{D}_i\equiv 0$ for some $\alpha_1, \cdots, \alpha_n\in\boldsymbol{Z}$, where $D:=\sum_{i=1}^n D_i$ and $\tilde{D}_i:=f_*D_i$ for $i=1, \cdots, n$. Then $f^*(\sum_{i=1}^n\alpha_i\tilde{D}_i)=\sum_{i=1}^n\alpha_i D_i+aE\equiv 0$ for some $a\in\boldsymbol{Z}$. We may assume $a\geqq 0$. If $a=0$, we get $\sum_{i=1}^n\alpha_i D_i\equiv 0$. So we have $\alpha_1=\cdots\alpha_n=0$ for $\bar{q}(V-D)=0$ implies that $D_1, \cdots, D_n$ are numerically independent. Suppose that $a>0$. After a suitable permutation of $\{1, \cdots, n\}$, we may assume that $\sum_{i=1}^n\alpha_i D_i=\sum_{i=1}^s a_i D_i-\sum_{j=s+1}^n b_j D_j$ with $a_i\geqq 0$ and $b_j\geqq 0$. Then we get $aE+\sum_{i=1}^s a_i D_i\equiv\sum_{j=s+1}^n b_j D_j$. Since $q(V)=0$, there exists an integer $N_1>0$

such that $N_1(aE+\sum_{i=1}^{s}a_iD_i)\sim N_1\sum_{j=s+1}^{n}b_jD_j$. Let $N_2=\mathrm{Max}\{b_{s+1},\cdots,b_n\}$. By the assumption that $a>0$, we have $N_2>0$. Then $N_1N_2D=N_1N_2(D_1+\cdots+D_s)+N_1N_2(D_{s+1}+\cdots+D_s)\sim N_1N_2(D_1+\cdots+D_s)+N_1(aE+\sum_{i=1}^{s}a_iD_i)+N_1\sum_{j=s+1}^{n}(N_2-b_j)D_j$. Since E appears in the right-hand side and does not appear in the left-hand side, we obtain $\dim|N_1N_2D|>0$. Since $|NK_V|\neq\phi$, we have $\dim|N_1N_2N(D+K_V)|\geqq\dim|N_1N_2ND|>0$, which is a contradiction because $\bar{\kappa}(V-D)=0$.

(2) It is easy. Q.E.D.

The following result due to Kawamata [5] is crucial.

Lemma 1.3. *Let (V,D) be a pair of a nonsingular projective surface V and an SNC divisor D on V. Suppose that $\bar{\kappa}(V-D)=0$ and that (V,D) is almost minimal. Then $n(D^{\#}+K_V)\sim 0$ for some $n\in\boldsymbol{N}$.*

Proof. See [6; Chap. II, Th. 2.2].

By using Lemma 1.3 and the results in [9], we verify the following lemma.

Lemma 1.4. *Suppose that (V,D) is a pair of a nonsingular projective surface V and an SNC divisor D on V. Suppose furthermore that $\kappa(V)=\bar{\kappa}(V-D)=0$ and that (V,D) is almost minimal. Thne the following are equivalent:*

(1) *(V,D) is a log $K3$-surface;*

(2) *V is a minimal $K3$-surface and D consists of (-2) rods and (-2) forks, where a (-2) rod (or (-2) fork, resp.) is a rod (or fork, resp.) whose irreducible components are (-2) curves, i.e., nonsingular rational curves with self-intersection (-2).*

(3) *$q(V)=0$, $p_g(V)=1$ and D consists of (-2) rods and (-2) forks.*

Proof. Suppose that (V,D) is almost minimal and that $\kappa(V)=\bar{\kappa}(V-D)=0$. Then, applying Lemma 1.3, we obtain $nK_V\sim 0$ for some $n>0$ and $D^{\#}=0$ since $\kappa(V)=0$. So we know that $\mathrm{Supp}D=\mathrm{Supp}BkD$, that D consists of (-2) rods and (-2) forks and that irreducible components of D are numerically independent. Hence $\bar{q}(V-D)=0$ iff $q(V)=0$. We know that $h^0(V,n(D+K_V))=h^0(V,[nD^{\#}+nK_V])=h^0(V,nK_V)$ for every $n>0$ (cf. [9; Lemma 1.10]). Then Lemma 1.4 is obvious. Q.E.D.

In the subsequent paragraphs of this section, we always assume that a pair (V,D) is an Iitaka surface. Then, since $h^0(V,[D^{\#}+K_V])=\bar{p}_g(V-D)=1$, just one of the following two cases takes place.

(1) There exists a curve $A\leqq[D^{\#}]$ with $p_a(A)\geqq 1$.

(2) Every curve $C\leqq[D^{\#}]$ is rational and the dual graph of $[D^{\#}]$ contains a rational loop A.

Moreover, we can show

Lemma 1.5. *Let (V, D) be an Iitaka surface. Then A is a connected component of D with $A+K_V \sim 0$ and $D^\sharp=[D^\sharp]=A$. Hence every connected component of D other than A is a (-2) rod or a (-2) fork. Furthermore, in case* (1), *we have $p_a(A)=1$, i.e., A is an elliptic curve.*

Proof. Since $|A+K_V| \neq \phi$ both in the cases (1) and (2), we get $D^\sharp=[D^\sharp]=A$ and $A+K_V \sim 0$ by virtue of Lemma 1.3. Hence A is a connected component of D because $D^\sharp=A$ implies that D contains no rational admissible twigs sprouting from A. In the case (1), we have $p_a(A)=1+\frac{1}{2}(A+K_V, A)=1$.

Q.E.D.

We know that the almost minimal model of a log $K3$-surface is an Iitaka surface; see the remark before Lemma 1.2. Conversely, we have the following lemma.

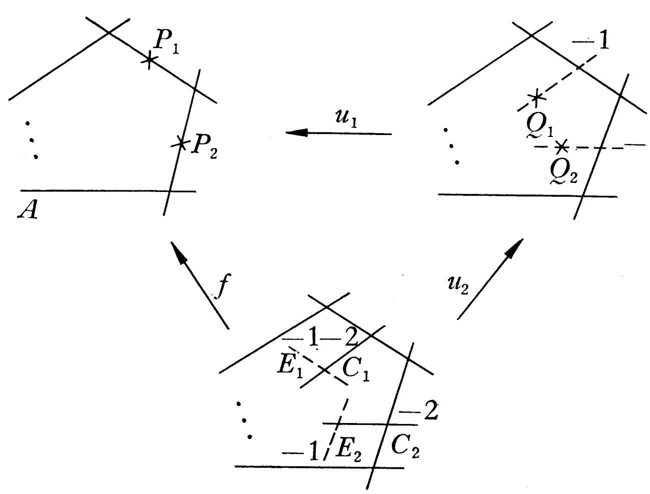
Lemma 1.6. *Let (V, D) be an Iitaka surface. Then we have:*

(1) *(V, D) is a log $K3$-surface provided that A is an elliptic curve.*

(2) *If A is a rational loop, there exists a birational morphism of pairs f: $(V^*, D^*) \to (V, D)$ such that (V^*, D^*) is a log $K3$-surface, (V, D) is an almost minimal model of (V^*, D^*) and f is the associated morphism.*

Proof. (1) is obvious (cf. Lemma 1.5).

(2) Suppose that (V, D) is an Iitaka surface and that A is a rational loop. Let u_1: $V_1 \to V$ be the blowing-ups of points P_1 and P_2 on A as shown in the picture below. Let u_2: $V^* \to V_1$ be the blowing-ups of points Q_1 and Q_2 on $u_1^*A - u_1'A$. Let $f=u_1 \circ u_2$ and $D^*=f'(D)+C_1+C_2$. Since $A+K_V \sim 0$, it is easy to see that $f'A+K_{V^*} \sim E_1+E_2$. We obtain easily $D^{*\sharp}=f'(A)+\frac{1}{2}(C_1+C_2)$. Since



$(D^{**}+K_{V^*}, E_1)=(E_1+E_2+\frac{1}{2}(C_1+C_2), E_1)=-\frac{1}{2}<0$ and since $E_1+\text{Supp}BkD$ is negative-definite, we must contract E_1 and C_1 to find an almost minimal model of (V^*, D^*); in fact it is (V, D). We see easily that irreducible components of D^* are numerically independent. Hence $\bar{q}(V^*-D^*)=0$ for $q(V^*)=q(V)=0$. we know that $\bar{p}_g(V^*-D^*)=\bar{p}_g(V-D)=1$ and $\bar{\kappa}(V^*-D^*)=\bar{\kappa}(V-D)=0$ (cf. [9; Lemma 1.10]). So (V^*, D^*) is a log $K3$-surface. Therefore, the assertion (2) is verified. Q.E.D.

We end this section with the following two lemmas.

Lemma 1.7. *Let (V, D) be an Iitaka surface. If there exists a (-1) curve E on V, we let u_1: $V\to V_1$ be the contraction of E and let $A_1=u_{1*}A$. Then $A_1+K_{V_1}\sim 0$ and A_1 is an NC (normal crossings) divisor. Moreover, A_1 is not an SNC divisor iff A is a loop consisting of two irreducible components, one of which is E.*

Proof. Note that $(A, E)=-(K_V, E)=1$, for $A+K_V\sim 0$. Lemma 1.7 is obvious. Q.E.D.

Lemma 1.8. *Suppose that (V, D) is an Iitaka surface. Then every nonsingular rational curve C on V has self-intersection more than (-3), unless C is a component of A.*

Proof. Since $A+K_V\sim 0$, $0\leqq(A, C)=(-K_V, C)=2-2p_a(C)+(C^2)=2+(C^2)$, i.e., $(C^2)\geqq -2$ for any nonsingular rational curve C with $C\not\subseteq\text{Supp}A$. Q.E.D.

2. Iitaka surfaces with $\rho(\bar{V})\geqq 2$

Fix an Iitaka surface (V, D) in the present section. Let ρ: $V\to\bar{V}$ be the contraction of BkD. Then $\bar{V}$ is a projective normal surface with only rational double points as singularities and there exists an $N\in\boldsymbol{N}$ such that $N\bar{F}$ is a Cartier divisor for every $\bar{F}\in\text{Div}(\bar{V})$ (cf. [9; Lemma 2.4]). Hence we have an intersection theory on $\bar{V}$. Furthermore, we have $K_V=\rho^*K_{\bar{V}}$ (cf. Artin [1; Th. 2.7]). We shall classify all Iitaka surfaces with $\rho(\bar{V})\geqq 2$.

DEFINITION 2.1. Let $N(\bar{V}):=\{\text{1-cycles}\}_{\boldsymbol{R}}/\{\text{numerical equivalence}\}$, and let $\overline{NE}(\bar{V})$:=the closure of the cone of effective 1-cycles $\{\sum_{i=1}^n a_i[\bar{C}_i];\ \bar{C}_i$: curve on $\bar{V}$, $[\bar{C}_i]\in N(\bar{V})$, $a_i\in\boldsymbol{R}_+$ and $n\in\boldsymbol{N}\}$ in $N(\bar{V})$, which is endowed with a usual Euclidean metric. An extremal rational curve $\bar{l}$ is a rational curve on $\bar{V}$ satisfying:

(i) $R:=\boldsymbol{R}_+[\bar{l}]$ is an extremal ray, i.e., $(K_{\bar{V}}, \bar{l})<0$ and $Z_1, Z_2\in R$ whenever $Z_1, Z_2\in\overline{NE}(\bar{V})$ with $Z_1+Z_2\in R$.

(ii) $-3\leqq(K_{\bar{V}}, \bar{l})<0$.

We take an ample Cartier divisor $\bar{L}$ on $\bar{V}$. We obtain the following lemma by using Th. 1.4 in Mori [10].

Lemma 2.2. *Suppose that $K_{\bar{V}}$ is not nef. Then, for an arbitrary $\varepsilon<0$, there exist extremal rational curves $\bar{l}_1, \cdots, \bar{l}_s$ such that $\overline{NE}(\bar{V})=\sum_{i=1}^{s}\boldsymbol{R}_+[\bar{l}_i]+\overline{NE}_\varepsilon(\bar{V})$, where $\overline{NE}_\varepsilon(\bar{V}):=\{\bar{Z}\in NE(\bar{V}); (\bar{Z}, K_{\bar{V}}+\varepsilon\bar{L})\geqq 0\}$.*

With the notations of §1, we have $A+K_V\sim 0$. Hence $\bar{A}+K_{\bar{V}}\sim 0$, where $\bar{A}:=\rho_*A$, and $K_{\bar{V}}$ is not nef. So, there is an extremal rational curve $\bar{l}$ on $\bar{V}$ by virtue of Lemma 2.2.

As in Mori [10; Lemma 3.7], there exists a nef divisor $\bar{H}$ on $\bar{V}$ such that $\bar{H}^\perp\cap\overline{NE}(\bar{V})=\boldsymbol{R}_+[\bar{l}]$, where $\bar{H}^\perp:=\{\bar{Z}\in\overline{NE}(\bar{V}); (\bar{Z}, \bar{H})=0\}$. Concerning $\bar{H}$, we consider the following three cases:

Case (1) $\bar{H}\equiv 0$. Then $\rho(\bar{V})=1$ and $-K_{\bar{V}}$ is ample.
Case (2) $\bar{H}\not\equiv 0$ and $(\bar{H}^2)=0$. Then $\bar{H}\in\boldsymbol{R}_+[\bar{l}]$ and $(\bar{l}^2)=0$.
Case (3) $(\bar{H}^2)>0$.

Set $l=\rho'\bar{l}$ and $H=\rho^*\bar{H}$. First of all, we consider the case (3) above. Namely we have

Lemma 2.3. *Let (V,D) be an Iitaka surface. In the case* (3) *where $(\bar{H}^2)>0$, $l+BkD$ is negative-definite and l is a (-1) curve on V.*

Proof. Sinec $(H^2)=(\bar{H}^2)>0$ and $(l,H)=(\rho_*l,\bar{H})=(\bar{l},\bar{H})=0$, we have $(l^2)<0$ by the Hodge index theorem. Note that $0>(\bar{l},K_{\bar{V}})=(\rho_*l,K_{\bar{V}})=(l,K_V)$ for $\bar{l}$ is extremal. Hence l is a (-1) curve. On the other hand, since $(l,H)=0$ and $(D_i,H)=(D_i,\rho^*\bar{H})=0$ for every $D_i\subseteq\mathrm{Supp}BkD$, we have only to show that $l,D_1,\cdots,D_r$ are numerically independent in order to verify that $l+BkD$ is negative-definite, where $\mathrm{Supp}BkD=\cup_{i=1}^{r}D_i$. Suppose that $al+\sum_{i=1}^{r}b_iD_i\equiv 0$ for some $a, b_1, \cdots, b_r\in\boldsymbol{R}$. After a suitable permutation of $\{1,\cdots,r\}$, we may assume that $a\geqq 0$, $b_1\geqq 0, \cdots, b_t\geqq 0$, $b_{t+1}<0, \cdots, b_r<0$. Hence $al+\sum_{i=1}^{t}b_iD_i\equiv-\sum_{j=t+1}^{r}b_jD_j$. Since BkD is negative-definite, $0\geqq(\sum_{j=t+1}^{r}(-b_j)D_j)^2=a(\sum_{j=t+1}^{r}(-b_j)D_j, l)+(\sum_{j=t+1}^{r}(-b_j)D_j, \sum_{i=1}^{t}b_iD_i)\geqq 0$. So, $(\sum_{j=t+1}^{r}(-b_j)D_j)^2=0$. Hence $b_{t+1}=\cdots=b_r=0$ because BkD is negative-definite. Therefore, we obtain $al+\sum_{i=1}^{t}b_iD_i\equiv 0$ and $a=b_1=\cdots=b_t=0$ for $a\geqq 0$ and $b_i\geqq 0$. Q.E.D.

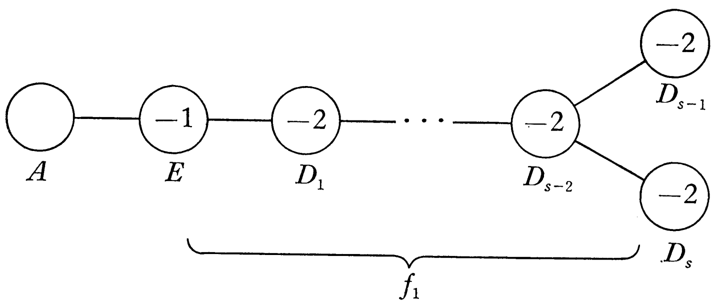
The following remark is useful, though obvious.

Remark 2.4. In the case (3) where $(\bar{H}^2)>0$, it is easy to see that $(l,\sum_{i=1}^{r}D_i)\leqq 1$ and that if $(D_i, l)=1$ for some $1\leqq i\leqq r$, then the connected component Δ of BkD containing D_i is a rod with D_i as a tip, where $\mathrm{Supp}\,BkD=\cup_{i=1}^{r}D_i$. This is a straightforward consequence of Lemma 2.3. Let $\sigma\colon V\to V'$ be the contraction of $l+\Delta$, where we set $\Delta=0$ when $(l,\sum_{i=1}^{r}D_i)=0$. Let $A'=\sigma_*A$ and $D'=\sigma_*D$. Then, unless A consists of two irreducible components, one of which is

l (in this case $(l, \sum_{i=1}^{r} D_i)=0$ because $A \cap \operatorname{Supp} BkD=\phi$), (V', D') is an Iitaka surface. In the above exceptional case, A' is a rational curve with one node.

Next we consider the case (2).

Lemma 2.5. *Let (V, D) be an Iitaka surface. Suppose that V is not isomorphic to F_m and that we are in the case (2) where $\bar{H} \not\equiv 0$ and $(\bar{H}^2)=0$. Then there exists a $\boldsymbol{P}^1$-fibration Φ: $V \to \boldsymbol{P}^1$ such that BkD is contained in the fibers of Φ. Moreover, a singular fiber f_1 of Φ has a configuration of the following type*:



where $f_1=2(E+D_1+\cdots+D_{s-2})+D_{s-1}+D_s (s \geqq 2)$, $\cup_{i=1}^{s} D_i \subseteq \operatorname{Supp} BkD$, and the integer in a circle is the self-intersection of the corresponding curve. Let u be the contraction of all (-1) curves in fibers of Φ. Then $u(V)=F_m$ for some $m \leqq 2$, $u_* A + K_{F_m} \sim 0$ and $u_* BkD$ consists of n fibers $f_1, \cdots, f_n$ of $\pi:=\Phi \circ u^{-1}$: $F_m \to \boldsymbol{P}^1$, where $n:=\#\{$singular fibers of $\Phi\}$.

Case. *A is an elliptic curve. Then $f_i (i=1, \cdots, n)$ passes through a ramification point of $\pi|_{u_*A}$. Hence $n \leqq 4$ and $k:=\#\{$connected components of BkD$\} \leqq 2n \leqq 8$.*

Case. *A is a rational loop. Then $m \leqq 1$, A consists of a nonsingular fiber l_1 and a 2-section F, and $f_i (i=1, \cdots, n)$ passes through a ramification point of $\pi|_{u_*F}$. Hence $n \leqq 2$ and $k \leqq 2n \leqq 4$.*

Proof. First of all, we shall construct a morphism Φ: $V \to \boldsymbol{P}^1$ as in [9; Lemma 2.8]. Define rational numbers $a_1, \cdots, a_r$ by the condition:

$$(l+\textstyle\sum_{i=1}^{r} a_i D_i, D_j) = 0 \quad \text{for } j = 1, \cdots, r$$

where Supp $BkD = \cup_{i=1}^{r} D_i$. Since $l \not\subseteq \operatorname{Supp} BkD$, we have $a_i \geqq 0$. We know that $N\bar{l}$ is a Cartier divisor; see the definition of N before Definition 2.1. Evidently $\rho^* N\bar{l} - N(l+\sum_{i=1}^{r} a_i D_i)$ is supported by SuppBkD. So we have:

$$\begin{aligned}(\rho^* N\bar{l} - N(l+\textstyle\sum_{i=1}^{r} a_i D_i))^2 &= (\rho^* N\bar{l}, \rho^* N\bar{l} - N(l+\textstyle\sum_{i=1}^{r} a_i D_i)) \\ &\quad - N(l+\textstyle\sum_{i=1}^{r} a_i D_i, \rho^* N\bar{l} - N(l+\textstyle\sum_{i=1}^{r} a_i D_i)) = 0\end{aligned}$$

by the definition of a_i's and by $(\rho^* \bar{l}, D_i)=0$. Hence $\rho^* N\bar{l} = N(l+\sum_{i=1}^{r} a_i D_i)$ because BkD is negative-definite.

We know that $h^2(V, n\rho^*N\bar{l})=h^0(V, K_V-n\rho^*N\bar{l})=0$ for $n>0$. So, by Riemann-Roch theorem we obtain:

$$h^0(V, n\rho^*N\bar{l})\geqq-\frac{n}{2}(\rho^*N\bar{l}, K_V)+\chi(\mathcal{O}_V) = -\frac{n}{2}(N\bar{l}, K_{\bar{V}})+1\to+\infty$$

as $n\to+\infty$ because $(\bar{l}, K_{\bar{V}})<0$. Hence, together with the fact that $q(V)=0$, $(\bar{l}^2)=0$ and $(K_{\bar{V}}, \bar{l})<0$ we know that there exists an $n\in \boldsymbol{N}$ such that $\Phi_{|n\rho^*N\bar{l}|}$ is composed of a $\boldsymbol{P}^1$-fibration Φ: $V\to\boldsymbol{P}^1$. There exists clearly a morphism ϕ: $\bar{V}\to \boldsymbol{P}^1$ such that $\Phi=\phi\circ\rho$. We first verify the following:

Claim 1. Every fiber of ϕ is irreducible, though it might be non-reduced.

Proof. Let $\bar{f}=\sum_{i=1}^k n_iF_i$ be a fiber of ϕ, where F_i is irreducible. Since $nN\bar{l}$ is a sum of fibers of ϕ, we have $(F_i, nN\bar{l})=0$. So $[F_i]\in\bar{l}^{\perp}\cap\overline{NE}(\bar{V})=\bar{H}^{\perp}\cap\overline{NE}(\bar{V})=\boldsymbol{R}_+[\bar{l}]$. Hence $(F_i^2)=0$. So $(\rho^*F_i^2)=0$ and ρ^*F_i is a rational multiple of the fiber $\rho^*(\bar{f})$. In particular, $\mathrm{Supp}\rho^*(\bar{f})=\mathrm{Supp}\rho^*F_i$. Therefore $k=1$.

Since $\Phi=\phi\circ\rho$, every connected component of BkD is contained in a singular fiber of Φ. Since such a fiber contains a (-1) curve (cf. [7; Chap. II, Lemma 2.2]), we conclude from the claim 1 the following

Claim 2. The support of every singular fiber f_1 of Φ is written as $E\cup(\cup_{i=1}^s B_i)$ for a (-1) curve E and irreducible components $B_1, \cdots, B_s (s\geqq 1)$ of BkD.

The claim 2 implies

Claim 3. There are no multiple fibers in Φ. If a fiber f_1 of Φ contains an irreducible component E of the part A of D, then $f_1=E$ and f_1 is a nonsingular fiber.

Proof. The first assertion is proven in M[7; Chap. II, Lemma 2.2]. Suppose that f_1 is a singular fiber of Φ containing an irreducible component E of A. With the notations of the claim 2, we have $\mathrm{Supp}\, f_1=E\cup(\cup_{i=1}^s B_i)$. The connectedness of f_1 implies that E meets $\cup_{i=1}^s B_i$, while this is impossible because $A\cap \mathrm{Supp}\, BkD=\phi$.

We can determine the configuration of a singular fiber as follows:

Claim 4. Let f_1 be a singular fiber of Φ. Then $f_1=2(E+B_1+\cdots+B_{s-2})+B_{s-1}+B_s\ (s\geqq 2)$ for a (-1) curve E and irreducible components $B_1, \cdots, B_s$ of BkD; see the configuration in the statement of this lemma.

Proof. From the claim 2 it follows that $f_1=aE+\sum_{i=1}^s a_iB_i$ for a (-1) curve E, irreducible components $B_1, \cdots, B_s$ of BkD and integers $a, a_1, \cdots, a_s$. Since $A\sim -K_V$ we have $2=(A, f_1)=a(A, E)=a$, i.e., $a=2$. Let u_1: $V\to V_1$ be the contraction of E. There apparently exists a morphism $\Phi_1: V_1 \to \boldsymbol{P}^1$ such that $\Phi=\Phi_1 \circ u_1$. It is easy to see that $u_{1*}A+K_{V_1} \sim 0$. So, $2=(u_{1*}A, u_{1*}f_1)=(u_{1*}A, \sum_{i=1}^{s} a_i u_{1*}B_i)$. Since $0 \leqq (u_{1*}A, u_{1*}B_i)=-(K_{V_1}, u_{1*}B_i)=2+(u_{1*}B_i^2)$, we have $(u_{1*}B_i^2)=-1$ or -2. Hence one of the following two cases takes place:

Case (i) There exists exactly one (-1) curve in $u_{1*}f_1$, say $u_{1*}B_1$; then $a_1=2$.

Case (ii) There exist exactly two (-1) curves in $u_{1*}f_1$, say $u_{1*}B_1$ and $u_{1*}B_2$; then $a_1=a_2=1$.

In the case (ii), we easily see that $f_1=2E+B_1+B_2$, $(E, B_i)=1$ $(i=1, 2)$ and $(B_1, B_2)=0$. In the case (i), we contract the unique (-1) curve $u_{1*}B_1$ in $u_{1*}f_1$ and have one of the above two cases. Continue this process untill the case (ii) takes place. So, $f_1=2(E+B_1+\cdots+B_{s-2})+B_{s-1}+B_s$ after a suitable change of indices $\{1, \cdots, s\}$. Its configuration is given in the statement of this lemma, where $D_i:=B_i$. After the contraction of $E, B_1, \cdots, B_{s-1}$, the proper transforms of A and B_s meet each other in a single point with contact of order 2. We have seen that every connected component of BkD is contained in a singular fiber of Φ. Hence, by the claim 4, we easily conclude

Claim 5. Every connected component of BkD is a rod of type A_1, a rod of type A_3 or a fork of type D_s $(s \geqq 4)$.

As in the proof of the claim 4, we contract all exceptional curves of the first kind contained in singular fibers of Φ. Then we have a birational morphism $u: V \to F_m$ onto a minimally ruled rational surface $\pi: F_m \to \boldsymbol{P}^1$ such that $\Phi=\pi \circ u$.

Suppose $BkD=\phi$. Then $V=\bar{V}$ is a minimally ruled rational surface. Then the configuration of $D=A$ is given in Lemma 2.6 below. Assume $BkD \neq \phi$. Then Φ contains at least one singular fiber f_1. By the observation in the claim 4, the fiber $u_* f_1$ touches an irreducible component, say A_1^*, of $A^*:=u_* A$ and meets none of the other components of A^*. Namely, the point $u_* f_1 \cap A_1^*$ is a ramification point of $\pi|_{A_1^*}: A_1^* \to \boldsymbol{P}^1$, and A_1^* is a 2-section of $\pi: F_m \to \boldsymbol{P}^1$. This implies that the irreducible component A_1^* of A^* is uniquely determined. We know also that u does not contract any irreducible component of A (cf. the claim 3). Hence we have:

$$\#\{\text{irreducible components of } A\}=\#\{\text{irreducible components of } A^*\}$$

We see easily that A^* is an SNC divisor with $A^*+K_{F_m} \sim 0$ (cf. Lemma 1.7). We consider the following two cases separately to verify the remaining assertions of Lemma 2.5.

Case. A is an elliptic curve.

Let M^* be a minimal cross-section of $\pi: F_m \to \boldsymbol{P}^1$ and let l^* be a general fiber of π. Then we have $0 \leqq (M^*, A^*)=(M^*, -K_{F_m})=(M^*, 2M^*+(m+2)l^*)=2-m$.

Hence $m \leqq 2$. On the other hand, since $(A^*, l^*)=2$, $\pi|_{A^*}: A^* \to \boldsymbol{P}^1$ is a double covering and hence it has exactly 4 ramification points. Thus, there are at most 8 connected components in BkD.

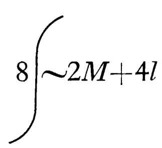
Case. A is a rational loop.

By the assumption that $V \neq F_m$, we know that Φ has a singular fiber. So, there is an irreducible component A_1^* of A^* such that $\pi|_{A_1^*}: A_1^* \to \boldsymbol{P}^1$ has a ramification point. On the other hand, we have $m \leqq 2$ as in the previous case. The case $m=2$ is excluded by virtue of Lemma 2.6 below. Thus, $m=0$ or 1, and A^* consists of a 2-section A_1^* and a fiber of Φ by the same lemma. Now, counting the number of ramification points of a double covering $\pi|_{A_1^*}: A_1^* \to \boldsymbol{P}^1$, one knows that there are at most two singular fibers in Φ and hence at most 4 connected components in BkD. Q.E.D.

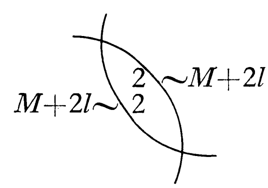
Lemma 2.6. *Let (V, D) be an Iitaka surface. Suppose that V is isomorphic to $\boldsymbol{P}^2$ or F_m. Then the configuration of D is given as follows, where if $V=F_m$ we denote by M the minimal section and by l a general fiber.*

(1) Case. $V=F_2$.

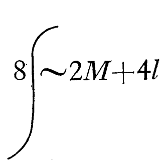
Case (a) $BkD \neq \phi$. *Then $SuppBkD=M$ and the configuration of A is one of the following*:



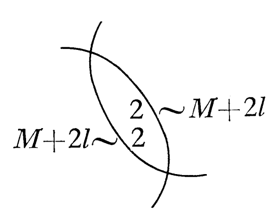
elliptic curve

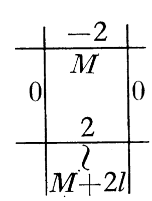


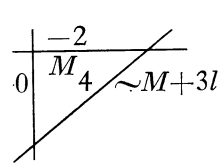
Case (b) $BkD=\phi$. *Then $D=A$ and the configuration of A is one of the following*:

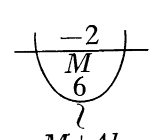


elliptic curve



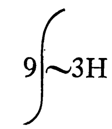




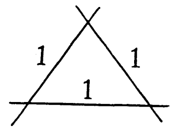


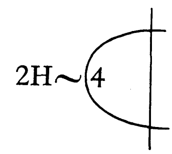
(2) Case. $V=\boldsymbol{P}^2$. *Then $BkD=\phi$ and $D=A$. The configuration of A is*

one of the following:



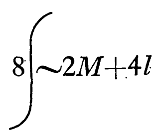
elliptic curve





where H is a line on $\boldsymbol{P}^2$.

(3) Case. $V=F_0$. *Then* $BkD=\phi$ *and* $D=A$. *The configuration of* A *is one of the following*:



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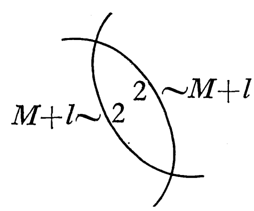
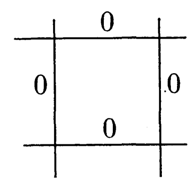
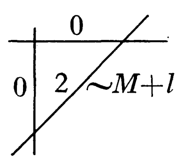
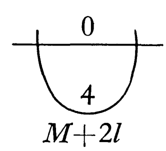
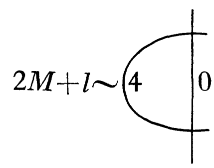


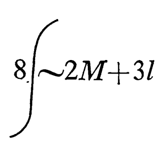




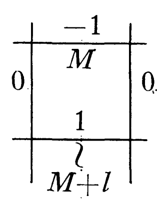


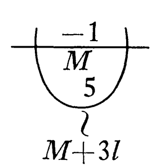


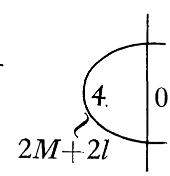
(4) Case. $V=F_1$. *Then* $BkD=\phi$ *and* $D=A$. *The configuration of* A *is one of the following*:



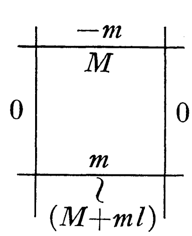
elliptic curve

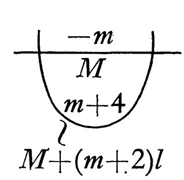






(5) Case. $V=F_m\,(m\geqq 3)$. *Then* $BkD=\phi$ *and* $D=A$. *The configuration of* A *is one of the following*:





Proof. Easy.

3. Iitaka surfaces with $\rho(\bar{V})=1$, the part (I)

In this section, we always assume that (V, D) is an Iitaka surface with $\rho(\bar{V})=1$. This case corresponds to the case where $\bar{H}\equiv 0$. We begin with

Lemma 3.1. *Let (V, D) be an Iitaka surface with $\rho(\bar{V})=1$. Then we have:*

(i) *$[A], [D_1], \cdots, [D_r]$ form a basis of $N(V)$, where $SuppBkD=\bigcup_{i=1}^r D_i$.*

(ii) *A is nef and, for any irreducible curve C on V, $(A, C)=0$ iff $C\subseteq Supp\ BkD$. In particular, every (-2) curve is contained in SuppBkD.*

(iii) *$(A^2)\geqq 1$. Hence $r=9-(A^2)\leqq 8$. Furthermore, $(A^2)\geqq 6$ if A is a rational loop.*

Proof. (i) is clear because $\rho(\bar{V})=1$. The assertion (ii) and the first part of the assertion (iii) are easy to verify. Note that $A+K_V\sim 0$, $\rho(V)+(K_V^2)=10$ and $\rho(V)=r+1$. Hence we obtain $r+(A^2)=9$. Suppose that $A=A_1+\cdots+A_t$ is a rational loop, where A_i is irreducible. We know that A is an SNC divisor, whence $t\geqq 2$. Since $A\cap \text{Supp}BkD=\phi$ and since every irreducible curve on $\bar{V}$ is ample, we know that $(A_i^2)\geqq 1$ for every i. Hence $(A^2)=\sum_{i=1}^t(A_i^2)+2\sum_{i<j}(A_i, A_j)\geqq 2+2\times 2=6$. Q.E.D.

Lemma 3.2. *Under the same hypothesis as in Lemma 3.1, the following assertions hold:*

Every (-1) curve E on V meets BkD. It is imopssible that E meets BkD in a single point on a tip D_1 of a rod R, which is a connected component of BkD.

Proof. Suppose that $E\cap \text{Supp}BkD=\phi$. Let $u_1\colon V\to V_1$ be the contraction of E. Write $\text{Supp}BkD=\bigcup_{i=1}^r D_i$. By virtue of Lemma 3.1, we have $\rho(V)=r+1$, whence $\rho(V_1)=r$. So, there exists $(a, b_1, \cdots, b_r)\in \boldsymbol{R}^{r+1}-(0, \cdots, 0)$ such that $au_{1*}A+\sum_{i=1}^r b_i u_{1*}D_i\equiv 0$. Since $E\cap \text{Supp}BkD=\phi$, we have $u_{1*}A\cap u_{1*}\ \text{Supp}BkD=\phi$ and hence $0=(u_{1*}A, au_{1*}A+\sum_{i=1}^r b_i u_{1*}D_i)=a(u_{1*}A^2)=a((A^2)+1)$. Then we obtain $a=0$ because $(A^2)+1\geqq 2$ by virtue of Lemma 3.1. Since $\sum_{i=1}^r u_{1*}D_i$ is obviously negative-definite, we must have $b_1=\cdots=b_r=0$, which is a contradiction. Hence E meets BkD.

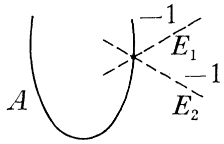
Suppose that $(E, D_1+\cdots+D_r)=(E, R)=(E, D_1)=1$, where D_1 is a tip of a rod R which is a connected component of BkD. We may write $R=D_1+\cdots+D_s$. Let $\sigma\colon V\to W$ be the contraction of $E+R$. Since σ_*BkD is contractible to points, we have $\rho(W)\geqq \#\{\text{irreducible components of } \sigma_*BkD\}+1=r-s+1$, while $\rho(W)=\rho(V)-(s+1)=(r+1)-(s+1)=r-s$. This is a contradiction. Q.E.D.

The following result guarantees the existence of a suitable $\boldsymbol{P}^1$-fibration in the present case.

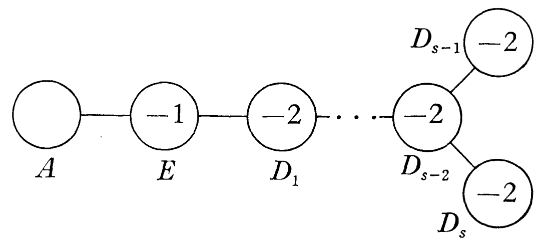
Lemma 3.3. *Let (V, D) be an Iitaka surface with $\rho(\bar{V})=1$. Assume that V*

is not isomorphic to $\boldsymbol{P}^2$ *or* F_m. *Then there exists a* $\boldsymbol{P}^1$*-fibration* $\Phi: V\to\boldsymbol{P}^1$ *satisfying the following conditions*:

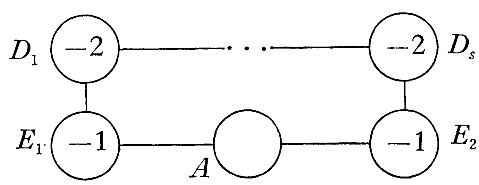
(1) *The configuration of any singular fiber* f_1 *of* Φ *is one of the following*:



$f_1=E_1+E_2$



$f_2=2(E+D_1+\cdots+D_{s-2})+D_{s-1}+D_s$ $(s\geqq 2)$



$f_1=E_1+E_2+D_1+\cdots+D_s$ $(s\geqq 0)$

where $\bigcup_{i=1}^{s}D_i\subseteq \mathrm{Supp}BkD$, *and the integer in a circle is the self-intersection of the corresponding curve.*

(2) *Let* u *be the contraction of all exceptional curves of the first kind in the fibers of* Φ. *Then* $u(V)$ *is a minimally ruled rational surface* F_m *with* $m\leqq 2$, *and we have* $u_*A+K_{F_m}\sim 0$ *and* $\#\{$*irreducible components of* $A\}=\#\{$*irreducible components of* $u_*A\}\leqq 4$.

Proof. Since $V\neq\boldsymbol{P}^2$ or F_m by the hypothesis, there exists a birational morphism $u(\neq id)$ from V to a minimally ruled rational surface F_m. Let $A^*=u_*A$ and $\Phi=\pi\circ u$, where $\pi: F_m\to\boldsymbol{P}^1$ is the $\boldsymbol{P}^1$-fibration on F_m. Let M^* be the minimal section on F_m and let l^* be a general fiber of π. We shall show $m\leqq 2$. Since $A^*+K_{F_m}\sim 0$ (cf. Lemma 1.7), we have $(M^*, A^*)=(M^*, 2M^*+(m+2)l^*)=2-m$. If A is an elliptic curve. we have $(M^*, A^*)\geqq 0$, whence $m\leqq 2$. Consider the case where A is a rational loop. Suppose that $m\geqq 3$. Then $(M^*, A^*)=2-m<0$, which implies that $M^*\leqq A^*$ and $u'M^*\leqq A$. On the other hand, $(u'M^{*2})\leqq(M^{*2})=-m\leqq -3$. This is impossible because $\rho(\bar{V})=1$ and $A\cap$ Supp $BkD=\phi$ (cf. Lemma 3.1). Hence $m\leqq 2$.

Let f_1 be a singular fiber of Φ. Since every irreducible component of f_1 has negative self-intersection and since every component of A has positive self-intersection, A and Supp f_1 have no common components. Hence f_1 consists of (-1) curves and (-2) curves by virtue of Lemma 1.8. We also have $\#\{$irreducible components of $A\}=\#\{$irreducible components of $u_*A\}$ because u does not contract any components in A by the above argument. Since $(A, f_1)=(-K_V, f_1)=2$, one of the following two cases takes place:

Case (i) $f_1=2E+B$ for a (-1) curve E and a divisor B whose irreducible components are all (-2) curves which are hence contained in SuppBkD by Lemma 3.1. As in the proof of Lemma 2.5, we can show $f_1=2(E+B_1+\cdots+B_{s-2})+B_{s-1}+B_s$ for some irreducible components $B_1, \cdots, B_s$ $(s\geqq 2)$ of BkD and that the configuration of f_1 is the second picture given in the statement of this lemma, where $D_i:=B_i$.

Case (ii) $f_1=E_1+E_2+B$ for two distinct (-1) curves E_1 and E_2, and a divisor B (which might be empty) whose irreducible components are (-2) curves contained in SuppBkD. By virtue of [7; Chap. II, Lemma 2.2], we easily see that only possible cases for the configuration of f_1 are those two given in the first picture and the third picture displayed in the statement of this lemma.

Thus, we completed the proof of Lemma 3.4.

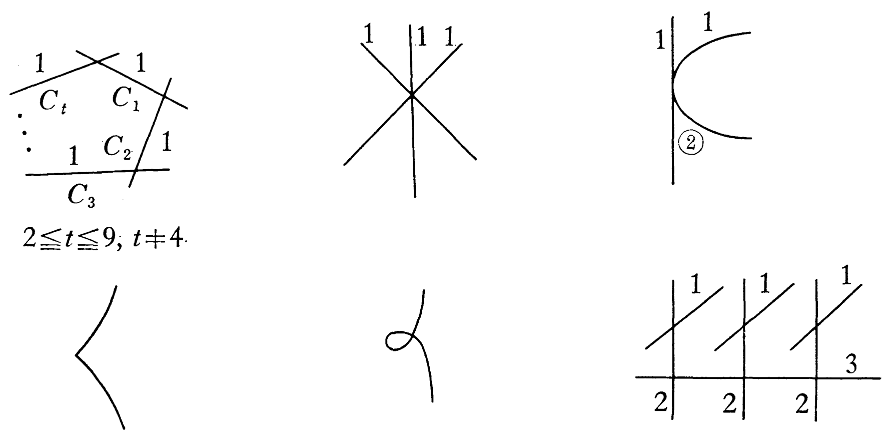
The following lemma is crucial.

Lemma 3.4. *Let (V, D) be an Iitaka surface with $\rho(\bar{V})=1$. Suppose that V is not isomorphic to $\boldsymbol{P}^2$ or F_m and that it satisfies the condition*:

(*) *For any irreducible component D_1 of BkD, there is no pair of an extremal rational curve $\bar{l}$ and a nef divisor $\bar{H}$ on $\bar{V}_1$ such that $\bar{H}^\perp\cap\overline{NE}(\bar{V}_1)=\boldsymbol{R}_+[\bar{l}]$, $\bar{H}\not\equiv 0$ and $(\bar{H}^2)=0$, where $g: V\to\bar{V}_1$ is the contraction of $Bk(D-D_1)$.*

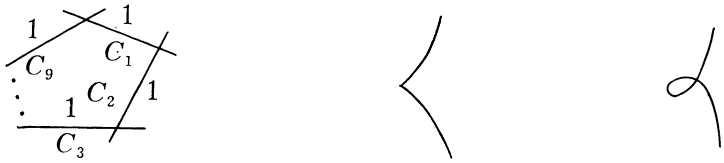
Then $(A^2)=1, 2$ or 3 and hence A is an elliptic curve. There exists a birational morphism $\sigma: V'\to V$ obtained by blowing up (A^2) points on A such that $\Phi_{|\sigma' A|}$ gives us an elliptic fibration from V' to $\boldsymbol{P}^1$ such that each of singular fibers is not a multiple fiber and has one of the following configurations.

Case. $(A^2)=1$.



where each nonsingular component is a (-2) *curve and the attached number indicates the multiplicity of the corresponding component in the fiber.*

Case. $(A^2)=2$ or 3.



Proof. Let (V, D) be an Iitaka surface satisfying the conditions stated in Lemma 3.4. We prove Lemma 3.4 in three steps.

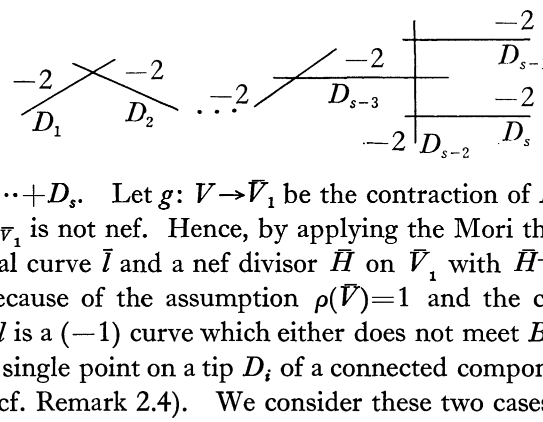
Step 1. We shall verify the following:

Claim 1. SuppBkD contains no forks of type D_s, E_7 or E_8.

Proof. Suppose, on the contrary, that there is a connected component F of BkD, which is a fork.

Case (i) F is a fork of type D_s.

Let D_1 be a tip of F as shown below:



where $F=D_1+\cdots+D_s$. Let g: $V\to\bar{V}_1$ be the contraction of $Bk(D-D_1)$. Since $K_{\bar{V}_1}\sim-g_*A$, $K_{\bar{V}_1}$ is not nef. Hence, by applying the Mori theory, we obtain an extremal ratiomal curve $\bar{l}$ and a nef divisor $\bar{H}$ on $\bar{V}_1$ with $\bar{H}^{\perp}\cap\overline{NE}(\bar{V}_1)=\boldsymbol{R}_+[\bar{l}]$ and $(\bar{H}^2)>0$, because of the assumption $\rho(\bar{V})=1$ and the condition $(*)$. Let $l=g'(\bar{l})$. Then l is a (-1) curve which either does not meet $Bk(D-D_1)$ or meets $Bk(D-D_1)$ in a single point on a tip D_i of a connected component of $Bk(D-D_1)$, which is a rod (cf. Remark 2.4). We consider these two cases separately.

Case (i-A) l meets $Bk(D-D_1)$.

Case (i-A-a) $l\cap D_1=\phi$.

By virtue of Remark 2.4 and Lemma 3.2, l must meet $F-D_1$, and $F-D_1$ is a rod, i.e., F is a fork of type D_4. We see easily that $D_i=D_3$ or D_4, say $D_i=D_3$. Then $|2(l+D_3+D_2)+D_1+D_4|$ gives us a $\boldsymbol{P}^1$-fibration from V to $\boldsymbol{P}^1$. Note that BkD is contained in the fibers of $\Phi_{|2(l+D_3+D_2)+D_1+D_4|}$. So $\rho(\bar{V})\geqq 2$, which is a contradiction.

Case (i-A-b) $l\cap D_1\neq\phi$. We consider first the following:

Case. D_i is a component of F. We shall verify the next

Claim. $(l, D_1)=(A^2)=1$ and $A\sim l+D_1+\cdots+D_i$, where $D_1+\cdots+D_i\leqq F$.

Proof. After a suitable change of indices $\{1, \cdots, s\}$, we may assume that

$D_1+\cdots+D_i$ is a rod containing D_1 and D_i. Let $x:=(A^2)\geqq 1$ and $L_n:=xl-A+n(D_1+\cdots+D_i)$. Since $(L_n, A)=0$ the Hodge index theorem implies:

$$0\geqq(L_n^2)\geqq -x^2+x-2in^2+2(-x+2nx+n^2(i-1))\,.$$

Note that the second inequality is an equality iff $(l,D_1)=1$, because we know that $(l, D_i)=1$. So $x^2-(4n-1)x+2n^2\geqq 0$ and, $x^2-(4n-1)x+2n^2=0$ iff $L_n\equiv 0$ and $(l, D_1)=1$. The last condition is equivalent to $x=n=1$. Indeed, if $L_n\equiv 0$ and $(l, D_1)=1$ hold, then $(L_n, l)=(L_n, D_1)=0$. So we obtain $-x-1+2n=x-2n+n=0$, i.e., $x=n=1$. Conversely, if $x=n=1$ then $x^2-(4n-1)x+2n^2=0$, whence $L_n\equiv 0$ and $(l, D_1)=1$. We show that $x=1$ always. In fact, if we set $n=2$, we must have $x^2-7x+8>0$, whence $x\neq 2$. If we set $n=4$, we must have $(x-8+\frac{1}{2})^2-(24+\frac{1}{4})>0$. Hence the only possible value for x is 1 because $1\leqq(A^2)=x\leqq 8$ and $x\neq 2$. So, we see $L_1\equiv 0$. Since V is rational, we have $L_1\sim 0$, i.e., $A\sim l+D_1+\cdots+D_i$. Thus the claim is verified.

However, this is impossible because $A\sim l+D_1+\cdots+D_i$ implies that $D_1+\cdots+D_i$ is a connected component of BkD containing D_1, while the connected component $F(\geqq D_1)$ of BkD is a fork.

Now we consider the next:

Case. D_i is not a component of F. Then $(l, D_1)=1$, cf. the case (i-B-b) below. We shall show that this contradicts the condition (*) where we take D_i as D_1 in the condition (*). Let $h: V\to\bar{V}_2$ be the contraction of $Bk(D-D_i)$. We obtain $\rho(\bar{V}_2)=2$ since $\rho(\bar{V})=1$. Let $\Phi=\Phi_{|2(l+D_1+\cdots+D_{s-2})+D_{s-1}+D_s|}: V\to \boldsymbol{P}^1$. There exists clearly a morphism $\psi: \bar{V}_2\to\boldsymbol{P}^1$ such that $\Phi=\psi\circ h$. Let $\bar{H}=2h_*l$, which is a fiber of ψ. Then $N(\bar{V}_2)=\boldsymbol{R}[h_*D_i]+\boldsymbol{R}[\bar{H}]$. Since $(h_*l, K_{\bar{V}_2})=-(h_*l, h_*A)=-1$, h_*l is an extremal rational curve. We easily see that $\bar{H}^\perp\cap\overline{NE}(\bar{V}_2)=\boldsymbol{R}_+[h_*l]$, $\bar{H}\not\equiv 0$ and $(\bar{H}^2)=0$. This contradicts the condition (*).

Case (i-B) l does not meet Supp $Bk(D-D_1)$. Then $(l, D_1)\geqq 1$ by virtue of Lemma 3.2.

Case (i-B-a) $(l, D_1)=1$.

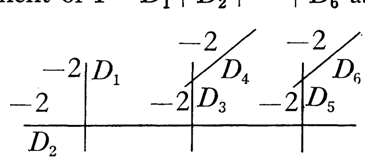
This leads to a contradiction as in the case (i-A-a).

Case (i-B-b) $(l, D_1)\geqq 2$.

Let $L_n=(A^2)\,l-A+nD_1$ as in the case (i-A-b). Then we see that $(A^2)=1$, $(l, D_1)=2$ and $A\sim l+D_1$. Hence D_1 is an isolated component of BkD, which is a contradiction. Therefore, we have proven that the case (i) does not occur.

Case (ii) F is a fork of type E_6.

Let D_1 be a component of $F=D_1+D_2+\cdots+D_6$ as shown below:



As in the case (i), we apply the Mori theory to the surface $\bar{V}_1$ obtained from V by contracting $Bk(D-D_1)$, which is a rod.

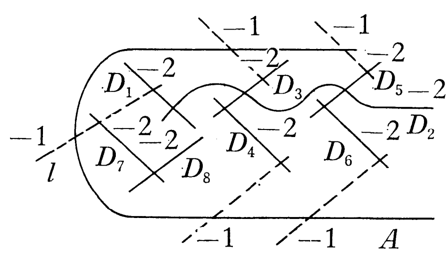
Case (ii-A) l meets $Bk(D-D_1)$.

Case (ii-A-a) $l\cap D_1=\phi$.

Then, by virtue of Remark 2.4 and Lemma 3.2, l meets $F-D_1$ in a single point on a tip D_i of a rod $F-D_1$. Thence $D_i=D_4$ or D_6, say $D_i=D_4$. As in the case (i-A-b), we can show that this contradicts the condition(*), where we take D_6 as D_1 in the condition(*).

Case (ii-A-b) $l\cap D_1\neq\phi$.

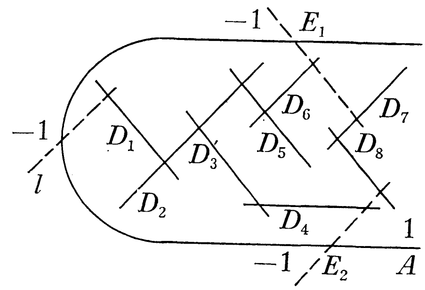
If D_i is a component of F, we would get a contradiction as in the case (i-A-b). So, we assume that D_1 and D_i are contained in distinct connected components of BkD. We may assume $i=7$. Let $R=D_7+\cdots+D_{7+t}$ be a connected component of BkD, which is a rod. If $t=0$, we would obtain a contradiction to the condition(*), where we take D_2 as D_1 in the condition(*). So, we assume that $t\geqq 1$. Then we have $\mathrm{Supp}BkD=\cup_{i=1}^{8}D_i$ by virtue of Lemma 3.1, (iii), and $R=D_7+D_8$. Let $\Phi=\Phi_{|2l+D_1+D_7|}: V\rightarrow \boldsymbol{P}^1$. We see easily that Φ is a $\boldsymbol{P}^1$-fibration, and that the singular fiber of Φ containing $D_3\cup D_4$ (or $D_5\cup D_6$, resp.) is given as follows (cf. Lemma 3.3):



Then we have $\rho(V)\geqq 10$, which is a contradiction because $\rho(V)=r+1=9$ (cf. Lemma 3.1).

Case (ii-B) l does not meet $Bk(D-D_1)$.

Then $(l, D_1)\geqq 1$ by Lemma 3.2. It is impossible that $(l, D_1)\geqq 2$ (cf. the case (i-B-b)). So, $(l, D_1)=1$. Let σ: $V'\rightarrow V$ be the blowing-up of the point $l\cap A$. Let $A'=\sigma'A$, $l'=\sigma'l$, $E=\sigma^{-1}(l\cap A)$ and $D_i'=\sigma'D_i$ for $i=1,\cdots,r$, where $\mathrm{Supp}BkD=\cup_{i=1}^{r}D_i$. Set $\Delta=3D_2'+2(D_1'+D_3'+D_5')+l'+D_4'+D_6'$. It is easy to see that $(\Delta^2)=(A', \Delta)=0$. We know that $(A'^2)\geqq 0$ by Lemma 3.1. Hence $(A'^2)=0$ by the Hodge index theorem. It is easy to check that $(A'-\Delta, E)=(A'-\Delta, A')=(A'-\Delta, D_i')=0$. So $A'\equiv\Delta$ because $E, A', D_1', \cdots, D_r'$ form a basis of $N(V')$ (cf. Lemma 3.1), whence $A'\sim\Delta$ because V' is rational. Therefore, we obtain an elliptic fibration $\Phi_{|A'|}: V'\rightarrow\boldsymbol{P}^1$. Since $(E, A')=1$, E is a cross-section, and any fiber of $\Phi_{|A'|}$ is not a multiple fiber. We also have $(A^2)=1$. Thence $r=9-(A^2)=8$, and A is an elliptic curve (cf. Lemma 3.1, (iii)). Consider a $\boldsymbol{P}^1$-fibration on V defined by $|(2(l+D_1+D_2)+D_3+D_5|$. By counting $\rho(V)$ $(=r+1=9)$, we can present the configuration of singular fibers of $\Phi_{|2(l+D_1+D_2)+D_3+D_5|}$ as follows:

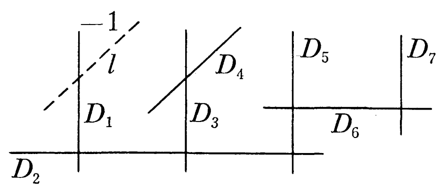


By virtue of Lemma 3.2, D_4 and D_6 are cross-sections meeting the singular fibers of $\Phi_{|2(l+D_1+D_2)+D_3+D_5|}$ as shown above.

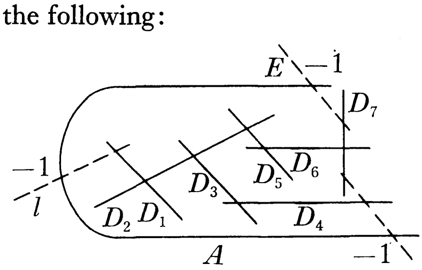
So, in this case, BkD consists of a fork of type E_6 and a rod with two irreducible components. Note that there is a (-1) curve l' on V meeting BkD only in the components D_7 and D_8. Indeed, consider a possible configuration of the singular fiber of $\Phi_{|A'|}$ containing D_7' and D_8'.

Case (iii). F is a fork of type E_7.

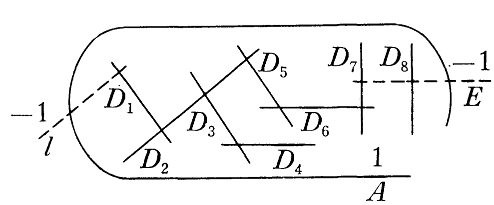
From the proof for the case (ii), it suffices to consider the following case: There exists a (-1) curve l such that $(l, D_1+\cdots+D_r)=(l, D_1)=1$, where $F=D_1+\cdots+D_7$, D_1 is a tip of F as shown below and Supp$BkD=\bigcup_{i=1}^{7}D_i$.



Consider a $\boldsymbol{P}^1$-fibration $\Phi:=\Phi_{|2(l+D_1+D_2)+D_3+D_5|}: V\to\boldsymbol{P}^1$. By virtue of Lemma 3.3, the configuration of singular fibers of Φ containing components of BkD is one of the following:



Supp$BkD\supseteq\bigcup_{i=1}^{7}D_i$,



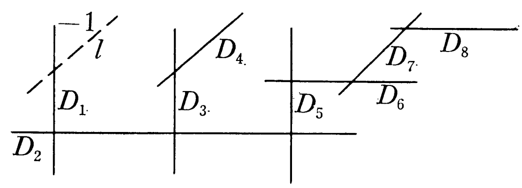
Supp$BkD=\bigcup_{i=1}^{8}D_i$.

The second csae leads to a contradiction, because $(D_4, 2E+D_7+D_8)=1$ implies $(D_4, D_7)=1$ or $(D_4, D_8)=1$, while D_8 is an isolated component of BkD. In the first case, we obtain a contradiction to the condition(*), where we take D_4 as D_1. Thus, the case (iii) does not occur.

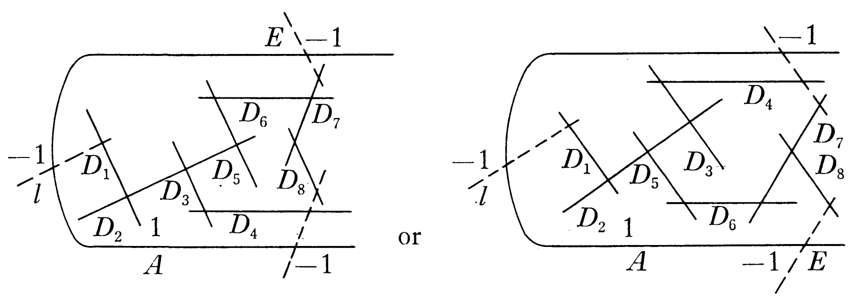
Case (iv) F is a fork of type E_8.

From the discussions in the case (ii), we know that it suffices to consider the case:

There exists a (-1) curve l such that $(l, D_1+\cdots+D_8)=(l, D_1)=1$, where $\mathrm{Supp}BkD=\mathrm{Supp}\ F=\cup_{i=1}^{8}D_i$ and D_1 is a tip of F as shown below:



Consider a P^1-fibration $\Phi:=\Phi_{|2(l+D_1+D_2)+D_3+D_5|}: V\to P^1$. By virtue of Lemma 3.3, the configuration of singular fibers is presented as follows:



If the first case (or the second case, resp.) takes place, we get a contradiction to the condition (*) where we take D_5 (or D_4, resp.) as D_1. So, the case (iv) does not occur.

Thus, we have verified the claim 1. Q.E.D.

Step 2. Our next claim is the following:

Claim 2. *BkD* contains no connected components consisting of three irreducible components.

Proof. Suppose that $R=D_1+D_2+D_3$ is a connected component of BkD. We assume that D_2 is the middle component of R. As in the step 1, there is a (-1) curve l such that either l does not meet $Bk(D-D_2)$ or l meets $Bk(D-D_2)$ in a single point on a tip D_i of a rod R_1, which is a connected component of $Bk(D-D_2)$. We consider these two cases separately.

Case (A) l meets $Bk(D-D_2)$.

Case (A-a) $l\cap D_2=\phi$.

By virtue of Lemma 3.2, R_1 is a part of R, whence $R_1=D_1$ or D_3. This is a contradiction (cf. Lemma 3.2).

Case (A-b) $l\cap D_2\neq\phi$.

If R_1 is a part of R, D_2 is a tip of R; see the proof for the claim 1, the case (i-A-b). This is a contradiction. So R_1 is not a part of R, whence $R_1 \cap R = \phi$. We also have $(D_2, l)=1$; see the case (B) below. Thus, we reach to a contradiction to the condition(*) where we take D_i as D_1.

Case (B) l does not meet $Bk(D-D_2)$.

Then $(l, D_2) \geqq 1$ by virtue of Lemma 3.2. If $(l, D_2)=1$, we reach to a contradiction as in the claim 1, the case (i-A-a). If $(l, D_2) \geqq 2$, one can show, by the arguments in the case (i-B-b) of the claim 1, that D_2 is an isolated component of BkD, which is a contradiction. Q.E.D.

Step 3. By virtue of the claim 1 and the case (ii), we may assume that BkD contains no forks. We know that $r := \#\{\text{irreducible components of } BkD\} = \rho(V)-1 \geqq 2$ by the hypothesis that V is not isomorphic to $\boldsymbol{P}^2$ or F_m. So, suppose that R is a rod which is a connected component of BkD. Let D_1 be a tip of R. As in the proof of the claim 1, there exists a (-1) curve l on V such that either l does not meet $Bk(D-D_1)$ or l meets $Bk(D-D_1)$ in a single point on a tip D_i of a connected component R_1 of $Bk(D-D_1)$, which is a rod. We consider these two cases separately.

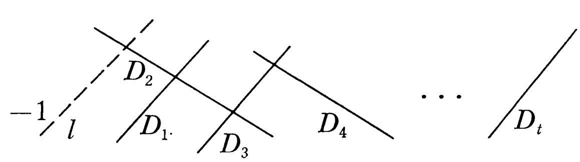
Case (B) l does not meet $Bk(D-D_1)$.

By virtue of Lemma 3.2, we have $(l, D_1) \geqq 2$. We can show that $(l, D_1) = 2$, $A \sim l + D_1$, $(A^2)=1$ (whence A is an elliptic curve by Lemma 3.1) and D_1 is an isolated component of BkD; see the proof for the claim 1, the case (i-B-b). Let $\sigma: V' \to V$ be the blowing-up of the point $l \cap A$. Then $\Phi_{|\sigma' A|}: V' \to \boldsymbol{P}^1$ is an elliptic fibration whose singular fiber is not a multiple fiber, and has one of the configurations listed in the statement of Lemma 3.4.

Case (A) l meets $Bk(D-D_1)$.

Case (A-a) $l \cap D_1 = \phi$.

By virtue of Lemma 3.2, R_1 is a part of R and l meets $R = D_1 + \cdots + D_t$ as shown below, where $t \geqq 3$.



The case $t=3$ leads to a contradiction as in the claim 1, the case (i-A-a). If $t \geqq 4$, we reach to a contradiction to the condition(*) where we take D_4 as D_1.

Case (A-b) $l \cap D_1 \neq \phi$.

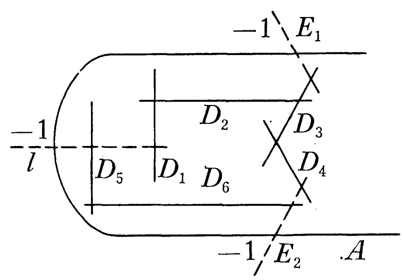
If D_i is a component of R, then D_i is a tip of R and, $(A^2)=1$ (whence A is an elliptic curve by Lemma 3.1), $(l, R)=2$ and $A \sim l+R$, cf. the claim 1, the case (i-A-b). Let $\sigma: V' \to V$ be the blowing-up of the point $l \cap A$. Then $\Phi_{|\sigma' A|}$ is an elliptic fibration whose singular fiber is not a multiple fiber, and has one of the configurations listed in the statement of Lemma 3.4.

Now we consider the case where D_i is not a component of R. We may assume that $R=D_1+\cdots+D_{i-1}$ and $R_1=D_i+\cdots+D_{i+t}$ $(t\geqq 0)$. By the assumption that $\rho(\bar{V})=1$ and the condition(*), we see $i\geqq 3$ and $t\geqq 1$. We consider the following cases separately. Namely, Case(α), where R or R_1, say R, consists of more than two components. Hence, by virtue of the claim 2, $i\geqq 5$. Then, Case(β), where R and R_1 consists of two irreducible components, i.e., $i=3$ and $t=1$.

We consider first:

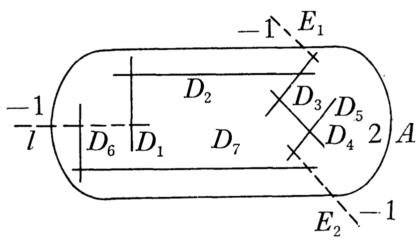
Case (α) Note that $t\neq 2$ by virtue of the claim 2. We know that $r\geqq 6$ and $r\leqq 8$ (cf. Lemma 3.1). We exhibit the configuration of singular fibers of $\Phi_{|2l+D_1+D_i|}$: $V\to\boldsymbol{P}^1$ as follows:

Case $i=5$. Note that D_6 meets E_2 and does not meet E_1 by virtue of Lemma 3.2; see the picture below.

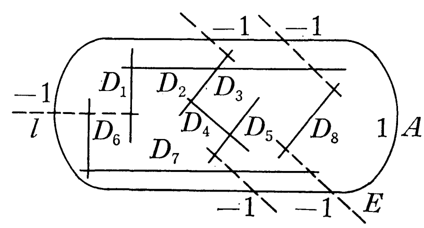


This contradicts the condition(*) where we contract $Bk(D-D_1)$. In fact, look at a $\boldsymbol{P}^1$-fibration defined by $|2(E_1+D_3)+D_2+D_4|$.

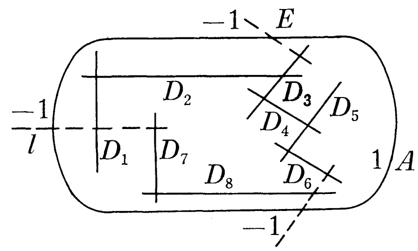
Case i$\geqq 6$, whence $r\geqq 7$. This case splits to the following three subcases; see the pictures below. In each of these three cases, A is an elliptic curve (cf. Lemma 3.1, (iii)).



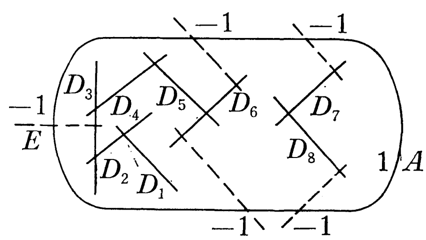
(1) $r=7$ and $(A^2)=9-r=2$.



(2) $r=8$, $i=6$ and $\rho(V)=r+1=9$.



(3) $r=8$, $i=7$ and $\rho(V)=r+1=9$.



(3)′

In the second case, we have $\rho(V)\geqq 10$, which is a contradiction. In the third case, we consider a $\boldsymbol{P}^1$-fibration $\Phi_{|2(E+D_3)+D_2+D_4|}$: $V\to\boldsymbol{P}^1$, instead of $\Phi_{|2l+D_1+D_7|}$. We present the configuration of singular fibers of $\Phi_{|2(E+D_3)+D_2+D_4|}$ in the picture (3)′ above. Then we reach to a contradiction as in the second case above. We now consider the first case. Let σ: $V'\to V$ be the blowing-ups of the points $l\cap A$ and $E_2\cap A$. It is easy to see that $\sigma'A\sim\sigma'(l+E_2+D_1+\cdots+D_7)$. Then $\Phi_{|\sigma'A|}$: $V'\to\boldsymbol{P}^1$ is an elliptic fibration whose singular fiber is not a multiple fiber because $\sigma^{-1}(l\cap A)$ is a cross-section of $\Phi_{|\sigma'A|}$, and has one of the configurations listed in the statement of Lemma 3.4.

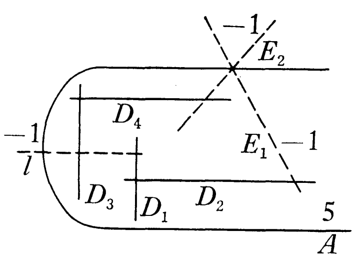
We consider next

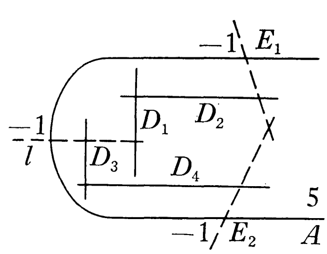
Case (β) By the discussions above, we may assume that every connected component of BkD consists of two components. In particular, we have:

$r:=\#\{\text{irreducible components of } BkD\}=2k$

for some $k\geqq 1$. Since we are in the case (β), we have $k\geqq 2$. By Lemma 3.1, we know that $k\leqq 4$. We shall show below that only the case $k=3$ takes place. We shall check all cases, one by one.

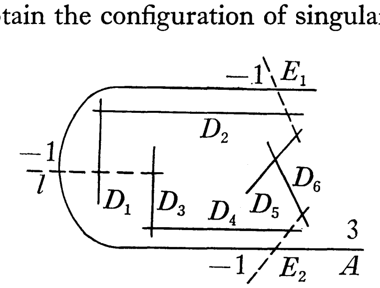
Case $k=2$. Then $\rho(V)=r+1=2k+1=5$. Let us consider a $\boldsymbol{P}^1$-fibration $\Phi:=\Phi_{|2l+D_1+D_3|}$: $V\to\boldsymbol{P}^1$. Computing $\rho(V)$ by counting the number of irreducible components in singular fibers of Φ, we see that there exists a singular fiber E_1+E_2 of Φ with two distinct (-1) curves, whose configuration is one of the following:





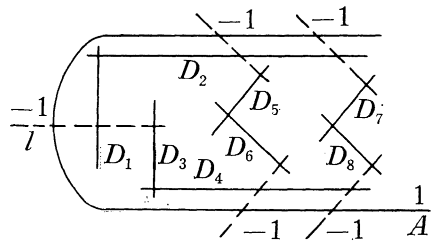
Both cases lead to a contradiction to Lemma 3.2. Therefore $k\neq 2$.

Case $k=3$. Then $(A^2)=9-r=9-2k=3$ and A is an elliptic curve (cf. Lemma 3.1). As in the case $k=2$, we consider the $\boldsymbol{P}^1$-fibration $\Phi:=\Phi_{|2l+D_1+D_3|}$: $V\to\boldsymbol{P}^1$. Note that $Bk(D-D_2-D_4)$ is contained in the singular fibers of Φ. By computing $\rho(V)$, we obtain the configuration of singular fibers of Φ.



By virtue of Lemma 3.2, D_2 and D_4 meet the singular fibers of Φ as shown above. Let σ: $V'\to V$ be the blowing-ups of the points $A\cap l$, $A\cap E_1$ and $A\cap E_2$. Then we see that $\sigma' A\sim\sigma'(l+E_1+E_2+D_1+\cdots+D_6)$ and, $\Phi_{|\sigma' A|}$: $V'\to \boldsymbol{P}^1$ is an elliptic fibration whose singular fiber is not a multiple fiber because $\sigma^{-1}(A\cap l)$ is a cross-section of $\Phi_{|\sigma' A|}$, and has one of the configurations listed in the statement of Lemma 3.4.

Case $k=4$. Then $\rho(V)=2k+1=9$. Since $Bk(D-D_2-D_4)$ is contained in the singular fibers of $\Phi:=\Phi_{|2l+D_1+D_3|}$, the singular fiber of Φ containing $D_5\cup D_6$ (or $D_7\cup D_8$, resp.) is the following:



By counting the number of irreducible components in the singular fibers of Φ, we see $\rho(V)\geqq 10$, which is a contradiction.

This completes the proof of Lemma 3.4.

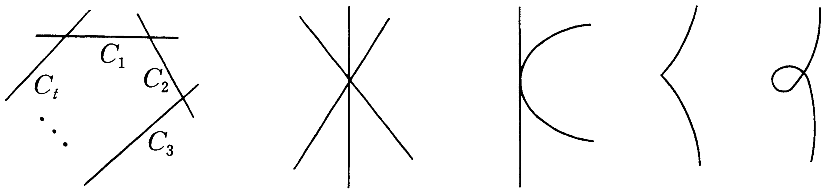
We can specify the configuration of BkD. Namely, we have:

Lemma 3.5. *Let the notations and assumptions be the same as in Lemma 3.4. Then all possibilities for BkD are exhausted by the following eight cases*:

(i) A_1+A_7, (ii) $2A_4$, (iii) $3A_2$, (iv) A_2+A_5, (v) A_8, (vi) A_2+E_6, (vii) $A_1+A_2+A_5$, (viii) $4A_2$.

There exists a birational morphism u: $V\to F_2$ *such that the configuration of* u_*D *corresponding to the case* (i) (*the case* (ii); *the case* (v); *the case* (vii); *the case* (iii), (iv) *and* (vi); *or the case* (viii); *resp.*) *is given in Fig.* 1 (*Fig.* 2; *Fig.* 3; *Fig.* 4; *Fig.* 5; *or Fig.* 6; *resp.*) *in the statement of Main Theorem, in which* $A^*=u_*A$ *is an elliptic curve. All these cases are realizable.*

Proof. By virtue of Lemma 3.4, the case where $(A^2)=2$ (or $(A^2)=3$, resp.) corresponds to the above case (iv) (or (iii), resp.) and the only possible case where BkD contains a fork is the above case (vi). We now assume that $(A^2)=1$ and that BkD contains no forks. Let $k=\#\{$connected components of $BkD\}$. We shall show $k\leqq 4$ by computing the Euler number $\chi(V')$. By Lemma 3.4, all possible singular fibers of $\Phi_{|\sigma' A|}$ are the following:



$2\leqq t\leqq 9,\ t\neq 4$ and $\chi=t$, $\chi=4$, $\chi=3$, $\chi=2$, $\chi=1$.

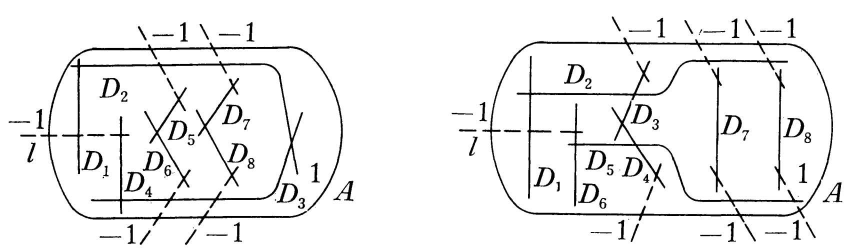
For the computation of χ, we refer to [11; Chap. IV, Lemma 4]. Note that a reducible singular fiber can occur in the cases (B) and (A-b) in the step (3) of the proof of Lemma 3.4. Hence if a singular fiber has s irreducible components with $s\geqq 2$ then $(s-1)$ of them are components of $\sigma'(D_1+\cdots+D_8)$, where Supp $BkD=\bigcup_{i=1}^{8}D_i$ (cf. Lemma 3.1, (iii)). Let $R_1, \cdots, R_k$ be all the connected components of BkD with $|R_1|\geqq\cdots\geqq|R_k|$, where by $|\Delta|$ we mean the number of irreducible components in an effective divisor Δ. Let $G_i (i=1,\cdots,k)$ be the singular fiber of $\Phi_{|\sigma'A|}$ containing $\sigma'R_i$. Then $\chi(G_i)\geqq|G_i|$ by the computation above. From the Noether formula and from [Sh 1; Chap. IV, Th. 6] we obtain the following inequality:

$$\text{(1)}\quad 12=12\chi(\mathcal{O}_{V'})-(K_{V'}^2)=\chi(V')=\sum_{f'}\chi(f')\geqq\sum_{i=1}^{k}\chi(G_i)$$
$$\geqq\sum_{i=1}^{k}|G_i|=\sum_{i=1}^{k}(|R_i|+1)=8+k\,.$$

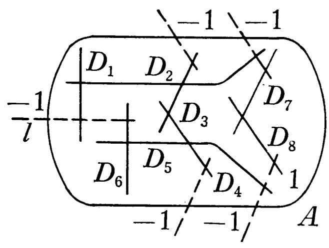
where f' ranges over all singular fibers of $\Phi_{|\sigma'A|}$. Hence we have $k\leqq 4$. The all possibilities for $(|R_1|, \cdots, |R_k|)$ are exhausted by the following cases (cf. the claim 2 in Lemma 3.4):

(8), (7,1), (6,2), (4,4), (6,1,1), (5,2,1), (4,2,2), (5,1,1,1), (4,2,1,1), (2,2,2,2).

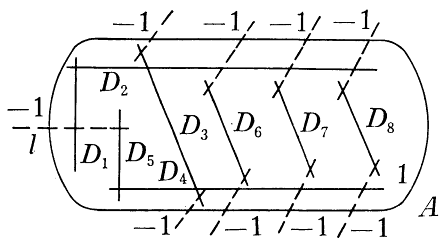
We shall verify that $(|R_1|, \cdots, |R_k|)\neq$(6,2), (6,1,1), (4,2,2), (5,1,1,1) or (4,2,1,1). Suppose, on the contrary, that one of the above five cases occurs. By the proof for the cases (B) and (A-b) in Lemma 3.4, we find a (-1) curve l such that $(l, D_1+\cdots+D_8)=(l, D_1+\cdots+D_s)=(l, D_1+D_s)=2(l, D_1)=2$, where D_1 and D_s are tips of a rod $R_1=D_1+\cdots+D_s$. We consider a $\boldsymbol{P}^1$-fibration $\Phi:=\Phi_{|2l+D_1+D_s|}\colon V\to \boldsymbol{P}^1$. All irreducible components of BkD, except two, say D_2 and D_{s-1}, are contained in the singular fibers of Φ. Note that D_2 and D_{s-1} are cross-sections. By virtue of Lemma 3.3, we obtain the following configuration of the singular fibers of Φ:



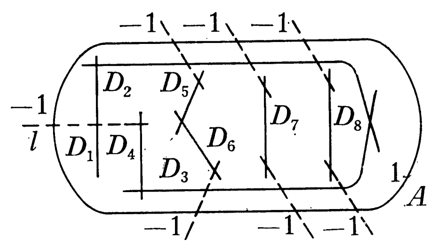
$(|R_1|,\cdots,|R_k|)=(6,2)$ and $\rho(V)\geqq 10$ $(|R|_1,\cdots,|R_k|)=(6,1,1)$ and $\rho(V)\geqq 11$



$(|R_1|, \cdots, |R_k|)=(4,2,2)$ and $\rho(V) \geqq 10$



$(|R_1|, \cdots, |R_k|)=(5,1,1,1)$ and $\rho(V) \geqq 12$



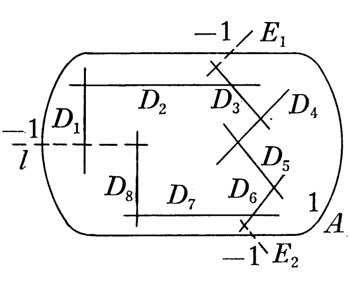
$(|R_1|, \cdots, |R_k|)=(4,2,1,1)$ and $\rho(V) \geqq 11$

where the inequality about $\rho(V)$ is obtained by counting the number of irreducible components in the singular fibers. Therefore we reach to a contradiction because $\rho(V)=9$ (cf. Lemma 3.1, (iii)). So, the all possibilities for *BkD* are those listed in the statement of Lemma 3.5.

Next, we want to find a suitable birational morphism $u: V \to F_2$ with the property required in the statement of Lemma 3.5.

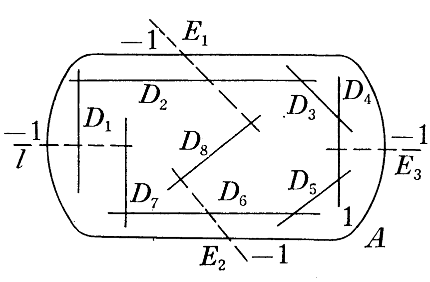
Case (v), (i), (vii) or (ii). Hence $(|R_1|, \cdots, |R_k|)=(8), (7,1), (5,2,1)$ or $(4,4)$.

By the same argument as above, we can find a $\boldsymbol{P}^1$-fibration $\Phi := \Phi_{|2l+D_1+D_s|}: V \to \boldsymbol{P}^1$ for a (-1) curve l such that all irreducible components, except D_2 and D_{s-1} are contained in the singular fibers and that D_2 and D_{s-1} are cross-sections of Φ, where $R_1 := D_1 + \cdots + D_s$. The computation of $\rho(V)$ by counting the number of irreducible components in the singular fibers shows that the configuration of singular fibers of ϕ is the following:



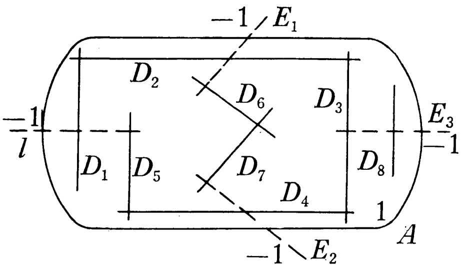
$(|R|_1, \cdots, |R_k|)=(8)$.

Fig. (3a)′



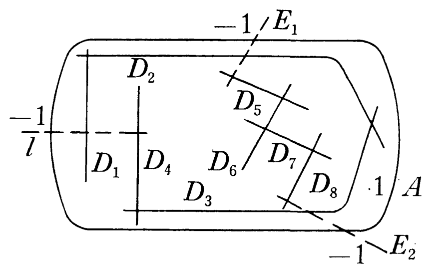
$(|R_1|, \cdots, |R_k|)=(7,1)$.

Fig. (1a)′



$(|R|_1, \cdots, |R_k|)=(5,2,1)$.

Fig. (4a)′



$(|R_1|, \cdots, |R_k|)=(4,4)$.

Fig. (2a)′

By virtue of Lemma 3.2, D_2 and D_{s-1} meet the singular fibers as shown above. In the case where $(|R_1|,\cdots,|R_k|)=(7,1)$ we consider a new $\boldsymbol{P}^1$-fibration $\Phi_{|2E_1+D_2+D_8|}$: $V\to\boldsymbol{P}^1$. Then $l+D_7+D_6+D_5+D_4+E_3$ is a fiber of $\Phi_{|2E_1+D_2+D_8|}$ and D_1, and D_3 are cross-sections of $\Phi_{|2E_1+D_2+D_8|}$. Let u: $V\to F_2$ be the contraction of l, D_8, E_1, E_2, D_6, D_5 and D_4 (E_1, D_8, E_3, l, D_7, D_6 and D_5; l, D_5, E_2, D_7, D_6, E_3 and D_8; l, D_4, E_2, D_8, D_7, D_6 and D_5; resp.) in the case where $(|R_1|,\cdots,|R_k|)=(8)$ ((7,1); (5,2,1); (4,4); resp.). The configurations of u_*D corresponding to the above four cases are the following (cf. the statement of Main Theorem):

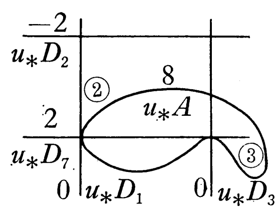


Fig. (3a)

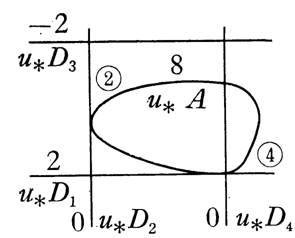


Fig. (1a)

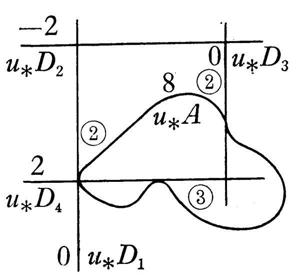
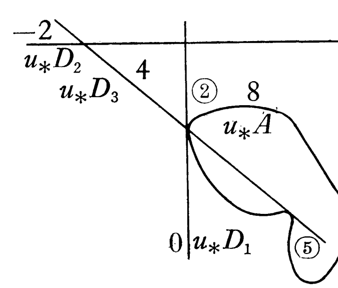


Fig. (4a)



(Fig. 2a)

Next, we consider the following

Case (vi), i.e., $(R_1,R_2)=(E_6,A_2)$. With the same notations as in Lemma 3.4, the case (ii-B), we consider the $\boldsymbol{P}^1$-fibration $\Phi_{|2(l+D_1+D_2)+D_3+D_5|}$: $V\to\boldsymbol{P}^1$. Let u: $V\to F_2$ be the contraction of l, D_1, D_2, D_3, E_2, D_8 and D_7. Then the configuration of u_*D is given below:

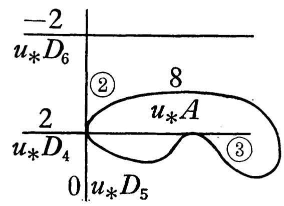


Fig. (5a)

Case (iii), i.e., $(R_1, R_2, R_3)=(A_2, A_2, A_2)$. Employing the notations in Lemma 3.4, the case (A-b-β), we consider the $\boldsymbol{P}^1$-fibration $\Phi_{|2l+D_1+D_3|}\colon V\to\boldsymbol{P}^1$. Let u: $V\to F_2$ be the contraction of l, D_3, E_2, D_6 and D_5. Then u_*D has a configuration given in Fig. (5a), in which the notations u_*D_6 and u_*D_5 are replaced by u_*D_2 and u_*D_1, respectively.

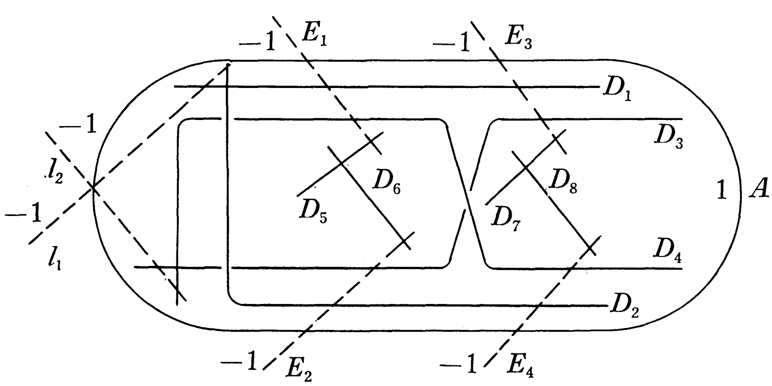
Case (iv), i.e., $(R_1, R_2)=(A_5, A_2)$. With the same notations in Lemma 3.4, the case (A-b-α), we consider a $\boldsymbol{P}^1$-fibration $\Phi_{|2(E_1+D_3)+D_2+D_4|}\colon V\to\boldsymbol{P}^1$. Let u: $V\to F_2$ be the contraction of E_1, D_3, D_4, E_2, D_7 and D_6. Then the configuration of u_*D is given in Fig. (5a), in which u_*D_6, u_*D_4 and u_*D_5 are replaced by u_*D_1, u_*D_5 and u_*D_2, respectively.

We now consider the following

Case (viii), i.e., $(R_1, R_2, R_3, R_4)=(A_2, A_2, A_2, A_2)$. Let $\Phi_{|\sigma' A|}\colon V'\to\boldsymbol{P}^1$ be the elliptic fibration considered in Lemma 3.4. Since the inequality (1) above becomes an equality: $12=8+4$, we have:

(2) $\chi(G_i) = |R_i|+1 = 3$ for $i = 1, \cdots, 4$

where $G_1, \cdots, G_4$ are all singular fibers of $\Phi_{|\sigma' A|}$. We write $\sigma_*G_i=R_i+l_i$ $(i=1, \cdots, 4)$ with a (-1) curve l_i. Note that l_1, l_2, l_3, l_4 and A share one and the same point. We consider a $\boldsymbol{P}^1$-fibration $\Phi_{|l_1+l_2|}\colon V\to\boldsymbol{P}^1$. We see that $Bk(D-R_1-R_2)$ is contained in the singular fibers and that $D_1, \cdots, D_4$ are cross-sections, where $R_1 := D_1+D_2$ and $R_2 := D_3+D_4$. So, we obtain the following configuration of the singular fibers:



Note that $(D_1+D_2, E_i)\leqq 1$ and $(D_3+D_4, E_i)\leqq 1$ for $i=1, \cdots, 4$. Indeed, suppose that $(D_1+D_2, E_i)\geqq 2$ for some i, say $i=1$. Then $A\sim E_1+D_1+D_2$, cf. Lemma 3.4, the case (i-A-b), while $(D_5, E_1+D_1+D_2)=1\neq(D_5, A)=0$. This is absurd. We can verify similarly $(D_3+D_4, E_i)\leqq 1$ for $i=1, \cdots, 4$. Hence, we may assume that D_1, D_2, D_3 and D_4 meet the singular fiber $E_1+E_2+D_5+D_6$ as shown in the picture above. Instead of $\Phi_{|l_1+l_2|}$, we consider a new $\boldsymbol{P}^1$-fibration $\Phi:=\Phi_{|2E_1+D_1+D_4|}: V\to \boldsymbol{P}^1$. D_2 and D_3 are cross-sections, D_5 is a 2-section of Φ and, D_6, D_7 and D_8 are contained in the singular fibers of Φ. The configuration of the singular fibers of Φ is given below:

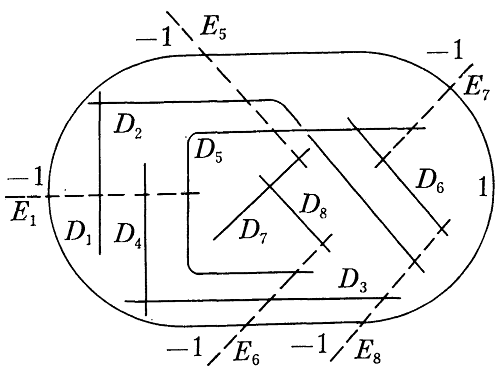


Fig. 6′

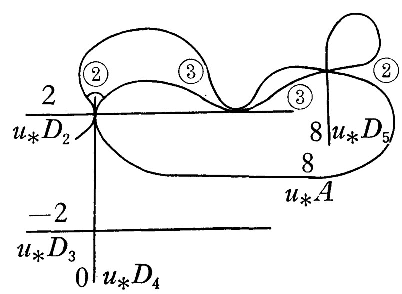


Fig. 6

We shall verify that D_2, D_3 and D_5 meet the singular fibers as shown in the picture above. Since D_5 is a 2-section, we may assume that $(D_5, E_7)=1$. Then $A\sim D_5+D_6+E_7$ (cf. Lemma 3.4, the case (i-A-b)), whence $(D_i, E_7)=(D_i, A-D_5-D_6)=0$ for $i=2, 3$. Let $\sigma_1: V_1\to V$ be the blowing-up of the point $A\cap E_7$. Then $\Phi_{|\sigma_1' A|}: V_1\to\boldsymbol{P}^1$ is an elliptic fibration. With the same argument as the one used to obtain the result (2) above, we see $\chi(\sigma_1'(D_5+D_6+E_7))=3$, whence $D_5\cap D_6\cap E_7=\phi$. Note that if $(D_5, E_5)=2$ or $(D_5, E_6)=2$, say $(D_5, E_5)=2$, then $A\sim D_5+E_5$ (cf. Lemma 3.4, the case (i-B-b)), while $(D_6, D_5+E_5)=1\neq(D_6, A)=0$. This is impossible. So, D_5 meets the singular fibers as shown above. We have seen $(D_2, E_7)=(D_3, E_7)=0$, whence $(D_2, E_8)=(D_3, E_8)=1$. Thence, $3A\sim 3E_8+2(D_2+D_3+D_6)+D_1+D_4+D_5$ and $3=(3A, E_5)=(3E_8+2(D_2+D_3+D_6)+D_1+D_4+D_5, E_5)=(2(D_2+D_3), E_5)+1$, i.e., $(D_2+D_3, E_5)=1$. We may assume that $(D_2, E_5)=1$ and $(D_3, E_5)=0$. Thence $(D_3, E_6)=1$. Therefore D_2 and D_3 meet the singular fibers as shown in the picture. Let $u: V\to F_2$ be the contraction of E_1, D_1, E_5, D_7, D_8, E_7 and D_6. Then the configuration of u_*D is given in Fig. 6 above.

In order to finish the proof, we have only to verify that there exist pictures Fig. (1a), $\cdots$, Fig. 5(a) and Fig. 6. First of all, we shall find the picture Fig. (2a) on F_2. Let M^* be the minimal esction and let l^* be a fiber of $\pi: F_2\to\boldsymbol{P}^1$. We fix an elliptic curve $A^*\in|-K_{F_2}|$. Note that $A^*\sim 2M^*+4l^*$. It is easy to show that the canonical restriction

$$H^0(F_2, \mathcal{O}(M^*+3l^*)) \simeq H^0(A^*, \mathcal{O}_{A^*}(M^*+3l^*))$$

is an isomorphism. Since A^* is a double covering of the base curve $\boldsymbol{P}^1$, there are four ramification points. Let P_0 be one of them. Since A^* is an elliptic curve, A^* has a group structure with P_0 as the origin. Choose a 5-torsion point $P \in A^*$ other than P_0, i.e., $5P \sim 5P_0$. Hence $5P+P_0 \sim 6P_0$, where $6P_0$ is cut out on A^* by a member $M^*+3l_1^*$ of $|M^*+3l^*|$, l_1^* being the fiber passing through P_0. Hence, by the above isomorphism, $5P+P_0$ is cut out on A^* by a member C^* of $|M^*+3l^*|$. Clearly, C^* is irreducible. Note that $(C^{*2})=4$. Thus, the divisor $M^*+l_1^*+A^*+C^*$ has a configuration Fig. (2a).

We can find the pictures Fig. (1a), Fig. (3a), Fig. (4a) and Fig. (5a) on F_2 in the same fashion.

Next, we shall find the picture Fig. 6 on F_2. Let A^* be a member of $|-K_{F_2}|$ such that A^* is a rational curve with a node Q. Note that A^*-Q is isomorphic to G_m which has a group structure. Then by an argument similar to the one in finding the picture Fig. (2a), we can find an irreducible curve C^* with $C^* \sim M^*+2l^*$ such that C^* meets A^* as shown below, where l_1^* is the fiber passing through a ramification point P_0 of $\pi|_{A^*}$ (cf. Fig. 5 in the statement of Main Theorem):

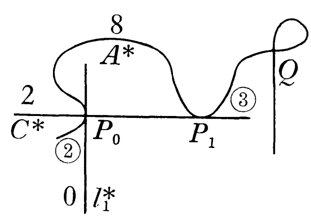


Fig. (5b)

Since $\dim|M^*+2l^*|=3$, there exists a member C_1^* of $|M^*+2l^*|$ such that C_1^* meets A^* once (or three times, resp.) at P_0 (or Q, resp.). Clearly, C_1^* is irreducible, $C^*+C_1^*$ is a member of $|-K_{F_2}|$, $C^*+C_1^*$ meets l_1^* twice at P_0 and its tangent at Q is one of the two tangents of A^*. Consider the linear system Λ generated by $C^*+C_1^*$ and A^*. Note that $Bs(\Lambda)=\{P_0, P_1, Q\}$. A general member of Λ can have singularities only at the base points of Λ by the Bertini theorem. But $C^*+C_1^* \in \Lambda$ is nonsingular at P_1 and Q, and $A^* \in \Lambda$ is nonsingular at P_0. Hence, a general member B^* of Λ is nonsingular everywhere and is irreducible because $A^* \in \Lambda$ is irreducible. Thus, the divisor $A^*+B^*+C^*+l_1^*+M^*$ gives us the picture Fig. 6. This completes the proof of Lemma 3.5.

4. Iitaka surfaces with $\rho(\bar{V})=1$, the part (II)

In the present section, we consider the case where the following conditions are satisfied:

There exist an irreducible component D_1 of BkD and an extremal rational curve $\bar{l}$ and a nef divisor $\bar{H}$ on the surface $\bar{V}_1$, obtained from V by contracting $Bk(D-D_1)$, such that $\bar{H}^{\perp}\cap\overline{NE}(\bar{V}_1)=\boldsymbol{R}_+[\bar{l}]$, $\bar{H}\not\equiv 0$ and $(\bar{H}^2)=0$.

By virtue of Lemma 2.5, there exists a $\boldsymbol{P}^1$-fibration Φ: $V\to\boldsymbol{P}^1$ such that every singular fiber has the configuration given in Lemma 2.5.

Furthermore, we assume that $\rho(\bar{V})=1$ and V is not isomorphic to $\boldsymbol{P}^2$ or F_m. Then we know by Lemma 2.5 that all irreducible components of BkD, except for D_1, are contained in the fibers of Φ.

Suppose D_1 is a cross-section of Φ. Then there exists a birational morphism u: $V\to F_2$ such that u_*D_1 is the unique (-2) curve on F_2 and that Supp u_*BkD is the union of less than four fibers of $\pi:=\Phi\circ u^{-1}$: $F_2\to\boldsymbol{P}^1$, each of which passes through a ramification point of $\pi_{|u_*A}$. Note that A, so u_*A is an elliptic curve (cf. Lemma 2.5) and that D_1 cannot meet more than three other components of BkD.

Lemma 4.1. *Let the notations and assumptions be the same as above. Suppose that* $(D_1, f)\geqq 2$ *for a fiber* f *of* Φ. *Then the following assertions hold true*:

(i) $(D_1, f)=2$.

(ii) $(A^2)=2$ *or* 1 *according to whether or not* D_1 *is an isolated component of* BkD. *Hence* A *is an elliptic curve by Lemma* 3.1.

(iii) *The following exhaust all possible configurations of* BkD:

Case $(A^2)=1$.

(1) $2A_1+2A_3$; (2) A_3+D_5; (3) D_8; (4) $2D_4$; (5) $2A_1+D_6$; (6) $4A_1+D_4$.

Case $(A^2)=2$.

(7) A_1+D_6; (8) $3A_1+D_4$; (9) A_1+2A_3; (10) $7A_1$.

Proof. Let f_1 be a singular fiber of Φ. Then f_1 is written as $2(E_1+D_2+\cdots+D_{s-2})+D_{s-1}+D_s$ for a (-1) curve E and irreducible components $D_2, \cdots, D_s$ of BkD with $s\geqq 3$ (cf. Lemma 2.5).

Claim 1. $(D_1, f)=2$.

Suppose, on the contrary, that $(D_1, f)\geqq 3$. We first consider

Case $s=3$. Since D is an SNC divisor and BkD is a tree, we have $(D_1, D_2)\leqq 1$ and $(D_1, D_3)\leqq 1$. Hence $3\leqq(D_1, f_1)=(D_1, 2E+D_2+D_3)\leqq 2+2(D_1, E)$ and $(D_1, E)\geqq 1$. If $(D_1, E)\geqq 2$, we have $A\sim D_1+E$ and $(D_1, E)=2$ (cf. Lemma 3.4, the case (i-B-b)). Hence $(D_1+E, D_2)=(A, D_2)$, while $(D_1+E, D_2)\geqq(E, D_2)=1$ and $(A, D_2)=0$. This is a contradiction. So $(D_1, E)=1$ and $(D_1, D_2+D_3)\geqq 1$. If $(D_1, D_2)=1$, then $A\sim D_1+D_2+E$ and if $(D_1, D_3)=1$ then $A\sim D_1+D_3+E$. However, if $(D_1, D_2)=1$ then $(D_1+D_2+E, D_3)\geqq(E, D_3)=1>(A, D_3)=0$. This is a contradiction. We have also a contradiction if $(D_1, D_3)=1$. We next consider:

Case $s\geqq 4$. Note that $(D_1, D_2+\cdots+D_s)\leqq 1$ because Supp BkD contains no loops. If $(D_1, D_2+\cdots+D_s)=0$, we have $3\leqq(D_1, f_1)=2(D_1, E)$ and $(D_1, E)\geqq 2$.

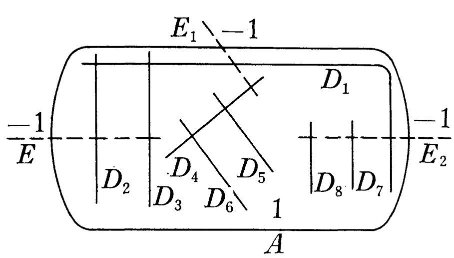
This leads to a contradiction as in the case $s=3$. If $(D_1, D_2+\cdots+D_s)=1$, then we must have $(D_1, E)=0$ for, otherwise, $D_1+\cdots+D_s$ is a rod with D_1 and D_2 as its tips (cf. Lemma 3.4, the case (i-A-b)), which is not the case. Thus we have shown $(D_1, f)=2$. Note that $(D_1, D_2+\cdots+D_s)=(D_1, D_2)=1$ provided $s\geqq 4$ and $(D_1, E)=0$, because $(D_1, f)=2$ and the connected component $D_1+\cdots+D_s$ of BkD is a fork.

We consider the case where D_1 is not an isolated component of BkD. Let $D_i\subseteq \mathrm{Supp}\ BkD$ be an irreducible component of BkD meeting D_1 and let f_1 be the fiber containing D_i.

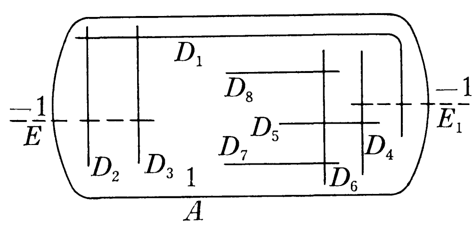
Claim 2. D_1 does not meet any irreducible component of BkD which is not contained in the fiber f_1. Moreover, $(A^2)=1$, and hence A is an elliptic curve by Lemma 3.1.

We consider first:

Case $s=3$, i.e., $f_1=2E+D_2+D_3$. Note that $(D_1, f)=2$ and a connected component of BkD is a rod or a fork. Hence, $(D_1, D_2)=(D_1, D_3)=1$, $A\sim E+D_1+D_2+D_3$ and $(A^2)=1$. Hence, the claim 2 follows, and by Lemma 3.1, BkD has eight irreducible components. Write $\mathrm{Supp}\ BkD=\cup_{i=1}^{8} D_i$. It is easy to check that all possible singular fibers of Φ are exhausted by the following:



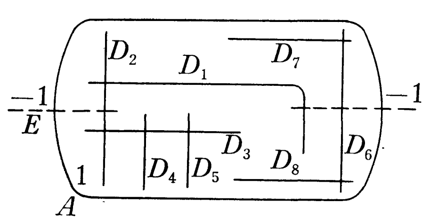
(1)



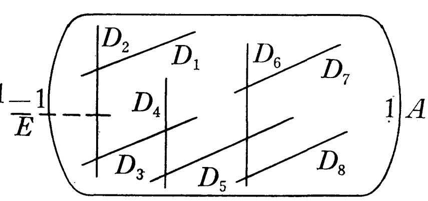
(2a)

We next consider:

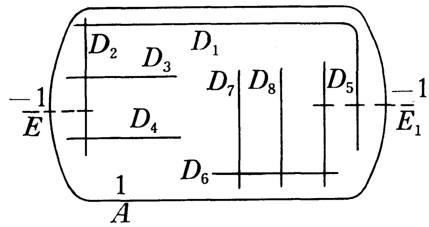
Case $s\geq 4$, i.e., $f_1=2(E+D_2+\cdots+D_{s-2})+D_{s-1}+D_s$. The configuration of D_1+f_1 is as shown below and $A\sim D_1+E+2(D_2+\cdots+D_{s-2})+D_{s-1}+D_s$. The first assertion of the claim 2 is now easily verified. Moreover, $(A^2)=1$ and hence A is an elliptic curve. We can easily exhaust all possibilities for singular fibers of Φ.



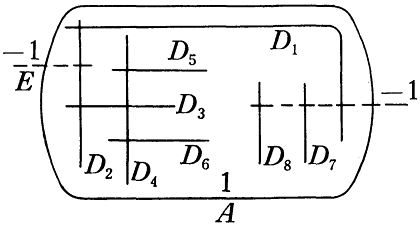
(2b)



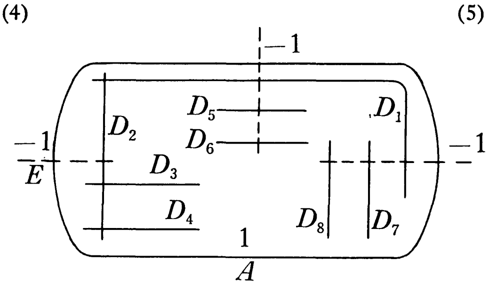
(3)



(4)

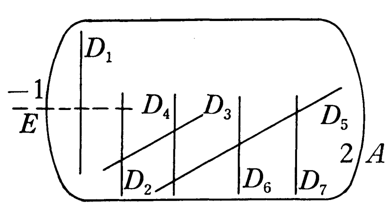


(5)

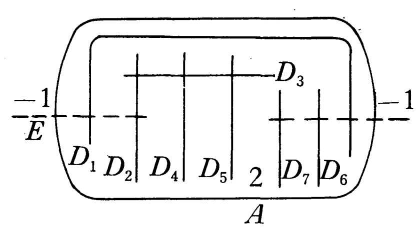


(6)

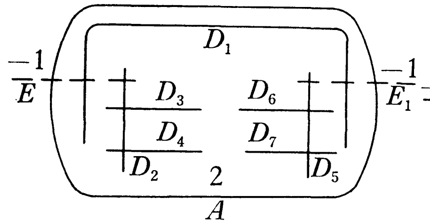
We now consider the case where D_1 is an isolated component of BkD. Write Supp $BkD=\bigcup_{i=1}^{r}D_i$. Let u: $V\to F_m$ be the contraction of all (-1) curves contained in fibers of Φ. Since D_1 meets the unique (-1) curve of each singular fiber of Φ, we easily show $(u_*D_1^2)=-2+r-1=r-3$. On the other hand, since u_*D_1 is a double section of π, write $u_*D_1\sim 2M^*+bl^*$, where M^* is the minimal cross-section of π: $F_m\to \boldsymbol{P}^1$ and l^* is a general fiber of π. Then we have $b\geqq 2m$ and $(u_*D_1^2)=4(b-m)\equiv 0 \mod (4)$. Therefore, $r=3$ or 7 for $r\leqq 8$ by Lemma 3.1. The case $r=3$ is impossible for, otherwise, we have $b=m=0$ and this contradicts the irreducibility of u_*D_1. Hence $(A^2)=9-r=2$ and A is an elliptic curve by Lemma 3.1. Moreover, $m\leqq 1$. We easily exhaust all possibilities for singular fibers of Φ as follows:



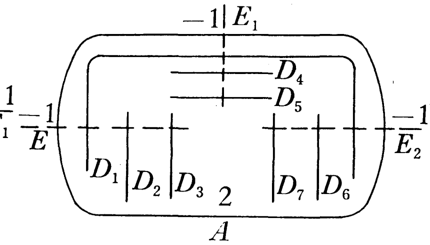
(7)



(8)



(9)

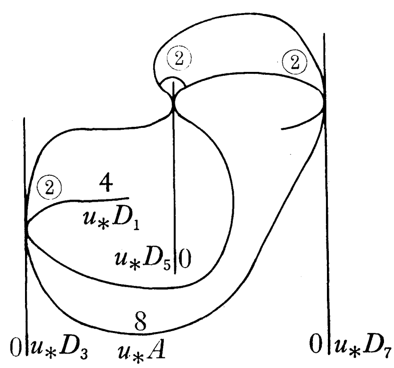


(10) Q.E.D.

Next, we verify the following

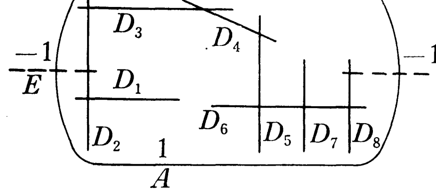
Lemma 4.2. *Let the notations and assumptions be the same as in Lemma* 4.1. *Then the cases* (6) *and* (10) *do not occur. For the cases* (3), (5), (7) *and* (8), *there exists a* $\boldsymbol{P}^1$*-fibration* Φ_1: $V \to \boldsymbol{P}^1$ *such that all irreducible components of BkD, except for one component, say* D_2, *are contained in the fibers of* Φ_1 *and that* D_2 *is a cross-section of* Φ_1. *Moreover, there exists a contraction* u: $V \to F_2$ *of* (-1) *curves contained in the fibers of* Φ_1 *such that Supp* u_*BkD *is the union of the unique* (-2) *curve and two or three fibers of* π: $F_2 \to \boldsymbol{P}^1$, *each of which passes through a ramification point of* $\pi|_{u_*A}$. *The cases* (1), (2a), (2b), (4) an *and* (9) *occur.*

Proof. *Cases* (6) *and* (10). First, we consider the case (10). Let u: $V \to F_m$ $(m \leqq 1)$ be the contraction of E, D_2, E_1, D_4, E_2 and D_6 (cf. the picture (10) in Lemma 4.1). The configuration of u_*D is given below:

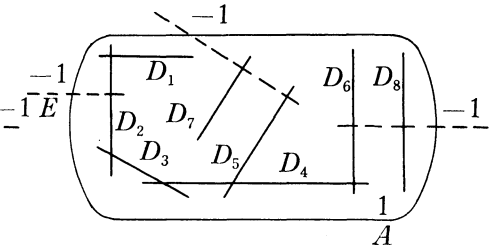


Let π: $F_m \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration for which u_*D_3, u_*D_5 and u_*D_7 are its fibers. We know that $\pi|_{u_*D_1}$ is a double covering. Since u_*D_1 is a nonsingular rational curve, it has exactly two ramification points. On the other hand, from the construction of u, we see that $u_*D_3 \cap u_*D_1$, $u_*D_5 \cap u_*D_1$ and $u_*D_7 \cap u_*D_1$ are three distinct ramification points of $\pi|_{u_*D_1}$. So, we reach to a contradiction. Therefore the case (10) does not occur. We can verify in the same way that the case (6) does not occur.

Case (3). Let $\Phi_1 = \Phi_{|2(E+D_2)+D_1+D_3|}$: $V \to \boldsymbol{P}^1$ (cf. the picture (3) in Lemma 4.1). Φ_1 is a $\boldsymbol{P}^1$-fibration. Note that $\rho(V)=9$ and that D_4 is a cross-section of Φ_1. So, we can exhibit easily the configuration of singular fibers of Φ_1 (cf. the picture (3)′ below). So, the assertion holds true if one takes D_4 as D_2 in the assertion.



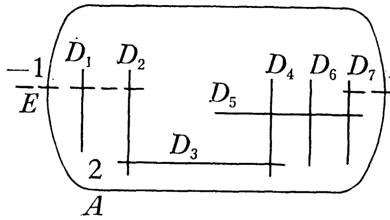
(3)′



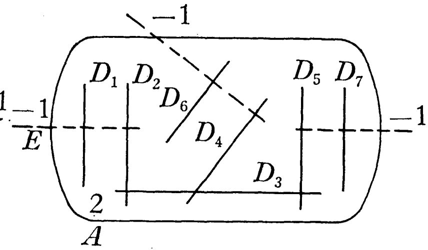
(5)′

Case (5). Instead of Φ, we consider a $\boldsymbol{P}^1$-fibration $\Phi_1:=\Phi_{|2(E+D_2)+D_1+D_3|}$: $V\to\boldsymbol{P}^1$ and can exhibit the singular fibers of Φ_1 by taking into consideration the fact that $\rho(V)=9$ (cf. the picture (5)′ above). So, the assertion holds true if one takes D_4 as D_2 in the assertion.

Case (7). Let $\Phi_1=\Phi_{|2E+D_1+D_2|}$: $V\to\boldsymbol{P}^1$ (cf. the picture (7) in Lemma 4.1). Φ_1 is a $\boldsymbol{P}^1$-fibration. Noting $\rho(V)=8$, we obtain the configuration of singular fibers of Φ_1 (cf. the picture (7)′ below). So, the assertion for the case (7) holds true if one takes D_3 as D_2 in the assertion.



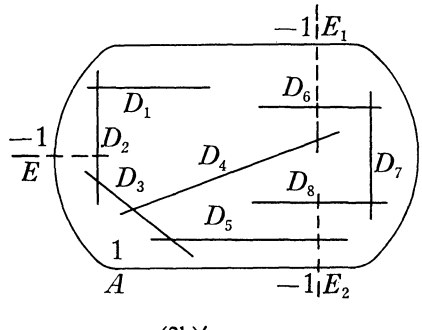
(7)′



(8)′

Case (8). Let $\Phi_1=\Phi_{|2E+D_1+D_2|}$: $V\to\boldsymbol{P}^1$, which is a $\boldsymbol{P}^1$-fibration (cf. the picture (8) in Lemma 4.1). The configuration of singular fibers of Φ_1 is given in the picture (8)′ above; note that $\rho(V)=8$, So, the assertion for case (8) holds true if one takes D_3 as D_2 in the assertion.

Case (2b). Instead of Φ, we consider a new $\boldsymbol{P}^1$-fibration $\Phi_1:=\Phi_{|2(E+D_2)+D_1+D_3|}$: $V\to\boldsymbol{P}^1$ (cf. the picture (2b) in Lemma 4.1). Since D_4 and D_5 are cross-sections of Φ_1, the singular fiber containing $D_6\cup D_7\cup D_8$ is given in the picture (2b)′ below:



(2b)′

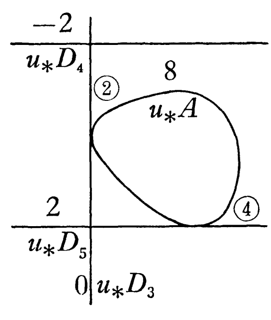
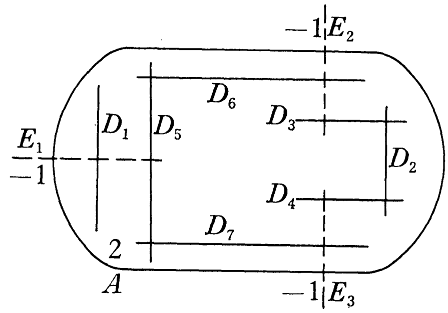


Fig. (9a)

Let u: $V\to F_2$ be the contraction of E, D_2, D_1, E_2, D_8, D_7 and D_6. Then the configuration of u_*D is given in Fig. (9a) above (cf. the statement of Main Theorem). We can find such a picture Fig. (9a) by the same proof as for Lemma 3.5, the case (i).

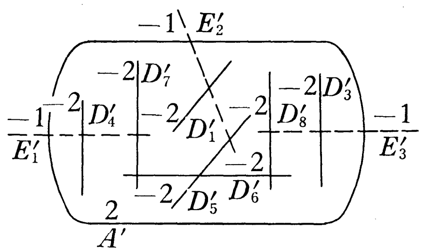
Case (2a). We consider a $\boldsymbol{P}^1$-fibration $\Phi_1:=\Phi_{|2(E_1+D_1)+D_2+D_3|}: V\to\boldsymbol{P}^1$ (cf. the picture (2a) in Lemma 4.1). We see that the configuration of singular fibers of Φ_1 is given in the picture (2b) in Lemma 4.1, in which the notations D_6, D_7, D_8, D_1, D_2, D_3, D_4 and D_5 are replaced by D_1, D_2, D_3, D_4, D_5, D_6, D_7 and D_8, respectively. So the case (2a) is nothing but the case (2b). Hence this case is realizable.

Case (9). Instead of Φ, we consider a new $\boldsymbol{P}^1$-fibration $\Phi_1:=\Phi_{|2E_1+D_1+D_5|}$: $V\to\boldsymbol{P}^1$ (cf. the picture (9) in Lemma 4.1). Note that D_6 and D_7 are then cross-sections of Φ_1. We see that the singular fiber containing $D_2\cup D_3\cup D_4$ is given as below:

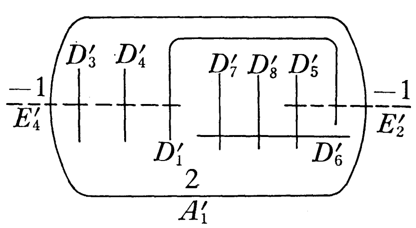


Let u: $V\to F_2$ be the contraction of E_1, D_1, E_3, D_4, D_2 and D_3. Then the configuration of u_*D is given in Fig. (9a) above in which u_*D_4, u_*D_5 and u_*D_3 are replaced by u_*D_6, u_*D_7 and u_*D_5, respectively. Hence the case (9) is realizable.

Case (4). We employ the notations in the picture (4) in Lemma 4.1. Let u: $V\to F_m$ $(m\leqq 1)$ be the contraction of E, D_2, D_3, E_1, D_5, D_6 and D_7. The configuration of u_*D is given in Fig. 8 in the statement of Main Theorem, in which $A^*:=u_*A$, $f_1:=u_*D_4$, $f_2:=u_*D_8$ and $C_1:=u_*D_1$. Next we shall show that the case (4) is realizable. We take an elliptic curve $A^*\in|-K_{F_2}|$. Let σ_1: $V'\to F_2$ be the blowing-ups of three distinct ramification points of $\pi|_{A^*}$: $A^*\to\boldsymbol{P}^1$ and their infinitely near points so that we obtain the following configuration (4)$'$ on V', where $A':=\sigma_1'A^*$ and D_6' is the proper transform of the minimal section on F_2:



(4)$'$



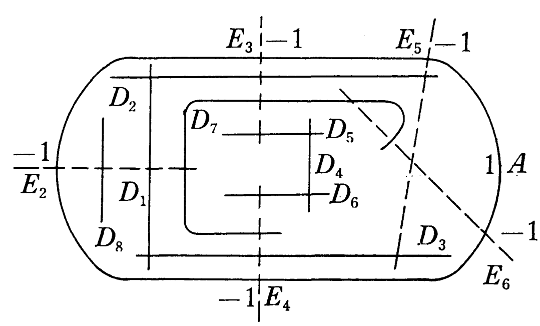
(4)$''$

Consider a $\boldsymbol{P}^1$-fibration $\Phi_{|2(E_2'+D_5'+D_6')+D_7'+D_8'|}$: $V'\to\boldsymbol{P}^1$ which has a singular fiber

$2E_4'+D_3'+D_4'$ with a (-1) curve E_4' because $\rho(V')=8$ (cf. the picture (4)″ above). Let σ_2: $V\to V'$ be the blowing-up of the point $A'\cap E_4'$. Let $A=\sigma_2'A'$, $D_2=\sigma_2'E_4'$ and $D_i=\sigma_2'D_i'$ $(i=1, 3, \cdots, 8)$. Then the pair (V, D) is an Iitaka surface such that the configuration of D is given in the picture (4) in Lemma 4.1. Hence the case (4) occurs.

Case (1). We use the notations in the picture (1) in Lemma 4.1. Let u: $V\to F_m$ $(m\leqq 1)$ be the contraction of $E, D_2, E_1, D_4, D_5, E_2$ and D_7. It is easy to see that the configuration of u_*D is given in Fig. 7 in the statement of Main Theorem, in which $A^*:=u_*A$, $f_1:=u_*D_3$, $f_2:=u_*D_6$, $f_3:=u_*D_8$ and $C_1:=u_*D_1$.

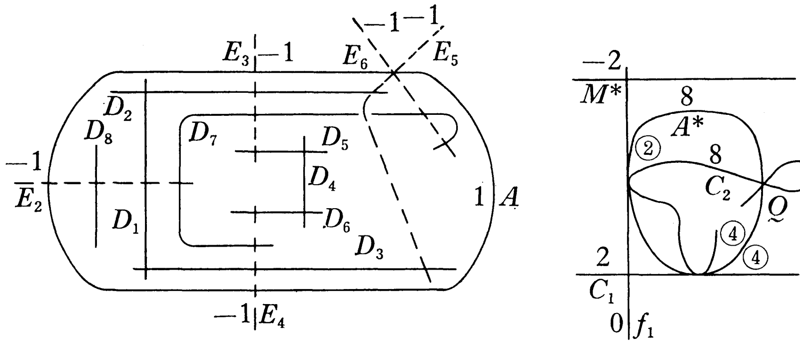
We shall construct an Iitaka surface (V, D) which fits to the case (1). Instead of Φ, we consider a $\boldsymbol{P}^1$-fibration $\Phi_1:=\Phi_{|2E_2+D_8+D_1|}$: $V\to\boldsymbol{P}^1$. Note that D_2 and D_3 are cross-sections and D_7 is a 2-section. Hence the singular fiber containing $D_4\cup D_5\cup D_6$ is given in the picture (1a)′ below:



(1a)′

Since $\rho(V)=9$, there exists a singular fiber in Φ_1 consisting of two (-1) curves E_5 and E_6. Then $E_5\cap E_6\cap A=\phi$ or a single point. First, we consider the case $E_5\cap E_6\cap A=\phi$. By the proof of Lemma 3.4 for the cases (i-A-b) and (i-B-b), we have $(D_7, E_i)\leqq 1$ and $(D_2+D_3, E_i)\leqq 1$ for $i=3, 4$. Indeed, suppose that $(D_7, E_i)\geq 2$ for some i, say $i=3$. Then $(D_7, E_3)=2$ and $A\sim E_3+D_7$. Hence $(A, D_5)=(E_3+D_7, D_5)$. This is impossible because $(A, D_5)=0$ and $(E_3+D_7, D_5)=1$. We can verify similarly $(D_2+D_3, E_i)\leqq 1$ for $i=3, 4$. Therefore D_2, D_3 and D_7 meet the singular fiber $E_3+E_4+D_5+D_6+D_4$ as shown in the above picture. Then we see $4A\sim 4E_4+3(D_3+D_6)+2(D_1+D_4+D_7)+D_2+D_5$ and $(E_i, 4E_4+3(D_3+D_6)+2(D_1+D_4+D_7)+D_2+D_5)=(E_i, 4A)=4$ for $i=5, 6$. So, we may assume $(E_6, D_7)=2$ and $(D_2, E_5)=(D_3, E_5)=1$. Therefore D_2, D_3 and D_7 meet the singular fibers of Φ_1 as shown in the above picture (1a)′. We shall see soon that this leads to a contradiction. Indeed, let σ_1: $V'\to V$ be the blowing-up of $A\cap E_6$. Since $A\sim E_6+D_7$ (cf. Lemma 3.4, the case (i-B-b)), $\Phi_{|\sigma_1'A|}$: $V'\to\boldsymbol{P}^1$ is an elliptic fibration. Let G be the singular fiber of $\Phi_{|\sigma_1'A|}$ containing $\sigma_1'(D_1+D_2+D_3)$. By the hypothesis that $E_5\cap E_6\cap A=\phi$, $\sigma_1'E_5$ is not a component of G. Since G is a fiber, we have $(\sigma_1'E_5, \sigma_1'A)=(\sigma_1'E_5, G)$. But $(\sigma_1'E_5, \sigma_1'A)=1$ and $(\sigma_1'E_5, G)\geqq(\sigma_1'E_5, \sigma_1'(D_1+$

$D_2+D_3))=2$. This is a contradiction. Hence we must have $E_5 \cap E_6 \cap A \neq \phi$. With the same argument as above, we obtain the following configuration:



(1b)′

where $\sigma: V \to F_2$ is the contraction of $E_2, D_8, E_4, D_6, D_4, D_5$ and E_6, and $A^*=\sigma_* A$, $M^*=\sigma_* D_2$, $f_1=\sigma_* D_1$, $C_1=\sigma_* D_3$ and $C_2=\sigma_* D_7$. The configuration of $\sigma_* D$ is given above, where the node Q of the rational curve C_2 is a ramification point of $\pi|_{A^*}: A^* \to \boldsymbol{P}^1$ if we let $\pi: F_2 \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration with f_1 as a fiber. We can find such a divisor $\sigma_* D$ on F_2 by an argument similar to the one in Lemma 3.5, the case (viii), where we take the linear system generated by C_2 and $C_1+M^*+f_1+\sigma_* E_5$ as Λ. In fact, we have only to show that for every elliptic curve $A^* \in |-K_{F_2}|$ meeting f_1, C_1 and C_2 as shown in the picture, the node Q of C_2 is a ramification point of $\pi|_{A^*}: A^* \to \boldsymbol{P}^1$. Indeed, if this is false for an elliptic curve $A^* \in |-K_{F_2}|$, by a suitable blowing ups of $A^* \cap f_1$, $A^* \cap C_1$ and Q and their infinitely near points, we obtain the picture (1a)′ above. This is absurd by the argument above. Therefore the case (1) is realizable.

We have finished the proof of Lemma 4.1.

We end this section by proving the following

Lemma 4.3. *Let the notations and assumptions be the same as at the begining of this section. If D_1 is a cross-section of Φ, then all possibilities for BkD are exhausted by the following*:

A_4; A_7· D_5; D_8; A_1+A_2; A_1+A_5; $2A_1+A_3$; A_1+D_6; $2A_1+D_6$; $3A_1+D_4$; E_6; E_7; E_8; A_1+E_7.

Proof. We denote by s the maximal number of irreducible components of BkD in a singular fiber of Φ. Then $s \geqq 2$ by Lemma 2.5 and $s \leqq \#$\{irreducible components in $BkD\}-1=9-(A^2)-1 \leqq 7$ by Lemma 3.1. We investigate each case according to the value of s, $2 \leqq s \leqq 7$, and note that a connected component of BkD is a rod or a fork of type D_n, E_6, E_7 or E_8. Then we obtain easily the result. Q.E.D.

5. The proof of Main Theorem

By virtue of Remark 2.4, it remains to study an Iitaka surface (V, D) of the following type:

A consists of two irreducible components, one of which is a (-1) curve l. Let $\tilde{u}$: $V \to \tilde{V}$ be the contraction of l. Let $\tilde{A}=u_*A$, $\tilde{D}=u_*D$ and $\tilde{D}_i=u_*D_i$ for every $D_i \subseteq \mathrm{Supp}\ BkD$. Note that so far we used the property $A+K_V \sim 0$ and did not use, from the begining of §2 to the first assertion in Remark 2.4, the property that A is nonsingular. We also apply the Mori theory to the pair $(\tilde{V}, \tilde{D})$ and see that there exists a birational morphism σ: $\tilde{V} \to W$ obtained by contracting all the divisors $\tilde{l}_i+\tilde{\Delta}_i$ of the type shown in Remark 2.4, where $\tilde{l}_i$ is a (-1) curve and $\tilde{\Delta}_i$ is a connected component of BkD which is a rod. Then, if we let $G=\sigma_*\tilde{A}$, $B=\sigma_*\tilde{D}$ and let g: $W \to \bar{W}$ be the contraction of $B-G$, there exists a pair $(\bar{E}, \bar{H})$ of an extremal rational curve $\bar{E}$ and a nef divisor $\bar{H}$ on $\bar{W}$ with $\bar{H}^{\perp} \cap \overline{NE}(\bar{W})=\boldsymbol{R}_+[\bar{E}]$ such that one of the following two cases occurs:

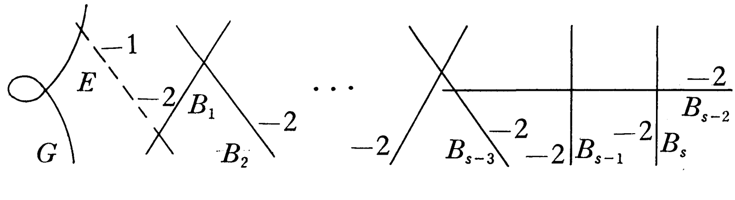
(1) $\bar{H} \equiv 0$ and $\rho(\bar{W})=1$.

(2) $\bar{H} \not\equiv 0$ and $(\bar{H}^2)=0$. Hence $\bar{H} \in \boldsymbol{R}_+[\bar{E}]$ and $(\bar{E}^2)=0$.

First of all, we consider the trivial cases where W is isomorphic to $\boldsymbol{P}^2$ or $F_m (m \leqq 2)$. Note that if $B \supsetneqq G$, $B-G$ consists of (-2) curves and (-2) forks. Hence, unless W is isomorphic to F_2, we have $B=G$ which is a rational curve with one node. If $W=F_2$, we have $B=G$ or $B=G+M$ with the minimal section M on F_2.

In the subsequent part of this section, we always assume that W is not isomorphic to $\boldsymbol{P}^2$ or F_m for any $m \geqq 0$. Using the arguments in the proof of Lemma 2.5 and noting that a double covering from a rational curve with one node to $\boldsymbol{P}^1$ has exactly two ramification points, we can prove:

Lemma 5.1. *Let the notations be as above. Suppose that W is not isomorphic to $\boldsymbol{P}^2$ or F_m and that the case where $\bar{H} \not\equiv 0$ and $(\bar{H}^2)=0$ occurs. Then there exists a $\boldsymbol{P}^1$ -fibration Φ: $W \to \boldsymbol{P}^1$ such that $B-G$ is contained in the fibers of Φ. Moreover, Φ has one or two singular fibers, each of which has a configuration of the following type*:



where $\bigcup_{i=1}^{s} B_i \subseteq Supp(B-G)$ and $f_1=2(E_1+B_1+\cdots+B_{s-2})+B_{s-1}+B_s$ is a singular fiber. Hence $B-G$ has at most four connected components. Let u: $W \to F_m (m \leqq 2)$ be the contraction of (-1) curves in the singular fibers of Φ. Then u_*G is a rational curve with one node and $u_*(B-G)$ is the union of one or two fibers of the $\boldsymbol{P}^1$-fibra-

tion $\pi:=\Phi\circ u^{-1}$: $F_m\to \boldsymbol{P}^1$ *which pass through ramification points of* $\pi|_{u_*G}$.

Next, we consider the case $\bar{H}\equiv 0$ and $\rho(\bar{W})=1$. By arguments similar to those used in the proof of Lemmas 3.3, 3.4, 3.5, 4.1 and 4.2, one can verify that one of the following cases occurs:

Case (E). There exist an irreducible component B_1 of $B-G$ and a $\boldsymbol{P}^1$-fibration Φ: $W\to \boldsymbol{P}^1$ such that $B-G-B_1$ is contained in the fibers of Φ and B_1 is a cross-section of Φ. In this case, let u: $W\to F_2$ be the contraction of (-1) curves in the singular fibers. Then $u_*(B-G)$ consists of the minimal section of $\pi:=\Phi\circ u^{-1}$: $F_2\to \boldsymbol{P}^1$ and one or two fibers of π: $F_2\to \boldsymbol{P}^1$, which pass through ramification points of a double covering $\pi|_{u_*G}$: $u_*G\to \boldsymbol{P}^1$.

Case (F). There exists a birational morphism u: $W\to F_m$ with $m\leqq 2$, such that u_*B has one of the nine configurations in the statement of Main Theorem, in which $A^*:=u_*G$ is a rational curve with one node Q and $Q\notin u_*G\cap u_*(B-G)$. We call these cooresponding pictures Fig. (1b), $\cdots$, Fig. (9b), respectively.

Lemma 5.2. *With the above notations, we suppose that W is not isomorphic to* $\boldsymbol{P}^2$ *or* F_m *and that the case* (*E*) *takes place. Then the following cases are all possibilities for* $B-G$:

A_4; A_7; D_5; D_8; A_1+A_2; A_1+A_5; $2A_1+A_3$; A_1+D_6; E_6; E_7; E_8; A_1+E_7.

Proof. We use the arguments in the proof of Lemma 4.3 and note that $\pi|_{u_*G}$: $u_*G\to \boldsymbol{P}^1$ has exactly two ramification points. Then Lemma 5.2 follows. Q.E.D.

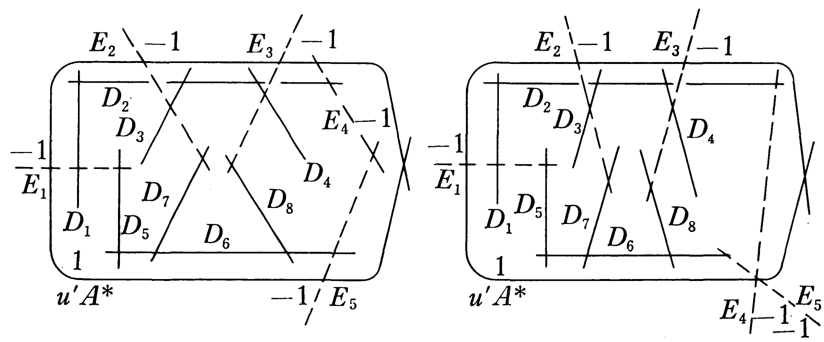
Lemma 5.3. *Let the notations be as above. Then Fig.* (6*b*), *Fig.* (7*b*) *and Fig.* (8*b*) *do not occur and Fig.* (1*b*), $\cdots$, *Fig.* (5*b*) *and Fig.* (9*b*) *are realizable.*

Proof. Suppose that there exists a picture Fig. (6b) on F_2. By a sequence of suitable blowing-ups u: $W\to F_2$, we obtain the configuration Fig. 6′ in Lemma 3.5, the case (viii), in which the elliptic curve A is replaced by a rational curve $u'A^*$ with one node. With the same notations as in Lemma 3.5, we let σ_1: $W'\to W$ be the blowing-up of the point $E_7\cap u'A^*$. Consider an elliptic fibration $\Phi_{|\sigma_1'u'A^*|}$: $W'\to \boldsymbol{P}^1$. Let G_1, G_2, G_3 or G_4 be the fiber of $\Phi_{|\sigma_1'u'A^*|}$ containing $\sigma_1'(D_1+D_2)$, $\sigma_1'(D_3+D_4)$, $\sigma_1'(D_5+D_6)$ or $\sigma_1'(D_7+D_8)$, respectively. By the Noether formula, we reach to a contradiction as follows:

$$12=12\chi(\mathcal{O}_{W'})-(K_{W'}{}^2)=\chi(W')\geqq \textstyle\sum_{i=1}^4\chi(G_i)+\chi(\sigma_1'u'A^*)=12+1=13\,.$$

We next suppose that there exists a picture Fig. (8b) on F_m $(m\leqq 1)$. Let u: $W\to F_m$ be a sequence of blowing-ups such that we obtain the picture (4) in Lemma 4.1, where we put $A=u'A^*$ and it is a rational curve with one node. Employing the notations in the picture (4), we consider a new $\boldsymbol{P}^1$-fibration

$\Phi_{|2E_1+D_1+D_5|}: W \to \boldsymbol{P}^1$. The computation of $\rho(W)$ by counting the number of irreducible components in the singular fibers of $\Phi_{|2E_1+D_1+D_5|}$ shows that the singular fibers are given in the picture below:



Let σ: $W \to F_2$ be the contraction of $E_1, D_1, E_2, D_3, E_3, D_4$ and E_4. Let $\pi = \Phi_{|2E_1+D_1+D_5|} \circ \sigma^{-1}$: $F_2 \to \boldsymbol{P}^1$. Then $\sigma_* u'A^* \cap \sigma_* D_5$, $\sigma_* u'A^* \cap \sigma_* D_7$ and $\sigma_* u'A^* \cap \sigma_* D_8$ are ramification points of $\pi|_{\sigma_* u'A^*}$. This is impossible because the double covering $\pi|_{\sigma_* u'A^*}$ has exactly two ramification points. So, there are no pictures like Fig. (8b) on F_m $(m \leqq 1)$. By the same reasoning, there are no pictures like Fig. (7b) on F_m $(m \leqq 1)$.

We have obtained the picture Fig. (5b) in Lemma 3.5, the case (viii). We can construct similarly the pictures Fig. (1b), $\cdots$, Fig. (4b) and Fig. (9b). Q.E.D.

We summarize Remark 2.4, Lemmas 2.5, 2.6, 3.5, 4.2, 5.1 and 5.3, the arguments at the begining of §4 and §5 and the argument before Lemma 5.2, and conlcude Main Theorem as stated at the begining of this paper.

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