Ma, H.
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ON SOME ASYMPTOTIC PROPERTIES OF THE SOLUTION FOR A STOCHASTIC DIFFERENTIAL EQUATION ON HILBERT SPACES

MA HONG

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1. Introduction

Let (H,p) and (K,q) be separable Hilbert spaces with inner products p and q respectively. $\sigma_2(H,K)$ denotes the Hilbert space of totality of Hilbert-Schmidt operators from H into K.

For a cylindrical Brownian motion B on H, we consider the following stochastic differential equation on K:

$$\begin{cases} dX(t) = G(X(t))dB(t)+b(X(t))dt \\ X(0) = x, \end{cases} \tag{1.1}$$

where $G: K \to \sigma_2(H, K)$ is Borel measurable and so is $b: K \to K$. Moreover G and b satisfy the following Lipschitz condition (A_1):

(A_1) There exists a positive constant α such that

$$\begin{cases} q(b(x)-b(y)) \leqq \alpha q(x-y) \\ \|G(x)-G(y)\| \leqq \alpha q(x-y), \end{cases}$$

where $\|\cdot\|$ mean the Hilbert-Schmidt norm in $\sigma_2(H, K)$ and $q^2(x)=q(x, x)$.

Therefore according to M. Yor [7] and Y. Miyahara [6], we have

Proposition. *There exists a unique solution X of* (1.1), *which is a diffusion with generator L;*

$$Lf(x) = q(f'(x), b(x))+(1/2)trace(G^*(x)f''(x)G(x)). \tag{1.2}$$

Moreover X has continuous paths, i.e. with probability 1,

$$q(X(t)-X(s)) \to 0 \quad as\ t-s \to 0. \tag{1.3}$$

In the finite dimensional case, A. Friedman [1] investigated various asymptotic properties of $X(t)$, and especially he gave recurrence and transience criteria in terms of G and b. The purpose of this paper is to extend Friedman's results on recurrence to the infinite dimensional case.

The following auxiliary theorem will be proved in §2.

Theorem 1. *Suppose the condition* (A_2).

(A_2) *For any* $R>0$, *there exist* $z_R \in K$, *and two positive constants* Γ_R *and* γ_R *such that*

(1.4) $$p^2(G^*(x)z_R)\Gamma_R+q(b(x), z_R)\geqq\gamma_R \quad \textit{whenever } q(x)\leqq R,$$

where G^* *is the conjugate operator of* G *and* $p^2(h)=p(h, h)$ *for* $h\in H$.

Then for any $x\in K$

(1.5) $$P_x\{\limsup_{t\to\infty} q(X(t))=\infty\}=1$$

holds.

According to A. Friedman [1], we define

$$A(x, y)=p^2(G^*(x)y)/q^2(y), \quad B(x)=||G^*(x)||^2,$$
$$C(x, y)=2q(b(x), y),$$
$$S(x, y)=\frac{B(x)+C(x, y)}{A(x, y)}-1 \text{ and } S(x)=S(x, x).$$

Let us introduce the non-degeneracy condition (A_3);

(A_3) $p(G^*(x)y)>0$ for any non-zero $y\in K$ and $x\in K$.

So the condition (A_3) implies that

(1.6) $$A(x, y)>0 \quad \text{for } y\neq 0.$$

In §3 we will prove the following Theorems 2 and 3, using the similar method as A. Friedman [1].

Theorem 2. *Besides* (A_2) *and* (A_3), *we assume* (A_4).

(A_4) *There exist a positive constant* R_0 *and a continuous function* ε_1 *on* $[0, \infty)$, *such that*

(1.7) $$S(x)\geqq 1+\varepsilon_1(q(x)) \quad \textit{whenever } q(x)\geqq R_0,$$

and

(1.8) $$\int_{R_0}^{\infty}(1/t)\exp\left[-\int_{R_0}^{t}\varepsilon_1(s)/s\, ds\right]dt<+\infty.$$

Then the solution X *is transient, i.e.*

(1.9) $$P_x(\lim_{t\to\infty} q(X(t))=\infty)=1 \quad \textit{for any } x\in K.$$

Theorem 3. *Besides* (A_2) *and* (A_3), *we assume* (A_5).

(A_5) *For any* $z\in K$, *there exist a positive constant* R_z *and a continuous function* $\mathcal{E}_2$ *on* $[0, \infty)$ *such that*

$$S(x, x-z)\leqq 1+\mathcal{E}_2(q(x-z)) \quad \textit{whenever } q(x-z)>R_z, \tag{1.10}$$

and for some $R^*>0$,

$$\int_{R^*}^{\infty}(1/t)\exp\left[-\int_{R^*}^{t}\mathcal{E}_2(s)/s\, ds\right]dt=\infty. \tag{1.11}$$

Then X *is recurrent, i.e. for any ball* $B_\alpha(z)=\{y: q(y-z)\leqq\alpha\}$, $\alpha>0$

$$P_x(X(t_n)\in B_\alpha(z) \quad \textit{for some } t_1<t_2<\cdots\uparrow\infty)=1 \quad \textit{for any } x\in K. \tag{1.12}$$

See Funaki [2] for a result related to Theorem 3. Moreover, we have from the separability of K

Corollary. *Under the conditions* (A_2), (A_3) *and* (A_5),

$$P_x\{\textit{closure of } \{X(t);\, t\in[0, \infty)\}=K\}=1 \quad \textit{for any } x\in K.$$

Finally, as a simple example, we treat an Ornstein-Uhlenbeck type process

$$dX(t)=GdB(t)-cX(t)dt,$$

where $G\in\sigma_2(H, K)$, and c is a real constant, and will discuss how the asymptotic behavior of $X(t)$ depends on the constant c.

2. Proof of Theorem 1

First of all, we will recall the definition of the solution of (1.1), according to M. Yor [7]. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A cylindrical Brownian motion B on H means a Wiener process $(0, p^2)$ on H, namely B: $[0, \infty)\times H\times\Omega\to\boldsymbol{R}^1$ satisfies the following conditions

(1) $B(0, \cdot, \cdot)=0$,

(2) $B(\cdot, h, \cdot)/p(h)$ is a one-dimensional Brownian motion for $h\neq 0$,

(3) for any $t\in[0, \infty)$ and λ, $\mu\in\boldsymbol{R}^1$,

$$B(t, \lambda h+\mu k, \cdot)=\lambda B(t, h, \cdot)+\mu B(t, k, \cdot) \quad \text{a.s.}.$$

Put $\mathcal{F}_t$:=the σ-field generated by $\{B(s, h, \cdot);\, s\leqq t, h\in H\}$. A K-valued process X is called a solution of (1.1) if

(1) X is $\mathcal{F}_t$-progressively measurable,

(2) $E\left[\int_0^T q^2(X(s))ds\right]<\infty$ for any $T>0$,

and

(3) for any t

$$X(t)=x+\int_0^t G(X(s))dB(s)+\int_0^t b(X(s))ds \quad \text{a.s.}$$

holds, where the second term is the stochastic integral and the third term is the Bochner integral.

Proof of Theorem 1. Fix $R>0$ arbitrarily. By (A_2) we can take z_R, γ_R and Γ_R of (1.4). So z_R is not zero. Put $\xi_1:=z_R/q(z_R)$ and define ϕ by

$$\phi(x) = A[e^{\alpha(R+1)}-e^{\alpha q(x,\xi_1)}]\,, \tag{2.1}$$

where A and α are some positive constants, which will be determined later. Then ϕ is Fréchet differentiable at any order and its first and second derivatives are given by

$$\phi'(x)\,[h] = -A\alpha e^{\alpha q(x,\xi_1)}q(h,\xi_1) \tag{2.2}$$

and

$$\phi''(x)\,[h_1,h_2] = -A\alpha^2 e^{\alpha q(x,\xi_1)}q(h_1,\xi_1)q(h_2,\xi_1)\,, \tag{2.3}$$

namely $\phi'(x)\in K^*$ can be regarded as

$$\phi'(x) = -A\alpha e^{\alpha q(x,\xi_1)}\xi_1\in K \tag{2.4}$$

and $\phi''(x)\in\mathcal{L}(K\to K^*)$ (linear map from K into K^*) can be regarded as

$$\phi''(x) = -A\alpha^2 e^{\alpha q(x,\xi_1)}\xi_1\otimes\xi_1\,. \tag{2.5}$$

Therefore $G^*(x)\phi''(x)G(x)\in\mathcal{L}(H\to H)$, and

$$\begin{aligned}(2.6)\qquad & \operatorname{trace}(G^*(x)\phi''(x)G(x)) \\ &= \sum_{i=1}^{\infty}p(e_i,G^*(x)\phi''(x)G(x)e_i) \\ &= -A\alpha^2 e^{\alpha q(x,\xi_1)}\sum_{i=1}^{\infty}p^2(G^*(x)\xi_1,e_i) \\ &= -A\alpha^2 e^{\alpha q(x,\xi_1)}p^2(G^*(x)\xi_1)\,,\end{aligned}$$

where $\{e_i;\,i=1,2,\cdots\}$ is an ONB in H. Hence we see that from (2.4) and (2.6)

$$\begin{aligned}(2.7)\qquad L\phi(x) &= q(\phi'(x),b(x))+(1/2)\operatorname{trace}(G^*(x)\phi''(x)G(x)) \\ &= -A\alpha e^{\alpha q(x,\xi_1)}[q(b(x),\xi_1)+(1/2)p^2(G^*(x)\xi_1)\alpha]\,.\end{aligned}$$

Put

$$\alpha = 2\Gamma_R q(z_R) \quad\text{and}\quad A = \exp\{2\Gamma_R q(z_R)R\}/2\Gamma_R\gamma_R\,. \tag{2.8}$$

Then recalling the definition of ξ_1 and (1.4), we have

$$L\phi(x)\leqq -1 \quad \text{whenever } q(x)\leqq R\,. \tag{2.9}$$

Since ϕ is smooth, Itô's formula derives

$$(2.10)\qquad \phi(X(t)) = \phi(x)+\int_0^t L\phi(X(s))ds+\int_0^t \langle G^*(X(s))\phi'(X(s)), dB(s)\rangle$$

where $\langle G^*(X(s))\phi'(X(s)), dB(s)\rangle=\sum_{i=1}^{\infty} p(G^*(X(s))\phi'(X(s)), e_i)dB(s, e_i)$.

Let $\tau=\tau_R$ be the first exit time from the ball $B_R:=\{y\in K; q(y)\leqq R\}$, i.e.

$$\tau = \begin{cases} \inf \{t>0;\ X(t)\notin B_R\}\ , \\ \infty \qquad \text{if the above set is empty.} \end{cases}$$

Then (2.10) yields

$$(2.11)\qquad E_x(\phi(X(t\wedge\tau))) = \phi(x)+E_x\left[\int_0^{t\wedge\tau} L\phi(X(s))ds\right]\leqq\phi(x)-E_x(t\wedge\tau)\,.$$

On the other hand, if $q(y)\leqq R$, then we have

$$(2.12)\qquad 0\leqq\phi(y)\leqq Ae^{\alpha(R+1)}\,,$$

the right hand side of which we denote by M. Therefore we can get from (2.11) and (2.12),

$$(2.13)\qquad E_x(t\wedge\tau)\leqq\phi(x)\leqq M \quad \text{for } x\in B_R\,.$$

Since $t\wedge\tau$ is increasing to τ as $t\to\infty$, the monotone convergence theorem implies

$$(2.14)\qquad E_x(\tau) = \lim_{t\to\infty} E_x(t\wedge\tau)\leqq M\,.$$

Hence we have

$$(2.15)\qquad P_x(\tau_R<\infty) = 1\,,$$

that is

$$(2.16)\qquad P_x(\sup_{t>0} q(X(t))\geqq R) = 1 \quad \text{for } x\in B_R\,.$$

Since R is arbitrary, we complete the proof.

3. Proof of Theorems 2 and 3

To prove Theorem 2, define functions θ, I, F and f by

$$\theta(r):= 1+\varepsilon_1(r) \quad r\geqq R_0\,, \quad I(s):= \int_{R_0}^{s} \theta(t)/t\, dt\,,$$
$$F(r):= \int_r^{\infty} e^{-I(s)}ds \quad r\geqq R_0\,, \quad f(x):= F(q(x)) \quad q(x)\geqq R_0\,.$$

Then the condition (A_4) means "$F(r)<\infty$" and we can easily calculate Freéhet derivatives f' and f'':

$$f'(x)\,[h] = F'(q(x))q(x, h)/q(x) \tag{3.1}$$

and

$$\begin{aligned} f''(x)\,[h_1, h_2] = {} & F''(q(x))q(x, h_1)q(x, h_2)/q^2(x) \\ & +F'(q(x))q(h_1, h_2)/q(x) \\ & -F'(q(x))q(x, h_1)q(x, h_2)/q^3(x)\,. \end{aligned} \tag{3.2}$$

Hence we get

$$\begin{aligned} & \mathrm{trace}(G^*(x)f''(x)G(x)) \\ & = \sum_{j=1}^{\infty} p(e_j, G^*(x)f''(x)G(x)e_j) \\ & = \sum_{j=1}^{\infty} q(G(x)e_j, f''(x)G(x)e_j) \\ & = [F''(q(x))/q^2(x)] \sum_{j=1}^{\infty} q^2(G(x)e_j, x) + [F'(q(x))/q(x)] \sum_{j=1}^{\infty} q^2(G(x)e_j) \\ & \quad -[F'(q(x))/q^3(x)] \sum_{j=1}^{\infty} q^2(G(x)e_j, x) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} Lf(x) & = (1/2)F''(q(x))A(x, x) + (1/2)F'(q(x))\{B(x) - A(x, x) + C(x, x)\}/q(x) \\ & = (1/2)A(x, x)\{F''(q(x)) + \theta(q(x))F'(q(x))/q(x)\} \\ & \quad +(1/2)A(x, x)F'(q(x))\{S(x) - \theta(q(x))\}/q(x) \\ & = (1/2)A(x, x)F'(q(x))\{S(x) - \theta(q(x))\}/q(x) \leqq 0\,. \end{aligned} \tag{3.4}$$

For any fixed $R_0 < \alpha < R < \beta$, we put

$$\begin{cases} \tau_{\alpha\beta} := \begin{cases} \inf\{t > 0;\ q(X(t)) \notin (\alpha, \beta)\} \\ \infty \qquad \text{if the above set is empty,} \end{cases} \\ \Omega^*(\alpha) := \{\omega \in \Omega;\ X(\tau_R + t_n, \omega) \in B_\alpha \quad \text{for some } t_1 < t_2 < \cdots \uparrow \infty\}\,, \\ \Omega(\alpha) := \{\omega \in \Omega;\ X(t, \omega) \in B_\alpha \quad \text{for some } t\}\,, \end{cases} \tag{3.5}$$

where τ_R is the exit time from the ball B_R.
Recalling Theorem 1, we have

$$P_y(\tau_{\alpha\beta} < \infty) = 1 \quad \text{for any } y \in K\,. \tag{3.6}$$

Hence, using Itô's formula and (3.4), we get for $\alpha < q(y) < \beta$

$$E_y\{f(X(\tau_{\alpha\beta}))\} - f(y) = E_y\{\int_0^{\tau_{\alpha\beta}} Lf(X(s))\,ds\} \leqq 0\,. \tag{3.7}$$

Appealing to the definition of f, we have for $\alpha < q(y) < \beta$

$$F(\alpha)P_y(q(X(\tau_{\alpha\beta})) = \alpha) + F(\beta)P_y(q(X(\tau_{\alpha\beta})) = \beta) \leqq F(q(y))\,. \tag{3.8}$$

Therefore we have

$$P_y(q(X(\tau_{\alpha\beta})) = \alpha) \leqq F(q(y))/F(\alpha) \quad \text{for } \alpha < q(y) < \beta .$$

Since $P_y(q(X(\tau_{\alpha\beta}))=\alpha)$ is increasing to $P_y(\Omega(\alpha))$ as $\beta \uparrow \infty$, we see that

$$P_y(\Omega(\alpha)) \leqq F(q(y))/F(\alpha) \quad \text{for } \alpha < q(y) . \tag{3.9}$$

Using the strong Markov property of X, we get

$$\begin{aligned} P_x(\Omega^*(\alpha)) &\leqq E_x[P_{X(\tau_R)}(\Omega(\alpha))] \leqq E_x[F(q(X(\tau_R)))/F(\alpha)] \\ &= E_x[F(R)/F(\alpha)] = F(R)/F(\alpha) \end{aligned} \tag{3.10}$$

for $q(x) < R$. Tending R to ∞, we conclude

$$P_x(\Omega^*(\alpha)) = 0 \quad \text{for any } x \in K , \tag{3.11}$$

since $\lim_{R\to\infty} F(R)=0$. Therefore we have

$$P_x(\liminf_{t\to\infty} q(X(t)) \geqq \alpha) = 1 \quad \text{for any } \alpha. \tag{3.12}$$

This completes the proof of Theorem 2.

For the proof of Theorem 3, we will show for simplicity

$$P_x(X(t_n) \in B_\alpha \quad \text{for some } t_1 < t_2 < \cdots \uparrow \infty) = 1 . \tag{3.13}$$

Because we can apply the similar argument for $B_\alpha(z)$, replacing $X(t)$ by $X(t)-z$. Define θ, I, F and f by

$$\theta(r) := 1+\varepsilon_2(r) \quad r \geqq R_0 , \quad I(s) := \int_{R_0}^{s} \theta(t)/t \, dt$$
$$F(r) := -\int_{R_0}^{r} e^{-I(s)} ds , \quad f(x) := F(q(x)) .$$

Then the condition (A_5) implies that

$$\lim_{r\to\infty} F(r) = -\infty . \tag{3.14}$$

Since f is twice Fréchet differentiable, in the same way as (3.4) we have

$$Lf(x) \geqq (1/2)A(x, x)F'(q(x))\{S(x)-\theta(q(x))\}/q(x) \geqq 0 . \tag{3.15}$$

From this, as we derived (3.8) from (3.6) and (3.7), we get

$$F(\alpha)P_y(q(X(\tau_{\alpha\beta})) = \alpha) + F(\beta)P_y(q(X(\tau_{\alpha\beta})) = \beta) \geqq F(q(y)) . \tag{3.16}$$

Tending β to ∞, we get from (3.14)

$$\lim_{\beta\to\infty} P_y(q(X(\tau_{\alpha\beta})) = \beta) = 0 . \tag{3.17}$$

Therefore (3.6) derives

$$P_y(\Omega(\alpha)) = \lim_{\beta\to\infty} P_y(q(X(\tau_{\alpha\beta})) = \alpha) = 1 \quad \text{for any } \alpha > 0 . \tag{3.18}$$

For any fixed $\alpha<R_1<R_2<\cdots\uparrow\infty$, we define stopping times t_m and σ_m by

$$t_1 := \inf\{t; X(t)\in B_\alpha\}, \quad \sigma_1 := \inf\{t; t>t_1 \text{ and } X(t)\in\partial B_{R_1}\},$$
$$\vdots$$
$$t_m := \inf\{t; t>\sigma_{m-1} \text{ and } X(t)\in B_\alpha\}, \ \sigma_m := \inf\{t; t>t_m \text{ and } X(t)\in\partial B_{R_m}\}.$$

Then we can easily see that

$$P_x(t_1<\infty) = P_x(\Omega(\alpha)) = 1 \quad \text{for any } x\in K \tag{3.19}$$

and that from Theorem 1, for $m=1, 2, \cdots$,

$$P_x(\sigma_m<\infty) = 1 \quad \text{for any } x\in K. \tag{3.20}$$

Again using the strong Markov property, we get by (3.19)

$$P_x(t_2<\infty) = E_x[P_{X(\sigma_1)}(t_1<\infty)] = 1 \quad \text{for any } x\in K. \tag{3.21}$$

Assume that $P_x(t_m<\infty)=1$ for any $x\in K$. Then (3.20) and the strong Markov property derive

$$P_x(t_{m+1}<\infty) = E_x[P_{X(\sigma_m)}(t_m<\infty)] = 1 \quad \text{for any } x\in K. \tag{3.22}$$

Therefore we see that for $m=1, 2, \cdots$,

$$P_x(t_m<\infty) = 1 \quad \text{for any } x\in K.$$

Since $t_1<\sigma_1<t_2<\sigma_2<\cdots$, we can show that

$$\lim_{m\to\infty} t_m = \infty \quad \text{a.s.} \tag{3.23}$$

by virtue of continuity (1.3) of X. Now we complete the proof.

EXAMPLE. Consider an Ornstein-Uhlenbeck type stochastic differential equation

$$dX(t) = GdB(t)-cX(t)dt. \tag{3.24}$$

We assume the condition (A_3), i.e.

$$p^2(G^*z)>0 \quad \text{if } z\neq 0. \tag{3.25}$$

For any fixed normalized $z\in K$, we put $\Gamma_R:=(1+|c|R)/p^2(G^*z)$. Then

$$p^2(G^*z)\Gamma_R-cq(x, z)\geqq 1+|c|R-|c|q(x)q(y)\geqq 1,$$

namely (A_2) holds.

Now, first assume that c is negative. Then we can easily see that

$$S(x) = (||G||^2-2cq^2(x))q^2(x)/p^2(G^*x)-1$$

$$\geqq (\|G\|^2 - 2cq^2(x))/\|G\|^2 - 1$$
$$= -2cq^2(x)/\|G\|^2 .$$

Putting $R_0 = \|G\|/\sqrt{|c|}$, we have

$$S(x) \geqq 2 \quad \text{whenever } q(x) \geqq R_0 ,$$

namely the constant function $\varepsilon_1 \equiv 1$ satisfies (1.7) and (1.8). So (A_4) holds in this case.

Next, consider the case when c is positive. Fix $z \in K$ arbitrarily. Then we have

$$A(x, x-z) = p^2(G^*(x-z))/q^2(x-z), \; C(x, x-z) = -2cq^2(x-z) - 2cq(z, x-z)$$

and

$$S(x, x-z) = [\|G^*\|^2 - 2cq^2(x-z) - 2cq(z, x-z)]q^2x-z)/p^2(G^*(x-z)) - 1 .$$

Hence

$$S(x, x-z) - 1 \leqq [\|G\|^2 - 2cq^2(x-z) - 2cq(z, x-z)]q^2(x-z)/p^2(G^*(x-z))$$
$$\leqq [\|G\|^2 - 2cq^2(x-z) + 2cq(z)q(x-z)]q^2(x-z)/p^2(G^*(x-z)) .$$

Consider the quadratic form $Q(\xi) = \|G\|^2 - 2c\xi^2 + 2cq(z)\xi$. Since c is positive, there exists $R_z > 0$ such that

$$Q(\xi) < 0 \quad \text{for } \xi > R_z .$$

Therefore $S(x, x-z) \leqq 1$ whenever $q(x-z) > R_z$. Setting $\varepsilon_2 = 0$ and $R^* = R_z$, we have (A_5).

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Department of Mathematics
Sichuan University
Chengdu, China