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A STOCHASTIC RESOLUTION OF A COMPLEX MONGE-AMPÈRE EQUATION ON A NEGATIVELY CURVED KÄHLER MANIFOLD

HIROSHI KANEKO

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1. Introduction

The Dirichlet problem for the complex Monge-Ampère equation on a strongly pseudo-convex domain of $\boldsymbol{C}^n$ was studied and solved by Bedford-Taylor [3]. The same problem for the Monge-Ampère equation on a negatively curved Kähler manifold has been recently proposed and solved by T. Asaba [2]. The main purpose of this paper is to solve the equation by using a method of the stochastic control presented by B. Gaveau [6].

Let M be an n-dimensional simply connected Kähler manifold with metric g whose sectional curvature K satisfies

$$-b^2 \leqq K \leqq -a^2$$

on M for some positive constants b and a. ω_0 denotes the associated Kahler form. We denote by $M(\infty)$ the Eberlein-O'Neill's ideal boundary of M and we always consider the cone topology on $\bar{M}=M \cup M(\infty)$ (see [4] for these notions). T. Asaba formulated the Monge-Ampère equation on M in the following manner:

We write $\mathrm{PSH}(D)$ for the family of locally bounded plurisubharmonic functions defined on a complex manifold D. When $u \in \mathrm{PSH}(D)$, the current $(dd^c u)^n = \underbrace{dd^c u \wedge \cdots \wedge dd^c u}_{n\text{-copies}}$ of type-(n, n) is defined as a positive Radon measure on D. Therefore, for given functions $f \in C(M)$ and $\varphi \in C(M(\infty))$, the complex Monge-Ampère equation

$$(1) \qquad \begin{cases} u \in \mathrm{PSH}(M) \cap C(\bar{M}) \\ (dd^c u)^n = f\omega_0^n/n! \qquad \text{on } M \\ u|_{M(\infty)} = \varphi \end{cases}$$

can be considered. T. Asaba found a unique solution of (1) by imposing the following condition on f: there exist two positive constants μ_0 and C_0 such that

$$(2) \qquad 0 \leqq f \leqq C_0 e^{-\mu_0 r}.$$

Here and in the sequel r stands for the distance function from a fixed point of M. Following a similar line to the proof performed by B. Gaveau [6], in which a stochastic proof of the existence of the Monge-Ampère equation on a strongly pseudo-convex domain of $\boldsymbol{C}^n$ was presented, we will prove not only the existence of the solution of (1) but also its uniqueness (§ 3, Thoerem B). Actually T. Asaba assumed condition (2) for a specific value of μ_0. In what follows, we assume the condition (2) on f holding for some $\mu_0>0$ and $C_0>0$.

In accordance with the first part of B. Gaveau [6], a certain transience behavior of the sample path of the conformal martingales on M need to be studied. It was conjectured by H. Wu [13] that M is biholomorphic to a bounded domain of $\boldsymbol{C}^n$ (cf. Y.T. Siu [11] and R.E. Greene [7]). If this would be true, then the conformal martingales of the type considered by B. Gaveau [6] must hit the boundary of M. In fact, we shall prove in Section 2 that the almost all sample paths of every non-degenerate conformal martingale converge to points of the ideal boundary $M(\infty)$. We use the method of J.J. Prat [10], in which the sample paths' property was proven for the Brownian motion on Riemannian manifolds with negative curvature bounded away from zero.

The basic estimates obtained in Section 2 will be further utilized after Section 3 in resolving the Monge-Ampère equation stochastically.

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2. Basic estimates for non-degenerate conformal martingales

We first define the notion of the conformal martingales on M.

DEFINITION. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geqq 0}$. An M-valued continuous stochastic process $(Z_t)_{0\leqq t<\zeta}$ defined up to a stopping time $\zeta>0$ is said to be a conformal martingale, if

(i) there exists $p\in M$ such that $Z_0=p$ a.s.

(ii) there exists a sequence of stopping times $(T_n)_{n=1}^{\infty}$ such that $T_n<\zeta$, $\lim T_n=\zeta$ and $(f(Z_{t\wedge T_n}))_{t\geqq 0}$ is a $\boldsymbol{C}$-valued bounded $(\mathcal{F}_t)$-martingale for every holomorphic function f on M (we need note that M is a Stein manifold and so M possessess enough holomorphic functions).

Noting the trivialty of the bundle of unitary frames, we choose smooth vector fields $X_1, \cdots, X_n$ of type-(1, 0) on M so that $g(X_\alpha, X_{\bar\beta})=\delta_{\alpha,\beta}$ on M. For a smooth function f defined on M, we write Lf for the Levi-form of f. The notion of conformal martingale is related to the Levi-form in the following way:

Proposition 1. *For each conformal martingale $(Z_t)_{0\leqq t<\zeta}$ on M, there is a non-negative hermitian matrix valued $(\mathcal{F}_t)$-adapted process $(\sum_{\alpha,\bar\beta}(t))_{0\leqq t<\zeta}$ such that* *it is increasing (in the sense that* $s \leqq t \Rightarrow \sum_{\alpha,\bar\beta}(s) \leqq \sum_{\alpha,\bar\beta}(t)$ *as hermitian matrices a.s.) and that, for each real valued function* $f \in C^2(M)$

$$f(Z_t)-f(Z_0)-\sum_{\alpha,\beta=1}^{n}\int_0^t Lf(X_\alpha, X_{\bar\beta})_{Z_s} d\Sigma_{\alpha,\bar\beta}(s)$$

is a local martingale.

Proof. Take countable local complex charts $(U_i; z_i^1, \cdots, z_i^n)_{i=1,2\cdots}$ of M and closed sets $V_i \subset U_i$ such that $\{V_i\}_{i=1}^\infty$ covers M. Since M is a Stein manifold, we may assume that $z_i^1, \cdots, z_i^n$ are the restrictions to U_i of certain holomorphic functions on M for every $i=1, 2, 3, \cdots$. Define a sequence of stopping times σ_k and random variables i_k successively as follows:

$$\begin{aligned}
\sigma_0 &= 0\\
i_0 &= \inf\{i;\ Z_0 \in V_i\}\\
\sigma_1 &= \inf\{t>0;\ Z_t \notin U_{i_0}\}\\
&\cdots\\
\sigma_k &= \inf\{t>\sigma_{k-1};\ Z_t \notin U_{i_{k-1}}\}\\
i_k &= \inf\{i;\ Z_{\sigma_k} \in V_i\}\\
&\cdots
\end{aligned}$$

By virtue of Ito's formula, we obtain

$$\begin{aligned}
f(Z_{t\wedge\sigma_k})-f(Z_{t\wedge\sigma_{k-1}}) &= \sum_{\beta=1}^{n}\int_{t\wedge\sigma_{k-1}}^{t\wedge\sigma_k} \partial f/\partial z^\alpha(Z_s)dz^\alpha(Z_s)\\
&+\sum_{\alpha=1}^{n}\int_{t\wedge\sigma_{k-1}}^{t\wedge\sigma_k} \partial f/\partial z^{\bar\alpha}(Z_s)dz^{\bar\alpha}(Z_s)\\
&+\sum_{\alpha,\beta=1}^{n}\int_{t\wedge\sigma_{k-1}}^{t\wedge\sigma_k} \partial^2 f/\partial z^\alpha \partial z^{\bar\beta}(Z_s)d\langle z^\alpha(Z_s),\ z^{\bar\beta}(Z_s)\rangle,
\end{aligned}$$

where $z^\alpha = z^\alpha_{i_{k-1}}$, $\alpha=1, 2, \cdots, n$, $k=1, 2, 3, \cdots$. Define a hermitian matrix valued process $\sigma(t)$ by $\sum_{\kappa-1}^{n} \sigma_\kappa^\alpha(t)(\partial/\partial z^\kappa|_{Z_t}) = X_\alpha|_{Z_t}$, $\alpha=1, 2, \cdots, n$ and set

$$\Sigma_{\alpha\bar\beta}(t) = \sum_{\kappa,\lambda=1}^{n}\int_0^t \sigma_\kappa^\alpha(s)\sigma_{\bar\lambda}^{\bar\beta}(s)d\langle z^\kappa(Z_s),\ z^{\bar\lambda}(Z_s)\rangle,$$

then this can be well defined, independently of the choice of local coordinates, and further

$$f(Z_{t\wedge\sigma_k})-f(Z_{t\wedge\sigma_{k-1}})-\sum_{\alpha,\beta=1}^{n}\int_{t\wedge\sigma_{k-1}}^{t\wedge\sigma_k} Lf(X_\alpha, X_{\bar\beta})_{Z_s} d\Sigma_{\alpha,\bar\beta}(s)$$

is a martingale. Since $\lim_{k=\infty}\sigma_k = \zeta$, the proof is completed. q.e.d.

For our investigation, it is enough to consider exclusively conformal martingales $(Z_t)_{0\leqq t<\zeta}$ for which the following stopping times τ_k ($k=0, 1, 2, 3, \cdots$) are finite almost surely:

$$\tau_0 = 0$$
$$\tau_1 = \inf\{t>0;\ \mathrm{dist}(Z_t, Z_0) = 1\}$$
$$\cdots$$
$$\tau_{k+1} = \inf\{t>\tau_k;\ \mathrm{dist}(Z_t, Z_{\tau_k}) = 1\}$$

We call such property "admissible" and in what follows τ_k means the above stopping time. Here, we present a basic estimate of the same type as in D. Sullivan [12].

Proposition 2. *For any $\mu\in(0, a)$, there exists a constant $C_1\in(0, 1)$ such that*

$$E[\exp(-\mu r(Z_{\tau_{k+1}}))]\leqq C_1 E[\exp(-\mu r(Z_{\tau_k}))]\,, \qquad k = 0, 1, 2, 3, \cdots,$$

for every admissible conformal martingale $(Z_t)_{0\leqq t<\zeta}$.

Proof. A Jacobi field estimate—the Hessian comparison theorem presented in [8; Theorem A] implies

$$L\exp(-\mu r)\leqq(\mu(\mu-a)/2)\exp(\mu r)g \qquad \text{in the sense [8]}.$$

By applying Proposition 1 to the function $\exp(-\mu r)$, we then have

$$\begin{aligned}
&E[\exp(-\mu r(Z_{\tau_{k+1}}))] = E[\exp(-\mu r(Z_{\tau_k}))]\\
&\quad +E[\sum_{\alpha,\beta=1}^{n}\int_{\tau_k}^{\tau_{k+1}} L\exp(-\mu r)(X_\alpha, X_\beta)_{Z_s} d\Sigma_{\alpha,\beta}(s)]\\
&\quad \leqq E[\exp(-\mu r(Z_{\tau_k}))]\\
&\quad +(\mu(\mu-a)/2)E[\int_{\tau_k}^{\tau_{k+1}}\exp(-\mu r(Z_s))d(\mathrm{trace}\,\Sigma_{\alpha,\beta}(s))]\,,\\
&\quad k = 0, 1, 2, \cdots.
\end{aligned}$$

While, taking conditional expectation, we have

$$\begin{aligned}
&E[\int_{\tau_k}^{\tau_{k+1}}\exp(-\mu r(Z_s))d(\mathrm{trace}\,\Sigma_{\alpha,\beta}(s))]\\
&= \int_M P(Z_{\tau_k}\in d\eta)E[\int_{\tau_k}^{\tau_{k+1}}\exp(-\mu r(Z_s))d(\mathrm{trace}\,\Sigma_{\alpha\beta}(s))\,|\,Z_{\tau_k} = \eta]\\
&\geqq \int_M P(Z_{\tau_k}\in d\eta)\exp(-\mu(r(\eta)+1))E[\int_{\tau_k}^{\tau_{k+1}} d(\mathrm{trace}\,\Sigma_{\alpha,\beta}(s))\,|\,Z_{\tau_k} = \eta]\,,
\end{aligned}$$

which is not less than $\exp(-\mu)C_2^{-1}E[\exp(-\mu r(Z_{\tau_k}))]$ by virtue of Lemma 1 stated below. Hence we arrive at the desired estimate with $C_1=1+((\mu(\mu-a)/2))C_2^{-1}\exp(-\mu)$. q.e.d.

In the above proof, we have used the next lemma, which also will be utilized in § 4.

Lemma 1. *There exists a positive constant* C_2 *depending only on a and b such that*

$$C_2^{-1} \leqq E\left[\int_{\tau_k}^{\tau_{k+1}} d(\text{trace} \sum_{\alpha,\bar{\beta}}(s)) \mid Z_{\tau_k} = \eta\right] \leqq C_2$$

holds $P(Z_{\tau_k} \in d\eta)$*-a.s.* $k=0, 1, 2, 3, \cdots$, *for every admissible conformal martingale* Z_t.

Proof. For $f \in C_b^2(M)$, we know from Proposition 1 that

$$E\left[f(Z_{\tau_{k+1}}) - f(Z_{\tau_k}) - \sum_{\alpha,\beta=1}^{n} \int_{\tau_k}^{\tau_{k+1}} Lf(X_\alpha, X_{\bar{\beta}})_{Z_s} d\sum_{\alpha,\bar{\beta}}(s) \mid Z_{\tau_k} = \eta\right] = 0$$

$$P(Z_{\tau_k} \in d\eta)\text{-a.s.}, \ k = 0, 1, 2, 3, \cdots.$$

Taking a countably dense subset of $C_b^2(M)$ and by the approximation procedure we know that the exceptional η-set in the above statement can be taken independently of $f \in C_b^2(M)$. Choose $f = f^{(\eta)}(p) \in C_b^2(M)$ which coincides with $\text{dist}(p, \eta)^2$ on a neighborhood of $\{p;\ \text{dist}(p, \eta) \leqq 1\}$. Then it turns out that

$$1 = E\left[\sum_{\alpha,\beta=1}^{n} \int_{\tau_k}^{\tau_{k+1}} Lf(X_\alpha, X_{\bar{\beta}})_{Z_s} d\sum_{\alpha,\bar{\beta}}(s) \mid Z_{\tau_k} = \eta\right] \qquad P(Z_{\tau_k} \in d\eta)\text{-a.s.}$$

Again by the Hessian comparison theorem, we find that there exists a constant C_2 depending only on the curvature bounds a and b such that

$$C_2 g \leqq Lf^{(\eta)} \leqq C_2^{-1} g \qquad \text{on } \{p;\ \text{dist}(p, \eta)\} \leqq 1,$$

so we have

$$C_2^{-1} \leqq E\left[\int_{\tau_k}^{\tau_{k+1}} d(\text{trace} \sum_{\alpha,\bar{\beta}}(s)) \mid Z_{\tau_k} = \eta\right] \leqq C_2$$

$$P(Z_{\tau_k} \in d\eta)\text{-a.s.} \qquad \text{q.e.d.}$$

The next theorem is an immediate consequence of Proposition 2 combined with the geometrical method employed by D. Sullivan [12] and J.J. Prat [10].

Theorem A. *For every admissible conformal martingale* $(Z_t)_{0 \leqq t < \zeta}$, *the following are true:*

(i) *The limit* $\lim_{t \uparrow \zeta} Z_t$ *exists in* $M(\infty)$ *a.s.*

(ii) *F any* $\xi \in M(\infty)$, $\varepsilon > 0$ *and neighborhood* $V \subset M(\infty)$ *of* ξ, *there exists a neighborhood* $U \subset \bar{M}$ *of* ξ *relative to the cone topology such that*

$$P(\lim_{t \uparrow \zeta} Z_t \in V) \geqq 1 - \varepsilon,$$

whenever Z_t *strats from a point of* U. *U does not depend on the choice of* $(Z_t)_{0 \leqq t < \zeta}$.

3. The stochastic solution of the Monge-Ampère equation—the statement of the main theorem

Let K_p be the family of all admissible conformal martingales $Z=(Z_t)_{0\leq t<\zeta(Z)}$ on M such that Z starts from $p\in M$ and the associate process $(\sum_{\alpha,\bar\beta}(t))_{0\leq t<\zeta(Z)}$ in Proposition 1 possesses a density $(A_{\alpha,\bar\beta}(t))_{0\leq t<\zeta(Z)}$ with respect to the Lebesgue measure dt with $\det A_{\alpha,\bar\beta}(t)\geqq 1$ for $t\geqq 0$ a.s. For $Z\in K_p$, set

$$w(p, Z) = E[-C(n)\int_0^{\zeta(Z)} f^{1/n}(Z_t)\,dt+\varphi(Z_{\zeta(Z)})]\,,$$

where $C(n)=n/8(n!)^{1/n}$. By virtue of Lemma 2 in the next section, we know that, if $Z=(Z_t)$ is the conformal diffusion generated by the Kahler mertic g on M, then $w(p, Z)$ is exactly the solution of the Dirichlet problem with boundary condition on the sphere at infinity:

$$\begin{cases} \Delta_g u/2 = C(n)f^{1/n} \\ u\,|_{M(\infty)} = \varphi \end{cases}$$

for the Laplace-Beltrami operator Δ_g related to g. Now, we can describe the solution of the Monge-Ampère equation (1), using the above stochastic notations.

Theorem B. *The function*

$$u(p) = \inf_{Z\in K_p} w(p, Z)\,, \qquad p\in M \tag{3}$$

is the unique solution of the Monge-Ampère equation (1).

In the following sections, we shall prove this theorem. The proof will be performed by the stochastic control method due to B. Gaveau [6].

4. Continuity of the stochastic solution

In this section, we shall prove the continuity of the function u defined by (3).

Proposition 3. *u can be extended to a continuous function on $\bar M$ and $u\,|_{M(\infty)}=\varphi$.*

We have to prepare several lemmas for the proof.

Lemma 2. *For each $Z\in K_p$, there exist positive constants ν and C_3 depending only on the constants μ_0, C_0 in* (2) *and the curvature bounds such that*

$$E[\int_0^{\zeta(Z)} f(Z_t)^{1/n}\,dt]\leqq C_2\exp(-\nu r(p))\,.$$

Proof. By the assumption (2) imposed on f, for $\nu\leqq\mu_0$, we know

$$
\begin{aligned}
&E[\int_0^{\zeta(Z)} f(Z_t)^{1/n}dt]\\
&\leqq C_0 E[\int_0^{\zeta(Z)} \exp(-\nu r(Z_t)/n)dt]\\
&\leqq C_0 \sum_{k=0}^{\infty} E[\int_{\tau_k}^{\tau_{k+1}} \exp(-\nu r(Z_t)/n)\,dt]\,,
\end{aligned}
$$

where $\tau_0=0$, $\tau_1=\inf\{t>0;\ \mathrm{dist}(Z_t, Z_0)=1\}$, $\cdots$, $\tau_{k+1}=\inf\{t>\tau_k;\ \mathrm{dist}(Z_t,Z_{\tau_k})=1\}$,$\cdots$. We may assume that ν is so small that ν/n is less than a. Because $E[\int_{\tau_k}^{\tau_{k+1}}\exp(-\nu r(Z_t)/n)\,dt]\leqq E[\int_{\tau_k}^{\tau_{k+1}}\exp(-\nu r(Z_t)/n)d(\mathrm{trace}\sum_{\alpha,\bar\beta}(t))]$, we have $E[\int_{\tau_k}^{\tau_{k+1}}\exp(-\nu r(Z_t)/n)dt]\leqq\exp(a)C_2E[\exp(-\nu r(Z_{\tau_k})/n)]$, in view of the proof of Proposition 2. Further by virtue of the basic estimate (Proposition 2) we know

$$\sum_{k=0}^{\infty} E[\exp(-\nu r(Z_{\tau_k})/n]\leqq(1-C_1)^{-1}\exp(-\nu r(p)/n)\,.$$

The desired inequality holds for $C_3=\exp(a)C_0C_2(1-C_1)$. q.e.d.

Combining this with the result on the weak convergence of the hitting distribution in Theorem A (ii), we know that for arbitrary $\xi\in M(\infty)$ and any $\varepsilon>0$, there exists a neighborhood U of ξ such that

$$p\in U\Rightarrow|w(p, Z)-\varphi(\xi)|<\varepsilon\,, \tag{4}$$

when $Z\in K_p$. Furthermore, we can show the following lemma.

Lemma 3. *For any $\varepsilon>0$, there exist a positive large constant R and a small constant γ_0 such that, if*

$$p\notin D_R=\{\eta\in M;\ r(\eta)<R\}$$

and $\mathrm{dist}(p, q)<\gamma_0$, *then*

$$|w(p, Z)-w(q, Z')|<\varepsilon\,,$$

for any $Z\in K_p$ and $Z'\in K_q$.

Proof. For any $\varepsilon>0$, there exist some points $\xi_1, \cdots, \xi_n\in M(\infty)$ and open sets $U_i\ni\xi_i$ such that

$$
\begin{aligned}
&p\in U_i \text{ and } Z\in K_p\\
&\quad\Rightarrow|w(p, Z)-\varphi(\xi_i)|<\varepsilon/2
\end{aligned}
$$

for all $i=1, 2, \cdots, n$ and $M(\infty)\subset\bigcup_{i=1}^{n} U_i$. Take a closed neighborhood $U'_i\subset U_i$ of ξ_i so that $M(\infty)\subset\bigcup_{i=1}^{n} U'_i$. Then, there exists $R>0$ satisfying $M\backslash D_R\subset\bigcup_{i=1}^{n} U'_i$. Therefore for sufficiently small, γ_0 we know that

$$\begin{aligned}&\operatorname{dist}(p, q)<\gamma_0,\ p\notin D_R\\ &\quad\Rightarrow |w(p, Z)-w(q, Z')|\\ &\quad\leqq |w(p, Z)-\varphi(\xi_i)|+|\varphi(\xi_i)-w(q, Z')|\\ &\quad<\varepsilon/2+\varepsilon/2=\varepsilon\,,\end{aligned}$$

whenever $Z\in K_p$ and $Z'\in K_q$, by choosing i so that $p\in U'_i$. q.e.d.

Because the holomorphic tangent bundle is holomorphically trivial, there exists a frame of holomorphic vector fields $Y_1, \cdots, Y_n$. Let $\Phi_z(p)=\operatorname{Exp}(\operatorname{Re}\sum_{i=1}^{n} z^i Y_i)(p)$, for $p\in M$ and $z=(z^1, \cdots, z^n)$ in $\boldsymbol{C}^n$. This transformation on M was considered in T. Asaba [2] and proven to enjoy the next property:

For any $R>0$, there exists $\Delta_\delta=\{z\in C^n;\ \sum_{i=1}^{n}|z^i|^2<\delta\}$ such that $\Phi_z(p)$ is a smooth mapping from $\Delta_\delta\times D_R$ to M satisfying the following properties (i), (ii) and (iii).

(i) For each $z\in\Delta_\delta$, Φ_z gives a biholomorphic mapping from the domain D_R to $\Phi_z(D_R)$.

(ii) Φ_0 is the identity transformation on D_R.

(iii) For $p\in D_R$, $\Phi_\cdot(p)$ defines a diffeomorphism from Δ_δ to some neighborhood of p.

Using this transformation Φ, we can prove the continuity of the stochastic solution u.

Lemma 4. *For any $\varepsilon>0$ and $R>0$, there exists $\gamma>0$ such that for each $Z\in K_p$ and q enjoying $p\in D_R$ and* $\operatorname{dist}(p, q)<\gamma$, *we can always find $Z'\in K_q$ so that*

$$|w(p, Z)-w(q, Z')|<\varepsilon\,.$$

Proof. To begin, replace R by a sufficiently large one and choose γ_0 so that the implication in Lemma 3 holds for $\varepsilon/2$ instead of ε. Fix $Z\in K_p$. We then consider the holomorphic local transformation Φ and the Kähler diffusion $B_t(\eta)$ on M starting from $\eta\in M$, independent of Z and measurable in t, z and ω. Let

$$(5)\qquad Z_t^{\Phi_z(p)}=\begin{cases}\Phi_z(Z_t)\,, & t\leqq\tau\\ B_{t-\tau}(\Phi_z(Z_\tau))\,, & t>\tau\,,\end{cases}$$

where $\tau=\inf\{t>0;\ Z_t\notin D_R\}$.

We next perform the time change by letting $\hat{Z}_t^{\Phi_z(p)}=Z_{\tau_t}^{\Phi_z(p)}$, up to the explosion time of $\hat{Z}^{\Phi_z(p)}=(\hat{Z}_t^{\Phi_z(p)})_{t\geq 0}$, where $\tau_t=\inf\{s>0;\ \int_0^s(\det A_{\alpha,\bar\beta}(u))^{1/n}du\geq t\}$, $(A_{\alpha,\bar\beta}(t))_{t\geq 0}$ being the density of the increasing process associated with $Z^{\Phi_z(p)}=(Z_t^{\Phi_z(p)})_{t\geq 0}$ according to Proposition 1.

On the other hand, taking conditional expectation, we have

$$w(p,\ Z)=W[-C(n)\int_0^\tau f^{1/n}(Z_t)dt]$$
$$+\int_{\partial D_R}E[C(n)\int_\tau^{\zeta(Z)}f^{1/n}(Z_t)dt+\varphi(Z_{\zeta(Z)})\,|\,Z_\tau=\eta]P(Z_\tau\in d\eta)\,.$$

If we set $W_t=Z_{t+\tau}$ and let

$$w(\eta,\ W)=E[-C(n)\int_0^{\zeta(Z)-\tau}f^{1/n}(W_t)dt+\varphi(Z_{\zeta(Z)})\,|\,Z_\tau=\eta]$$

for $W=(W_t)_{0\leq t<\zeta(Z)-\tau}$, then

$$w(p,\ W)=E[-C(n)\int_0^\tau f^{1/n}(Z_t)dt]+\int_{\partial D_R}w(\eta,\ W)P(Z_\tau\in d\eta)\,.$$

Similarly, letting σ be the first exit time from $\Phi_z(D_R)$ of $\hat{Z}^{\Phi_z(p)}$, we set $W_t^{\Phi_z(p)}=\hat{Z}_{t+\sigma}^{\Phi_z}$, $0\leq t<\zeta(\hat{Z}^{\Phi_z(p)})-\sigma$ and then, for $W^{\Phi_z(p)}=(W_t^{\Phi_z(p)})_{t\geq 0}$,

$$w(\eta,\ W^{\Phi_z(p)})=E[-C(n)\int_0^{\zeta(W^{\Phi_z(p)})}f^{1/n}(W_t^{\Phi_z(p)})dt$$
$$+\varphi(W^{\Phi_z(p)}_{\zeta(W^{\Phi_z(p)})})\,|\,\hat{Z}_\sigma^{\Phi_z(p)}=\eta]\,.$$

Then

$$w(\Phi_z(p),\ \hat{Z}^{\Phi_z(p)})=E[-C(n)\int_0^\sigma f^{1/n}(\hat{Z}_t^{\Phi_e(p)})dt]$$
$$+\int_{\partial\Phi_z(D_R)}w(\eta',\ W^{\Phi_e(p)})P(\hat{Z}_\sigma^{\Phi_z(p)}\in d\eta')$$

Therefore, after all we have that

$$w(p,\ Z)-w(\Phi_z(p),\ Z^{\Phi_z(p)})=E[-C(n)(\int_0^\tau f^{1/n}(Z_t)\,dt-\int_0^\sigma f^{1/n}(\hat{Z}_t^{\Phi_e(p)})dt)]$$
$$+\int_{\partial D_R}\{w(\eta,\ W)-w(\Phi_z(\eta),\ W^{\Phi_z(p)})\}P(Z_\tau\in d\eta)\,.$$

From Lemma 2, there exists $\delta>0$ such that the absolute value of the second term of the right hand side is less than $\varepsilon/2$ for every $z\in\Delta_\delta$. While the continuity of $f^{1/n}$ shows that the first term of the right hand side is less than $\varepsilon/2$ in the abo absolute value, whenever $z\in\Delta_\delta$.

Because, for sufficiently small γ, the γ-neighborhood of each $p\in D_R$ is contained in the image of Δ_δ by the mapping $\Phi.(p)$, for $q=\Phi_z(p)$, $Z'=\hat{Z}^{\Phi_z(p)}$ is the required conformal martingale in our lemma. q.e.d.

Proof of Proposition 3. The last inequality in Lemma 4 implies $w(p, Z)\geqq u(q)-\varepsilon$. Taking the infimum over $Z\in K_p$, we can conclude that $u(p)\geqq u(q)-\varepsilon$, whenever p, $q\in D_R$ and $\mathrm{dist}(p, q)<\gamma$. Exchanging the role of p and q, we see that u is a continuous function on M. Recalling the estimate (4) noted after Lemma 2, we know that $\lim_{p\to\xi} u(p)=\varphi(\xi)$ for each $\xi\in M(\infty)$. This completes the proof. q.e.d.

5. The Bellman principle

The purpose of this section is to establish the Bellman principle in order to localize the stochastic expression of the function u defined by (3).

Proposition 4. *For every bounded domain D of M and $p\in D$, we obtain*

$$u(p) = \inf_{Z\in K_p} E[-C(n)\int_0^{\tau_D(Z)} f^{1/n}(Z_t)dt+u(Z_{\tau_D(Z)})]\,,$$

where $\tau_D(Z)=\inf\{t>0;\ Z_t\notin D\}$.

Proof. Fix $\varepsilon>0$ and take R so that $D_R\supset\bar{D}$. For each $q\in\partial D$ there exist $\delta>0$ and $Z\in K_q$ such that, for $z\in\Delta_\delta$,

$$|w(\Phi_z(q), \hat{Z}^{\Phi_z(q)})-u(q)|>\varepsilon\,,$$

where $Z^{\Phi_z(q)}$ is the conformal martingale defined by (5). Therefore, we can select several points $q_1, \cdots, q_n\in\partial D$ and their neighborhoods $\Delta(q_1), \cdots, \Delta(q_n)$ so that $\partial D\subset\bigcup_{i=1}^n\Delta(q_i)$ (disjoint union), the image of $\Phi.(q_i)$ contains $\Delta(q_i)$ and

$$|w(\Phi_z(q_i), \hat{Z}^{\Phi_z(q_i)})-u(q_i)|<\varepsilon\,,$$

whenever $Z^{\Phi_z(q_i)}$ is in $\Delta(q_i)$, $i=1, 2, \cdots, n$.

For each $Z\in K_p$, we set

$$Z_t^* = \begin{cases} Z_t\,, & \text{if } t\leqq\tau_D(Z) \\ \hat{Z}^{\Phi_z(q_i)}_{t-\tau_D(Z)} & \text{if } t>\tau_D(Z),\ Z_{\tau_D(Z)}\in\Delta(q_i) \text{ and} \\ & \Phi_z(q_i) = Z_{\tau_D(Z)},\ i = 1, 2, \cdots, n\,, \end{cases}$$

where we take $Z^{\Phi_z(q_i)}$ and Z to be independent. Then $Z^*=(Z_t^*)\in K_p$. By the same method of B. Gaveau [6; pp. 400–403], we can prove that

$$u(p)-\varepsilon \leqq E[-C(n)\int_0^{\tau_D} f^{1/n}(Z_t)dt+u(Z_{\tau_D})]$$
$$\leqq E[-C(n)\int_0^{\zeta(Z)} f^{1/n}(Z_t)dt+\varphi(Z_{\zeta(Z)})]\,.$$

Since $\varepsilon>0$ is arbitrary, the proof is completed. q.e.d.

6. Proof of the main theorem

Finaly, we shall finish the proof of the main theorem by showing the next two propositions.

Proposition 5. *u is a plurisubharmonic function and $(dd^c u)^n=f\omega_0^n/n!$ on M.*

Proposition 6. *If u_0 is a solution of* (1), *then*

$$u_0(p)=\inf_{Z\in K_p} E[-C(n)\int_0^{\zeta} f^{1/n}(Z_t)dt+\varphi(Z_\zeta)]\,.$$

In particular, (1) *has a unique solution.*

Proof of Proposition 5. Let p be an arbitrary point of M. Choose a complex local coordinate system $(D, z^1, \cdots, z^n)$ around p such that $\psi=(z^1, \cdots, z^n)$ defines a biholomorphic mapping from D to the complex unit ball $B=\{(z^1, \cdots, z^n)\in \boldsymbol{C}^n;\ \sum_{i=1}^{n}|z^i|^2<1\}$. For the push forward function $U(z)=(\psi_* u)(z)=u(\psi^{-1}(z))$,

$$U(z)=\inf_{Z\in K_z} E[-C(n)\int_0^{\tau_B(Z)} (\psi_*(f\det(g)))_{i\bar{j}}^{1/n}(Z_t)dt+U(Z_{\tau_B(Z)})]\,,$$

where $g_{i\bar{j}}=g(\partial/\partial z^i, \partial/\partial z^{\bar{j}})$) and K_z is the family of all $\boldsymbol{C}^n$-valued conformal martingales Z which start from $z\in B$ such that $a_{i\bar{j}}(t)=d\langle z^i(Z_t), z^{\bar{j}}(Z_t)\rangle/dt$ satisfy $\det(a_{i\bar{j}}(t))\geqq 1$, $t\geqq 0$ a.s.

Consider the following Monge-Ampère equation

$$(6)\qquad \begin{cases} v\in PSH(B)\cap C(\bar{B}) \\ (dd^c v)^n=\psi_*(f\det(g_{i\bar{j}}))dV \\ v|_{\partial B}=U|_{\partial B}\,, \end{cases}$$

where dV stands for the Lebesgue measure on $\boldsymbol{C}^n$. Because of the strongly pseudo-convexity of B, we see that Theorem 4 and Remark of B. Gaveau [6; pp. 402–403] ensure the following stochastic description of the solution v_0 of (6):

$$v_0(z)=\inf_{Z\in K_z} E[-C(n)\int_0^{\tau_B(Z)} (\psi_*(f\det(g_{i\bar{j}})))^{1\ n}(Z_t)dt$$
$$+U(Z_{\tau_B(Z)})]\,,\qquad z\in B\,.$$

Hence, we know that $v_0=U$ on B and $u(p)=\psi_* v_0(p)\in PSH(D)$ and that $(dd^c u)^n = f\omega_0^n/n!$ on D. q.e.d.

Proof of Proposition 6. To begin, take the countable family of charts $(U_i; z_i^1, \cdots, z_i^n)_{i=1}^{\infty}$ appeared in the proof of Proposition 1, we may assume that each $\psi_i=(z_i^1, \cdots, z_i^n)$ gives a biholomorphic mapping between U_i and the unit ball $B\subset C^n$. By virtue of Theorem 4 of B. Gaveau [6], for any $\varepsilon>0$, there exists a $Z^{(1)}\in K_p$ such that

$$E[-C(n)\int_0^{\sigma_1} f^{1/n}(Z_t)dt+u_0(Z_{\sigma_1}^{(1)})]\leqq u_0(p)+\varepsilon/2\,,$$

where σ_1 is the stopping time for $Z^{(1)}$ defined in the proof of Proposition 1. For each $q\in\partial U_i$ there exists $\delta>0$ and $Z\in K_q$ such that

$$w(\Phi_z(q),\ \hat{Z}^{\Phi_z(q)})<u_0(q)+\varepsilon/2^2\,,$$

whenever $z\in\Delta_\delta$. Using the same argument as in the proof of Proposition 4, we can construct $Z^{(2)}\in K_p$ which satisfies

$$Z_{t\wedge\sigma_1}^{(1)}=Z_{t\wedge\sigma_1}^{(2)}$$

and

$$E[-C(n)\int_0^{\sigma_2} f^{1/n}(Z_t^{(2)})dt+u_0(Z_{\sigma_2}^{(2)})]\leqq u_0(p)+\varepsilon/2+\varepsilon/2^2\,,$$

where σ_2 is defined for $Z^{(2)}$ in the same way as above. Repeating this procedure, we obtain a sequence $(Z^{(k)})_{k=1}^{\infty}\subset K_p$ so that $Z_{t\wedge\sigma_{k-1}}^{(k-1)}=Z_{t\wedge\sigma_{k-1}}^{(k)}$, $t\geqq 0$. a.s. and that

$$E[-C(n)\int_0^{\sigma_k} f^{1/n}(Z_t^{(k)})dt+u_0(Z_{\sigma_k}^{(k)})]\leqq u_0(p)+\sum_{i=1}^{k}\varepsilon/2^i\,,$$

where σ_k is defined for $Z^{(k)}$ as above.

Define $Z_t=Z_t^{(k)}$, if $t<\sigma_k$. Then we can easily check that $Z=(Z_t)\in K_p$ and that $\lim_{k\to\infty}\sigma_k=\zeta(Z)$. Hence, we know

$$E[-C(n)\int_0^{\zeta} f^{1/n}(Z_t)dt+\varphi(Z_\zeta)]\leqq u_0(p)+\varepsilon\,.$$

Letting $\varepsilon\to 0$, we can conclude that

$$u_0(p)\geqq \inf_{Z\in K_p} E[-C(n)\int_0^{\zeta} f^{1/n}(Z_t)dt+\varphi(Z_\zeta)]\,.$$

On the other hand, we can inductively obtain, for each $Z\in K_p$,

$$u_0(p)\leqq E[-C(n)\int_0^{\sigma_k} f^{1/n}(Z_t)dt+u_0(Z_{\sigma_k})]\,,\qquad k=1,2,3,\cdots,$$

and so we have the opposite inequality, by letting $k\to\infty$. q.e.d.

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Department of Mathematical Science
College of Engineering
University of Osaka Prefecture
Mozu-umemachi, Sakai-city
Osaka 591, Japan