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A TOPOLOGICAL INVARIANT RELATED TO THE NUMBER OF ORTHOGONAL GEODESIC CHORDS

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1. Introduction

A geodesic on a Riemannian manifold with boundary is called an *orthogonal geodesic chord* if it connects two points of the boundary and intersects with the boundary orthogonally at both end points. For orthogonal geodesic chords, Lyusternik and Schnirelmann [7] prove the existence of n such chords of any convex body in $\boldsymbol{R}^n$ (compact convex C^∞ submanifold with boundary of $\boldsymbol{R}^n$ with an interior point) and Bos [1] extends it to locally convex disks of dimension n (the precise definition is given later).

In this note, we denote by M a compact Riemannian C^∞ manifold of dimension n with boundary, define a topological invariant integer $\nu(M)$ and show that there exist at least $\nu(M)$ non-constant orthogonal geodesic chords of M if the boundary is *locally convex* with respect to the Riemannian metric given on M, namely if there exists a positive number η such that for any two points p and q of the boundary with $d(p, q)<\eta$, where d is the distance derived from the Riemannian metric, there is the unique geodesic in M w.r.t. the Riemannian metric, connecting the two points p and q. Furthermore we show $\nu(D^n)=n$, where D^n is the n dimensional disk, and

$$\nu(M)\geq n$$

if M is contractible ($n=dim\ M$).

For $n=1$ and 2, we know

(*) "compact contractible manifold with boundary is always homeomorphic to a disk."

For $n=3$, this statement is equivalent to the Poincaré Conjecture, that is, we have (*) iff the conjecture for three dimension is true. For $n\geq 4$, there are examples of compact contractible manifolds with boundary, which are not homeomorphic to a disk [8]. As a corollary, we have a generalization of Bos' result to compact contractible manifolds (Locally convex disk is a D^n with a Riemannian metric, w.r.t. which the boundary is locally convex).

The result obtained here are closely related to the existence problem of periodic solutions of Hamiltonian systems (Cf. [2] [4] [11]).

2. Numbers determined by compact manifold with boundary

Let M be a compact C^∞ manifold with boundary $B\equiv\partial M\neq\emptyset$.

We define four numbers $\nu(M)$, $\nu_\pi(M)$, $\nu_H(M)$, and $\nu_{\mathrm{II}}(M)$ as follows. We consider the set

$$Y_M\equiv\{\omega\colon [0, 1]\to M;\ \text{continuous and } \omega(0),\ \omega(1)\in B\}$$

endowed with compact open topology and identify B as a subset of Y_M by identifying a point of B with the constant path. In the following, homology and cohomology groups are considered with the coefficient field $\boldsymbol{Z}_2$.

(i) $$\nu_\pi(M)=\begin{cases}1 & \text{if } \pi_k(Y_M, B)\neq 0 \text{ for some } k\geq 1\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

(ii) $\nu_H(M)$ is the maximal number of $k\geq 0$ for which there exist $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}\in H^*(Y_M)$ with deg $\alpha_j>0$ $(j=1, \cdots, k-1)$ and $a\in H_*(Y_M, B)$ such that $(\alpha_1\cup\cdots\cup\alpha_{k-1})\cap a\neq 0$.

We remark that

$$\nu_H(M)\geq 1 \quad \text{iff} \quad H_*(Y_M, B)\neq 0\,. \tag{2.1}$$

(iii) Let $\xi\colon Y_M\to Y_M$ be the involution defined by

$$\xi\omega=\omega^{-1}\,,$$

where $\omega^{-1}(\cdot)=\omega(1-\cdot)$. This defines a $\boldsymbol{Z}_2$-action on Y_M and we can define $\boldsymbol{Z}_2$-equivariant (co)homology groups $H_{\mathrm{II}}{}^*(Y_M)$ and $H_*{}^{\mathrm{II}}(Y_M, B)$ (see [3] [10]). As $\nu_H(M)$, we define:

$\nu_{\mathrm{II}}(M)$ is the maximal number of $k\geq 0$ for which there exist $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}\in H_{\mathrm{II}}{}^*(Y_M)$ with deg $\alpha_j>0$, and $a\in H_*{}^{\mathrm{II}}(Y_M, B)$ such that $(\alpha_1\cup\cdots\cup\alpha_{k-1})\cap a\neq 0$.

Finally we define a nonnegative integer $\nu(M)$ by

$$\nu(M)=\mathrm{Max}\{\nu_\pi(M),\ \nu_H(M),\ \nu_{\mathrm{II}}(M)\}\,. \tag{iv}$$

This number $\nu(M)$ is obviously a topological invariant number associated to M.

Lemma 2 of [4] shows that $\nu_\pi(M)\geq 1$ or $\nu_H(M)\geq 1$, hence we have

$$\nu(M)\geq 1\,. \tag{2.2}$$

Lemma 1. *Let L be a compact manifold and N a compact manifold with boundary. Then, if $\nu_H(N)\geq 1$, we have*

$$\nu_H(N\times L)\geq(\text{cup length } L)+\nu_H(N)\,.$$

Proof. Let $k=\nu_H(N)$. Then there exist $\alpha_1, \cdots, \alpha_{k-1}\in H^*(N)$, $\deg\alpha_j>0$, and $a\in H_*(N, \partial N)$ with

$$a'\equiv(\alpha_1\cup\cdots\cup\alpha_{k-1})\cap a\neq 0\,.$$

Let $l=$cup length L and let $\beta_1, \cdots, \beta_l\in H^*(L)$, $\deg\beta_j>0$, with $\beta'\equiv\beta_1\cup\cdots\cup\beta_l\neq 0$ (Recall that *cup length* is the number which is maximal with this property). We choose $b\in H_*(L)$ so that $\langle\beta', b\rangle=1$, then $\beta'\cap b\neq 0$. Now $a\times b\in H_*((N, \partial N)\times L)=H_*(N\times L, \partial(N\times L))$.

Then we have

$$\begin{aligned}
&((1\times\beta_1)\cup\cdots\cup(1\times\beta_l)\cup(\alpha_1\times 1)\cup\cdots\cup(\alpha_{k-1}\times 1))\cap(a\times b)\\
&=(1\times\beta')\cap(((\alpha_1\times 1)\cup\cdots\cup(\alpha_{k-1}\times 1))\cap(a\times b))\\
&=(1\times\beta')\cap(a'\times b)\\
&=a'\times(\beta'\cap b)\\
&\neq 0 \quad \text{in} \quad H_*((N\times L), \partial(N\times L))\,.
\end{aligned}$$

This implies the inequality of the lemma. Q.E.D.

REMARK: We do not know whether or not $\nu_H(N)\geq 1$ for any compact manifold N.

3. Orthogonal geodesic chords

Theorem 1. *Let M be a compact Riemannian manifold with locally convex boundary. Then there exist at least $\nu(M)$ non-constant orthogonal geodesic chords of M.*

Before giving the proof, we set a path space and distances on it. Let M be as in Theorem 1 and Λ be the set of all piecewise C^∞ curve λ: $[0, 1]\to M$ with $\lambda(0)$ and $\lambda(1)\in B$. For $\lambda_1, \lambda_2\in\Lambda$, we define

$$d_\infty(\lambda_1, \lambda_2)\equiv\operatorname*{Max}_{0\leq t\leq 1} d(\lambda_1(t), \lambda_2(t))\,,$$

where d is the distance derived from the Riemannian metric on M, and

$$d_1(\lambda_1, \lambda_2)\equiv d_\infty(\lambda_1, \lambda_2)+\left(\int_0^1(|\dot\lambda_1(t)|-|\dot\lambda_2(t)|)^2dt\right)^{1/2},$$

where $|\cdot|$ is the length of tangent vector.

Let E be the energy functional $E\colon \Lambda \to [0, \infty)$ defined by

$$E(\lambda) \equiv \frac{1}{2} \int_0^1 |\dot{\lambda}(t)|^2 dt .$$

We know that E is continuous w.r.t. d_1 (§ 16 in [9]). A *critical value* of E is a number κ for which there is an orthogonal geodesic chord λ with $E(\lambda)=\kappa$. This naming is reasonable because orthogonal geodesic chords are regarded as *critical points* of E.

Now we have a sequence of continuous mappings

$$(\Lambda, d_1) \overset{i}{\subset} (\Lambda, d_\infty) \overset{j}{\subset} (Y_M, d_\infty) .$$

Both these maps i and j give homotopy equivalences by Theorem 17.1 in [9].

For $K \geq 0$, we put

$$\Lambda^K \equiv \{\lambda \in \Lambda;\ E(\lambda) \leq K\} .$$

Lemma A.1.4 in [6] is given for closed curves on a compact manifold without boundary. For our case, the manifold is with boundary and curves connect two points of the boundary. But in our case, the boundary is locally convex, hence we can define a deformation like one in Lemma A.1.4 of [6], by the method used in [2] [4]. Therefore we have an analogue of Lemma A.1.4 in [6]:

Lemma 2. *We take the distance d_∞ on Λ. For given $K \geq 0$, there is a deformation (continuous map)*

$$D\colon \Lambda^K \times [0, 1] \to \Lambda^K$$

satisfying:

(i) *$D(\cdot, 0)$ is the identity map*: $\Lambda^K \to \Lambda^K$,

(ii) *D is E-non-increasing, that is*

$$E(D(\lambda, t)) \leq E(D(\lambda, s)) \qquad \text{if} \quad t \geq s ,$$

(iii) *$D(\lambda, 1)=\lambda$ if and only if λ is an orthogonal geodesic chord (or constant curve on B).*

(iv) *Let $0<\kappa<K$ and C_κ be the set of all orthogonal geodesic chords of energy κ. Then for any open neighborhood W of C_κ (one may choose $W=\emptyset$ in the case $C_\kappa=\emptyset$), there exists $\rho>0$ such that*

$$D_1 \Lambda^{\kappa+\rho} \subset W \cup \Lambda^{\kappa-\rho}$$

where $D_1 \equiv D(\cdot, 1)\colon \Lambda^K \to \Lambda^K$.

(v) *$D(\cdot, t)\colon \Lambda^K \to \Lambda^K$ is $\boldsymbol{Z}_2$-equivariant for any $t \in [0, 1]$.*

This lemma means that the convexity of the boundary plays the role of the condition (C) of Palais-Smale.

Proof of Theorem 1. First we prove that $\nu_\pi(M)=1$ implies the existence of at least one non-constant orthogonal geodesic chord. Since $\Lambda^\varepsilon \cong \Lambda^0$ for sufficiently small $\varepsilon>0$, we have

$$\pi_k(\Lambda, \Lambda^\varepsilon) \cong \pi_k(\Lambda, \Lambda^0) \cong \pi_k(Y_M, B) .$$

Let a be the non-trivial element of $\pi_k(\Lambda, \Lambda^\varepsilon)$. We put

$$\kappa_a = \inf_{f\in a} \sup E(f(D^k)) . \tag{3.1}$$

Although E is not continuous w.r.t. d_∞, this number is finite, because the inclusion i: $(\Lambda, d_1)\subset(\Lambda, d_\infty)$ is a homotopy equivalence as remarked above. We can choose a representative f: $D^k \to \Lambda$ of a, which is continuous w.r.t. d_1. Now we show that κ_a is a critical value with $\kappa_a \geq \varepsilon$. This follows from the fact that a is non-trivial in the relative homotopy $\pi_k(\Lambda, \Lambda^\varepsilon)$, because $\kappa_a<\varepsilon$ implies $f(D^k)\subset\Lambda^\varepsilon$ for some $f\in a$, which means $a=0$ in $\pi_k(\Lambda, \Lambda^\varepsilon)$.

Assume that κ_a is not a critical value, that is, $C_{\kappa_a}=\emptyset$. We put $K=\kappa_a+1$. Then, choosing $W=\emptyset$ in (iv) of Lemma 2, there exists $\rho>0$ such that $D_1\Lambda^{\kappa_a+\rho} \subset\Lambda^{\kappa_a-\rho}$. By the definition of κ_a, we can choose $f\in a$ with $f(D^k)\subset\Lambda^{\kappa_a+\rho}$, hence $D_1 f(D^k)\subset\Lambda^{\kappa_a-\rho}$. Since D_1 is homotopic to the identity, we have $D_1 f\in a$ so

$$\kappa_a \leq \sup E(D_1 f(D^k)) \leq \kappa_a-\rho .$$

This is the contradiction showing that κ_a is a critical value.

This means that $\nu_\pi(M)=1$ implies that there exist at least one non-constant orthogonal geodesic chord. We also have at least one such chord if $\nu_H(M) \geq 1$, that is $H_k(\Lambda, \Lambda^\varepsilon)\neq 0$ (see (2.1)). In this case, the non-zero element of $H_k(\Lambda, \Lambda^\varepsilon)$ plays the role of a above (written also as a) and, instead of (3.1), $\inf_{z\in a} \sup E(|z|)$ is a critical value with $\geq\varepsilon$ (the definition of $|z|$ is given below).

Next we consider the equivariant case, that is, we shall prove that we have at least $\nu_{\mathrm{II}}(M)$ non-constant orthogonal geodesic chords. We put $\Lambda_{\mathrm{II}} \equiv S^\infty \underset{\mathrm{II}}{\times} \Lambda$, the orbit space of $S^\infty\times\Lambda$ under the involution $(\zeta, \lambda)\mapsto(-\zeta, \xi\lambda)$, where $S^\infty = \bigcup_{k=0}^\infty S^k$, S^k is the k dimensional sphere. Since E is invariant under ξ, we can define the map

$$\tilde{E}\colon \Lambda_{\mathrm{II}} \to [0, \infty)$$

by $\tilde{E}[\zeta, \lambda]=E(\lambda)$, where $[\zeta, \lambda]$ is the equivalence class represented by (ζ, λ). Let a be the nontrivial element of $H_*^{\mathrm{II}}(\Lambda, \Lambda^\varepsilon)$, which is equal to $H_*(\Lambda_{\mathrm{II}}, \Lambda_{\mathrm{II}}^\varepsilon)$. We put

$$\tilde{\kappa}_a = \inf_{z\in a} \sup \tilde{E}(|z|) ,$$

where $|z| = \bigcup_i \operatorname{Im} \sigma_i$, $z = \sum_i \sigma_i$, σ_i: $\Delta_q \to \Lambda_{\mathrm{II}}$, singular simplices. Then, by Lemma 2 in [5] (We take $K = \tilde{\kappa}_a + 1$, then $D(\cdot, t)$ plays the role of ϕ_s in the lemma.), $C_{\tilde{\kappa}_a} \neq \emptyset$ and $\tilde{\kappa}_a \geq \varepsilon$. Thus $\nu_{\mathrm{II}}(M) \geq 1$ yields the existence of at least one nonconstant orthogonal geodesic chord.

Furthermore, assume that there exists $\alpha \in H^*(\Lambda_{\mathrm{II}})$ with $\deg \alpha > 0$, such that $b \equiv \alpha \cap a \neq 0$ in $H_*(\Lambda_{\mathrm{II}}, \Lambda_{\mathrm{II}}{}^{\varepsilon})$, that is, $\nu_{\mathrm{II}}(M) \geq 2$. Then $\tilde{\kappa}_b$ is also a critical value with $\tilde{\kappa}_b \geq \varepsilon$. In this case, in general, $\tilde{\kappa}_b \leq \tilde{\kappa}_a$ and if $\tilde{\kappa}_b = \tilde{\kappa}_a$, then there are infinitely many critical points on the level (a version of Lemma 2 in [5]). In any case, we have at least two distinct non-constant orthogonal geodesic chords. Thus $\nu_{\mathrm{II}}(M) \geq 2$ implies the existence of at least two such chords.

If $\nu_{\mathrm{II}}(M) \geq 3$, then we have α_1 and $\alpha_2 \in H^*(\Lambda_{\mathrm{II}})$, with $\deg \alpha_j > 0$, satisfying $\alpha_1 \cap (\alpha_2 \cap a) = (\alpha_1 \cup \alpha_2) \cap a \neq 0$. Putting $b_1 = \alpha_2 \cap a$ and $b_2 = \alpha_1 \cap b_1$, we have

$$\tilde{\kappa}_a \geq \tilde{\kappa}_{b_1} \geq \tilde{\kappa}_{b_2} \geq \varepsilon > 0 .$$

In this case we have at least three distinct non-constant orthogonal geodesic chords. Repeatedly, we have at least $\nu_{\mathrm{II}}(M)$ such chords.

The case of $\nu_H(M)$ is the standard one (cf. Corollary 2.1.11 of [6]). We don't need S^∞ and similar argument as above, removing $\sim$ or Π (that is, consider E, H^*, H_* instead of $\tilde{E}$, $H_{\mathrm{II}}{}^*$, $H_*{}^{\mathrm{II}}$, etc.) yields the existence of at least $\nu_H(M)$ non-constant orthogonal geodesic chords. Q.E.D.

By (2.2), we get

Corollary (Theorem C of [2]). *Any compact Riemannian manifold with locally convex boundary has at least one non-constant orthogonal geodesic chord.*

REMARK: There is an example of compact Riemannian manifold with boundary with no orthogonal geodesic chords [1].

Theorem 2. *Let M be a compact contractible manifold with locally convex boundary. Then there exist at least n non-constant orthogonal geodesic chords of M, where $n = \dim M$.*

To prove Theorem 2, we need

Lemma 3 ([3], Theorem 2). *Let B be a closed manifold of dimension m. Put $B^2 = B \times B$ and define the involution ξ: $B^2 \to B^2$ by*

$$\xi(x, y) = (y, x) .$$

Then there exist $a \in H_{2m}{}^{\mathrm{II}}(B^2, \Delta)$ and $\theta \in H_{\mathrm{II}}{}^1(B^2)$ with $\theta^m \cap a \neq 0$ in $H_m{}^{\mathrm{II}}(B^2, \Delta)$, where Δ denotes the diagonal set.

In this note, we apply this lemma for the case where B is the boundary of a manifold.

Proof of Theorem 2. By Theorem 1, it is sufficient to show

$$\nu_{\mathrm{II}}(M) \geq n . \tag{3.2}$$

Recall that $B = \partial M$. Let π: $Y_M \to B^2$ be the projection defined by $\omega \mapsto (\omega(0), \omega(1))$. Since the mapping π': $S^\infty \times Y_M \to S^\infty \times B^2$ defined by $(\zeta, \omega) \to (\zeta, \pi(\omega))$ is $\boldsymbol{Z}_2$-equivariant, we have the mapping

$$\tilde{\pi}: S^\infty \underset{\mathrm{II}}{\times} Y_M \to S^\infty \underset{\mathrm{II}}{\times} B^2 .$$

Then it is easy to see that $\tilde{\pi}$ gives a fibre space.

We fix base points $b \in B$ and $* \in S^\infty$ and take $\tilde{*} = [*, (b, b)]$ as the base point of $S^\infty \underset{\mathrm{II}}{\times} B^2$. Then we can identify the fibre $\tilde{\pi}^{-1}(\tilde{*})$ with ΩM, the space of loops in M starting from and ending at b. Thus we have a fibration

$$\Omega M \to S^\infty \underset{\mathrm{II}}{\times} Y_M \overset{\tilde{\pi}}{\to} S^\infty \underset{\mathrm{II}}{\times} B^2 . \tag{3.3}$$

Now we use the assumption that M is contractible. Since $M \simeq b$, we have $\Omega M \simeq b$. Hence the spectral sequence of the (co)homology associated to the fibration degenerates, so the homology group and the cohomology ring of the base space $S^\infty \underset{\mathrm{II}}{\times} B^2$ is isomorphic to those of $S^\infty \underset{\mathrm{II}}{\times} Y_M$. Also, using 5-lemma, we see that $H_*{}^{\mathrm{II}}(Y_M, B)$ is isomorphic to $H_*{}^{\mathrm{II}}(B^2, \Delta)$. Therefore, by Lemma 3 and the naturality of the (co)homology theory, there exist $\tilde{a} \in H_{2m}{}^{\mathrm{II}}(Y_M, B)$ and $\tilde{\theta} \in H^1{}_{\mathrm{II}}(Y_M)$ with $\tilde{\theta}^m \cap \tilde{a} \neq 0$ $(m = n-1)$. This means $\nu_{\mathrm{II}}(M) \geq n$. Q.E.D.

Remark that we have

$$\nu(D^n) = n . \tag{3.4}$$

In fact, (3.2) implies $\nu_{\mathrm{II}}(D^n) \geq n$. Recall that the solid ellipsoid, whose lengths of axes are all different and with standard Riemannian metric, has exactly n non-constant orthogonal geodesic chords. Thus Theorem 1 excludes the posibility that $\nu(D^n) > n$, giving (3.4).

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