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ON FOX'S CONGRUENCE CLASSES OF KNOTS

Dedicated to Professor Shin'ichi Kinoshita for his 60th birthday

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R.H. Fox introduced the notion of congruence classes of knots in [3], and he gave a necessary condition for congruence in terms of Alexander matrices and polynomials. In this note we will improve his condition and discuss some of its consequences.

1. Congruence classes of knots

In this note we only consider 1-dimensional tame oriented knots k in an oriented 3-sphere S^3. Two knots k and k' are said to be *equivalent*, iff there is an orientation preserving homeomorphism from (S^3, k) onto (S^3, k'), and each equivalence class of knots is called a *knot type*. A knot k is called *trivial* (or *unknotted*) iff there exists a disk D in S^3 with $\partial D=k$.

DEFINITION (Fox [3]). Let n and q be non-negative integers. The knot types κ and λ are said to be *congruent modulo n,q*, written $\kappa\equiv\lambda$ (mod n, q), iff there are knots $k_0, k_1, k_2, \cdots, k_l$, integers $c_1, c_2, \cdots, c_l$, and trivial knots $m_1, m_2, \cdots, m_l$ such that

(1) k_{i-1} and m_i are disjoint,

(2) k_i is obtained from k_{i-1} by $1/c_in$-surgery along m_i (see [9, 10] for a/b-surgery),

(3) the linking number $lk(k_{i-1}, m_i)\equiv 0 \pmod q$, and

(4) k_0 represents κ, and k_l represents λ.

DEFINITION. Two knot types κ and λ are said to be *q-congruent modulo n*, written $\kappa \underset{q}{\equiv} \lambda \pmod n$, iff they satisfy the conditions (1), (2), (4) in the above and the following condition (3′):

(3′) $lk(k_{i-1}, m_i) = q\,.$

We note that these relations are equivalence relations.

Fox [3] pointed out that congruence modulo 0, q is just the knot equivalence, and that any two knot types are congruent modulo 1, q, because if a knot is obtained from another by changing an overpass to underpass, they belong to the same congruence class modulo 1, q (see Fig. 1).

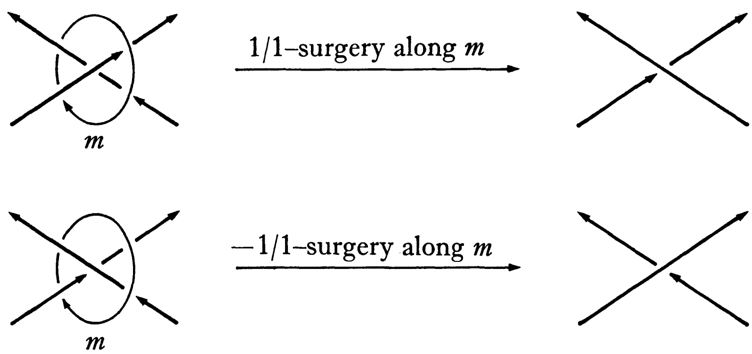

Fig. 1

First we give a necessary condition for congruence modulo n, q. This condition, rather effective if $n>1$, uses the Alexander polynomial $\Delta_\kappa(t)$ of a knot type κ. An Alexander matrix of κ is denoted by $A_\kappa(t)$ and $(1-t^n)/(1-t)=1+t+t^2+\cdots+t^{n-1}$ by $\sigma_n(t)$.

Theorem 1. *If* $\kappa \underset{q}{\equiv} \lambda \pmod n$, *then, for properly chosen* $A_\kappa(t)$ *and* $A_\lambda(t)$, *we have*

$$A_\kappa(t) \equiv A_\lambda(t) \bmod (1-t)\sigma_n(t^q)\,,$$

and hence

$$\Delta_\kappa(t) \equiv \pm t^r \Delta_\lambda(t) \bmod (1-t)\sigma_n(t^q)\,.$$

Further, we have similar statements for the elementary ideals of deficiency greater than 1.

Theorem 2. *If* $\kappa \equiv \lambda \pmod{n,\ q}$, *then, for properly chosen* $A_\kappa(t)$ *and* $A_\lambda(t)$, *we have*

$$A_\kappa(t) \equiv \quad A_\lambda(t) \bmod \begin{Bmatrix} n(1-t) = (1-t)\sigma_n(t^{0\times q})\,, & (1-t)\sigma_n(t^{1\times q})\,, \\ (1-t)\sigma_n(t^{i_1\times q}),\ \cdots,\ (1-t)\sigma_n(t^{i_*\times q})\,, & \end{Bmatrix}$$

and hence

$$\Delta_\kappa(t) \equiv \pm t^r \Delta_\lambda(t) \bmod \begin{Bmatrix} n(1-t) = (1-t)\sigma_n(t^{0\times q})\,, & (1-t)\sigma_n(t^{1\times q})\,, \\ (1-t)\sigma_n(t^{i_1\times q}),\ \cdots,\ (1-t)\sigma_n(t^{i_*\times q})\,, & \end{Bmatrix}$$

where $i_1, \cdots, i_*$ *are all divisors of* n *and* $1<i_1<\cdots<i_*<n$. *Further, we have similar statements for the elementary ideals of deficiency greater than* 1.

In the above, $f(t)\equiv g(t) \bmod \{h_1(t), h_2(t), \cdots, h_j(t)\}$ means that $f(t)$ and $g(t)$ are in the same class of the quotient $\boldsymbol{Z}\langle t\rangle/(h_1(t), h_2(t), \cdots, h_j(t))$, where $(h_1(t), h_2(t), \cdots, h_j(t))$ is the ideal generated by $h_1(t), h_2(t), \cdots, h_j(t)$ in $\boldsymbol{Z}(t)$.

We will prove Theorems 1 and 2 in the next section.

REMARK. In [6], Kinoshita proved some theorems similar to Theorem 1, but in a more special setting.

Corollary. *If n or q is even, and $\kappa \equiv \lambda$ (mod n, q), then we have*

$$\Delta_\kappa(-1) \equiv \Delta_\lambda(-1) \qquad (\mathrm{mod}\, 2n)\,.$$

Proof of Corollary. When q is even, $[\sigma_n(t^{i\times q})]_{t=-1}$ is equal to n. When q is odd, and n is even, $[\sigma_n(t^{i\times q})]_{t=-1}$ is equal to 0. Since each $[(1-t)\sigma_n(t^{i\times q})]_{t=-1}$ is equal to $2n$ or 0, Theorem 2 implies Corollary.

Applying Theorems 1 and 2, we can find infinitely many knot types that are incongruent modulo n, q.

Theorem 3. *Let n be an integer greater than* 1 *and q a non-negative integer such that $(n, q) \neq (2, 1)$ nor $(2, 2)$. For congruence modulo n, q, there exist infinitely many distinct classes.*

The proof of Theorem 3 will be given in §3.

For the remaining two cases $(n, q)=(2, 1)$ or $(2, 2)$, we show the following.

Theorem 4. *For any knot type κ, we have*

$$\Delta_\kappa(t) \equiv \pm t^r \cdot 1 \bmod \{2(1-t),\ (1-t)\sigma_2(t^2)\}\,,$$

and hence

$$\Delta_\kappa(t) \equiv \pm t^r \cdot 1 \bmod \{\, 2(1-t),\ (1-t)\sigma_2(t)\}\,.$$

Proof. It is well-known that the Alexander polynomial $\Delta_\kappa(t)$ of a knot type is characterized by the conditions (1) $\Delta_\kappa(t)=t^{2s}\Delta_\kappa(t^{-1})$ for some integer s and (2) $\Delta_\kappa(1)=\pm 1$ [7, 9, 11]. So, we can assume that

$$\Delta_\kappa(t) = c_s t^s + c_{s-1}t^{s-1} + \cdots + c_1 t + c_0 + c_1 t^{-1} + \cdots + c_{s-1}t^{-s+1} + c_s t^{-s}\,.$$

Deforming $\Delta_\kappa(t)$ symmetrically by $(1-t)\sigma_2(t^2)=1-t+t^2-t^3$, we have $\Delta_\kappa(t) \equiv ct+(2c\pm 1)+ct^{-1} \bmod (1-t)\sigma_2(t^2)$. When c is even, $ct+(2c\pm 1)+ct^{-1}+(c/2) \times(1-t^{-1})\times(2(1-t))=\mp 1$. When c is odd, $ct+(2c\pm 1)+ct^{-1}+((c\pm 1)/2)(1-t^{-1}) \times(2(1-t))\pm t^{-1}\times(1-t+t^2-t^3)=\pm t^2$. Therefore, we have $\Delta_\kappa(t)\equiv \pm t^r \cdot 1 \bmod \{2(1-t),\ (1-t)\sigma_2(t^2)\}$. By $(1-t)\sigma_2(t^2)=(1+t)\times(1-t)\sigma_2(t)-t\times 2(1-t)$, the ideal $(2(1-t),\ (1-t)\sigma_2(t^2))$ is contained in $((2(1-t),\ (1-t)\sigma_2(t))$. So we have $\Delta_\kappa(t)\equiv \pm t^r \cdot 1 \bmod \{2(1-t),\ (1-t)\sigma_2(t)\}$. Hence, the proof is complete.

By our experiments, we could not find distinct knot types that are incongruent modulo 2, 1 or 2, 2. Hence, we raise the following conjectures:

CONJECTURE C: All knots are congruent modulo 2, 1.

CONJECTURE B: All knots are congruent modulo 2, 2.

CONJECTURE A: All knots are deformable to a trivial knot by a finite sequence of operations τ^2's, which are shown in Fig. 2.

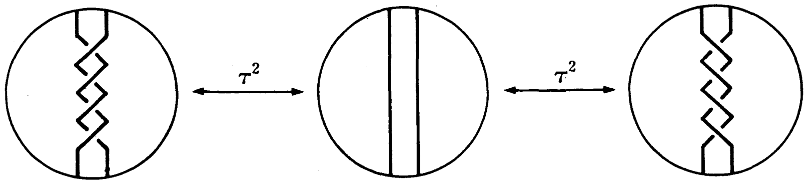

Fig. 2

NOTE. If Conjecture A is true, then Conjecture B is true. If Conjecture B is true, then Conjecture C is true.

Conjectures A and B are true for all (at most) 10 crossing knots, all Montesinos knots (which contain all 2-bridge knots and all pretzel knots), many closed 3-braid knots $(\sigma_1\sigma_2^{-1})^{3n\pm1}$, and so on. Conjecture C is true for all (at most) 10 crossing knots, all Montesions knots, all torus knots, all closed 3-braid knots, and so on. (See [12].)

QUESTION. For any Alexander polynomial $\Delta(t)$, does there exist a knot type κ such that the Alexander polynomial of κ is $\Delta(t)$, and κ is congruent to a trivial knot type modulo 2, 1 or 2, 2?

2. Proofs of Theorems 1 and 2

To prove Theorem 1, it is sufficient to show the following Lemma.

Lemma. *Let n and q be non-negative integers, and k and k' knots. Let m be a trivial knot disjoint from k such that $lk(k, m)=q$. Suppose that k' is obtained from k by $1/n$-surgery along m. Then, for properly chosen Alexander matrices $A_k(t)$ and $A_{k'}(t)$ of k and k', we have*

$$A_k(t) \equiv A_{k'}(t) \bmod (1-t)\sigma_n(t^q),$$

and hence

$$\Delta_k(t) \equiv \pm t^r \Delta_{k'}(t) \bmod (1-t)\sigma_n(t^q).$$

Further, we have similar statements for the elementary ideals of deficiency greater than 1.

Proof. We prove this Lemma after Fox [3].

Fig. 3 illustrates the neighbourhood of m in (S^3, k).

We can choose generators $x_1, x_2, \cdots, x_{2l}$, A, B of the fundamental group $\pi_1(S^3-k-m)$ as shown in Fig. 3, and further we choose other generators

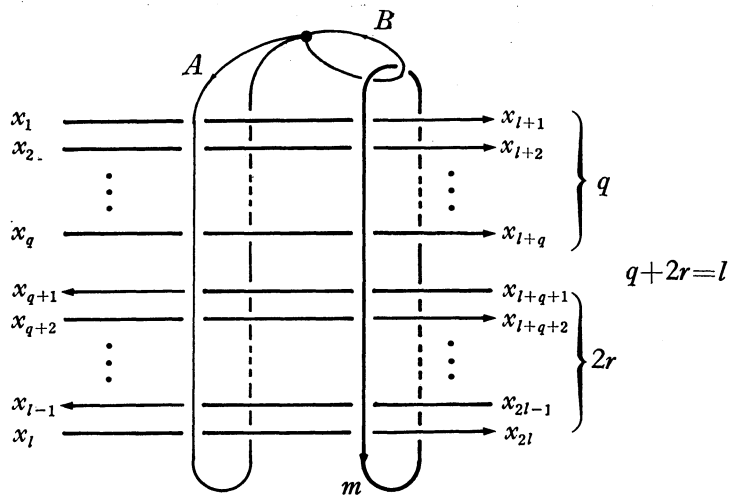

Fig. 3

$x_{2l+1}, \cdots$ as usual. Then, we have a group presentation of $\pi_1(S^3-k-m)$:

$$\left|\begin{array}{l|l} A,\ B, & A = w(x_1, x_2, \cdots, x_l)\,, \\ x_1, x_2, \cdots, x_{2l}, & x_{l+i} = B^{-1}x_iB \qquad (i = 1, 2, \cdots, l)\,, \\ x_{2l+1}, \cdots & r_j \ \textit{(relations corresponding to other crossings)} \end{array}\right| .$$

Hence, we have a group presentation of $\pi_1(S^3-k)$:

$$\left|\begin{array}{l|l} A, B, & A = w(x_1, x_2, \cdots, x_l)\,, \quad B = 1\,, \\ x_1, x_2, \cdots, x_{2l}, & x_{l+i} = B^{-1}x_iB \qquad (i = 1, 2, \cdots, l)\,, \\ x_{2l+1}, \cdots & r_j \end{array}\right|$$

$$= \left|\begin{array}{l|l} A, & A = w(x_1, x_2, \cdots, x_l)\,, \\ x_1, x_2, \cdots, x_{2l}, & x_{l+i} = x_i \qquad (i = 1, 2, \cdots, l)\,, \\ x_{2l+1}, \cdots & r_j \end{array}\right| ,$$

and a group presentation of $\pi_1(S^3-k')$:

$$\left|\begin{array}{l|l} A, B, & A = w(x_1, x_2, \cdots, x_l)\,, \quad BA^n = 1\,, \\ x_1, x_2, \cdots, x_{2l}, & x_{l+i} = B^{-1}x_iB \qquad (i = 1, 2, \cdots, l)\,, \\ x_{2l+1}, \cdots & r_j \end{array}\right|$$

$$= \left|\begin{array}{l|l} A, & A = w(x_1, x_2, \cdots, x_l)\,, \\ x_1, x_2, \cdots, x_{2l}, & x_{l+i} = A^n x_i A^{-n} \qquad (i = 1, 2, \cdots, l)\,, \\ x_{2l+1}, \cdots & r_j \end{array}\right| .$$

We use Fox's free differential calculus [2]. Since

$$\begin{aligned}\alpha\phi(\partial A^n x_i A^{-n}/\partial A) &= \alpha\phi((1+A+\cdots+A^{n-1})-A^n x_i(A^{-1}+A^{-2}+\cdots+A^{-n}))\\ &= (1+t^q+\cdots+(t^q)^{n-1})-t((t^q)^{n-1}+(t^q)^{n-2}+\cdots+1)\\ &= (1-t)\sigma_n(t^q),\end{aligned}$$

we have the Alexander matrices $A_k(t)$ and $A_{k'}(t)$ of k and k', respectively, as follows:

$$A_k(t)=\left|\begin{array}{c|cccc|cccc|cc} 1 & * & * & \cdots & * & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \hline 0 & 1 & & & & -1 & & & & & \\ 0 & & 1 & & & & -1 & & & & \\ \cdot & & & \cdot & & & & \cdot & & & \\ \cdot & & & & \cdot & & & & \cdot & & O \\ \cdot & & & & \cdot & & & & \cdot & & \\ 0 & & & & 1 & & & & -1 & & \\ \hline O & \multicolumn{10}{c}{\alpha\phi(\partial r_j/\partial x_i)} \end{array}\right|,$$

and

$$A_{k'}(t)=\left|\begin{array}{c|cccc|cccc|cc} 1 & * & * & \cdots & * & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \hline (1-t)\sigma_n(t^q) & t^{qn} & & & & -1 & & & & & \\ (1-t)\sigma_n(t^q) & & t^{qn} & & & & -1 & & & & \\ \cdot & & & \cdot & & & & \cdot & & & \\ \cdot & & & & \cdot & & & & \cdot & & O \\ \cdot & & & & \cdot & & & & \cdot & & \\ (1-t)\sigma_n(t^q) & & & & t^{qn} & & & & -1 & & \\ \hline O & \multicolumn{10}{c}{\alpha\phi(\partial_j r/\partial x_i)} \end{array}\right|.$$

From $1-t^{qn}=(1-t^q)\sigma_n(t^q)=\sigma_q(t)(1-t)\sigma_n(t^q)$, it follows that $A_k(t)\equiv A_{k'}(t)$ mod $(1-t)\sigma_n(t^q)$ and $\Delta_k(t)\equiv\pm t^r\Delta_{k'}(t)$ mod $(1-t)\sigma_n(t^q)$. Hence, the proof is complete.

Proof of Theorem 2. For the case $lk(k,m)=sq$, let d be the greatest common divisor of s and n. Notice that the collection of integers $\{0\times sq, 1\times sq, 2\times sq, \cdots, (n-1)sq\}$ is equal to the collection of integers $\{0\times dq, 1\times dq, 2\times dq, \cdots, (n-1)dq\}$ modulo nq. So, the following two polynomials coincide modulo $1-t^{nq}=(1-t^q)\sigma_n(t^q)$:

$$\sigma_n(t^{sq}) = 1+t^{sq}+t^{2sq}+\cdots+t^{(n-1)sq}, \quad \text{and}$$
$$\sigma_n(t^{dq}) = 1+t^{dq}+t^{2dq}+\cdots+t^{(n-1)dq}.$$

Hence $(1-t)\sigma_n(t^{sq})$ is contained in the ideal

$$((1-t)\sigma_n(t^{0\times q}),\ (1-t)\sigma_n(t^{1\times q}),\ (1-t)\sigma_n(t^{i_1\times q}),\ \cdots,\ (1-t)\sigma_n(t)^{i_*\times q})\,,$$

and the proof is complete.

3. Proof of Theorem 3

We divide the proof of Theorem 3 into three lemmas.

Lemma 1. *For congruence modulo* 2,0, *there exist infinitely many distinct classes.*

Proof. For a non-negative integer $n\in \boldsymbol{N}_0$, let κ_n be the $(2n+1, 2)$-torus knot. Then the Alexander polynomial $\Delta_{\kappa_n}(t)$ of κ_n is

$$\Delta_{\kappa_n}(t) = t^{2n}-t^{2n-1}+t^{2n-1}-\cdots-t+1\,.$$

The quotient $\boldsymbol{Z}\langle t\rangle/(2(1-t),\ \Delta_{\kappa_n}(t))$ has an abelian group presentation:

$$\begin{aligned}&|t^i\,|\,2(1-t)t^i = 0,\ t^i\Delta_{\kappa_n}(t) = 0\ (i=\cdots, -1, 0, ,1\ 2, \cdots)|\\ &\cong|(1), (1-t), \cdots, (1-t^{2n-1})\,|\,2(1-t^i) = 0\ (i=1, 2, \cdots, 2n-1)|\\ &\cong \boldsymbol{Z}\oplus(\boldsymbol{Z}_2)^{2n-1}\,.\end{aligned}$$

Therefore, $\Delta_{\kappa_n}(t)$ and $\pm t^r\Delta_{\kappa_n'}(t)$ are in distinct classes of $\boldsymbol{Z}\langle t\rangle/(2(1-t))$ if $n\neq n'$. Hence, the congruence classes modulo 2, 0 of $\kappa_n(n\in \boldsymbol{N}_0)$ are mutually distinct. Hence, the proof is complete.

For a non-negative integer $n\in \boldsymbol{N}_0$, let λ_n be the connected sum of n copies of a trefoil knot. (If it is desired, by a theorem in [8], we can choose a prime knot λ_n' whose Alexander matrix is same to that of λ_n.) Then, the ith elementary ideal $E_i(t)$ of λ_n is $((t^2-t+1)^{n+1-i})$ for $1\leqq i\leqq n$ and is (1) for $i\geqq n+1$.

From now on, we consider the congruence classes of $\{\lambda_n\}$. (Of course, their classes modulo 2, 0 are mutually distinct by the proof of Lemma 1 and Theorem 2.)

Lemma 2. *For congruence modulo* 2, q *with* $q\geqq 3$, *there exist infinitely many distinct classes.*

Proof. If we show that (t^2-t+1) and $\pm t^r\cdot 1$ are in distinct classes of the quotient $\boldsymbol{Z}\langle t\rangle/(2(1-t), (1-t)\sigma_2(t^q)=1-t+t^q-t^{q+1})$, then the sequences of the elementary ideals of $\{\lambda_n\}$ $(n\in N_0)$ are mutually distinct mod $\{2(1-t), (1-t)\sigma_2(t^q)\}$. Hence, by Theorem 2, there exist infinitely many distinct classes for congruence mod 2, q $(q\geqq 3)$. Now we show that (t^2-t+1) and $\pm t^r\cdot 1$ are in distinct classes of the quotient $\boldsymbol{Z}\langle t\rangle/(2(1-t), \sigma_2(t^q)=1+t^q)$. The quotient $\boldsymbol{Z}\langle t\rangle/(2(1-t), 1+t^q)$ has an abelian group presentation:

$$\begin{aligned}&|t^i|2(1-t)t^i=0,\ (1+t^q)t^i=0\ (i=\cdots,-1,0,1,2,\cdots)|\\ &\quad\cong|(1),(1-t),\cdots,(1-t^{q-1})|4(1)=0,\ 2(1-t^i)=0(i=1,2,\cdots,q-1)|\\ &\quad\cong \boldsymbol{Z}_4\oplus(\boldsymbol{Z}_2)^{q-1}.\end{aligned}$$

Since $(t^2-t+1)\pm t^r\cdot 1=(t^2-1)-(t-1)\pm(t^r-1)+(1\pm 1)$ is of order 2, (t^2-t+1) and $\pm t^r\cdot 1$ are in distinct classes of $\boldsymbol{Z}\langle t\rangle/(2(1-t),1+t^q)$. Therefore, they are in distinct classes of $\boldsymbol{Z}\langle t\rangle/(2(1-t),(1-t)\sigma_2(t^q))$. Hence, the proof is complete.

Lemma 3. *For congruence modulo n, q with $n\geqq 3$, there exist infinitely many distinct classes.*

Proof. By the definition, $\kappa\equiv\lambda\ (\mathrm{mod}\ n,q)$ implies $\kappa\equiv\lambda\ (\mathrm{mod}\ n,1)$, and $\kappa\equiv\lambda\ (\mathrm{mod}\ n,q)$ implies $\kappa\equiv\lambda\ (\mathrm{mod}\ p,q)$ if p is a divisor of n. We have, therefore, only to verify the lemma for the case (n is an odd prime integer) or (n is an even integer greater than or equal to 4), and $q=1$.

For the case that n is an odd prime integer, we will show that $(t^2-t+1)\not\equiv \pm t^r\cdot 1$ mod $\{n(1-t),(1-t)\sigma_n(t)=1-t^n\}$. The quotient $\boldsymbol{Z}\langle t\rangle(n(1-t),1-t^n)$ has an abelian group presentation:

$$\begin{aligned}&|t^i|n(1-t)t^i=0,\ (1-t^n)t^i=0(i=\cdots,-1,0,1,2,\cdots)|\\ &\quad\cong|(1),(1-t),\cdots,(1-t^{n-1})|n(1-t^i)=0\ (i=1,2,\cdots,n-1)|\\ &\quad\cong \boldsymbol{Z}\oplus(\boldsymbol{Z}_2)^{n-1}.\end{aligned}$$

Therefore, $(t^2-t+1)\pm t^r\cdot 1=(t^2-1)-(t-1)\pm(t^r-1)+(1\pm 1)$ is of order 2 or of infinite order. Hence, (t^2-t+1) and $\pm t^r\cdot 1$ are in distinct classes of $\boldsymbol{Z}\langle t\rangle/(n(1-t),1-t^n)$.

For the case that n is an even integer greater than or equal to 4, we see $[t^2-t+1]_{t=-1}=3\not\equiv 1\ (\mathrm{mod}\ 2n)$. So, it is clear that $(t^2-t+1)\not\equiv\pm t^r\cdot 1$ mod $\{n(1-t),(1-t)\sigma_n(t)\}$ (cf. the proof of Corollary). Hence, the proof is complete.

4. Remarks

4.1. There is an error in Fox [3]. He confused "congruence modulo n, q" with "q-congruence modulo n" in the sense of this note, and "(and B into 1)" ([3], p. 38, the bottom line) should be read as "(and B into 1 or t^{nq})". So, we should read each phrase "congruence modulo n,q" in his paper as "q-congruence modulo n".

4.2. A proof of a theorem in Kinoshita [5] following the same pattern as Fox [3] is also in error (mentioned in [6]). Here we give a counter-example to the theorem in [5].

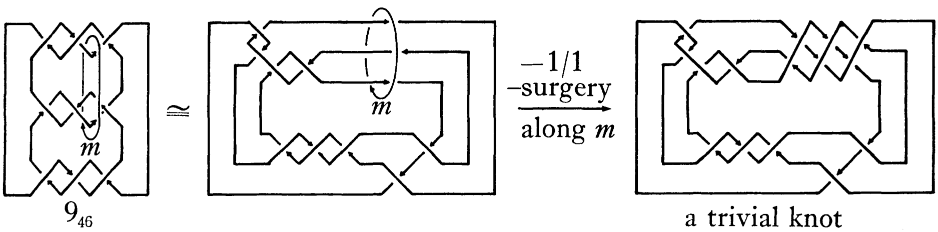

Fig. 4

We consider the knot 9_{46} and a trivial knot m as in Fig. 4. By $-1/1$-surgery along m, we obtain a trivial knot. So, we have $\bar{\bar{s}}(9_{46})=1$ in the sense of [5]. The 2-fold branched covering space Σ_2 of 9_{46} has the first integral homology group $H_1(\Sigma_2)\cong \boldsymbol{Z}_3\oplus\boldsymbol{Z}_3$, so $e_2=2$ in the sense of [5]. This is a contradiction to the formula $e_g\leqq(g-1)\cdot\bar{\bar{s}}(k)$ in [5].

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