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YANG-MILLS CONNECTIONS AND MODULI SPACE

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0. Introduction

There are many researches on deformations of Yang-Mills connections over 4-dimensional manifolds. In this paper, we generalize the results into higher dimensional cases. In 4-dimensional case, the Hodge $*$-operator acts on $\Lambda^2 M$ and the notion of (anti-)self dual connection is introduced, which brings beautiful results in Atiyah, Singer and Hitchin [1]. Therefore, in higher dimensional cases, we have to assume some properties on the base riemannian manifold. In [1], it is already pointed out that if M is a (2-dimensional) complex manifold, an anti-self dual connection defines a holomorphic structure of the bundle. Itoh [6] considers in full this situation, which we will generalize by the notion of "Einstein holomorphic connection" over a Kähler manifold. However, the moduli space of Yang-Mills connections over a higher dimensional Kähler manifold may have many singularities, and probably we can not expect that the moduli space becomes a manifold.

The fundamental notions in this paper come from [1], and fundamental idea comes from Koiso [9]. It is remarkable that the results for the moduli space of Einstein metrics and that of Yang-Mills connections are quite analogous. In fact we will get the following results.

Theorem 2.7 (c.f. [9, Theorem 3.1]). *The local pre-moduli space is a finite dimensional real analytic set.*

Corollary 6.5 (c.f. [9, Theorem 10.5]). *If the initial structure (Einstein metric or Yang-Mills connection) is compatible with a complex structure, then also around structures are compatible with some complex structures.*

Theorem 9.3 (c.f. [9, Theorem 12.3]). *Under some assumption, the local pre-moduli space has a canonical Kähler structure.*

However, there is an important difference. For Einstein metrics, we have no effective obstruction spaces for deformation ([9, Proposition 5.4]), but for Yang-Mills connections we have one (Theorem 6.9).

1. Yang-Mills connections

Let (M, g) be a compact riemannian manifold, G a compact Lie group, P a principal G-bundle over M. Denote by $\mathfrak{g}$ the Lie algebra of G and by G_P (resp. $\mathfrak{g}_P$) the associated fiber bundle $P\times_{\mathrm{Ad}_G}G$ (resp. $P\times_{\mathrm{Ad}_G}\mathfrak{g}$). The space C of all connections of P is an affine space whose standard vector space is $C^\infty(\Lambda^1\otimes\mathfrak{g}_P)$, where Λ^p denotes the vector bundle of p-forms on M (see [1, p430]). We fix an effective representation $G\to GL(V)$ and identify a connection of P with a covariant derivation on $P\times_G V$ or $P\times_G \mathrm{End}(V)$. In this sence, for a connection ∇ of P and an element A of $C^\infty(\Lambda^1\otimes\mathfrak{g}_P)$ the curvature tensor is transformed as

$$R^{\nabla+A}=R^\nabla+d^\nabla A+[A\wedge A]\,, \tag{1.0.1}$$

where d^∇ and $[\cdot\wedge\cdot]$ are defined by

$$(d^\nabla A)(X, Y)=(\nabla_X A)(Y)-(\nabla_Y A)(X) \tag{1.0.2}$$

and

$$[A\wedge B](X, Y)=\frac{1}{2}([A(X), B(Y)]-[A(Y), B(X)])\,. \tag{1.0.3}$$

We fix a G-invariant inner product on $\mathfrak{g}$. Then the vector bundle $\mathfrak{g}_P$ admits a canonical fiber inner product $(\cdot, \cdot)$ and the vector space $C^\infty(\mathfrak{g}_P)$ admits a (global) inner product $\langle\cdot, \cdot\rangle$. We denote by $\|\cdot\|$ the L_2-norm defined by $\langle\cdot, \cdot\rangle$. Define an action integral F_{YM} for connections by

$$F_{YM}(\nabla)=\frac{1}{2}\|R^\nabla\|^2\,. \tag{1.0.4}$$

Definition 1.1. The function F_{YM} on C is called *the Yang-Mills functional*, its Euler-Lagrange equation is called *the Yang-Mills equation* and its solution is called a *Yang-Mills connection.*

Let us represent the Yang-Mills equation by a tensor equation. Let ∇_t be a 1-parameter family of connections on P and set $\nabla=\nabla_0$ and $A=(d/dt)_0\nabla_t$. Then

$$\begin{aligned}\frac{d}{dt}\Big|_0 F_{YM}(\nabla_t)&=\langle\frac{d}{dt}\Big|_0 R^{\nabla_t}, R^\nabla\rangle\\ &=\langle d^\nabla A, R^\nabla\rangle=\langle A, (d^\nabla)^*R^\nabla\rangle\\ &=2\langle A, \delta^\nabla R^\nabla\rangle\,,\end{aligned}$$

where $(\cdot)^*$ denotes the formal adjoint and the operator δ^∇ from $C^\infty(\Lambda^p\otimes\mathfrak{g}_P)$ to $C^\infty(\Lambda^{p-1}\otimes\mathfrak{g}_P)$ is defined by

$$(\delta^\nabla s)_{i_1\cdots i_{p-1}}=-\nabla^l s_{li_1\cdots i_{p-1}}\,. \tag{1.1.1}$$

Thus Yang-Mills equation becomes

$$E_{YM}(\nabla) \equiv \delta^{\nabla} R^{\nabla} = 0 \,. \tag{1.1.2}$$

Next, we consider infinitesimal deformations of a Yang-Mills connection. From now on, we enlarge the space of C^∞-sections to the space of H^s-sections, and denote by $H^s(E)$ the space of all H^s-sections of a fiber bundle E over M, where s is assumed to be sufficiently large. H^s-norm will be denoted by $\|\cdot\|_s$. The completion of the space $\mathcal{C}$ etc. with respect to H^s-topology will be denoted by $\mathcal{C}^s$ etc.

DEFINITION 1.2. Let ∇ be a Yang-Mills connection. A solution of the equation

$$(E_{YM})'_{\nabla}(A) = 0 \tag{1.2.1}$$

is called *a Yang-Mills infinitesimal deformation*, where $'$ denotes the Fréchet derivative. The space of all Yang-Mills H^s-infinitesimal deformations is *denoted by* $\mathrm{YMID}^s(\nabla)$.

Lemma 1.3. $(E_{YM})'_{\nabla}(A) = \delta^{\nabla} d^{\nabla} A + \mathrm{tr}[R^{\nabla}, A]$,

where $\mathrm{tr}[R^{\nabla}, A]_i = g^{kl}[R^{\nabla}_{ki}, A_l]$.

Proof. $$\frac{d}{dt}\bigg|_0 (\delta^{\nabla_t} R^{\nabla_t})_i = -\frac{d}{dt}\bigg|_0 \nabla^l_t R^{\nabla_t}_{li}$$
$$= -[A^l, R^{\nabla}_{li}] - \nabla^l\Big(\frac{d}{dt}\bigg|_0 R^{\nabla_t}_{li}\Big),$$

where $A = \dfrac{d}{dt}\bigg|_0 \nabla_t$. Q.E.D.

The automorphism group $\mathcal{G} = C^\infty(G_P)$ of the bundle P is called the gauge group of P, and it acts on $\mathcal{C}$ by pull-back as

$$\gamma^*\nabla = \nabla + \gamma^{-1}\nabla\gamma \quad (\gamma \in \mathcal{G},\ \nabla \in \mathcal{C}) \,. \tag{1.3.1}$$

If ∇ is a Yang-Mills connection, then $\gamma^*\nabla$ is so. In particular, if γ_t is a 1-parameter family of gauge transformations such that $\gamma_0 = \mathrm{id}_P$, then $(d/dt)_0 \gamma_t^* \nabla = \nabla((d/dt)_0 \gamma_t)$ becomes a Yang-Mills infinitesimal deformation of ∇.

DEFINITION 1.4. Let ∇ be a Yang-Mills connection. A Yang-Mills infinitesimal deformation is said to be *trivial* if it coincides with ∇v for some $v \in H^{s+1}(\mathfrak{g}_P)$. A Yang-Mills infinitesimal deformation is said to be *essential* if it is orthogonal to all trivial Yang-Mills infinitesimal deformations. The space of all essential Yang-Mills infinitesimal deformations is *denoted by* $\mathrm{YMEID}(\nabla)$.

By definition, a Yang-Mills infinitesimal deformation $A\in H^s(\Lambda^1\otimes\mathfrak{g}_P)$ is essential if and only if $\langle\nabla v, A\rangle=0$ for any $v\in H^{s+1}(\mathfrak{g}_P)$, which is equivalent to that $\delta^\nabla A=0$. Thus the defining equation of the space YMEID(∇) becomes

$$\delta^\nabla d^\nabla A+\mathrm{tr}[R^\nabla, A]=0 \tag{1.4.1}$$

and

$$\delta^\nabla A=0\,. \tag{1.4.2}$$

This system is elliptic, and so the space YMEID(∇) is finite dimensional and each element is C^∞.

The following lemma will be used later.

Lemma 1.5. *For any connection ∇, equality*

$$\delta^\nabla E_{YM}(\nabla)=0 \tag{1.5.1}$$

and decomposition

$$H^s(\Lambda^1\otimes\mathfrak{g}_P)=\mathrm{Im}(\nabla\,|\,H^{s+1})\oplus\mathrm{Ker}\;\delta^\nabla \tag{1.5.2}$$

hold. If ∇ is a Yang-Mills connection, then the sequence

$$C^\infty(\mathfrak{g}_P)\underset{\nabla}{\rightarrow}C^\infty(\Lambda^1\otimes\mathfrak{g}_P)\xrightarrow[(E_{YM})'_\nabla]{}C^\infty(\Lambda^1\otimes\mathfrak{g}_P)\underset{\delta^\nabla}{\rightarrow}C^\infty(\mathfrak{g}_P) \tag{1.5.3}$$

is an elliptic complex. In particular, the following decompositions as Hilbert space hold.

$$H^s(\Lambda^1\otimes\mathfrak{g}_P)=\mathrm{Im}(E_{YM}'\,|\,H^{s+2})\oplus YMEID\oplus\mathrm{Im}(\nabla\,|\,H^{s+1})\,, \tag{1.5.4}$$

$$\mathrm{Ker}(E_{YM}'\,|\,H^s)=YMEID\oplus\mathrm{Im}(\nabla\,|\,H^{s+1})\,, \tag{1.5.5}$$

$$\mathrm{Ker}(\delta^\nabla\,|\,H^s)=YMEID\oplus\mathrm{Im}(E_{YM}'\,|\,H^{s+2})\,. \tag{1.5.6}$$

Proof. Equality (1.5.1) is easy to check directly, but here we show it using an idea from variation. Since the function F_{YM} on C is invariant under the action of the group $\mathcal{G}$, we see that

$$(F_{YM})'_\nabla(\nabla v)=0\qquad\text{for any}\quad v\in C^\infty(\mathfrak{g}_P)\,,$$

i.e.,

$$\langle E_{YM}(\nabla), \nabla v\rangle=0\,,$$

which implies (1.5.1). Decomposition (1.5.2) follows from Lemma 13.1. Let ∇ be a Yang-Mills connection. Then the space Im∇ is the space of trivial infinitesimal deformations, hence $(E_{YM})'_\nabla\circ\nabla=0$. From equality (1.5.1) we derive the equality

$$(\delta^\nabla)'\cdot E_{YM}(\nabla)+\delta^\nabla(E_{YM})'_\nabla=0\,.$$

Thus sequence (1.5.3) is a complex, and its ellipticity is easy to check. Therefore we have decomposition

$$(1.5.7) \qquad H^s(\Lambda^1\otimes\mathfrak{g}_P) = \mathrm{Im}(\nabla\,|\,H^{s+1})\oplus\mathrm{Im}((E_{YM})'^{*}_{\nabla}\,|\,H^{s+2}) \oplus\mathrm{Ker}\,\nabla^*\cap\mathrm{Ker}(E_{YM})'_{\nabla}\,.$$

Here we have $\nabla^*=\delta^\nabla$ and so we get (1.5.4) if we show that $(E_{YM})'_\nabla$ is self-adjoint. But we see

$$\langle(E_{YM})'_\nabla(A), B\rangle = (\mathrm{Hess}\,F_{YM})\,(A, B)\,,$$

regarding F_{YM} as a function on C^s, hence $(E_{YM})'_\nabla$ is symmetric with respect to $\langle\cdot, \cdot\rangle$. Since the space $\mathrm{Im}(\nabla\,|\,H^{s+1})$ is closed in $H^s(\Lambda^1\otimes\mathfrak{g}_P)$, decomposition (1.5.5) is reduced to the definition of the space $YMEID(\nabla)$. If we remark that the space $\mathrm{Ker}(\delta^\nabla\,|\,H^s)$ is the orthogonal complement of the space $\mathrm{Im}(\nabla\,|\,H^{s+1})$, then (1.5.6) follows from (1.5.4). Q.E.D.

2. Moduli space of Yang-Mills connections

To define "local pre-moduli space", we need some preparation. We use some basic facts on C^∞-maps in Hilbert space category. (See Lemmas 13.2, 13.3.)

Remark that the space $\mathcal{G}^s=H^s(G_P)$ is a C^∞-(infinite dimensional) Lie group. In fact, if we take a complexification G^C of G and set $G^C_P=P\times_{\mathrm{Ad}_G}G^C$, then to multiply and to get inverse element are extended to maps: $H^s(G^C_P)\times H^s(G^C_P)\to H^s(G^C_P)$, $H^s(G^C_P)\to H^s(G^C_P)$ so that the restriction on each fiber is holomorphic. Therefore, by Lemma 13.3, they are C^∞.

Let ∇ be a connection and $\mathcal{G}^s_\nabla$ the group of isotropy, i.e., $\mathcal{G}^s_\nabla=\{\gamma\in\mathcal{G}^s;\ \nabla\gamma=0\}$ (see (1.3.1)). Since ∇ is an elliptic operator, we see that $\mathcal{G}^s_\nabla=\mathcal{G}^\infty_\nabla$ and so we simply denote it by $\mathcal{G}_\nabla$. The exponential map $\exp\colon\mathfrak{g}\to G$ defines a C^∞-map $\exp^s\colon H^s(\mathfrak{g}_P)\to\mathcal{G}^s$ (by Lemma 13.3) and we can easily check that the quotient space $\mathcal{G}_\nabla\backslash\mathcal{G}^s$ admits a C^∞-structure and that there exists a local cross section $\chi^s\colon\mathcal{G}_\nabla\backslash\mathcal{G}^s\to\mathcal{G}^s$ so that the domain U^s is uniform on s, i.e., equations $U^{s+1}=(\mathcal{G}_\nabla\backslash\mathcal{G}^{s+1})\cap U^s$ and $\chi^{s+1}=\chi^s\,|\,U^{s+1}$ hold for any s. Define a C^∞-map

$$\mathcal{A}^s\colon U^{s+1}\times(\nabla+\mathrm{Ker}(\delta^\nabla\,|\,H^s))\to C^s$$

by

$$\mathcal{A}^s(u, \nabla_1) = \chi^{s+1}(u)^*\nabla_1\,.$$

Its derivative at ([id], ∇) is given by

$$(v, A)\to\nabla(\chi'_{[\mathrm{id}]}(v))+A\,,$$

and is bijective by decomposition (1.5.2). Therefore there exists a local inverse map $(\mathcal{A}^s)^{-1}=q^s\times p^s\colon C^s\to U^{s+1}\times(\nabla+\mathrm{Ker}(\delta^\nabla\,|\,H^s))$. By an analogous way with Ebin's Slice theorem in [4, Theorem 7.1], we get the following

Proposition 2.1. *Let $\nabla \in C^\infty$. There exist a neighbourhood U^{s+1} of* [id] *in $\mathcal{G}_\nabla \backslash \mathcal{G}^{s+1}$, a neighbourhood V^s of ∇ in $\nabla + \mathrm{Ker}(\delta^\nabla | H^s)$ and a neighbourhood W^s of ∇ in C^s so that*

$$\mathcal{A}^s\colon U^{s+1} \times V^s \to W^s$$

is a C^∞-diffeomorphism. Moreover if $\gamma \in \mathcal{G}_\nabla$ then $\gamma^*(V^s) = V^s$, and $\gamma^*(V^s) \cap V^s \neq \phi$ if and only if $\gamma \in \mathcal{G}_\nabla$.

Proof. Only the last statement is not shown. Since $\mathcal{G}_\nabla$ is a compact group and preserves Ker γ^∇, taking $\bigcap_{\gamma \in \mathcal{G}_\nabla} \gamma^*(V^s)$ if necessary, we may assume that $\gamma^*(V^s) = V^s$ if $\gamma \in \mathcal{G}_\nabla$. We now show that if $\gamma^*(V^s) \cap V^s \neq \phi$ then $\gamma \in \mathcal{G}_\nabla$. If $[\gamma]$ belongs to U^{s+1}, then bijectivity of $\mathcal{A}^s$ implies that $\gamma \in \mathcal{G}_\nabla$. Hence we assume that for any V^s there is $\gamma \in \mathcal{G}^{s+1}$ such that $\gamma^*(V^s) \cap V^s \neq \phi$ but $[\gamma] \notin U^{s+1}$. This means that there are a sequence $\{\gamma_i\}$ in $\mathcal{G}^{s+1}$ and sequences $\{\nabla_{1i}\}$ and $\{\nabla_{2i}\}$ in $(\nabla + \mathrm{Ker}(\delta^\nabla | H^s))$ which converge to ∇ such that $\gamma_i^* \nabla_{1i} = \nabla_{2i}$ and $[\gamma_i] \notin U^{s+1}$. Then by the following lemma, a subsequence of $\{\gamma_i\}$ converges to an element γ_∞ in $\mathcal{G}^{s+1}$, and so $\gamma_\infty \in \mathcal{G}_\nabla$ and $[\gamma_i] \in U^{s+1}$ for some i, which contradicts the assumption. Q.E.D.

Lemma 2.2. *Let $\{\gamma_i\}$, $\{\nabla_{1i}\}$ and $\{\nabla_{2i}\}$ be as above. Then a subsequence of $\{\gamma_i\}$ converges in $\mathcal{G}^{s+1}$.*

Proof. The equation $\gamma_i^* \nabla_{1i} = \nabla_{2i}$ is equivalent to the equation $\gamma_i^{-1} \nabla_{1i} \gamma_i = \nabla_{2i} - \nabla_{1i}$. Set $A_i = \nabla_{1i} - \nabla$ and $B_i = \nabla_{2i} - \nabla$. Then we see that $\nabla \gamma_i = \gamma_i (B_i - A_i) - A_i \gamma_i$. In general, we have

$$\|\gamma_i\|_t < C_1 \|\nabla \gamma_i\|_{t-1} + C_2 \|\gamma_i\|_0$$

for some real number C_1 and C_2, and $\|\gamma_i\|_0 < C_3$ since G is compact. Therefore

$$\|\gamma_i\|_t < C_1 \|\gamma_i (B_i - A_i) - A_i \gamma_i\|_{t-1} + C_2 \cdot C_3 .$$

Since the multiplication: $H^s \times H^t \to H^t$ for $t \leq s$ is continuous (see [12, Section 9]), we see that

$$\|\gamma_i\|_t < C_1 \|\gamma_i\|_{t-1} (\|A_i\|_s + \|B_i\|_s) + C_2 \cdot C_3 \quad (t-1 \leq s) .$$

Thus we see by induction that the sequence $\|\gamma_i\|_{s+1}$ is bounded, and so a subsequence of $\{\gamma_i\}$ converges in H^s, which we replace by $\{\gamma_i\}$. Then we have

$$\nabla(\gamma_i - \gamma_j) = (\gamma_i (B_i - A_i) - A_i \gamma_i) - (\gamma_j (B_j - A_j) - A_j \gamma_j) ,$$

and so

$$\|\gamma_i - \gamma_j\|_{s+1} < C_4 \|\gamma_i - \gamma_j\|_s + C_5 \|\gamma_i - \gamma_j\|_0$$

for some C_4 and C_5, and $\{\gamma_i\}$ is a Cauchy sequence in H^{s+1}-topology. Q.E.D.

DEFINITION 2.3. The manifold V^s in Proposition 2.1 is called *the slice at* ∇ and is *denoted by* $\mathcal{S}^s_\nabla$.

A priori, the slice may degenerate for $s\to\infty$. But we have following lemmas, which say that we can take slices "uniformly" and they are "natural".

Lemma 2.4. *Let* $t\geq s$ *and set* $U^{t+1}=U^{s+1}\cap(\mathcal{G}_\nabla\backslash\mathcal{G}^{t+1})$, $V^t=V^s\cap C^t$ *and* $W^t=W^s\cap C^t$. *Then Proposition* 2.1 *holds when s is replaced by t.*

Proof. It is sufficient to prove for $t=s+1$. The map

$$\mathcal{A}^{s+1}\colon\ U^{s+2}\times V^{s+1}\to W^{s+1}$$

is a C^∞-injective immersion.

(surjectivity) Let $\nabla_1\in W^{s+1}$. Then there is $\gamma\in\pi^{-1}(U^{s+1})$ so that $\gamma^*\nabla_1\in V^s$. Set $A_1=\nabla_1-\nabla$ and $A_2=\gamma^*\nabla_1-\nabla$. Then $A_1\in H^{s+1}(\mathfrak{g}_P)$, $A_2\in H^s(\mathfrak{g}_P)$, and $\nabla\gamma=\gamma A_2-A_1\gamma$. Since $\delta^\nabla A_2=0$, we have

$$\delta^\nabla\nabla\gamma=\operatorname{tr}(\nabla\gamma\otimes A_2)-\delta^\nabla(A_1\gamma)\,,$$

where $\nabla\gamma\otimes A_2\in H^s$ and $\delta^\nabla(A_1\gamma)\in H^s$. Thus $\gamma\in\mathcal{G}^{s+2}$.

(surjectivity of derivative) Let $u_0\in U^{s+2}$ and $\nabla+A_0\in V^{s+1}$. Then the derivative of the map $\mathcal{A}^s$ is given by

$$\begin{aligned}&(\mathcal{A}^s)'_{(u_0,\nabla+A_0)}(u',A')\\&\quad=\chi(u_0)^*\{\nabla(\varphi'(u'))+[A_0,\varphi'(u')]+A'\}\end{aligned}$$

where φ is defined by $\varphi(u)=\chi(u)\cdot\chi(u_0)^{-1}$. Let B be any element of $H^{s+1}(\mathfrak{g}_P)$. Then there are $u'\in T_{u_0}U^{s+1}$ and $A'\in T_{A_0}V^s$ so that

$$\chi(u_0)^*\{\nabla(\varphi'(u'))+[A_0,\varphi'(u'))]+A'\}=B\,.$$

This implies that

$$\delta^\nabla\nabla(\varphi'(u'))=\delta^\nabla(\chi(u_0)^{-1*}B-[A_0,\varphi'(u')])\,.$$

where the right hand side belongs to H^{s-1}. Thus $\varphi'(u')\in H^{s+1}$, and so the right hand side belongs to H^s, and $\varphi'(u')\in H^{s+2}$. Therefore $u'\in H^{s+2}$ and $A'\in H^{s+1}$.

Q.E.D.

Lemma 2.5. *Let* $\nabla_1\in\mathcal{S}^s_\nabla$. *If there is* $\gamma\in\mathcal{G}^{s+1}$ *such that* $\gamma^*\nabla_1\in C^t$, *then* $\nabla_1\in\mathcal{S}^t_\nabla$. *In particualr, if* $\gamma^*\nabla_1\in C^\infty$, *then* $\nabla_1\in C^\infty$.

Proof. Let $\{\gamma_i\}$ be a sequence in $\mathcal{G}^{t+1}$ which converges to γ in H^{s+1}-topology. Then $\gamma_i^{-1*}\gamma^*\nabla_1\to\nabla_1$ in C^s, and so for some i $\gamma_i^{-1*}\gamma^*\nabla_1$ belongs to W^s in Proposition 2.1. But here $\gamma_i^{-1*}\gamma^*\nabla_1\in C^t$. Therefore by Lemma 2.4 $\gamma_i^{-1*}\gamma^*\nabla_1\in W^t$, and so $\pi(\gamma\gamma_i^{-1})\in U^{t+1}$ and $\nabla_1\in\mathcal{S}^t_\nabla$. Q.E.D.

Corollary 2.6. *Let $\nabla_1 \in \mathcal{S}^s_\nabla$ be a Yang-Mills connection. Then $\nabla_1 \in \mathcal{S}^\infty_\nabla$.*

Proof. By Theorem 12.1, ∇_1 satisfies the condition in Lemma 2.5. Q.E.D.

Theorem 2.7. *Let ∇ be a Yang-Mills connection. There are a neighbourhood U^s of ∇ in $\mathcal{S}^s_\nabla$ and a closed C^∞-submanifold Z of U^s whose tangent space at ∇ coincides with $YMEID(\nabla)$ such that the set $YMLPM(\nabla)$ of all Yang-Mills connections in U^s is a real analytic set of Z. Moreover, the spaces Z and $YMLPM(\nabla)$ do not depend on s.*

Proof. Set $\varphi^s = E_{YM} | \mathcal{S}^s_\nabla$. Then by (1.5.2) we see

$$\operatorname{Im} \varphi^{s\prime}_\nabla = E_{YM}{}'_\nabla(\operatorname{Ker}(\delta^\nabla | H^s)) = \operatorname{Im}(E_{YM}{}'_\nabla | H^s)\,.$$

On the other hand, from (1.5.4) and (1.5.5) we have

$$H^{s-2}(\Lambda^1 \otimes \mathfrak{g}_P) = \operatorname{Im}(E_{YM}{}' | H^s) \oplus \operatorname{Ker}(E_{YM}{}' | H^{s-2})\,.$$

Let p^s (resp. q^s) be the projectoin to the first (resp. second) component. Then the C^∞-map $p^s \circ \varphi^s$ has surjective derivative at ∇ and by the implicit function theorem there is a neighbourhood U^s of ∇ in $\mathcal{S}^s_\nabla$ so that the set $Z = \{\nabla_1 \in U^s \mid p^s \circ \varphi^s(\nabla_1) = 0\}$ is a C^∞-submanifold of U^s. The tangent space $T_\nabla Z$ coincides with the space $YMEID(\nabla)$ and the set $YMLPM(\nabla)$ is the zero of the map $q^s \circ \varphi^s$ on Z.

Next we have to show that if we set $Z^t = Z \cap \mathcal{S}^t_\nabla$ and $U^t = U^s \cap \mathcal{S}^t_\nabla$ for $t \geq s$ then Z^t coincides with Z as manifold and $p^t \circ \varphi^t$ has surjective derivative at any point of Z^t. Let $\nabla + A \in Z$. Then by the definition of Z and Lemma 1.5 we have

$$\delta^\nabla A = 0\,, \quad E_{YM}{}'_\nabla(E_{YM}(\nabla + A)) = 0\,.$$

Since this is an elliptic system, A is C^∞, and so $Z^t = Z$ as set. Let $\nabla_1 \in Z^t$. Since $p^s \circ \varphi^s$ has surjective derivative at ∇_1, for any $A \in \operatorname{Im}(E_{YM}{}'_{\nabla_1} | H^t)$ there are $B \in \operatorname{Ker}(\delta^\nabla | H^s)$ and $C \in \operatorname{Ker}(E_{YM}{}'_\nabla | H^s)$ so that $(\varphi^s)'_{\nabla_1}(B) = A + C$. Then

$$E_{YM}{}'_\nabla \circ (\varphi^s)'_{\nabla_1}(B) = E_{YM}{}'_\nabla(A) \in H^{t-4}\,,$$

and $\delta^\nabla B = 0$. Therefore $B \in H^t$, which implies that $p^t \circ \varphi^t$ has surjective derivative at ∇_1, and so Z^t is a closed C^∞-submanifold of U^t. Moreover, the identity: $Z^t \to Z$ is bijective and its derivative also, hence is a diffeomorphism. Q.E.D.

Definition 2.8. The set $YMLPM(\nabla)$ is called *the local pre-moduli space of Yang-Mills connections around* ∇ and the set Z is called its *support manifold.*

We may summarize results as

Theorem 2.9. *Let ∇ be a Yang-Mills connection. The local pre-moduli space $YMLPM(\nabla)$ of Yang-Mills connections has the following properties. a)*

$YMLPM(\nabla)\subset\mathcal{S}_{\nabla}^{\infty}$. b) If ∇_1 is a Yang-Mills connection sufficiently close to ∇, then there is $\gamma\in\mathcal{G}^{s+1}$ so that $\gamma^*\nabla_1\in YMLPM(\nabla)$. c) If $\gamma^* YMLPM(\nabla)\cap YMLPM(\nabla)\neq\phi$ for $\gamma\in\mathcal{G}^{s+1}$, then $\gamma^*\nabla=\nabla$, i.e., $\nabla\gamma=0$.

REMARK 2.10. The global moduli space $\mathcal{G}\backslash\{$Yang-Mills connections$\}$ is locally homeomorphic with the coset space $\mathcal{G}_{\nabla}\backslash YMLPM(\nabla)$. Since $\mathcal{G}_{\nabla}$ is a compact Lie group, almost all local properties of the global moduli space is reduced to that of $YMLPM(\nabla)$.

Corollary 2.11. (1) *Let ∇ be a Yang-Mills connection. If $YMEID(\nabla)=0$, then $[\nabla]$ is isolated in the global moduli space of Yang-Mills connections.* (2) *The Yang-Mills functional F_{YM} is constant on the space YMLPM, and locally constant on the global moduli space.* (3) *If a connection ∇ minimizes the functional F_{YM} on C, then any Yang-Mills connection sufficiently close to ∇ also minimizes F_{YM}.* (4) *Any Yang-Mills connection sufficiently close to a flat connection is flat.* (5) *Let M be 4-dimensional. Any Yang-Mills connection sufficiently close to a self-dual (resp. anti self-dual) connection is self-dual (resp. anti self-dual).*

Proof. (1) The assumption implies that the support manifold Z is a point. (2), (3) The set *YMLPM* forms a real analytic set and its points are critical points of F_{YM}. (4) A connection ∇ is flat if and only if $F_{YM}(\nabla)=0$. (5) A connection ∇ is (anti) self-dual if and only if $F_{YM}(\nabla)$ coincides with a topological invariant of the principal bundle (see [1, p. 432]). Q.E.D.

3. The obstruction for deformations

We have shown that the local pre-moduli space $YMLPM(\nabla)$ is a real analytic set of the support manifold. Therefore we want to know when *YMLPM* coincides with the support manifold. In this section we introduce a notion which will be used later.

Let $\mathcal{P}$ be an open set of a Hilbert space, $\mathcal{Q}$ and $\mathcal{R}$ Hilbert spaces. Let E: $\mathcal{P}\to\mathcal{Q}$ and I: $\mathcal{P}\times\mathcal{Q}\to\mathcal{R}$ be C^∞-maps and define I_p: $\mathcal{Q}\to\mathcal{R}$ for each fixed $p\in\mathcal{P}$.

DEFINITION 3.1. If I_p is linear for each $p\in\mathcal{P}$ and $I_p(E(p))=0$ for all $p\in\mathcal{P}$, then I is called *an identity for E*.

If I is an identity for E and $E(p)=0$, then we see that $I_p\circ E'_p=0$, i.e., $\operatorname{Im} E'_p\subset\operatorname{Ker} I_p$.

DEFINITION 3.2. Let I be an identity for E and assume that $E(p)=0$. The space $\operatorname{Ker} I_p/\operatorname{Im} E'_p$ is called *the obstruction space for E-deformations of p with respect to I.*

Lemma 3.3. *Let I be an identity for E and $p\in E^{-1}(0)$. If the obstruction space* $\operatorname{Ker} I_p/\operatorname{Im} E'_p$ *vanishes, then the set $E^{-1}(0)$ around p forms a manifold whose* *tangent space at p coincides with* Ker E'_p, *provided that one of the following conditions is satisfied.* (1) *The map E is C^∞.* (2) *The space* Im I_p *is closed in* $\mathcal{R}$.

Proof. By a similar way as in Proof of Theorem 2.7, we see that there exists the "support manifold" Z whose tangent space at p coincides with Ker E'_p such that $E^{-1}(0)=(p_C\circ E|Z)^{-1}(0)$ around p, where p_C is the projection to a complement C of Im E'_p in Q. Set $\hat{E}=E|Z$: $Z\to C$ and $\hat{I}=I|(Z\times C)$: $Z\times C\to \mathcal{R}$. It is enough to prove that $\hat{E}=0$. Remark that Ker $\hat{I}_p=$Ker $I_p\cap C=$Im $E'_p\cap C=0$ and so $\hat{I}_p$ is injective. Assume condition (2). Then Im $\hat{I}_p=I_p(C)=I_p(\mathrm{Im}\,E'_p\oplus C)=$Im I_p, and so Im $\hat{I}_p$ is closed in $\mathcal{R}$, hence $\hat{I}_p$ is an isomorphism from C into $\mathcal{R}$. Therefore $\hat{I}_{p_1}$ is injective if $p_1\in Z$ is sufficiently close to p. But here we know that $\hat{I}_{p_1}(\hat{E}(p_1))=0$. Thus $\hat{E}(p_1)=0$.

Next we assume condition (1) and show the r-th derivative $\hat{E}^{(r)}$ vanishes for all $r\geq 0$ by induction. By taking r-th derivative of the identity $\hat{I}_{p_t}(\hat{E}(p_t))=0$ and setting $v=\left.\frac{d}{dt}\right|_0 p_t$, we get

$$\hat{I}_p(\hat{E}^{(r)}_p(v,\underset{(r)}{\cdots},v))=-\sum_{i=1}^r\binom{r}{i}\left(\frac{d}{dt}\right)^i\Big|_0\hat{I}_{p_t}\cdot\left(\frac{d}{dt}\right)^{r-i}\Big|_0\hat{E}(p_t)\,.$$

By induction we may assume that the right hand side vanishes, and so the left hand side vanishes. But we know that $\hat{I}_p$ is injective, hence $\hat{E}^{(r)}_p=0$. Q.E.D.

Remark 3.4. This Lemma essentially is "Kuranishi's method" ([8]).

4. The deformation of Yang-Mills connection caused by a deformation of base metric

We want apply Lemma 3.3 to a deformations of Yang-Mills connection. Unfortunately, it is not possible if we use Yang-Mills equation itself. In 6, we will introduce the notion of "Einstein holomorphic connection" and apply Lemma 3.3.

Now, by equation (1.5.1), δ^∇ is an identity for E_{YM}, and the obstruction space

$$\mathrm{Ker}\,\delta^\nabla/\mathrm{Im}\,E_{YM}{}'_\nabla\cong YMEID(\nabla)$$

by equation (1.5.6).

Proposition 4.1. *Let ∇ be a Yang-Mills connection. The obstruction space for E_{YM}-deformation of ∇ with respect to δ^∇ is isomorphic with the space $YMEID(\nabla)$ of essential infinitesimal deformations.*

Hence we apply Lemma 3.3 to the situation that we deform the metric g on M and Yang-Mills connection follows it. Denote by $\mathcal{M}^s$ the space of all H^s-riemannian metrics on M and define maps E, I by

$$E\colon \mathcal{M}^{s+1}\times C^{s+2}\to H^s(\Lambda^1\otimes\mathfrak{g}_p);\quad (g,\nabla)\to\delta_g^\nabla R^\nabla\,,$$

$$I\colon \mathcal{M}^{s+1}\times C^{s+2}\times H^s(\Lambda^1\otimes\mathfrak{g}_P)\to H^s(\mathfrak{g}_P);\quad (g,\nabla,\varphi)\to\delta^\nabla\varphi\,.$$

Then I is an identity for E.

Theorem 4.2. *Let ∇ be a Yang-Mills connection over (M,g), i.e., $E(g,\nabla)=0$. If $YMEID(\nabla)=0$, then for any deformation g_t of g there exists a 1-parameter family of connections ∇_t so that each ∇_t is a Yang-Mills connection with respect to g_t, provided that $|t|$ is small.*

Proof. The obstruction space for E-deformation of (g,∇) with respect to I coincides with the space

$$\operatorname{Ker}\delta_g^\nabla/\operatorname{Im}E'_{(g,\nabla)}\,,$$

which is a quotient space of the space $YMEID(\nabla)$ by equation (1.5.6), and vanishes by assumption. Therefore by Lemma 3.3 the set $E^{-1}(0)$ arround (g,∇) forms a manifold whose tangent space at (g,∇) coincides with Ker $E'_{(g,\nabla)}$. But here the projection map from Ker $E'_{(g,\nabla)}$ to $T_g\mathcal{M}^{s+1}$ is surjective, which completes the proof by the implicit function theorem. In fact, for any $h\in T_g\mathcal{M}^{s+1}$, we get $E'_{(g,\nabla)}(h,0)\in\operatorname{Ker}\delta_g^\nabla$, therefore by equation (1.5.6) and assumption, there is $A\in T_\nabla C^{s+2}$ such that Ker $E'_{(g,\nabla)}(h,0)=E_{YM}{'}_\nabla(A)$, i.e., $E'_{(g,\nabla)}(h,-A)=0$. Q.E.D.

5. Holomorphic structures

Let M be a compact complex manifold and P^C a principal G^C-bundle, where G^C is a complexification of G. A G^C-invariant almost complex structure on P^C is called an almost holomorphic structure of P^C. If it is integrable, then it is called a holomorphic structure. An almost holomorphic structure can be regarded as a first order differential operator

$$\bar\partial\colon C^\infty(\mathfrak{g}_P^C)\to C^\infty(\Lambda^{0,1}\otimes\mathfrak{g}_P^C)\,,\tag{5.0.1}$$

and it is a holomorphic structure if and only if the torsion $T(\bar\partial)$ of $\bar\partial$ vanishes, where $T(\bar\partial)\in C^\infty(\Lambda^{0,2}\otimes\mathfrak{g}_P^C)$ is defined by

$$T(\bar\partial)(X,Y)v\equiv\bar\partial_X\bar\partial_Y v-\bar\partial_Y\bar\partial_X v-\bar\partial_{[X,Y]}v\,.\tag{5.0.2}$$

An almost holomorphic structure $\bar\partial$ extends to the operators

$$\bar\partial^p\colon C^\infty(\Lambda^{0,p}\otimes\mathfrak{g}_P^C)\to C^\infty(\Lambda^{0,p+1}\otimes\mathfrak{g}_P^C)\,,\tag{5.0.3}$$

and if $\bar\partial$ is a holomorphic structure, then they defines an elliptic complex and the cohomology groups

$$H^{0,p}(\mathfrak{g}_P^C)\equiv\operatorname{Ker}\bar\partial^p/\operatorname{Im}\bar\partial^{p+1}\tag{5.0.4}$$

are defined.

We can study deformations of holomorphic structures by a similar way as deformations of complex structures on manifold (c.f. [13, pp 172–176]), but we use here notations similar with [6].

The space $\mathcal{AH}$ of all almost holomorphic structures forms an affine space with standard vector space $C^\infty(\Lambda^{0,1}\otimes\mathfrak{g}_P^C)$, and the complex gauge group $\mathcal{G}^C=C^\infty(G_P^C)$ acts on it. Let $\mathcal{G}_{\bar\partial}^C$ be the group of isotropy, i.e., $\mathcal{G}_{\bar\partial}^C=\{\gamma\in\mathcal{G}^C \mid \gamma^*\bar\partial=\bar\partial\}$. Then the H^{s+1}-gauge group $\mathcal{G}^{C,s+1}$ acts on $\mathcal{AH}^s$ holomorphically and the coset space $\mathcal{G}_{\bar\partial}^C\backslash\mathcal{G}^{C,s+1}$ forms a complex analytic manifold. The following proposition is proved by a similar manner as Proposition 2.1.

Proposition 5.1. *Let $\bar\partial\in\mathcal{AH}$. There exist a neighbourhood $\mathcal{S}_{\bar\partial}^{C,s}$ of $\bar\partial$ in $\bar\partial+\mathrm{Ker}\,(\bar\partial^*\,|\,H^s)$, a neighbourhood $U^{C,s+1}$ of $[id]$ in $\mathcal{G}_{\bar\partial}^C\backslash\mathcal{G}^{C,s+1}$ and a neighbourhood $W^{C,s}$ of $\bar\partial$ in $\mathcal{AH}^s$ so that the action*

$$\mathcal{A}^{C,s}\colon U^{C,s+1}\times\mathcal{S}_{\bar\partial}^{C,s}\to W^{C,s}$$

becomes a complex analytic diffeomorphism. Here, the formal adjoint $\bar\partial^*$ of $\bar\partial$ is defined by some (and fixed) hermitian inner product of $\mathfrak{g}_P^C$.

Let $\bar\partial$ be a holomorphic structure of P^C. For $A\in C^\infty(\Lambda^{0,1}\otimes\mathfrak{g}_P^C)$ we see that $T(\bar\partial+A)=T(\bar\partial)+\bar\partial^1A+[A\wedge A]$. Therefore the equation of infinitesimal deformation of holomorphic structure of $\bar\partial$ is given by

$$\bar\partial^1A=0\,. \tag{5.1.1}$$

The space of all essential infinitesimal deformations of $\bar\partial$ is given by

$$EHID(\bar\partial)=\mathrm{Ker}\,\bar\partial^1\cap\mathrm{Ker}\,\bar\partial^*\,. \tag{5.1.2}$$

By a similar way as Theorem 2.7, we have

Theorem 5.2. *Let $\bar\partial$ be a holomorphic structure. There are a neighbourhood $U^{C,s}$ of $\bar\partial$ in $\mathcal{S}_{\bar\partial}^{C,s}$ and a complex analytic submanifold Z^C of $U^{C,s}$ so that the set of all H^s-holomorphic structures in $U^{C,s}$ forms a complex analytic set of Z^C.*

DEFINITION 5.3. The set of all H^s-holomorphic structures in $U^{C,s}$ is called *the local pre-moduli space of holomorphic structures around* $\bar\partial$ and *denoted by* $HLPM(\bar\partial)$. The manifold Z^C is called its *support manifold.*

Equalities (5.1.2) and (5.0.4) mean that the space $HEID(\bar\partial)$ is canonically isomorphic to $H^{0,1}(\mathfrak{g}_P^C)$. Moreover, for any $\bar\partial\in\mathcal{AH}$ we have

$$\bar\partial^2T(\bar\partial)=0\,, \tag{5.3.1}$$

which means that $\bar\partial^2$ is an identity for T. Therefore, by Lemma 3.3, we get

Theorem 5.4. *Let $\bar{\partial}$ be a holomorphic structure. If $H^{0,2}(\mathfrak{g}_P^C)=0$, then the space $HLPM(\bar{\partial})$ forms a (complex) manifold whose tangent space at $\bar{\partial}$ coincides with the space $HEID(\bar{\partial})$.*

REMARK 5.5. Since the action of $\mathcal{G}^{C,s+1}$ on $\mathcal{AH}^s$ is complex analytic, the complex structure of the above space $HLPM(\bar{\partial})$ is canonical. I.e., if $\bar{\partial}_1\in HLPM(\bar{\partial})$, then the "projection map": $HLPM(\bar{\partial}_1)\rightarrow HLPM(\bar{\partial})$ defined by Proposition 5.1 is complex analytic.

REMARK 5.6. The space $HLPM(\bar{\partial})$ has similar properties as $YMLPM(\nabla)$ in Theorem 2.9. But property (c) does not hold for $HLPM(\bar{\partial})$, because G^C is not compact. Therefore the quatient space $\mathcal{G}_{\bar{\partial}}^C\backslash HLPM(\bar{\partial})$ is not necessarily identified with an open set of global moduli space of holomorphic structures.

6. Einstein holomorphic connections

Let (M, g) be a compact Kähler manifold, ω its Kähler form. Then the $(0, 1)$ component of a connection ∇ on P is a almost holomorphic structure $\bar{\partial}$ of P^C. Since $T(\bar{\partial})$ coincides with the $(0, 2)$ component of R^∇, $\bar{\partial}$ is a holomorphic structure if and only if R^∇ is of type $(1, 1)$.

DEFINITION 6.1. A connection ∇ of P is said to be *holomorphic* if the $(0, 1)$ component of ∇ is a holomorphic structure, or equiavlently, if R^∇ is of type $(1, 1)$. (Remark that this definition is not exactly the same with [6].)

Denote by R_H^∇ (resp. R_S^∇) the hermitian (resp. skew-hermitian) part of R^∇. Elements of Lie algebra $\mathfrak{z}$ of the center $Z(G)$ of G define parallel sections of $C^\infty(\mathfrak{g}_P)$, and are denoted also by $\mathfrak{z}$.

DEFINITION 6.2. A holomorphic connection ∇ is called *an Einstein holomorphic connection* if $(\omega, R^\nabla)\in\mathfrak{z}$ as section.

For example, if $G=U(r)$, a connection ∇ is an Einstein holomorphic connection if and only if ∇ is an Einstein hermitian connection for some holomorphic structure.

Lemma 6.3 (Itoh, Personal communication). *An Einstein holomorphic connection takes the minimum value of the Yang-Mills functional F_{YM} on C. Conversely a connection which takes the value is an Einstein holomorphic connection.*

Proof. Let ∇ be a connection of P and consider the characteristic classes of P. For each $c\in\mathfrak{z}$, the classes represented by (c, R^∇) and $\mathrm{Tr}(R^\nabla\wedge R^\nabla)$ do not depend on ∇, and so the values $\int_M (c, R^\nabla)\wedge\omega^{n-1}$ and $\int_M \mathrm{Tr}(R^\nabla\wedge R^\nabla)\wedge\omega^{n-2}$ are constant for ∇. Therefore there are $c_0\in\mathfrak{z}$ and a real number C such that equalities

(6.3.1) $$\langle \omega\otimes c, R^{\nabla}\rangle = \langle c_0, c\rangle$$

and

(6.3.2) $$\|R^{\nabla}_H\|^2 - \|R^{\nabla}_S\|^2 - \|(\omega, R^{\nabla})\|^2 = C$$

hold for all ∇. Let $(\omega, R^{\nabla}) = c_1 + v$ where $c_1 \in \mathfrak{z}$ and v is orthogonal to $\mathfrak{z}$ with respect to the global inner product. Then

$$\langle c_0, c\rangle = \langle c, (\omega, R^{\nabla})\rangle = \langle c, c_1\rangle$$

for any $c \in \mathfrak{z}$. Therefore $c_1 = c_0$ and

$$\begin{aligned}\|R^{\nabla}\|^2 &= \|R^{\nabla}_H\|^2 + \|R^{\nabla}_S\|^2 \\ &= C + 2\|R^{\nabla}_S\|^2 + \|(\omega, R^{\nabla})\|^2 \\ &\geq C + \|(\omega, R^{\nabla})\|^2 \\ &= C + \|c_0\|^2 + \|v\|^2 \\ &\geq C + \|c_0\|^2 .\end{aligned}$$

The equality $\|R^{\nabla}\|^2 = C + \|c_0\|^2$ holds if and only if $R^{\nabla}_S = 0$ and $v = 0$, i.e., $(\omega, R^{\nabla}) \in \mathfrak{z}$. Q.E.D.

Remark 6.4. We saw that if ∇ is an Einstein holomorphic connection then $(\omega, R^{\nabla}) = c_0$.

Corollary 6.5. *All Einstein holomorphic connections are Yang-Mills connections. Conversely all Yang-Mills connections which are sufficiently close to an Einstein holomorphic connection are Einstein holomorphic connections.*

Proof. Easy to see by Corollary 2.11 (3). Q.E.D.

Next we consider infinitesimal deformations of Einstein holomorphic connections. Define a map $E_{EH}\colon C^s \to H^{s-1}(\Lambda^{0,2}\otimes \mathfrak{g}^{C}_P) \oplus H^{s-1}(\mathfrak{g}_P)$ by

(6.5.1) $$\nabla \to (p^{0,2}R^{\nabla}, (\omega, R^{\nabla}) - c_0),$$

where $p^{0,r}$ is the projection map from Λ^r to $\Lambda^{0,r}$ and $c_0 \in \mathfrak{z}$ is defined in Proof of Lemma 6.3. By Remark 6.4, a connection ∇ is an Einstein holomorphic connection if and only if $E_{EH}(\nabla) = 0$.

Definition 6.6. Let ∇ be an Einstein holomorphic connection. An element A of $H^s(\Lambda^1\otimes \mathfrak{g}_P)$ is called *an Einstein holomorphic infinitesimal deformation of* ∇ if $E_{EH}{}'_{\nabla}(A) = 0$. An Einstein holomorphic infinitesimal deformation is said to be *essential* if it is orthogonal to all trivial infinitesimal deformations of ∇, and the space of all Einstein holomorphic essential infinitesimal deformations is *denoted by* $EHEID(\nabla)$.

By a similar way as Theorem 2.7, we get

Theorem 6.7. *Let ∇ be an Einstein holomorphic connection. There are a neighbourhood U^s of ∇ in S^s_∇ and a closed C^ω-submanifold Z of U^s whose tangent space at ∇ coincides with $EHEID(\nabla)$ such that the set $EHLPM(\nabla)$ of all Einstein holomorphic connections in U^s is a real analytic set of Z.*

Moreover, the combination of an obvious inclusion: $EHLPM(\nabla)\subset YMLPM(\nabla)$ and the converse inclusion $YMLPM(\nabla)\subset EHLPM(\nabla)$ by Corollary 6.5 means that $EHLPM(\nabla)=YMLPM(\nabla)$. Let ∇ be a connection. Define a map $I_\nabla\colon H^{s-1}(\Lambda^{0,2}\otimes\mathfrak{g}_P^C)\oplus H^{s-1}(\mathfrak{g}_P)\to H^{s-2}(\Lambda^{0,3}\otimes\mathfrak{g}_P^C)\oplus\mathfrak{z}$ by

$$I_\nabla(P,\eta)=(p^{0,3}(d^\nabla P),\ \mathfrak{z}\text{-part of }\eta)\,.$$

Lemma 6.8. *The map I is an identity for E_{EH}.*

Proof. For any ∇, we see

$$p^{0,3}(d^\nabla(p^{0,2}R^\nabla))=p^{0,3}(d^\nabla R^\nabla)=0\,,$$

and (6.3.1) means that $\mathfrak{z}$-part of $(\omega, R^\nabla)-c_0$ vanishes. Q.E.D.

Therefore if ∇ is an Einstein holomorphic connection and $\operatorname{Ker} I_\nabla/\operatorname{Im} E_{EH}'{}_\nabla$ vanishes, then the local pre-moduli space $EHLPM(\nabla)$ of Einstein holomorphic connections forms a manifold with tangent space $EHEID(\nabla)$ at ∇.

Theorem 6.9. *Let ∇ be an Einstein holomorphic connection. In general, the space $EHLPM(\nabla)$ forms a real analytic set of the support manifold Z whose tangent space at ∇ is isomorphic with the cohomology group $H^{0,1}(M,\mathfrak{g}_P^C)$. If $H^{0,2}(M,\mathfrak{g}_P^C)=0$ and $H^0(M,\mathfrak{g}_P^C)\cong\mathfrak{z}^C$, then the psace $EHLPM(\nabla)$ coincides with the support manifold.*

Proof. We must show that $EHEID(\nabla)\cong H^{0,1}(M,\mathfrak{g}_P^C)$ and $\operatorname{Ker} I_\nabla/\operatorname{Im} E_{EH}'{}_\nabla\cong H^{0,2}(M,\mathfrak{g}_P^C)\oplus H^0(M,\mathfrak{g}_P)/\mathfrak{z}$, where $H^0(M,\mathfrak{g}_P)$ denotes the vector space of all parallel sections of $\mathfrak{g}_P$. First we see that the sequence

$$(6.9.1)\qquad C^\infty(\mathfrak{g}_P)\xrightarrow[\nabla]{} C^\infty(\Lambda^1\otimes\mathfrak{g}_P)\xrightarrow[E_{EH}'{}_\nabla]{} C^\infty(\Lambda^{0,2}\otimes\mathfrak{g}_P^C)\oplus C^\infty(\mathfrak{g}_P)\xrightarrow[\mathrm{pr}\circ I_\nabla]{} C^\infty(\Lambda^{0,3}\otimes\mathfrak{g}_P^C)$$

is an elliptic complex. Therefore

$$(6.9.2)\qquad \begin{aligned}(\operatorname{Ker} I_\nabla/\operatorname{Im} E_{EH}'{}_\nabla)\oplus\mathfrak{z}&\cong\operatorname{Ker}(\mathrm{pr}\circ I_\nabla)/\operatorname{Im} E_{EH}'{}_\nabla\\ &\cong\operatorname{Ker}(\mathrm{pr}\circ I_\nabla)\cap\operatorname{Ker}(E_{EH}'{}_\nabla)^*\,.\end{aligned}$$

Let $(P,\eta)\in\operatorname{Ker}(\mathrm{pr}\circ I_\nabla)\cap\operatorname{Ker}(E_{EH}'{}_\nabla)^*$. We easily see that

$$(6.9.3)\qquad (\omega, d^\nabla A)=4\operatorname{Re}(\sqrt{-1}\,\nabla^{\bar\alpha}A_{\bar\alpha})\,.$$

Thus $(P,\eta)\in\operatorname{Ker}(E_{EH}'{}_\nabla)^*$ means that

(6.9.4) $$\langle (P, \eta), (p^{0,2}(d^{\nabla}A), 4\mathrm{Re}(\sqrt{-1}\,\nabla^{\bar{\alpha}}A_{\bar{\alpha}}))\rangle = 0$$

for all $A \in C^{\infty}(\Lambda^1 \otimes \mathfrak{g}_P)$, from which we have

(6.9.5) $$-\nabla^{\bar{\beta}}P_{\bar{\beta}\bar{\alpha}} + 2\sqrt{-1}\,\nabla_{\bar{\alpha}}\eta = 0\,.$$

Here we know that $\nabla^{\bar{\alpha}}\nabla^{\bar{\beta}}P_{\bar{\beta}\bar{\alpha}}=0$ since ∇ is Einstein. Therefore we see that

(6.9.6) $$-\nabla^{\bar{\beta}}P_{\bar{\beta}\bar{\alpha}} = 0 \quad \text{and} \quad \nabla\eta = 0\,.$$

Combining with the assumption that $(P, \eta) \in \mathrm{Ker}(\mathrm{pr}\circ I_{\nabla})$, we see that P is harmonic and η is parallel. The converse is obvious, and we get

(6.9.7) $$\mathrm{Ker}(\mathrm{pr}\circ I_{\nabla}) \cap \mathrm{Ker}(E_{EH}{}'_{\nabla})^* \cong H^{0,2}(M, \mathfrak{g}_P^{\mathbf{C}}) \oplus H^0(M, \mathfrak{g}_P)\,.$$

Let $A \in$ *EHEID*(∇). Then by definition and equality (6.9.3) we get

(6.9.8) $$\nabla_{\bar{\alpha}}A_{\bar{\beta}} - \nabla_{\bar{\beta}}A_{\bar{\alpha}} = 0\,,$$

(6.9.9) $$\nabla^{\bar{\alpha}}A_{\bar{\alpha}} \in C^{\infty}(\mathfrak{g}_P)$$

and

(6.9.10) $$\nabla^{\bar{\alpha}}A_{\bar{\alpha}} + \nabla^{\alpha}A_{\alpha} = 0\,.$$

Thus we see that $p^{0,1}A$ is harmonic and so the first isomorphism holds. Q.E.D.

The above results are resumed as follows.

Theorem 6.10. *Let ∇ be an Einstein holomorphic connection. The space EHLPM(∇) coincides with the space YMLPM(∇) around ∇, which is a real analytic set of the support manifold Z whose tangent space at ∇ coincides with the space EHEID(∇). If $H^{0,2}(M, \mathfrak{g}_P^{\mathbf{C}})=0$ and $H^0(M, \mathfrak{g}_P^{\mathbf{C}})=\mathfrak{z}^{\mathbf{C}}$, then the space EHLPM$(\nabla)$ coincides with the support manifold Z.*

Remark 6.11. The above statement suggests the equality *YMEID*$(\nabla)=$ *EHEID*(∇), which in fact holds.

7. The deformation of Einstein holomorphic connections caused by a deformation of complex structure of the base manifold

In this section we discuss on deformations of Einstein holomorphic connections in a similar situation as in section 4. Let (M, J, g) be the base Kähler manifold and (J_t, g_t) be a one-parameter family of Kähler structure such that $(J_0, g_0)=(J, g)$. Define maps E, I

$$E: (-\varepsilon, \varepsilon)\times C^s \to H^{s-1}(\Lambda^{0,2}\otimes\mathfrak{g}_P^{\mathbf{C}})\oplus H^{s-1}(\mathfrak{g}_P)\,,$$

$$I_{(t,\nabla)}: H^{s-1}(\Lambda^{0,2}\otimes\mathfrak{g}_P^{\mathbf{C}})\oplus H^{s-1}(\mathfrak{g}_P) \to H^{s-2}(\Lambda^{0,3}\otimes\mathfrak{g}_P^{\mathbf{C}})\oplus\mathfrak{z}$$

by

$$E(t, \nabla) = (p^{0,2}R^{\nabla_t}, (\omega_t, R^{\nabla_t})_t - c_t)$$

and

$$I_{(t,\nabla)}(P, \eta) = (p^{0,3}(d^{\nabla_t}P), \mathfrak{z}\text{-part of } \eta),$$

where all operators and $c_t \in \mathfrak{z}$ depending on base Kähler structure are defined by (J_t, g_t). Then we know that I is an identity for E.

Theorem 7.1. *Let ∇ be an Einstein holomorphic connection on (M, J, g). If $H^{0,2}(M, \mathfrak{g}_P^C)=0$ and $H^0(M, \mathfrak{g}_P)=\mathfrak{z}$, then for any deformation (J_t, g_t) of Kähler structures of (J, g) there exists a one-parameter family of connections ∇_t of P so that each ∇_t is an Einstein holomorphic connection over (M, J_t, g_t), provided that $|t|$ is sufficiently small. Moreover, each local pre-moduli space $EHLPM(\nabla_t)$ over (M, J_t, g_t) forms a manifold of the same dimension.*

Proof. The obstruction space for E-deformation of $(0, \nabla)$ with respect to I coincides with the space

$$\operatorname{Ker} I_\nabla / \operatorname{Im} E'_{(0,\nabla)},$$

where I_∇ is introduced before Lemma 6.8. It is a quotient space of the space $\operatorname{Ker} I_\nabla / \operatorname{Im} E_{EH}'{}_\nabla$ and vanishes by assumption. Therefore by Lemma 3.3 the set $E^{-1}(0)$ around $(0, \nabla)$ forms a manifold whose tangent space at $(0, \nabla)$ is given by $\operatorname{Ker} E'_{(0,\nabla)}$. But here the projection map from $\operatorname{Ker} E'_{(0,\nabla)}$ to $T_0(-\mathcal{E}, \mathcal{E})$ is surjective, which completes the proof by the implicit function theorem. In fact, for $u \in T_0(-\mathcal{E}, \mathcal{E})$, we get $E'_{(0,\nabla)}(u, 0) \in \operatorname{Ker} I_\nabla$, therefore by assumption there is $A \in T_\nabla C^s$ such that $E'_{(0,\nabla)}(u, 0) = E_{EH}'{}_\nabla(A)$, i.e., $E'_{(0,\nabla)}(u, -A)=0$. Q.E.D.

8. Einstein holomorphic connections and holomorphic structures

For a connection ∇ of P we denote by $\Psi(\nabla)$ the $(0, 1)$-part of ∇, which is an almost holomorphic structure of P^C. Remark that the map Ψ commutes with the action of the gauge group $\mathcal{G}$. Therefore Ψ induces a map from the moduli space of Einstein holomorphic connections to the moduli space of holomorphic structures. This map locally corresponds to a map φ: $EHLPM(\nabla) \to HLPM(\bar\partial)$, where ∇ is an Einstein holomorphic connection and $\bar\partial = \Psi(\nabla)$.

Theorem 8.1. *Let ∇ be an Einstein holomorphic connection. If $H^0(M, \mathfrak{g}_P) \cong \mathfrak{z}$, then the map $p \circ \Psi$ gives a bijection between $EHLPM(\nabla)$ and $HLPM(\Psi(\bar\partial))$ around ∇, where the map p: $W^{C,s} \to \mathcal{S}^{C,s}_{\Psi(\nabla)}$ is defined by Propotition* 5.1.

Proof. Set $\mathcal{E} = \{\nabla_1 \in \mathcal{S}^s_\nabla;\ (\omega, R^{\nabla_1}) - c_0 = 0\}$. The derivative of the map f: $\nabla_1 \to (\omega, R^{\nabla_1}) - c_0$ at ∇ is given by

$$A \to 2\sqrt{-1}\,(\nabla^{\bar\alpha} A_{\bar\alpha} - \nabla^{\alpha} A_{\alpha})\,.$$

Set $A_{\bar\alpha} = \sqrt{-1}\,\nabla_{\bar\alpha}\psi$ for $\psi \in H^{s+1}(\mathfrak{g}_P)$. Then

$$A \in T_\nabla \mathcal{S}^s_\nabla$$

and

$$2\sqrt{-1}\,(\nabla^{\bar\alpha} A_{\bar\alpha} - \nabla^{\alpha} A_{\alpha}) = 2\nabla^*\nabla\psi\,.$$

Therefore the image of the derivative of the map f from $\mathcal{S}^s_\nabla$ is closed in $H^{s-1}(\mathfrak{g}_P)$, and coincides with the orthogonal complement of $H^0(M, \mathfrak{g}_P)$. Therefore by assumption and Lemma 6.8, the map f from $\mathcal{S}^s_\nabla$ to the orthogonal complement of $\mathfrak{z}$ has surjective derivative, from which we see that $\mathcal{E}$ is a manifold whoes tangent space at ∇ coincides with the space

$$\{A \in H^s(\Lambda^1 \otimes \mathfrak{g}_P);\ -\nabla^{\bar\alpha} A_{\bar\alpha} = 0\}\ .$$

Since the derivative of the map $p\circ\Psi$ from $\mathcal{E}$ is nothing but the correspondence: $A \to (0, 1)$-part of A, $p\circ\psi$ gives a local diffeomorphism from $\mathcal{E}$ to $\mathcal{S}^s_{\bar\partial}$. If $\nabla_1 \in EHLPM(\nabla)$ then $p\circ\Psi(\nabla_1) \in HLPM(\Psi(\nabla))$, conversely, if $\bar\partial_1 \in HLPM(\Psi(\nabla))$ then $(p\circ\Psi\,|\,\mathcal{E})^{-1}(\bar\partial_1)$ is Einstein holomorphic by definition of $\mathcal{E}$. Q.E.D.

Remark 8.2. Theorem 8.1 and Theorem 5.4 give another proof of Theorem 6.10.

Combining with Theorem 6.10, we get the following

Theorem 8.3. *Let ∇ be an Einstein holomorphic connection and set $\bar\partial = \Psi(\nabla)$. Then there exists a natural correspondence*

$$YMLPM(\nabla) = EHLPM(\nabla) \to HLPM(\bar\partial)\,,$$

where $\to$ is an injection, and becomes a bijection if $\operatorname{Ker}\nabla = \mathfrak{z}$.

9. A structure on the moduli space

Let ∇ be an Einstein holomorphic connection and set $\bar\partial = \Psi(\nabla)$. Assume that $H^0_{\bar\partial}(\mathfrak{g}^C_P) = \mathfrak{z}^C$ and $H^2_{\bar\partial}(\mathfrak{g}^C_P) = 0$. Then the manifolds $EHLPM(\nabla)$ and $HLPM(\bar\partial)$ are isomorphic by Theorem 8.1, and become complex manifolds by Theorem 5.4. The complex structures are realized by the almost complex structures given by multiplying $\sqrt{-1}$ on $T_{\bar\partial} HLPM(\bar\partial)$ and $\tilde J$ on $T_\nabla EHLPM(\nabla)$, where $\tilde J$ is defined by $(\tilde J A)_i = -A_j J^j{}_i$. In fact, we see that

$$\Psi(\tilde J A) = (\tilde J A)^{(0,1)} = -A_{\bar\beta} J^{\bar\beta}{}_{\bar\alpha} = \sqrt{-1}\,A_{\bar\alpha} = \sqrt{-1}\,\Psi(A)\,.$$

On the other hand, the space C^s has the riemannian metric $\langle\cdot,\cdot\rangle$, which is invariant under the action of $\mathcal{G}^{s+1}$. Therefore the manifold $EHLPM(\nabla)$ has a canonical riemannian metric, which is given as follows. Let $\nabla_1 \in EHLPM(\nabla)$ and $A, B \in T_{\nabla_1} EHLPM(\nabla)$. The elements A and B are Einstein holomorphic infinitesimal deformations of ∇_1, and are decomposed into the essential parts A_E, B_E and trivial parts A_T, B_T (see (1.5.2)). We define the inner product of A and B by $\langle A_E, B_E \rangle$. From Lemma 13.1, we see that this inner product becomes a C^ω-riemannian metric.

DEFINITION 9.1. The above riemannian metric on $EHLPM(\nabla)$ is called *the natural riemannian metric.*

REMARK 9.2. Let ∇_1 and ∇_2 be Einstein holomorphic connections and assume that there are $\nabla_0 \in EHLPM(\nabla_1)$ and $\gamma \in \mathcal{G}^{s+1}$ such that $\gamma^*\nabla_0 \in EHLPM(\nabla_2)$. Then for each $\nabla \in EHLPM(\nabla_1)$ sufficiently close to ∇_0 there is $\gamma \in \mathcal{G}^{s+1}$ so that $\gamma^*\nabla \in EHLPM(\nabla_2)$, and this correspondence: $\nabla \rightarrow \gamma^*\nabla$ becomes an isometry. Therefore we may say that the canonical riemannian metric is independent of ∇.

Theorem 9.3. *Let ∇ be an Einstein holomorphic connection and set $\bar\partial = \Psi(\nabla)$. If $H^0_{\bar\partial}(\mathfrak{g}^C_P) = \mathfrak{z}^C$ and $H^2_{\bar\partial}(\mathfrak{g}^C_P) = 0$, then the canonical riemannian metric on $EHLPM(\nabla)$ is a Kähler metric with respect to the complex structure on $HLPM(\bar\partial)$.*

Proof. We easily see that the canonical riemannian metric is a hermitian metric. We have to show that the Kähler form is closed. We replace ∇ by ∇_0 and denote by ∇ elements of $HLPM(\nabla_0)$ regarded as variable. Conisder the fiber bundle $p\colon P \times EHLPM \rightarrow EHLPM$. In general, a diffeomorphism from a fiber to another fiber which commutes with the action of G and fixes M pull backs a G-invariant structure, and so if a vector field v on $P \times EHLPM$ is p-projectable, G-invarant and $\pi^*v = 0$, where π is the projection to M, then the Lie derivation $\mathcal{L}_v$ on a family of G-invariant structures is defined. For example,

$$\mathcal{L}_v \nabla \equiv \frac{d}{ds}\bigg|_0 (\exp sv)^*\nabla .$$

If we decompose v into the P-part v_P and the $EHLPM$-part v_M, we see that

$$\mathcal{L}_v \nabla = v_M[\nabla] + L_{v_P}\nabla .$$

Now, we denote the almost complex structure on $EHLPM$ by J^E, the canonical riemannian metric by g^E and the Kähler form by ω^E. Decompose $v \in T(EHLPM)$ into v_E and v_T so that $\mathcal{L}_{v_E}\nabla$ is essential and $\mathcal{L}_{v_T}\nabla$ is trivial. This decomposition is not unique, but we may assume that it depends C^∞-ly on v by Lemma 13.1. Then we see that

$$\begin{aligned}
&\mathcal{L}_{(J^E v)_E}\nabla = \tilde{J}\mathcal{L}_{v_E}\nabla ,\\
&g^E(v, w) = \langle \mathcal{L}_{v_E}\nabla, \mathcal{L}_{w_E}\nabla \rangle ,\\
&\omega^E(v, w) = g^E(J^E v, w) = \langle \tilde{J}\mathcal{L}_{v_E}\nabla, \mathcal{L}_{w_E}\nabla \rangle ,
\end{aligned}$$

where $\tilde{J}$ is defined in the first paragraph of this section. We may assume that $[v, w]=[w, z]=[z, v]=0$ without loss of generality, and see that

$$\begin{aligned}(d\omega^E)(v, w, z) &= v\cdot\omega^E(w, z)+\text{alternating terms}\\ &= v\cdot\langle\tilde{J}\mathcal{L}_{w_E}\nabla, \mathcal{L}_{z_E}\nabla\rangle+\text{alt}.\\ &= \langle\tilde{J}\mathcal{L}_{v_E}\mathcal{L}_{w_E}\nabla, \mathcal{L}_{z_E}\nabla\rangle+\langle\tilde{J}\mathcal{L}_{w_E}\nabla, \mathcal{L}_{v_E}\mathcal{L}_{z_E}\nabla\rangle+\text{alt}.\\ &= -\langle\tilde{J}\mathcal{L}_{v_E}\mathcal{L}_{w_E}\nabla, \tilde{J}\mathcal{L}_{z_E}\nabla\rangle+\langle\mathcal{L}_{v_E}\mathcal{L}_{z_E}, \tilde{J}\mathcal{L}_{w_E}\nabla\rangle+\text{alt}.\\ &= -\langle[\mathcal{L}_{v_E}, \mathcal{L}_{w_E}]\nabla, \tilde{J}\mathcal{L}_{z_E}\nabla\rangle+\text{alt}.\\ &= -\langle\mathcal{L}_{[v_E,w_E]}\nabla, \tilde{J}\mathcal{L}_{z_E}\nabla\rangle+\text{alt}.\end{aligned}$$

But here $p_*[v_E, w_E]=[v, w]=0$ and so $[v_E, w_E]$ is vertical, which implies that $\mathcal{L}_{[v_E,w_E]}\nabla$ is trivial. Q.E.D.

10. Example I

Let M be a flat torus T^2, P the trivial principal $U(2)$-bundle and ∇_0 the canonical connection of P. ∇_0 is a flat connection, and so an Einstein holomorphic connection. Therefore, by Lemma 6.3, all Einstein holomorphic connections of P are flat. Fix a point x in M and an element p in P_x. Any closed curve c ($c(0)=c(1)=x$) in M is horizontally lifted to a curve $\tilde{c}$ in P so that $\tilde{c}(0)=p$, and we get an element $\tilde{c}(1)$ in P_x. Let a be an element of $U(2)$ such that $\tilde{c}(1)=p\cdot a$. Since ∇ is flat, this mapping: $c\to a$ induces a homomorphism: $\pi_1(M)\to U(2)$, defined by $[c]\to a$. Taking generators $\{[c_1], [c_2]\}$ of $\pi_1(M)$, we get corresponding elements $\{a_1, a_2\}$ in $U(2)$ such that $a_1^{-1}\cdot a_2^{-1}\cdot a_1\cdot a_2=\mathrm{id}$. Denote by $f(\nabla)$ this pair (a_1, a_2). We see that by a gauge transformation η of P, $f(\nabla)=(a_1, a_2)$ is transformed as

$$f(\eta^*\nabla) = (b^{-1}\cdot a_1\cdot b, b^{-1}\cdot a_2\cdot b)\,, \tag{10.1}$$

where $b\in U(2)$ is defined by $\eta(x)\cdot p=p\cdot b$.

Thus the global moduli space of Einstein holomorphic connections is identified with the quotient space $\{\text{commuting pair in } U(2)\times U(2)\}/\sim$, where $\sim$ is defined by $(b^{-1}\cdot a_1\cdot b, b^{-1}\cdot a_2\cdot b)\sim(a_1, a_2)$ for $b\in U(2)$. By diagonalization, this space becomes the space $T^2\times T^2/\sim$, where

$$\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \begin{pmatrix}\gamma & 0\\ 0 & \delta\end{pmatrix}\right)\sim\left(\begin{pmatrix}\alpha' & 0\\ 0 & \beta'\end{pmatrix}, \begin{pmatrix}\gamma' & 0\\ 0 & \delta'\end{pmatrix}\right)$$

if and only if they coinside or $\beta'=\alpha$, $\alpha'=\beta$, $\delta'=\gamma$ and $\gamma'=\delta$.

On the other hand, *the space $EHEID(\nabla_0)$ is the space of harmonic sections of $\Lambda^1\otimes\mathfrak{u}(2)$, and is isomorphic with $\boldsymbol{R}^2\otimes\mathfrak{u}(2)$.* Let $A\in EHEID(\nabla_0)$ and consider the connection ∇_0+A. Since $\nabla_0 A=0$, we see that

$$E_{EH}(\nabla_0+A) = (0, 2\,[\overline{A^{(0,1)}}, A^{(0,1)}])\,,$$

and

$$(\omega, d^{\nabla_0}B) = 2\sqrt{-1}\,(\nabla_0^{\bar{\alpha}}B_{\bar{\alpha}}-\nabla_0^{\alpha}B_{\alpha})\,,$$

which implies that ∇_0+A is an element of the support manifold of $EHLPM(\nabla_0)$. Thus we see that *the support manifold is locally isomorphic with* $\boldsymbol{R}^2\otimes\mathfrak{u}(2)$. Moreover ∇_0+A belongs to $EHLPM(\nabla_0)$ if and only if

$$[\overline{A^{(0,1)}}, A^{(0,1)}] = 0\,.$$

Therefore *the space* $EHLPM(\nabla_0)$ *is a proper subset of the support manifold.* Moreover, the group $\mathcal{G}_{\nabla_0}\cong U(2)$ acts on the space $EHLPM(\nabla_0)$ analogously as (10.1), and we see that

$$\mathcal{G}_{\nabla_0}\backslash EHLPM(\nabla_0)\cong\boldsymbol{R}^2\times\boldsymbol{R}^2/\sim\,.$$

By a similar way we see that the space $HEID(\bar\partial_0)$ is canonically isomorphic with the space $\boldsymbol{C}\otimes\mathfrak{gl}(2,\boldsymbol{C})$, and $\bar\partial_0+HEID(\bar\partial_0)$ is the support manifold of $HLPM(\bar\partial_0)$. In this case, the space $HLPM(\bar\partial_0)$ is an open set of the support manifold. We can see more details as follows. The group $\mathcal{G}^{\boldsymbol{C}}_{\bar\partial_0}$ acts on the space $HLPM(\bar\partial_0)$, and

$$\mathcal{G}^{\boldsymbol{C}}_{\bar\partial_0}\backslash HLPM(\bar\partial_0)\cong GL(2,\boldsymbol{C})\backslash\mathfrak{gl}(2,\boldsymbol{C})\,,$$

whose elements are classified using Jordan's normal form. An element of $\mathfrak{gl}(2,\boldsymbol{C})$ corresponds to an Einstein holomorphic connection if and only if it is diagonalizable. Thus

$$\mathcal{G}_{\nabla_0}\backslash EHLPM(\nabla_0)\subsetneqq\mathcal{G}^{\boldsymbol{C}}_{\bar\partial_0}\backslash HLPM(\bar\partial_0)\,.$$

Remark that the space $\mathcal{G}^{\boldsymbol{C}}_{\bar\partial_0}\backslash HLPM(\bar\partial_0)$ is *not* a Hausdorff space. In fact, any neighbourhood of the element $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ in $U(2)$ implies some $\begin{pmatrix}\lambda & t\\ 0 & \lambda\end{pmatrix}$ $(t\neq 0)$, which is conjugate with $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$.

11. Example II

Let (M,g) be an Einstein-Kähler manifold with Ricci tensor$=e\cdot g$, ∇ an Einstein holomorphic connection and $\bar\partial=\Psi(\nabla)$. Then we can see that

$$\begin{aligned}(11.0.1)\qquad \{(\bar\partial^*\bar\partial+2\bar\partial\bar\partial^*)\,A\}_{\bar\alpha}&=(\nabla^*\nabla A)_{\bar\alpha}+eA_{\bar\alpha}+2[R^{\nabla\bar\beta}{}_{\bar\alpha}, A_{\bar\beta}]\\ &= 2\{-\nabla^{\bar\beta}\nabla_{\bar\beta}A_{\bar\alpha}+eA_{\bar\alpha}+[R^{\nabla\bar\beta}{}_{\bar\alpha}, A_{\bar\beta}]\}\end{aligned}$$

for $\mathfrak{g}_P^{\boldsymbol{C}}$-valued (0,1)-form A,

$$(11.0.2)\qquad \{(\frac{2}{3}\bar\partial^*\bar\partial+2\bar\partial\bar\partial^*)A\}_{\bar\alpha\bar\beta}$$
$$= (\nabla^*\nabla A)_{\bar\alpha\bar\beta}+2eA_{\bar\alpha\bar\beta}+2[R^{\nabla\bar\gamma}{}_{\bar\alpha}, A_{\bar\gamma\bar\beta}]+2[R^{\nabla\bar\gamma}{}_{\bar\beta}, A_{\bar\alpha\bar\gamma}]$$

for $\mathfrak{g}_P^C$-valued (0,2)-form A. Therefore, to see whether $H^1_{\bar\partial}$ and $H^2_{\bar\partial}$ vanish, we have to get eigenvalues of these operators.

Let M be a homogeneous space K/H and P the principal G-bundle $K\times_\rho G$, where ρ is a homomorphism: $H\to G$. Then we have

$$G_P = K\times_{\mathrm{Ad}_\rho} G\,,\quad \mathfrak{g}_P=K\times_{\mathrm{Ad}_\rho}\mathfrak{g}\,.$$

As usual, we identify $C^\infty(\mathfrak{g}_P)$ with $C^\infty(K,\mathfrak{g})_H$. Let $\mathfrak{k}=\mathfrak{h}+\mathfrak{m}$ be an H-invariant decomposition and define a differential operator D: $C^\infty(K,\mathfrak{g})_H\to C^\infty(K,\mathfrak{m}^*\otimes\mathfrak{g})_H$ by

$$(D\mathfrak{s})(X) = (X\mathfrak{s})\,.$$

Then this operator D gives a covariant derivative of $\mathfrak{g}_P$, which is identified with the standard connection ∇ of P. Let C_K (resp. C_H) be the Cassimir operator of the K-module (resp. H-module) $C^\infty(K,\mathfrak{g})_H$. We can check that

$$\nabla^*\nabla = C_K - C_H$$

and

$$R^\nabla(X, Y) = -\rho[X, Y]\qquad\text{for}\quad X, Y\in\mathfrak{m}\,.$$

(See e.g., [10, Proposition 5.3].)

Therefore the eigenvalues of operators (11.0.1) and (11.0.2) are calculated explicitly by the representation theory. The calculation is easy but complicated, and we omit the detail. See e.g. [10, §7].

Let $M=P^n(\boldsymbol{C})=SU(n+1)/S(U(n)\times U(1))$ and P the unitary frame bundle of T^+M. Then $\mathfrak{g}=\mathfrak{m}^-\otimes\mathfrak{m}^+$, and the operator (11.0.1) has only positive eigenvalues. Thus $H^1_{\bar\partial}(M,\mathfrak{g}_P^C)=0$.

Proposition 11.1. *The standard connection of the unitary frame bundle of $T^+P^n(\boldsymbol{C})$ is isolated in the moduli space.*

Next, let P be the unitary frame bundle of the symmetric tensor product S^2T^+M of T^+M. Then $\mathfrak{g}=(S^2\mathfrak{m}^-)\otimes(S^2\mathfrak{m}^+)$. In this case the operator (11.0.1) has 0 as an eigenvalue, ard all eigenvalues of the operator (11.0.2) are poitive. Moreover, we can easily check that $H^0_{\bar\partial}(M,\mathfrak{g}_P^C)=\mathfrak{z}^C$. Thus by Theorem 6.10, we get the following

Proposition 11.2. *The local pre-moduli space around the standard connection of the unitary frame bundle of $S^2T^+P^n(\boldsymbol{C})$ $(n\geq 2)$ forms a non-trivial manifold.*

12. Regularity of Yang-Mills connections

In this section we consider not a family of connections but one connection. Let ∇ be a Yang-Mills $C^{2+\alpha}$-connection of P $(0<\alpha<1)$. I.e., if we represent ∇ by a local frame $\{\xi_p\}$ of $\mathfrak{g}_P$ as

$$\nabla_{\partial_i}\xi_p = \Gamma^q_{ip}\,\xi_q$$

then Γ^q_{ip} are $C^{2+\alpha}$. A local section ξ of $\mathfrak{g}_P$ is said to be harmonic if $\nabla^*\nabla\xi=0$. The defining equation of harmonic section is a linear elliptic differential equation with $C^{1+\alpha}$-coefficients. Therefore we can take a local frame by harmonic sections, which are $C^{3+\alpha}$ ([2, p. 228 Theorem 1]). The coefficients Γ^q_{ip} with respect to the frame are $C^{2+\alpha}$. But we know that $\{\Gamma^q_{ip}\}$ satisfies Yang-Mills equation:

$$g^{kl}\partial_k(\partial_e\Gamma^q_{ip}-\partial_i\Gamma^q_{lp})+\text{lower terms} = 0\,,$$

and harmonic equation

$$g^{kl}\partial_k\Gamma^q_{lp}+\text{lower terms} = 0\,,$$

which is quasi-linear elliptic system with C^∞-coefficients. Thus Γ^q_{ip} are C^∞([11, Theorem 6.8.1]). If (M, g) is a C^ω-riemannian manifold, then Γ^q_{ip} are C^ω ([11, Theorem 6.7.6]).

Theorem 12.1. *Let (M, g) be a C^∞ (resp. C^ω) riemannian manifold and ∇ a Yang-Mills C^3-connection. Then there exists a C^3-gauge transformation γ so that $\gamma^*\nabla$ is C^∞* (resp. C^ω).

Corollary 12.2. *Let (M,g) be a simply connected C^ω-riemannian manifold. Let ∇_1 and ∇_2 be Yang-Mills connections on M. Assume that there is an open set U of M and a gauge transformation γ on U such that $\gamma^*\nabla_1=\nabla_2$. Then γ extends to a global gauge transformation $\tilde{\gamma}$ so that $\tilde{\gamma}^*\nabla_1=\nabla_2$ on M.*

Proof. We may assume that $\gamma=id$ on U and ∇_1 is C^ω. For $x\in U$ and $y\in M$, take a joining geodesic c: $[0, 1]\to M$ and a C^ω-tubular neighbourhood $V\cong(-\varepsilon, 1+\varepsilon)\times D^{n-1}$ of $c[0, 1]$. Take a C^ω-frame of $\mathfrak{g}_P$ on $\{0\}\times D^{n-1}$ and take the parallel extension $\{\xi_p\}$ (resp. $\{\bar{\xi}_p\}$) for the direction $(-\varepsilon, 1+\varepsilon)$ with respect to ∇_1 (resp. ∇_2). Let $\tilde{\gamma}$ be the gauge transformation on V which transforms $\{\xi_p\}$ to $\{\bar{\xi}_p\}$. Since ∇_2 is C^ω with respect to $\{\bar{\xi}_p\}$, $\tilde{\gamma}^{-1*}\nabla_2$ is C^ω with respect to $\{\xi_p\}$. But here $\tilde{\gamma}=id$ on U, which implies that $\tilde{\gamma}^{-1*}\nabla_2=\nabla_1$ on V by analyticity. Moreover the extension of γ to $\tilde{\gamma}$ is unique and well-defined since M is simply connected. Q.E.D.

REMARK 12.3. This is an analogy of the unique extension theorem of Einstein metrics in [3, Section 5].

13. Some basic lemmas

Lemma 13.1 ([8, Lemma 4.3]). *Let v_t be a family of volume elements on M, E_t, F_t families of vector bundles over M with fiber metrics g_t^E, g_t^F and Q_t: $C^\infty(E_t) \to C^\infty(F_t)$ a family of differential operators of order k with injective symbol. Assume that v_t, E_t, F_t, g_t^E, g_t^F and Q_t depend C^∞-ly* (resp. *real analytically*) *on t. That is, there are bundle isomorphism e_t: $E_0 \to E_t$ and f_t: $F_0 \to F_t$ such that the coefficients of $e_t^* g_t^E$, $f_t^* g_t^F$ and $(f_t^{-1})_* \circ Q_t \circ (e_t)_*$ depend C^∞-ly* (resp. *real analytically*) *on t. Then the dimension of the space* Ker Q_t *is upper semicontinuous. If the dimension of the space* Ker Q_t *is constant, then the decompositions*

$$H^s(E_t) = Q_t^*(H^{s+k}(F_t)) \oplus \mathrm{Ker}\, Q_t\,, \tag{13.1.1}$$

$$H^s(F_t) = Q_t(H^{s+k}(E_t)) \oplus \mathrm{Ker}\, Q_t^* \tag{13.1.2}$$

depend C^∞-ly (resp. *real analytically*) *on t, where Q_t^* is the formal adjoint operator of Q_t with respect to g_t^E, g_t^F and v_t. Moreover the isomorphisms*

$$Q_t^*+1\colon Q_t(H^{s+2k}(E_t)) \oplus \mathrm{Ker}\, Q_t \to H^s(E_t)\,, \tag{13.1.3}$$

$$Q_t+1\colon Q_t^*(H^{s+2k}(F_t)) \oplus \mathrm{Ker}\, Q_t^* \to H^s(F_t) \tag{13.1.4}$$

also depend C^∞-ly (resp. *real analytically*) *on t.*

Lemma 13.2 ([4, Theorem 3.12]). *In the real analytic category in Banach spaces, the implicit function theorem holds.*

Lemma 13.3 ([8, Lemma 13.7]). *Let E and F be vector bundles over M and E^C, F^C their complexifications. Let f be a C^∞-cross section of E and ψ: $E \to F$ a fiber preserving C^∞-map defined on an open set of E which contains the image of f. Assume that ψ has an extension to a fiber preserving map ψ^C: $E^C \to F^C$ defined on an open set of E^C such that the restriction ψ_x^C to each fiber E_x^C is holomorphic. Then the map Ψ: $H^s(E) \to H^s(F)$ defined by*

$$\Psi(u) = \Psi \circ u\,, \tag{13.3.1}$$

defined on an open neighbourhood of f, is real analytic provided that $s > [n/2]+1$.

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