

GENERALIZATIONS OF NAKAYAMA RING IV

(LEFT SERIAL RINGS WITH $(*, I)$)

YOSHITOMO BABA AND MANABU HARADA

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Let R be an algebra over an algebraically closed field K with finite dimension. Under an assumption $J^4=0$, we have studied a left serial algebra with $(*, 1)$: the radical of any hollow right R -module is always a direct sum of hollow modules, in [3], where J is the Jacobson radical of R . In this case eJ/eJ^2 is square-free, i.e., a direct sum of simple modules, which are not isomorphic to one another. We shall give, in this note, a complete characterization of a left serial ring with $(*, 1)$ under the assumption: eJ/eJ^2 square-free. In the forthcoming paper [5], we shall study a left serial ring with $(*, 1)$ in general.

1. Definitions and preliminaries

In this note we only deal with a left and right artinian ring R with identity. We assume that every R -module M is a unitary right (or left) R -module and denote its Jacobson radical and socle by $J(M)$ and $\text{Soc}(M)$, respectively. $|M|$ means the length of a composition series of M . If M has a unique composition series, we call M a *uniserial module*. If, for each primitive idempotent e , eR is uniserial as a right R -module, we call R a *right serial ring (Nakayama ring)*.

We obtained a characterization of a right serial ring in terms of submodules in a direct sum of uniserial modules [1]. As a generalization of the above result, we studied the following property:

$(*, n)$ *Every maximal submodule of a direct sum of n hollow modules is also a direct sum of hollow modules [2].*

In this note we shall study a ring with $(*, 1)$, i.e., every factor module of eJ is a direct sum of hollow modules for each primitive idempotent e , where $J=J(R)$. Concerning $(*, 1)$ we got

Lemma A ([4], Theorem 4). *Let R be a right artinian ring. Then R satisfies $(*, 1)$ for any hollow right R -module if and only if the following two conditions are fulfilled:*

1) $eJ = \sum_{i=1}^m \oplus A_i$, where e is any primitive idempotent in R and the A_i are hollow.

2) Let $C_i \supset D_i$ be two submodules of A_i such that C_i/D_i is simple. If $f: C_i/D_i \cong C_j/D_j$ for $i \neq j$, f or f^{-1} is extendible to an element in $\text{Hom}_R(A_i/D_i, A_j/D_j)$ or $\text{Hom}_R(A_j/D_j, A_i/D_i)$.

On the other hand T. Sumioka found the following remarkable result:

Lemma B ([6], Corollary 4.2). *Let R be a left serial ring, then eJ^i is a direct sum of hollow modules as right R -modules for any i .*

We shall study, in this paper, only left serial rings, and so we denote the content of Lemma B by the following diagram:

$$(1) \quad \begin{array}{ccccccc} & & & & & & eR \\ & & & & & & | \\ & & & & & & A_1 \quad A_2 \quad \cdots \quad A_n \\ & & & & & & | \quad | \quad \cdots \quad | \\ & & & & & & \overline{A_{11}} \cdots \overline{A_{1n_1}} \quad \overline{A_{21}} \cdots \overline{A_{2n_2}} \quad \cdots \quad \overline{A_{n1}} \cdots \overline{A_{nn_n}} \\ & & & & & & | \quad | \quad \cdots \quad | \end{array} \quad \begin{array}{l} eJ \\ eJ^2 \end{array}$$

The diagram means that $J(eR) = eJ = \sum_{i=1}^n \oplus A_i$, $J(A_k) = \sum_{i=1}^{n_k} \oplus A_{ki}$ ($k = 1, 2, \dots, n$). (cf. [2], § 2).

Further we continue to observe (1).

$$(2) \quad \begin{array}{ccc} & A_{11} & \\ & | & \\ \overline{A_{111}} \overline{A_{112}} \cdots \overline{A_{11t_1}} & \cdots & J(A_{11}) \\ | \quad | \quad \cdots \quad | & & \\ & A_{12} & \\ & | & \\ \overline{A_{121}} \overline{A_{122}} \cdots \overline{A_{12t_2}} & \cdots & J(A_{12}) \\ | \quad | \quad \cdots \quad | & & \end{array}$$

and repeat this process. We sometime denote $A_{i_1 i_2 \cdots i_t}$ by A_α , and define $|\alpha| = t$. Let $\alpha = i_1 i_2 \cdots i_t$ and $\beta = j_1 j_2 \cdots j_{t'}$. We define $\alpha > \beta$ if $t < t'$ and $i_i = j_i \cdots, i_t = j_t$, which is nothing but $A_\alpha \supset A_\beta$. We note that $\alpha \not\geq \beta$ if and only if $A_\alpha \cap A_\beta = 0$.

Let x be an element in eJ . Then $x = x_1 + x_2 + \cdots + x_n$; $x_i \in A_i$. If $x_1 \in J(A_1)$, $x_1 = x_{11} + x_{12} + \cdots + x_{1n_1}$; $x_{1i} \in A_{1i}$. Repeating this process, we obtain finally

$$(3) \quad x = z_1 + z_2 + \cdots + z_t, \text{ where } z_i \in A_{\alpha_i} - J(A_{\alpha_i}), \text{ and } \alpha_i \not\geq \alpha_j \text{ if } i \neq j, \text{ i.e., } \sum_{i=1}^t A_{\alpha_i} = \sum \oplus A_{\alpha_i}.$$

Finally, let e and f be primitive idempotents. By $T(eJ^i f)$ (resp. $T(J^i f)$) we denote the set $eJ^i f - eJ^{i+1} f$ (resp. $J^i f - J^{i+1} f$). For a hollow module A , \bar{A} means $A/J(A)$.

2. Main Theorem

We shall give a characterization of a left serial ring with $(*, 1)$ as a right R -module.

Theorem. *Let R be a left serial ring such that eJ/eJ^2 is square-free for each primitive idempotent e . Then*

(*, 1) holds for any hollow right R -module if and only if we have the following condition: If $A_\alpha/J(A_\alpha) \approx A_\beta/J(A_\beta)$ for $\alpha = i_1 i_2 \cdots i_k$, $\beta = j_1 j_2 \cdots j_s$ ($1 < k < s$ and $i_1 \neq j_1$), $A_{\beta'}/J(A_{\beta'})$ is uniterial, where $\beta' = j_1 j_2 \cdots j_{s-k+1}$ and A_α , A_β and $A_{\beta'}$ are hollow modules in (1). (See [5], for the general case.)

Corollary. Let R be a left serial algebra over an algebraically closed field of finite dimension. Then (*, 1) holds for any hollow right R -module if and only if the condition in Theorem is satisfied.

Proof. This is clear from Theorem and [3], Lemma 3.

We shall study some properties of a left serial ring. T. Sumioka has communicated us the following lemma.

Lemma 1 (T. Sumioka). Let R be left serial. Let x (resp. y) be in $T(eJ^i f)$ (resp. $T(fJ^j g)$), where e , f and g are primitive idempotents. If $J^{i+j}g \neq 0$, $xy \in T(eJ^{i+j}g)$.

Proof. Assume that $xy \in eJ^{i+j+1}g$. Since R is left serial, $Rx = J^i f$ and $Ry = J^j g$. Hence $J^{i+j}g = J^i Ry = J^i f y = Rxy \subset ReJ^{i+j+1}g \subset J^{i+j+1}g$. Therefore $J(J^{i+j}g) = J^{i+j}g$, which is a contradiction to the assumption $J^{i+j}g \neq 0$.

From now on for a left serial ring R $A_\alpha, A_\beta \cdots$ are hollow modules in the diagram (1) and $A_{\alpha(i)}$ means a submodule of A_i . When we need to specify $|\alpha(i)| = t$, we denote $A_{\alpha(i)}$ by $A_{\alpha(i,t)}$.

Lemma 2. Let R be a left serial ring and X, Y hollow right submodules in R . If $f: X/X_1 \approx Y/Y_1$ for some $X_1 \subset X$ and $Y_1 \subset Y$, there exists d in R such that $dX = Y$ (or $dY = X$). If $X_1 = J(X)$ and $Y_1 = J(Y)$, d_i , left-sided multiplication of d , induces f . In general, if d_i induces f , $dX_1 \subset Y_1$. In particular, if $\bar{A}_{\alpha(i)} \approx \bar{A}_{\beta(j)}$ ($|\alpha(i)| \leq |\beta(j)|$), we can find such d in A_j .

Proof. Since X is hollow, we can find a generator x of X with $x e = x$ for some primitive idempotent e . Put $f(x + X_1) = y + Y_1$ ($y \in Y$). f being an isomorphism, y is a generator of Y and we may assume $y e = y$. Further we may assume $x \in T(J^i)$, $y \in T(J^j)$ and $i \leq j$ (if $j > i$, replace X by Y). Then, since Re is uniserial, there exists d in R such that $dx = y$. If $X_1 = J(X) = xJ$, $dX_1 = dxJ = yJ = Y_1$. Let x_1 be an element in X_1 and $x_1 = xr$; $r \in R$. Then $dx_1 = dxr = yr \in Y$ and $dx_1 + Y_1 = yr + Y_1 = f(xr + X_1) = Y_1$, provided that d_i induces f . Hence $dX_1 \subset Y_1$ (see the proof of [3], Theorem 3), If $\bar{A}_{\alpha(i)} \approx \bar{A}_{\beta(j)}$, $da = b$, where $A_{\alpha(i)} = aR$, $A_{\beta(j)} = bR$ and $d \in eJe$. Let $d = \sum d_i$; $d_i \in A_i$. Since $b = da = \sum d_i a \in A_j$, $b = d_j a$.

Lemma 3. Let R be a left serial ring. Assume $\bar{A}_{\alpha(i,t)} \approx \bar{A}_{\beta(j,t)}$ for $A_{\alpha(i,t)} \subset A_i$ and $A_{\beta(j,t)} \subset A_j$. Then $A_{\alpha'(i,p)} \approx A_{\beta'(j,p)}$, provided $A_{\alpha'(i,p)} \supset A_{\alpha(i,t)}$ and $A_{\beta'(j,p)} \supset A_{\beta(j,t)}$.

Proof. We show $\bar{A}_{\alpha'(i,p)} \approx \bar{A}_{\beta'(j,p)}$. From Lemma 2, there exists an element d in eRe such that $dA_{\alpha(i,t)} = A_{\beta(j,t)}$. Let $\pi_{\gamma(s)}$ be the projection of eJ^p to $A_{\gamma(s,p)}$. Then $d_l | A_{\alpha'(i,p)} = (\sum_s \pi_{\gamma(s)} d_l) | A_{\alpha'(i,p)}$, where d_l means the left-sided multiplication of d . If $\pi_{\beta'(j,p)} d_l | A_{\alpha'(i,p)}$ is an epimorphism, $\bar{A}_{\alpha'(i,p)} \approx \bar{A}_{\beta'(j,p)}$. Assume $\pi_{\beta'(j,p)}(dA_{\alpha'(i,p)}) \subset J(A_{\beta'(j,p)})$. Let $A_{\alpha(i,t)} = a_t R$ and $A_{\alpha'(i,p)} = a_p R$. Then $a_t = a_p x$; $x \in T(J^{t-p})$. $(\pi_{\beta'(j,p)}(eJ^p) \supset) A_{\beta(j,t)} = dA_{\alpha(i,t)} = da_t R = \Sigma(\pi_{\gamma(s)} da_t) R = \pi_{\beta'(j,p)}(da_p x R) \subset J(A_{\beta'(j,p)}) x R \subset eJ^{t+1}$, a contradiction. Hence $\bar{A}_{\alpha'(i,p)} \approx \bar{A}_{\beta'(j,p)}$, and so $A_{\alpha'(i,p)} \approx A_{\beta'(j,p)}$ by Lemma 2 (note $A_{\alpha'(i,p)}, A_{\beta'(j,p)}$ are contained in eJ^p but not in eJ^{p+1}).

Lemma 4. *Let R be a left serial ring. Assume that $eJ = \Sigma \oplus A_i$, where the A_i are hollow. Then there are no non-zero elements d_1, d_2 in eJe such that $d_1 \in A_i, d_2 \in A_j (i \neq j)$ provided $\bar{A}_i \approx \bar{A}_j$.*

Proof. We may assume that $d_1 \in T(A_{\alpha(i)})$ and $d_2 \in T(A_{\alpha(j)})$. Put $t_1 = |\alpha(i)|$ and $t_2 = |\alpha(j)|$. Then $t_1 \neq t_2$ by assumption and Lemma 3, since $\bar{d}_i R \not\approx eR$ for $i=1, 2$. We can choose a pair (d'_1, d'_2) such that $t'_1 + t'_2$ is minimal ($t'_1 < t'_2$). Then there exists d_0 in $A_j \cap eJ^{t'_2 - t'_1}$ such that $d_0 d'_1 = d'_2$ from Lemma 2, and so (d'_1, d_0) gives the contradiction to the minimality.

Lemma 5. *Let A_i be as above. Assume that any pair of $\{A_1, A_2, A_3\}$ is not isomorphic to one another. Then, for any primitive idempotent g , there are no three elements $\{a_{ij}\}$ as follows: 1) $a_{1j_1} \in T(A_1 J^{j_1}), a_{2j_2} \in T(A_2 J^{j_2})$ and $a_{3j_3} \in T(A_3 J^{j_3})$ such that $a_{ij_i} g = a_{ij_i}$ and $j_1 < \min(j_2, j_3)$ or 2) $a_{ij_i} \in A_i$ and $a_{ij_i} g = a_{ij_i}$ for $i=1, 2, 3$.*

Proof. Since $Ra_{ij_i} \subset Rg$ and $ea_{ij_i} = a_{ij_i}$, this is clear from Lemmas 2 and 4.

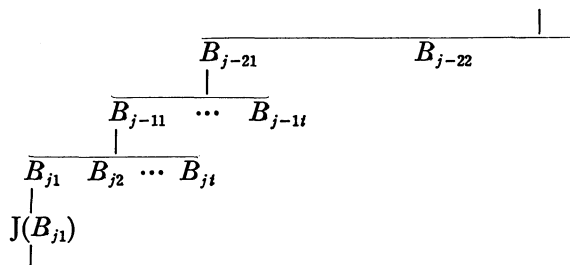
Now we study a right artinian ring with $(*, 1)$. For the sake of simplicity, we change the expression of a direct decomposition of eJ^i .

The following lemma is the “only if” part of Theorem.

Lemma 6. *Assume that R is a right artinian ring with $(*, 1)$ as right R -modules. Put $eJ = A_1 \oplus B_1 \oplus \dots, eJ^i = \sum_{k=1}^{n_i} \oplus A_{ik} \oplus \sum_{k=1}^{n'_i} \oplus B_{ik} \oplus \dots$, where the $A_1, B_1, A_{ik}, B_{ik} \dots$ are hollow and $A_1 \supset A_{ik}, B_1 \supset B_{ik}, \dots$. If $\bar{A}_{i1} \approx \bar{B}_{j1}$ for $1 < i < j, B_{j-i+11} / J(B_{j1})$ is uniserial.*

Proof. Put $C_1 = \sum_{k=2}^{n_2} \oplus A_{2k} + \sum_{k=2}^{n_3} \oplus A_{3k} + \dots + \sum_{k=1}^{n_i} \oplus A_{ik}$ and $D_1 = \sum_{k=2}^{n_2} \oplus A_{2k} + \dots + \sum_{k=2}^{n_{i-1}} \oplus A_{i-1k} + J(A_{i1}) + \sum_{k=2}^{n_i} \oplus A_{ik}$. Then $f: C_1 / D_1 \approx A_{i1} / J(A_{i1}) \approx B_{j1} / J(B_{j1})$. Assume that f^{-1} is extendible to a g' in $\text{Hom}_R(B_{j1} / J(B_{j1}), A_{i1} / D_1)$. Since $B_{j1} J^i \supset B_{j1} J^{i-1} \supset B_{j1}, g'(B_{j1} J^i / J(B_{j1})) \supset g'(B_{j1} / J(B_{j1})) = f^{-1}(B_{j1} / J(B_{j1})) \neq 0$. On the other

hand, $g'(B_1J^i/J(B_{j_1})) \subset (A_1/D_1)J^i = 0$ for $A_1J^i \subset D_1$, which is a contradiction. Hence f is extendible to a g in $\text{Hom}_R(A_1/D_1, B_1/J(B_{j_1}))$ by Lemma A. Since A_1/D_1 is uniserial, $\text{Soc}(A_1/D_1) = (A_{i1} + D_1)/D_1 \approx A_{i1}/(J(A_{i1}))$ and $g|_{A_{i1}/(J(A_{i1}))} = f$ is an isomorphism, g is a monomorphism. Consider a uniserial module $\tilde{B}_{i-1} = g((A_{i-11} + D_1)/D_1) \supset B_{j_1}/J(B_{j_1})$. Let $J(B_{j-11}) = B_{j_1} \oplus B_{j_2} \oplus \dots \oplus B_{j_t}$. Assume $B_{j_2} \neq 0$.



Then $\text{Soc}(B_1/B_{j_1}) = \sum_{k=2}^t \oplus \text{Soc}(B_{j_k}) \oplus \sum_{k=2}^{t'} \oplus \text{Soc}(B_{j_{-1k}}) \oplus \dots$. Since $\text{Soc}(\tilde{B}_{i-1}) = B_{j_1}/J(B_{j_1})$, \tilde{B}_{i-1} is not uniserial. Hence $B_{j_2} = B_{j_3} = \dots = B_{j_t} = 0$. Considering the same situation for $g((A_{i-21} + D_1)/D_1)$, we obtain similarly $B_{j-12} = \dots = B_{j-1t} = 0$, where $J(B_{j-21}) = B_{j-11} \oplus \dots \oplus B_{j-1t}$. Repeating this argument, we know $B_{j-i+11}/J(B_{j_1})$ is uniserial.

In Lemma 6, if $i=j$, $B_1/J(B_{i1})$ is not uniserial in general (see Example 1 in [5]).

Lemma 7. *Let R be left serial and $A_{\alpha(i,t)}$, $A_{\beta(i,t)}$ hollow modules in (1). Assume $\alpha(i,t) \neq \beta(i,t)$. Then there are no d in J such that $dA_{\alpha(i,t)} = dA_{\beta(i,t)} \neq 0$.*

Proof. Let $A_{\alpha(i,t)} = aR$, $A_{\beta(i,t)} = bR$ and assume that there exists d such that $daR = dbR$. Then there exists r in R such that $dar = db$, and so $d(ar - b) = 0$. On the other hand, $ar - b \neq 0$ by assumption, and there exists a primitive idempotent g such that $db = dbg$. Further $d \in T(J^s)$ for some s and $0 \neq db \in T(J^{t+s}g)$ by Lemma 1. Hence $d(ar - b) \in T(J^{t+s}g)$, a contradiction to Lemma 1.

The following is the ‘‘if’’ part of Theorem.

Lemma 8. *Let R be a left serial ring with \overline{eJ} square-free. If R satisfies the result in Lemma 6, then $(*, 1)$ holds for any hollow module.*

Proof. First we shall show from the assumption that

(4) there are no three distinct hollow modules $A_{\alpha(i,s)}$, $A_{\beta(i,s)}$ and $A_{\gamma(j,t)}$ such that $\bar{A}_{\alpha(i,s)} \approx \bar{A}_{\beta(i,s)} \approx \bar{A}_{\gamma(j,t)}$ and $i \neq j$, $s < t$.

Contrarily we assume that there exist such modules. Put $A_{\alpha(i,s)} = aR$, $A_{\beta(i,s)} =$

bR and $A_{\gamma(j,t)}=cR$. $A_{\gamma'(j,t-s+1)}/J(A_{\gamma(j,t)})$ is uniserial for some $\gamma'(j,t-s+1) > \gamma(j,t)$ by assumption. We may assume that $ag=a, bg=b$ and $cg=c$ for some primitive idempotent g . Then there exists d' in $T(eJ^{t-s}e)$ such that

$$(5) \quad d'a = c$$

by Lemmas 1 and 2. Let $A_i=a_iR$ and $d'a_i=\sum_j v_j$ as in (3) ($a_ih=a_i, v_jh=v_j$ for some primitive idempotent h). a being in $A_i, a=a_i r$ for some r in J . Then $c=d'a=d'a_i r=\sum_j v_j r$. Hence from the observation after (2)

$$(6) \quad c=v_k r \text{ for some } k \text{ (say } 1) \text{ and } v_{k'} r = 0 \text{ for } k' \neq 1.$$

$c=v_1 r$ implies that $v_1 \in A_{\delta(j)}$ and $\delta(j) > \beta(j,t)$. Further $a=a_i r$ implies $r \in T(hJ^{s-1}g)$, and so $v_1 \in T(J^{t-s+1}h)$ by Lemma 1. Hence $v_1 \in A_{\gamma'(j,t-s+1)}$ (note $A_{\gamma(j,t)} \subset A_{\gamma'(j,t-s+1)}$). Now there exists d in $T(eJ^{t-s}e) \cap A_j$ with $v_1=da_i$, and so

$$(7) \quad da = da_i r = v_1 r = c \text{ and } da_i = v_1$$

by (6). Let $b=a_i r'$ for some r' in R . Then $r' \in T(J^{s-1})$ and $db=da_i r' = v_1 r' \in A_{\gamma'(j,t-s+1)} \cap T(J^t) = A_{\gamma(j,t)}$, since $A_{\gamma'(j,t-s+1)}/J(A_{\gamma(j,t)})$ is uniserial. Hence $dA_{\alpha(i,s)}=dA_{\beta(i,s)} \neq 0$, which is a contradiction to Lemma 7. Thus we have shown (4). Now let $A_1 \supset C_1 \supset D_1$ and $A_2 \supset C_2 \supset D_2$ be submodules such that $f: C_1/D_1 \cong C_2/D_2$ and C_1/D_1 is simple. Let $c_i = \sum_k z_{ik}$ be a generator of C_i for $i=1, 2$, where $z_{ik} \in T(A_{\alpha_{ik}})$ from (3) and $z_{ik}g = z_{ik}$ for some primitive idempotent g . We choose a generator $c_i \in C_i$ with $\min\{\min_k |\alpha_{ik}|\}$. We may assume that $c_1 = \sum_k z_{1k}$ and $|\alpha_{11}| = 11 \cdots 1$ is minimal. Then $c_1 = z_{11}$ ($=x$) from (4) and Lemmas 3 and 5. We shall take the following D_1^* similarly to D_1 in the proof of Lemma 6: $D_1^* = A_{12} \oplus A_{13} \oplus \cdots \oplus A_{112} \oplus A_{113} \oplus \cdots \oplus A_{11 \cdots 12} \oplus \cdots (11 \cdots 1 = \alpha_{11})$. We shall show

$$(8) \quad D_1 \subset J(A_{\alpha_{11}}) \oplus D_1^*.$$

Let y be an element in D_1 and $y = \sum y_i; y_i \in A_{\alpha_i}$ as in (3). Assume $\alpha_1 \geq \alpha_{11} = 11 \cdots 1$. Then there exists r in Rg such that $y_1 r = x$. $x - yr = \sum_{i \geq 2} -y_i r$ is also a generator of C_1 and contained in D_1^* . Since $A_{\alpha_{11}} \cap D_1^* = 0$, we obtain a contradiction from (4) and Lemma 5. Hence $\alpha_1 < \alpha_{11}$, and so (8) is true. We choose a representative w in C_2 of $f(x)$ such that $wg = w$. Let $w = \sum_{i=1}^r w_i (w_i g = w_i)$ as in (3); $w_i \in A_{\beta(2)}$. Since $w_1 g = w_1$ and $|\alpha_{11}|$ is minimal, there exists d'_1 in A_2 with $d'_1 x = w_1$. Now we have the same situation as (5). Hence from the argument after (5), similarly to (7) there exists d_1 in $T(eJ^{t-s}e) \cap A_2$ such that $d_1 a_1 \in T(A_{\beta'(t,t-s+1)})$ and $d_1 x = w_1$, where $t = |\beta_1(2)|, s = |\alpha_{11}|$ and $\beta'(2, t-s+1) > \beta_1(2)$. Let $\gamma(1, q) = 11 \cdots 1$ and $\gamma'(1, q)$ be two distinct indices with $1 < q \leq s$. Let

$p \in D_1^* \cap A_{\gamma(1,q)}$ and a_q a generator of $A_{\gamma(1,q)}$. Then $x = a_q r'$ for some r' in R . $d_1 a_q r' = d_1 x = w_1 \neq 0$, and so $d_1 a_q (\neq 0) \in T(eJ^{t-s+q})$. If $d_1 p \neq 0$, $d_1 p \in T(eJ^{t-s+q})$ by Lemma 1 and $d_1 p$ and $d_1 a_q$ generate a same submodule between $A_{\beta'(2,t-s+1)} = d_1 A_1$ and $A_{\beta_1(2)}$, since $A_{\beta'(2,t-s+1)}/J(A_{\beta_1(2)})$ is uniserial, which is a contradiction to Lemma 7. Therefore $d_1 p = 0$ and so $d_1(D_1^*) = 0$. Similarly we can find $d_i \in A_2$ such that $d_i x = w_i$ for each w_i and $d_i(D_1^*) = 0$. Put $d^* = \sum_{i=1}^r d_i$. Then $d^* x = \sum d_i x = \sum w_i = w$. Let u be any element in D_1 , then from (8) $u = u_1 + u_2$, where $u_1 \in J(A_{\alpha_{11}})$, $u_2 \in D_1^*$. We denote $u_1 = xj$; $j \in J$. $d^* u = d^* u_1 + d^* u_2 = d^* u_1 = d^* xj = wj \in C_2 J \subset D_2$. Therefore d^* , left-sided multiplication of d^* , is the desired extension of f .

Proposition. *Let R be a left serial ring with $(*, 1)$. Let $\alpha(1, s) = 11 \cdots 1$, $\beta(2, t) = 211 \cdots 1$ ($s < t$) and $|A_{\beta(2,t)}| = k$. If $\bar{A}_{\alpha(1,s)} \approx \bar{A}_{\beta(2,t)}$, then $A_{\beta'(t-s+1)} \approx A_1/D$, where $\beta'(2, t-s+1) = 21 \cdots 1$ and after renumbering $A_{\alpha(1,p)}$ for all p , $D = A_{12} \oplus A_{13} \oplus \cdots \oplus A_{112} \oplus A_{113} \oplus \cdots \oplus J(A_{\gamma(1,s+k-1)})$; $\gamma(1, s+k-1) = 11 \cdots 1$. Hence $A_{\beta'(2,t-s+1)}$ is uniserial.*

Proof. Since $\bar{A}_{\alpha(1,s)} \approx \bar{A}_{\beta(2,t)}$, there exists d in A_2 such that $dA_{\alpha(1,s)} = A_{\beta(2,t)}$ by Lemma 2, and so $dJ(A_{\alpha(1,s)}) = J(A_{\beta(2,t)})$. Let $J(A_{\alpha(1,s)}) = A_{11 \cdots 1} \oplus A_{11 \cdots 12} \oplus \cdots$ and $J(A_{\beta(2,t)}) = A_{21 \cdots 1} \oplus A_{21 \cdots 12} \oplus \cdots$. Since $dJ(A_{\alpha(1,s)}) = J(A_{\beta(2,t)})$ and $A_{21 \cdots 1}$ is hollow, $\bar{A}_{21 \cdots 1} \approx \bar{A}_{11 \cdots 1k}$ for some k . Then $A_{21 \cdots 1k'} = 0$ for $k' \neq 1$ by Theorem. Repeating this argument, we know that $A_{\beta(2,t)}$ is uniserial and $A_{\beta''(2,t+k-1)}$ is the socle of $A_{\beta(2,t)}$ and is isomorphic to $A_{\alpha'(1,s+k-1)}$. There exists d' in J from (7) such that $d'A_1 = A_{\beta'(2,t-s+1)}$ and $d'A_{\alpha'(1,s+k-1)} = A_{\beta''(2,t+k-1)}$. Therefore we obtain the proposition from the last part of proof of Theorem.

References

[1] M. Harada: *Serial rings and direct decompositions*, J. Pure Appl. Algebra, **31** (1984), 55–61.
 [2] ———: *On maximal submodules of a finite direct sum of hollow modules III*, Osaka J. Math. **22** (1985), 81–98.
 [3] ———: *Note on left serial algebras*, Tsukuba J. Math. **10** (1986), 193–197.
 [4] ———: *Generalizations of Nakayama ring III*, Osaka J. Math. **23** (1986), 523–539.
 [5] ———: *Generalizations of Nakayama ring V*, to appear.
 [6] T. Sumioka: *Tachikawa's theorem on algebras of left colocal type*, Osaka J. Math. **22** (1984), 629–648.

Department of Mathematics
 Osaka City University
 Sugimoto-3, Sumiyoshi-ku
 Osaka 558, Japan

