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ISOMORPHISM OF GROUP RINGS OF INFINITE NILPOTENT GROUPS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Introduction

Let RG denote the group ring of a group G over a commutative ring R with identity. Extending the terminology of Roggenkamp [8] to infinite groups, we say here that R is *G-adapted* if R is an integral domain of characteristic 0 in which no element $g\neq 1$ of G has order invertible.

Recently, Röhl [10] has introduced a new notion of the torsion-length for nilpotent groups and has shown that if G is nilpotent and $\boldsymbol{Z}G\cong\boldsymbol{Z}H$ for some group H, then H is nilpotent, too. He has also shown in [9] that every circle group of a nilpotent ring is characterized by its integral group ring. Both of these results are mainly based on the well known fact that central units of finite order in any integral group ring $\boldsymbol{Z}G$ are trivial. As an application of Theorem 1.7 (due to Sehgal) of this paper, we show that the triviality of central units of finite order is still true in a more general context, namely if we replace $\boldsymbol{Z}$ by a G-adapted ring R. Thus we have at once generalizations of Röhl's results. More precisely, Theorem 1.9 states that if R is an integral domain of characteristic 0, then every circle group G of a nilpotent R-algebra is characterized by RG. Also, Lemma 2.1 states that if G is nilpotent, and if $RG\cong RH$ as R-algebras, where R is a G-adapted ring, then H is nilpotent, too.

The main purpose of this paper is to consider the question whether, under the hypotheses of Lemma 2.1, the nilpotence class $cl(G)$ of G coincides with that of H. This is discussed in Section 4 and as a main theorem, we prove that if $cl(G/TG)\leqq 5$, where TG is the torsion subgroup of G, then $cl(G)=cl(H)$. It is also proved that if G is metabelian and nilpotent, then $cl(G)=cl(H)$. These results heavily depend on the results of Sandling [11] on Lie dimension subgroups (Lemma 3.3). Section 1 includes some preliminary results on the congruence subgroup $U(1+I)$ of the unit group $U(RG)$ with respect to an ideal I of RG; that is, $U(1+I)=U(RG)\cap(1+I)$. In Section 2, under the assumptions of Lemma 2.1 we investigate the normal subgroup correspondence and show that the isomorphism $RG\cong RH$ yields an isomorphism between the lattice of periodic normal subgroups of G and that of H (Theorem 2.4).

In this paper, R always denotes a commutative ring with identity and, unless otherwise stated, G denotes an arbitrary group.

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1. The torsion elements of normalized units

We start with an elementary observation. Let S be a ring with identity, and let $U(S)$ denote the unit group of S. If I is a (two-sided) ideal of S, then $U(1+I)=U(S)\cap(1+I)$ is the kernel of the natural homomorphism of $U(S)$ into $U(S/I)$, and hence a normal subgroup of $U(S)$.

Lemma 1.1. *Let I be an ideal of S. If a rational prime p is a unit in S, then the factor group $U(1+I)/U(1+\bigcap_{n=1}^{\infty} I^n)$ is p-torsion-free.*

Proof. It suffices to show that for each $n\geqq 1$, $U(1+I^n)/U(1+I^{n+1})$ is p-torsion-free. From the identity $ab-1=a-1+b-1+(a-1)(b-1)$ for $a, b\in S$, we see that the map

$$U(1+I^n)\ni a \to a-1+I^{n+1}\in I^n/I^{n+1}$$

is a homomorphism whose kernel is $U(1+I^{n+1})$. Thus we have a monomorphism $U(1+I^n)/U(1+I^{n+1})\to I^n/I^{n+1}$ of abelian groups. Since $p\in U(S)$, the additive group I^n/I^{n+1} is p-torsion-free and therefore, so is $U(1+I^n)/U(1+I^{n+1})$. Hence the result follows.

Let RG be the group ring of G over R, and $\Delta_R(G)$ the augmentation ideal of RG. For any normal subgroup N of G, we write $\Delta_R(G, N)$ for the kernel of the natural map $RG\to R(G/N)$.

Let P be a group property. Recall that G is said to be *residually* P if, to each element $x\neq 1\in G$, there is a normal subgroup N_x of G such that $x\notin N_x$ and G/N_x is P.

Lemma 1.2. *Let R be an integral domain of characteristic* 0, *and let N be a normal subgroup of G. If N is residually 'torsion-free nilpotent', then $U(1+\Delta_R(G, N))$ is torsion-free.*

Proof. Let K be the quotient field of R. Then any rational prime is a unit in KG and thus by Lemma 1.1, $U(1+\Delta_K(G, N))/U(1+\bigcap_{n=1}^{\infty}\Delta_K(G, N)^n)$ is torsion-free. Now, since N is residually 'torsion-free nilpotent', the augmentation ideal $\Delta_K(N)$ is residually nilpotent, i.e., $\bigcap_{n=1}^{\infty}\Delta_K(N)^n=0$ (see [7, p. 90]). By using [13, p. 35, Lemma 3.13], we have $\bigcap_{n=1}^{\infty} \Delta_K(G, N)^n = (\bigcap_{n=1}^{\infty} \Delta_K(N)^n)KG = 0$, so it follows that $U(1+\Delta_K(G, N))$ and hence $U(1+\Delta_R(G, N))$ is torsion-free.

For any group X, we denote by TX the set of torsion elements of X. The next result is known for the case when N is a torsion central subgroup of G (see e.g. [4, Lemma 4]).

Lemma 1.3. *Let R be an integral domain of characteristic 0 in which no rational prime is invertible. If N is a central subgroup of G, then $TU(1+\Delta_R(G, N)) = TN$.*

Proof. We need only prove that $TU(1+\Delta_R(G, N)) \subseteq TN$, the reverse inclusion being trivial. Let $u = \sum_g u(g) g \in TU(1+\Delta_R(G, N))$, and let $^-: RG \to R(G/TN)$ be the natural map. Then it is clear that $\bar{u}$ is in $TU(1+\Delta_R(\bar{G}, \bar{N}))$. However, since $\bar{N} = N/TN$ is torsion-free abelian, Lemma 1.2 shows that $TU(1+\Delta_R(\bar{G}, \bar{N})) = \{1\}$. Thus, $\bar{u}=1$ i.e. $u-1 \in \Delta_R(G, TN)$ and so we can choose an element $x \in TN$ with $u(x) \neq 0$. Then $v = x^{-1}u$ is a unit of finite order such that $v(1) \neq 0$, so it follows from [13, p. 45, Corollary 1.2] that $v=1$ and hence $u = x \in TN$. This completes the proof.

For any two subgroups A and B of G, we write (A, B) for the subgroup generated by all commutators $(a, b) = a^{-1}b^{-1}ab$, $a \in A$, $b \in B$. Also, we denote by $D_{n,R}(G) = G \cap (1+\Delta_R(G)^n)$ the n-th dimension subgroup of G over R. The following lemma is well known for the case $R = \boldsymbol{Z}$, the ring of rational integers (see [13, p. 100]).

Lemma 1.4. *Let N be a normal subgroup of G and assume that one of the following two conditions holds:*

(a) *R is G-adapted and (N, G) is periodic,*

(b) *R is an integral domain of characteristic 0 in which no rational prime is invertible.*

Then, $(N, G) = G \cap (1+\Delta_R(G, N)\, \Delta_R(G) + \Delta_R(G)\, \Delta_R(G, N))$.

Proof. Assume first (a). That the left-hand side is contained in the right-hand side follows from the identity

$$(x, g) = 1 + x^{-1}g^{-1}\{(x-1)(g-1) - (g-1)(x-1)\}$$

for $x \in N$ and $g \in G$. The opposite inclusion can be seen as follows. Since (N, G) is periodic, R is also $G/(N, G)$-adapted and hence by considering $G/(N, G)$ it suffices to verify that if R is G-adapted and N is abelian, then $G \cap (1+\Delta_R(G)\Delta_R(G, N)) = \{1\}$. This statement immediately follows from [7, Theorem II. 2.1, Proposition V. 5.3] together with the fact that $G \cap (1+\Delta_R(G)$

$\Delta_R(G, N))=D_{2,R}(N)$. Under condition (b), R is G-adapted for any G, so it is clear that the same conclusion is obtained.

REMARK. The above result does not hold with only the hypothesis that R is G-adapted. In fact, if we take, especially, $N=G$, then the equality above implies that $G'=D_{2,R}(G)$ where G' is the derived group of G, which is not necessarily true if only R is G-adapted. For example, let R be a field of characteristic 0, and let G be a torsion-free group such that $T(G/G')\neq\{1\}$. (By example on p. 79 in [5], such a G exists.) Then it is sure that R is G-adapted. However, as in the proof of Lemma 1.1, $U(1+\Delta_R(G))/U(1+\Delta_R(G)^2)$ is torsion-free and hence, so is $G/D_{2,R}(G)$. Thus $G'\subsetneqq D_{2,R}(G)$. This example also disproves Proposition 4.24 on p. 97 in [13].

As a corollary of the last two lemmas we observe the following interesting fact.

Corollary 1.5. *Let R be as in Lemma 1.3. Let N be a normal subgroup of G, and let I be an ideal of RG such that $\Delta_R(G, N)\Delta_R(G)+\Delta_R(G)\Delta_R(G, N)\subseteq I\subseteq\Delta_R(G, N)$. Then $U(1+I)/U(1+\Delta_R(G, G\cap(1+I)))$ is torsion-free.*

Proof. Under the natural map $\bar{\ }: RG\rightarrow R(G/G\cap(1+I))$ we see easily that $\bar{G}\cap(1+\bar{I})=\{1\}$. Thus we may assume, by going mod $G\cap(1+I)$, that $G\cap(1+I)=\{1\}$. Then $(N, G)=\{1\}$, by Lemma 1.4, so we conclude from Lemma 1.3 that $TU(1+I)\subseteq G\cap(1+I)=\{1\}$, proving the corollary.

We now record a similar result to Lemma 1.3, which will be used in Section 2. The center of a group X will be denoted by $\zeta(X)$.

Lemma 1.6. *Let G be a nilpotent group and R a G-adapted ring. If N is a normal subgroup of G with $TN\subseteq\zeta(G)$, then $TU(1+\Delta_R(G, N))=TN$.*

Proof. Since N/TN is torsion-free nilpotent, $U(1+\Delta_R(G, N))/U(1+\Delta_R(G, TN))$ is torsion-free, by Lemma 1.2. Moreover, we know from [2, Lemma 1.2] that $TU(1+\Delta_R(G, TN))=TN$. Hence the result follows.

We shall denote by $V(RG)$ the group of normalized units of RG, that is, $V(RG)=U(1+\Delta_R(G))$. Recall that a unit of RG is said to be *trivial* if it is of the form rg, $r\in U(R)$, $g\in G$.

Theorem 1.7 (Sehgal). *Let G be a polycyclic-by-finite group and R a G-adapted ring. If $V(RG)$ has an element $u=\sum_g u(g)g$ of order p^α, p a prime, $\alpha>0$, then G has an element g_0 of order p^α with $u(g_0)\neq 0$.*

REMARK. Theorem 1.7 can be proved in the same way as Theorem 2.1 on p. 177 in [13], so we shall omit the proof. In the proof, however, it should be noted that the prime p could not be invertible in R. In fact, if $p\in U(R)$, then by taking an element v of $V(RG)$ of order p we have a nontrivial idempotent $e=\frac{1}{p}(1+v+\cdots+v^{p-1})$ in RG. But this is a contradiction, since RG has no nontrivial idempotents (see [13, p. 25, Theorem 2.20]).

Corollary 1.8. *If R is a G-adapted ring, then any central unit of finite order in RG is trivial.*

Proof. Let u be a central unit of finite order in RG. Then u can be written as $u=rv$, $r\in U(R)$, $v\in T\zeta(V(RG))$ and it remains to show that $v=g$ for some $g\in G$. To this end, we may suppose that $v=\sum_g v(g)g$ has order p^α, p a prime. Let H be the subgroup of G generated by $\mathrm{supp}(v)=\{g\in G\mid v(g)\neq 0\}$. Then since v is central in RG, $\mathrm{supp}(v)$ is a finite normal subset of G. Thus H is a finitely generated FC-group and hence a finite extension of its center $\zeta(H)$ (see e.g. [12, p. 442]). So it follows from [13, p. 37, Proposition 4.1] that $\zeta(H)$ is finitely generated. Consequently, we deduce that $\zeta(H)$ is polycyclic and hence that H is polycyclic-by-finite. Now, since $v\in V(RH)$, by Theorem 1.7 there is an element $g\in\mathrm{supp}(v)$ with $o(g)=p^\alpha$. Then $w=g^{-1}v$ is a unit of finite order in RH such that $w(1)\neq 0$ and thus by [13, p. 45, Corollary 1.4], we have $w=1$ i.e. $v=g$, as required.

We close this section by extending a result of Röhl [9] which states that every circle group of a nilpotent ring is characterized by its integral group ring. Let R be an integral domain of characteristic 0, A a nilpotent R-algebra, and $G=(A, \circ)$ the circle group of A; that is, A is considered as a group under the operation $a\circ b=a+b+ab$. Then the n-th dimension subgroup $D_{n,R}(G)$ of G over R is always contained in $(A^n, \circ)$, so that $D_{n,R}(G)$ vanishes for some n. Hence, we see from [3, Lemma 1.1] that R is G-adapted and, from Corollary 1.8, that any central unit of finite order in RG is trivial. Consequently, as Röhl remarked in [9], the argument employed in the proof of [9, 2.4. Theorem] is valid for any integral domain R of characteristic 0. Thus we have

Theorem 1.9. *If R is an integral domain of characteristic 0 and A is a nilpotent R-algebra, then the circle group $G=(A, \circ)$ is characterized by RG.*

2. Normal subgroup correspondence

Throughout this section, R will denote a G-adapted ring.

Let us first recall the "torsion-length" of a nilpotent group, which is introduced by Röhl [10]. Let $\{1\}=\zeta_0(G)\subseteqq\zeta_1(G)\subseteqq\cdots\subseteqq\zeta_i(G)\subseteqq\cdots$ be the upper central series of G and write $T_i(G)=TG\cap\zeta_i(G)$ for $i\geqq 0$. If TG forms a subgroup of G, then $\{T_i(G)\}_{i\geqq 0}$ is an ascending series of normal subgroups of

G with the property that $T_i(G/T_j(G))=T_{i+j}(G)/T_j(G)$ for all $i, j\geqq 0$. This formula can be seen by induction on i, keeping j fixed. In case G is nilpotent (of class n), we have a finite series $\{1\}=T_0(G)\subseteqq T_1(G)\subseteqq\cdots\subseteqq T_n(G)=TG$ and the torsion-length $t(G)$ of G is defined to be the number of different terms $T_i(G)$ $(\neq\{1\})$. Clearly, $t(G)=0$ if and only if G is torsion-free. Moreover, since every nontrivial normal subgroup of a nilpotent group G has a nontrivial intersection with the center $\zeta(G)=\zeta_1(G)$, we observe from the above formula that if $t(G)>0$, then $t(G)=t(G/T_1(G))+1$.

The induction argument on the torsion-length $t(G)$ immediately gives us the following result whose proof is identical to that of [10, Proposition on p. 138].

Lemma 2.1. *Let G be a nilpotent group and suppose that $RG\cong RH$ as R-algebras for some group H. Then, H is nilpotent with $t(G)=t(H)$, and R is H-adapted.*

Proof. Let θ: $RG\rightarrow RH$ be the given R-algebra isomorphism, then we may assume that θ is augmented. If $h\in H$ is an element of prime order p, then $\theta^{-1}(h)$ is an element of $V(RG)$ of order p. Since the supporting subgroup of $\theta^{-1}(h)$ is polycyclic, we see from Theorem 1.7 that G has an element of order p and hence p is not a unit of R. Therefore R is H-adapted. Now, the proof of the first part proceeds by induction on $t(G)$. If $t(G)=0$, then G is torsion-free and hence by [13, p. 166, Corollary 1.7], we have $V(RG)=G$. It therefore follows that $\theta(G)=H$ and the statement is trivial. Assume $t(G)>0$ so that $t(G)>t(G/T_1(G))$. Then since Corollary 1.8 says that $T_1(G)=T\zeta(V(RG))$, we obtain $\theta(T_1(G))=T_1(H)$, so $\theta(\Delta_R(G, T_1(G)))=\Delta_R(H, T_1(H))$. Thus we have $R(G/T_1(G))\cong R(H/T_1(H))$, and since R is $G/T_1(G)$-adapted we conclude by induction that $H/T_1(H)$ is nilpotent with $t(G/T_1(G))=t(H/T_1(H))$. Consequently, H is nilpotent because $T_1(H)$ is central in H, and we get $t(G)=t(H)$ as desired.

The above result shows that in the case where G is periodic, the nilpotence class of G is equal to that of H (see Remark on p. 138 in [10]). For the general nilpotent case, however, we need to investigate a correspondence between the set of normal subgroups of G and that of H.

Lemma 2.2. *Let G and H be nilpotent groups, and assume that we have an augmented epimorphism f: $RH\rightarrow RG$ of R-algebras such that $H\cap(1+Ker\,f)=\{1\}$. Then $U(1+Ker\,f)$ is torsion-free.*

Proof. We proceed by induction on $t(G)$. Let $t(G)=0$. Then, as in the proof of Lemma 2.1, we have $V(RG)=G$ and so the restriction of f to H is an embedding into G. This implies that f is an isomorphism, and the result is trivial. Assume that $t(G)>0$, and set $W=T_1(G)$, $K=H\cap(1+f^{-1}(\Delta_R(G, W)))$, where $f^{-1}(X)$ means the inverse image in RH of a subset X of RG. Then, $\Delta_R(H, K)\subseteq f^{-1}(\Delta_R(G, W))$, and the natural epimorphism $RH/\Delta_R(H, K)\to RH/f^{-1}(\Delta_R(G, W))$ induces an augmented epimorphism θ: $R(H/K)\to R(G/W)$ with $(H/K)\cap(1+Ker\ \theta)=\{1\}$. As $t(G/W)<t(G)$, the induction hypothesis ensures that $U(1+Ker\ \theta)$ is torsion-free.

We claim that $TK\subseteq\zeta(H)$ and R is H-adapted. Since $f(TK)$ is a periodic subgroup of $U(1+\Delta_R(G, W))$, Lemma 1.6 shows that $f(TK)\subseteq W$, so $f((TK, H))=\{1\}$. Hence $(TK, H)=\{1\}$, i.e., $TK\subseteq\zeta(H)$. That R is H-adapted follows from Theorem 1.7, because H can be embedded in $V(RG)$ via f.

To complete the proof of the lemma, let $u\in TU(1+Ker\ f)$. Then since $u-1\in f^{-1}(\Delta_R(G, W))$ we observe that $\bar{u}$ is contained in $U(1+Ker\ \theta)$ under the natural map $\bar{\ }$: $RH\to R(H/K)$. On the other hand, $U(1+Ker\ \theta)$ is torsion-free as above, so $\bar{u}=1$, that is, $u\in U(1+\Delta_R(H, K))$. Thus, by Lemma 1.6, we have $u=k$ for some $k\in TK$. This implies that $u=1$ because $u-1\in Ker\ f$, and the proof is complete.

Lemma 2.3. *Let G and H be nilpotent groups, let I be an ideal of RH contained in $\Delta_R(H)$, and assume that $RH/I\cong RG$ as R-algebras. Then $U(1+I)/U(1+\Delta_R(H, H\cap(1+I)))$ is torsion-free.*

Proof. Recall that the (canonical) augmentation map ε_H: $RH\to R$ is given by $\varepsilon_H(\sum_h \alpha(h)h)=\sum_h \alpha(h)$. Since $I\subseteq\Delta_R(H)$, ε_H induces naturally an R-algebra homomorphism $\bar{\varepsilon}_H$: $RH/I\to R$, $\alpha+I\to\varepsilon_H(\alpha)$, so that RH/I may be viewed as an augmented algebra with the augmentation map $\bar{\varepsilon}_H$. Let θ: $RH/I\to RG$ be the given isomorphism. Then setting for $g\in G$, $\lambda(g)=(\bar{\varepsilon}_H\theta^{-1})(g)g\in RG$ and extending R-linearly we have an R-algebra automorphism λ of RG satisfying $\varepsilon_G\lambda=\bar{\varepsilon}_H\theta^{-1}$. Note that the isomorphism $\lambda\theta$: $RH/I\to RG$ is compatible with augmentation maps; thus we may assume, without loss of generality, that θ is augmented.

Set $K=H\cap(1+I)$. Then, by composing the natural epimorphism $RH/\Delta_R(H, K)\to RH/I$ with θ, we get an augmented epimorphism f: $R(H/K)\to RG$ such that $(H/K)\cap(1+Ker\ f)=\{1\}$. So it follows from Lemma 2.2 that $U(1+Ker\ f)$ is torsion-free. On the other hand, the natural map $\bar{\ }$: $RH\to R(H/K)$ yields a group monomorphism

$$U(1+I)/U(1+\Delta_R(H, K)) \to U(1+\bar{I})$$

and furthermore, $\bar{I}$ is clearly equal to $Ker\ f$. Therefore, $U(1+\bar{I})$ is torsion-free and the lemma is proved.

Let us now suppose that we have an augmented isomorphism θ: $RG\to RH$ of R-algebras. Following [1], for any normal subgroup N of G we define the normal subgroup ϕN of H as $\phi N = H \cap (1+\theta(\Delta_R(G, N)))$. Similarly, for any normal subgroup K of H, define $\phi^* K = G \cap (1+\theta^{-1}(\Delta_R(H, K)))$. It is well known [13, pp. 94–95] that if R is an integral domain of characteristic 0 in which no rational prime number has an inverse, then ϕ yields an isomorphism between the lattice of finite normal subgroups of G and that of H. However it seems to be hard to know if ϕ is an isomorphism between the lattice of general normal subgroups of G and that of H. At any rate, in the case when G is nilpotent we shall collect some properties of ϕ below, which will be applied in the subsequent argument. For convenience, we denote by $L_{PN}(G)$ the lattice of periodic normal subgroups of G.

Theorem 2.4. *If G is a nilpotent group, then the following hold:*

(a) *For $N \in L_{PN}(G)$, $\phi N \in L_{PN}(H)$ and $\theta(\Delta_R(G, N)) = \Delta_R(H, \phi N)$.*

(b) *For $N \in L_{PN}(G)$ and $K \in L_{PN}(H)$, we have $\phi^* \phi N = N$ and $\phi \phi^* K = K$. Consequently, ϕ induces an isomorphism between $L_{PN}(G)$ and $L_{PN}(H)$.*

(c) *Let I be an ideal of RG and set $N = G \cap (1+I)$, $K = H \cap (1+\theta(I))$. If $K \in L_{PN}(H)$, then $\phi N = K$.*

(d) *For $N \in L_{PN}(G)$, $\phi(N, G) = (\phi N, H)$.*

Proof. (a) As in the proof of [2, Lemma 2.3], we obtain $\theta(\Delta_R(G, TG)) = \Delta_R(H, TH)$ so that $\phi TG = TH$. Since ϕ is inclusion preserving it follows $\phi N \in L_{PN}(H)$. For the next part we have only to verify that $\theta(\Delta_R(G, N)) \subseteq \Delta_R(H, \phi N)$, the reverse inclusion being obvious from the definition of ϕ. By virtue of Lemma 2.1, H is also nilpotent and since $RH/\theta(\Delta_R(G, N)) \cong R(G/N)$ as R-algebras, Lemma 2.3 shows that $U(1+\theta(\Delta_R(G, N)))/U(1+\Delta_R(H, \phi N))$ is torsion-free. On the other hand, $\theta(N)$ is a periodic subgroup of $U(1+\theta(\Delta_R(G, N)))$, and so we get $\theta(N) \subseteq U(1+\Delta_R(H, \phi N))$, which implies that $\theta(\Delta_R(G, N)) \subseteq \Delta_R(H, \phi N)$.

(b) By (a), $\Delta_R(G, N) = \theta^{-1}(\Delta_R(H, \phi N))$, so $\phi^* \phi N = G \cap (1+\theta^{-1}(\Delta_R(H, \phi N))) = N$. Similarly, we have $\phi \phi^* K = K$. Thus it is clear that the restriction of the mapping ϕ to $L_{PN}(G)$ is an isomorphism between $L_{PN}(G)$ and $L_{PN}(H)$.

(c) As $\Delta_R(G, N) \subseteq I$ and $\Delta_R(H, K) \subseteq \theta(I)$, it follows that $\phi N \subseteq K$ and $\phi^* K \subseteq N$. Moreover, since $K \in L_{PN}(H)$, we have $K = \phi \phi^* K$ by (b), so that $K \subseteq \phi N$. Hence $\phi N = K$.

(d) From (a), (c) and Lemma 1.4, it follows that

$$\begin{aligned} \phi(N, G) &= H \cap (1+\theta\{\Delta_R(G, N)\, \Delta_R(G) + \Delta_R(G)\, \Delta_R(G, N)\}) \\ &= H \cap (1+\Delta_R(H, \phi N)\, \Delta_R(H) + \Delta_R(H)\, \Delta_R(H, \phi N)) \\ &= (\phi N, H). \end{aligned}$$

3. Lie dimension subgroups

Define the Lie powers $\Delta_R^{(n)}(G)$ of the augmentation ideal $\Delta_R(G)$ as $\Delta_R^{(1)}(G)$

$=\Delta_R(G)$ and, inductively, $\Delta_R^{(n+1)}(G)=[\Delta_R^{(n)}(G), \Delta_R(G)]RG$ for $n\geqq 1$, where $[\Delta_R^{(n)}(G), \Delta_R(G)]$ is the R-submodule of RG generated by $\alpha\beta-\beta\alpha$, $\alpha\in\Delta_R^{(n)}(G)$, $\beta\in\Delta_R(G)$. Then the n-th Lie dimension subgroup $D_{(n),R}(G)$ of G over R is defined by $D_{(n),R}(G)=G\cap(1+\Delta_R^{(n)}(G))$. The series $\{D_{(n),R}(G)\}_{n\geqq 1}$ is a descending central series of G (see [7]). We denote by $\gamma_n(G)$ the n-th term of the lower central series of G starting with $\gamma_1(G)=G$. Note that $\gamma_n(G)\subseteqq D_{(n),R}(G)\subseteqq D_{n,R}(G)$ for all $n\geqq 1$, since $\Delta_R^{(n)}(G)\subseteqq\Delta_R(G)^n$.

For the main theorem of the next section, we need the following three lemmas on Lie dimension subgroups, the third of which was proved by Sandling. The first one is analogous to a result of [2] on dimension subgroups.

From now on, we assume that the ring R is G-adapted.

Lemma 3.1. *Suppose $RG\cong RH$ as R-algebras. Then $D_{(n),R}(G)=\{1\}$ implies $D_{(n),R}(H)=\{1\}$.*

Proof. Let θ: $RG\to RH$ be an augmented isomorphism, and let ϕ be as in Theorem 2.4. If $D_{(n),R}(G)=\{1\}$, then G is nilpotent and thus by [2, Lemma 2.3], $G/TG\cong H/TH$. Therefore $\gamma_n(H/TH)=\{1\}$ i.e. $\gamma_n(H)\subseteqq TH$ and furthermore, we have $D_{(n),R}(H)\subseteqq TH$ because $D_{(n),R}(H)/\gamma_n(H)$ is periodic (see [7, pp. 44–45]). Since $\theta(\Delta_R^{(n)}(G))=\Delta_R^{(n)}(H)$, it follows from Theorem 2.4 (c) that $\phi D_{(n),R}(G)=D_{(n),R}(H)$ and hence that $D_{(n),R}(H)=\{1\}$.

Lemma 3.2. *If $\gamma_n(G)$ is periodic, then $D_{(n),R}(G)=D_{(n),Z}(G)$.*

Proof. The case $n=1$ is trivial, so let $n\geqq 2$. Then we know from [6, Theorem 6.1] that

$$D_{(n),R}(G)=\prod_{p\in\pi(R)}\gamma_2(G)\cap\tau_p\ (G \bmod D_{(n),Z}(G))$$

where $\pi(R)$ is the set of primes p which are invertible in R and $\tau_p(G \bmod D_{(n),Z}(G))$ stands for the p-torsion subgroup of G mod $D_{(n),Z}(G)$ (see also [7, p. 18]). (Note that R is an integral domain of characteristic 0.) Since $\gamma_n(G)$ is periodic, so is $D_{(n),Z}(G)$ and hence, R is $G/D_{(n),Z}(G)$-adapted, since it is G-adapted. Thus, $\tau_p(G \bmod D_{(n),Z}(G))=D_{(n),Z}(G)$ for all $p\in\pi(R)$ and since $\gamma_2(G)=D_{(2),Z}(G)\supseteqq D_{(n),Z}(G)$, we conclude that $D_{(n),R}(G)=D_{(n),Z}(G)$.

Lemma 3.3 (Sandling [11]). (1) *For all $n\leqq 6$, we have $D_{(n),Z}(G)=\gamma_n(G)$.* (2) *If G is a metabelian group, then $D_{(n),Z}(G)=\gamma_n(G)$ for all $n\geqq 1$.*

4. Main theorem

We are now in a position to prove our main theorem. For a nilpotent group G, $cl(G)$ will denote its nilpotence class; that is, $cl(G)$ is the smallest integer c such that $\gamma_{c+1}(G)=\{1\}$. The ring R is assumed to be G-adapted in this section also.

Theorem 4.1. *Let G be a nilpotent group and suppose that $RG\cong RH$ as R-algebras. If $cl(G/TG)\leqq 5$, then $cl(G)=cl(H)$.*

Proof. We assume first that $cl(G)\leqq 5$ and write $c=cl(G)$. Then $\gamma_{c+1}(G)=\{1\}$ and $c+1\leqq 6$, so, combining Lemma 3.2 with Lemma 3.3 (1), we obtain $D_{(c+1),R}(G)=\{1\}$. Thus by Lemma 3.1, we have $D_{(c+1),R}(H)=\{1\}$ so that $\gamma_{c+1}(H)=\{1\}$. If $\gamma_c(H)=\{1\}$, then the same argument shows that $\gamma_c(G)=\{1\}$, which is a contradiction. Therefore, $cl(G)=cl(H)$.

Returning to the general case, let $c=cl(G/TG)$. Then $\gamma_{c+1}(G)$ is periodic, and so is $\gamma_{c+1}(H)$ because $G/TG\cong H/TH$. Let $\theta: RG\rightarrow RH$ be an augmented isomorphism, let ϕ and ϕ^* be as in Theorem 2.4. Then $\theta(\Delta_R(G, \gamma_{c+1}(G)))=\Delta_R(H, \phi\gamma_{c+1}(G))$ by Theorem 2.4 (a), which yields $R(G/\gamma_{c+1}(G))\cong R(H/\phi\gamma_{c+1}(G))$. Since $cl(G/\gamma_{c+1}(G))=c\leqq 5$, the first argument shows that $cl(H/\phi\gamma_{c+1}(G))=c$ and hence that $\gamma_{c+1}(H)\subseteq\phi\gamma_{c+1}(G)$. Furthermore, we do have $\gamma_{c+1}(G)\subseteq\phi^*\gamma_{c+1}(H)$ by symmetry. Therefore we conclude by Theorem 2.4 (b) that $\phi\gamma_{c+1}(G)=\gamma_{c+1}(H)$. Consequently, the induction process together with Theorem 2.4 (d) shows that $\phi\gamma_{c+i}(G)=\gamma_{c+i}(H)$ for all $i\geqq 1$, and therefore we have $cl(G)=cl(H)$ as desired.

As in the proof of Lemma 1.4 we observe that if the second derived group G'' of G is periodic, then $G''=G\cap(1+\Delta_R(G)\,\Delta_R(G, G'))$. The application of this fact and Lemma 3.3 (2) gives us the following

Proposition 4.2. *Let G be a metabelian and nilpotent group, and suppose that $RG\cong RH$ as R-algebras. Then H is metabelian and nilpotent, and $cl(G)=cl(H)$.*

Proof. Since $G/TG\cong H/TH$, H'' is periodic and so we have $H''=H\cap(1+\Delta_R(H)\,\Delta_R(H, H'))$. (Note that R is also H-adapted.) Moreover, since $\Delta_R(G, G')=\Delta_R^{(2)}(G)$ it follows that $\theta(\Delta_R(G)\,\Delta_R(G, G'))=\Delta_R(H)\,\Delta_R(H, H')$ under the augmented isomorphism $\theta: RG\rightarrow RH$. Thus we deduce from Theorem 2.4 (c) that $H''=\{1\}$ and hence H is metabelian. Therefore, by applying Lemma 3.3 (2) instead of Lemma 3.3 (1), the same argument as in Theorem 4.1 shows that $\gamma_{c+1}(G)=\{1\}$ if and only if $\gamma_{c+1}(H)=\{1\}$. This implies that $cl(G)=cl(H)$ and the proof is complete.

We close this paper by providing an improvement of Theorem 2.4 (a).

Proposition 4.3. *If, under the hypotheses of Theorem 2.4, N is a normal subgroup of G, then $\theta(\Delta_R(G, TN))=\Delta_R(H, T\phi N)$.*

Proof. By Theorem 2.4 (a), we have $\theta(\Delta_R(G, TN))=\Delta_R(H, \phi TN)\subseteq\Delta_R(H, \phi N)$, so that $\theta(TN)$ is a periodic subgroup of $U(1+\Delta_R(H, \phi N))$. On the other hand, since $\phi N/T\phi N$ is torsion-free nilpotent we see from Lemma

1.2 that $U(1+\Delta_R(H, \phi N))/U(1+\Delta_R(H, T\phi N))$ is torsion-free. Thus we obtain $\theta(TN)\subseteqq U(1+\Delta_R(H, T\phi N))$, so $\theta(\Delta_R(G, TN))\subseteqq\Delta_R(H, T\phi N)$. As $\theta^{-1}(\Delta_R(H, T\phi N))\subseteqq\Delta_R(G, N)$, the same argument shows that $\theta^{-1}(\Delta_R(H, T\phi N))\subseteqq\Delta_R(G, TN)$, and hence the result follows.

REMARK. In the context of Theorem 2.4, it would be nice to know if it is true that, given a torsion-free normal subgroup N of G, there always exists a normal subgroup K of H such that $\theta(\Delta_R(G, N))=\Delta_R(H, K)$; because in that case, one can see, by going mod TN, that $\theta(\Delta_R(G, N))=\Delta_R(H, \phi N)$ for any normal subgroup N of G and hence that ϕ is an isomorphism between the lattice of all normal subgroups of G and that of H.

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