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Osaka J. Math.
24 (1987), 61–76

A NON-TANGENTIAL LIMIT THEOREM

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(Received January 13, 1986)

Introduction

The classical theorem of Fatou states that any positive harmonic function on the unit disk possesses a non-tangential limit at almost all boundary points. In 1959, Doob [13] showed that even the quotient of $\frac{u}{h}$ two positive harmonic functions on the half space possesses a non-tangential limit at μ_h-almost all boundary points, where μ_h is the representing measure of h in the Riesz-Herglotz Theorem. In the subsequent time there appeared many papers (the most far reaching are cited in sect. 4) generalizing the above two results in two directions

1. More general domains than balls and half spaces.
2. More general elliptic differential operators than the Laplace operator.

In this paper we prove a non-tangential limit theorem which is most general in both directions. We work in the setting of Brelot harmonic spaces satisfying the uniform Harnack inequality. Best results can be obtained by combining some "hard analysis" estimates with the abstract Fatou-Naïm-Doob Theorem about existence of fine limits. This was first done by Brelot, Doob [8]. In their case the "hard analysis" estimates were comparatively simple since the Green function of the half space is known explicitely. In the subsequent development generalizations of two estimates of Carleson [10] and Dahlberg [12], Lemma 1, were essential. Unfortunately Dahlberg's estimate doesn't make sense in our setting and its proof is of limited generalisability because it involves the very special behaviour of the Green function near its singularity and also specific properties of Lebesgue measure. In our previous paper [35] we have proved a different estimate which can be used to prove our non-tangential limit theorem by modifying the Brelot-Doob approach.

The plan of our paper is as follows. In sect. 1 we recall some basic notions and present some general examples of domains. In sect. 2 Poisson kernels and fine limit theory are recalled. In sect. 3 the non-tangential limit theorem is stated and proved. In sect. 4 many examples from the theory of partial differential equations including non-elliptic operators are presented.

1. NTA-domains

Throughout the whole paper $(X, \mathcal{H})$ will be a Brelot harmonic space (cf. Constantinescu, Cornea [11]) and for any $U \subset X$ open $\mathcal{H}(U)$ (resp. $\mathcal{S}(U)$) will be the set of harmonic (resp. superharmonic) functions on U. The subset of non-negative elements in $\mathcal{H}(U)$ (resp. $\mathcal{S}(U)$) will be denoted by $\mathcal{H}_+(U)$ (resp. $\mathcal{S}_+(U)$). For an arbitrary set $A \subset X$, $\bar{A}$ denotes its closure and ∂A its boundary. The harmonic measure of a resolutive open set $U \subset X$ and a point $x \in U$ will be denoted by μ_x^U, i.e.

$$y \to \int_{\partial U} f(x) \mu_x^U(dx)$$

is the Perron-Wiener-Brelot solution of the Dirichlet-problem for any resolutive function f on ∂U.

An open set $U \subset X$ will be called *admissible* if U is resolutive, relatively compact and if there exist $V \supset \bar{U}$ open and $h_0 \in \mathcal{H}_+(V)$ with

$$0 < \inf_{y \in V} h_0(y) \leq \sup_{y \in V} h_0(y) < \infty$$

Remark 1.1 below will show why we have introduced this new notion.

In order to state non-tangential limit theorems we need some geometry on X: (Y, d) will always be a metric space such that X is an open subspace of Y and the potential theory on X is linked with the metric d by the *uniform Harnack inequality*, i.e. there exists $C_H > 0$, $R_H > 0$ such that for any $x \in X$, $0 < r \leq R_H$ with $\bar{B}(x, 2r) \subset X$ we have

$$u(y_1) \leq C_H u(y_2) \qquad (u \in \mathcal{H}_+(B(x, 2r)),\ y_1, y_2 \in \bar{B}(x, r)),$$

where

$$B(x, r) := \{y \in Y : d(x, y) < r\}, \ \bar{B}(x, r) := \{y \in Y : d(x, y) \leq r\}.$$

We assume also that $B(x, r)$ is resolutive for any $x \in X$, $0 < r \leq R_H$ with $\bar{B}(x, r) \subset X$ and in order to simplify notation we write μ_x^r instead of $\mu_x^{B(x,r)}$.

Let us now specify the class of domains $U \subset X$ with which we will deal. A finite sequence $(z_i)_{0 \leq i \leq n}$ is called a *Harnack chain* of length n in an open set $U \subset X$ if

$$d(z_{i-1}, z_i) \leq \frac{1}{2} d(z_i, Y \setminus U),\ d(z_{i-1}, z_i) \leq \frac{1}{2} R_H \qquad (1 \leq i \leq n),$$

where

$$d(z, A) := \inf \{d(z, y) : y \in A\}$$

The name "Harnack" is justified by

$$C_H^{-n} u(z_0) \leq u(z_n) \leq C_H^n u(z_0) \qquad (U \subset X,\ u \in \mathcal{H}_+(U)).$$

An open set $U \subset Y$ is said to satisfy the *Harnack chain condition* if for any $y_1, y_2 \in U$ and $k \in \boldsymbol{N}$ with

$$d(y_1, y_2) \leq 2^k \min(d(y_1, Y \backslash U), d(y_2, Y \backslash U))$$

there exists a Harnack chain $(z_i)_{0 \leq i \leq n}$ with

$$z_0 = y_1,\ z_n = y_2,\ n \leq C_U k$$

$C_U > 0$ and $0 < R_U < R_H$ will always be constants depending only on U. We say that U satisfies the *interior corkscrew condition* if for any $x \in \partial U$, $0 < r \leq R_U$ there exists $y \in U$ with

$$d(x, y) \leq r,\ d(y, Y \backslash U) \geq C_U^{-1} r\,.$$

Finally U is said to satisfy the *complement condition* if for any $x \in \partial U$, $0 < r \leq R_U$ there exists $C_U^{-1} r \leq r' \leq r$ with $\bar{B}(x, r') \subset X$ and

$$\mu_x^{r'}(X \backslash U) \geq C_U^{-1}$$

Any admissible domain $U \subset X$ satisfying the above three conditions will be called an *NTA-domain*. By [11], p. 118 and the complement condition any NTA-domain is regular.

REMARK 1.1. Let U be an NTA-domain and $V \supset \bar{U}$ open $h_0 \in \mathcal{H}_+(V)$ with $0 < \inf_{y \in V} h_0(y) \leq \sup_{y \in V} h_0(y) < \infty$. Setting

$$\tilde{X} := V,\ \tilde{\mathcal{H}}(G) := \{u/h_0\colon u \in \mathcal{H}(G)\},\ G \subset \tilde{X} \text{ open}$$

then U is an NTA-domain in the harmonic space $(\tilde{X}, \tilde{\mathcal{H}})$ where $\tilde{C}_U$, $\tilde{C}_H$ may be larger and $\tilde{R}_U$ smaller. In addition the function 1 is $\tilde{\mathcal{H}}$-harmonic on $\tilde{X}$. Hence if $Y = \tilde{X}$ then we would be exactly in the realm of our previous paper [35]. But as already remarked in [35], p. 420 all the results of [35] hold in the present slightly more general context without any change of proof.

The complement condition involves harmonic measure of balls. Hence our definition of NTA-domain is not purely geometric. Now the subsequent proposition shows that under a mild condition on the harmonic space we may replace the complement condition by the *exterior corkscrew condition*, i.e. for any $x \in \partial U$, $0 < r \leq R_U$ we have $\bar{B}(x, r) \subset X$ and there exists $y \in X \backslash U$ with $d(x, y) \leq r$, $d(y, \bar{U}) \geq C_U^{-1} r$. An admissible domain $U \subset X$ satisfying the Harnack chain, the interior and the exterior corkscrew condition is called a *geometric NTA-domain*. If $X = Y = \boldsymbol{R}^n$ and if d is the Euclidean metric then this class coincides with that introduced by Jones [23] and Jerison, Kenig [22].

Proposition 1.2. *Assume that*
(HB) *for any $\beta > 0$ there exist a constant $C_\beta > 0$ such that*

$$\mu_z^r(B(z, \beta r)) \geq C_\beta \qquad (z \in \partial B(x, r),\ x \in X,\ 0 < r \leq R_H,\ \bar{B}(x, r) \subset X)$$

Then the exterior corkscrew condition implies the complement condition. In particular any geometric NTA-domain is an NTA-domain.

Proof. Let $x \in \partial U$ and $0 < r \leq R_U$. By the exterior corkscrew condition there exists $x \in X \setminus \bar{U}$ with

$$r' := d(x, y) \leq r,\ d(y, \bar{U}) \geq C_{\bar{U}}^{-1} r \geq C_{\bar{C}}^{-1} r'$$

and therefore, applying (HB) with $\beta := C_{\bar{U}}^{-1}$

$$\mu_x^{r'}(X \setminus U) \geq \mu_x^{r'}(B(x, \beta r')) \geq C_\beta$$

Replacing C_U by $C_U' := \max(C_U, C_\beta^{-1})$ the complement condition follows.

We now give concrete examples for NTA-domains in the case when (Y, d) is Euclidean space. A relatively compact domain $U \subset \boldsymbol{R}^n$ is called a *Lipschitz domain* if for any $x \in \partial U$ there exist a C^1-diffeomorphism $F := (f_1, \cdots, f_n)$ from an open neighbourhood W of x into $\boldsymbol{R}^n$ and a Lipschitz continuous function $\varphi\colon \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ such that

$$W \cap U = \{y \in W : f_n(y) > \varphi(f_1(y), \cdots, f_{n-1}(y))\}.$$

If φ is continuous and satisfies

$$(*) \qquad \sup\left\{\frac{|\varphi(x+h)+\varphi(x-h)-2\varphi(x)|}{2|h|} : x \in \boldsymbol{R}^{n-1},\ h \in \boldsymbol{R}^{n-1},\ h \neq 0\right\}$$

instead of Lipschitz continuity then U is called a *Zygmund domain* [22]. Note that this class is considerably more general than Lipschitz domain since there exist nowhere differentiable functions satisfying (*) and since any Lipschitz continuous function is almost everywhere differentiable. We now have

Main Example 1.3 (Jerison-Kenig [22], p. 94). If (Y, d) is a Euclidean space then any Zygmund domain $U \subset \bar{U} \subset X$ is a geometric NTA-domain.

Some more examples are given in [22], sect. 2.

2. Poisson kernels and minimal thinness

Let $U \subset X$ be a relatively compact, resolutive domain. A continuous function $p\colon U \times \partial U \to \boldsymbol{R}_+$ is called a *Poisson kernel* of U if

(i) $y \to p(y, x)$ is harmonic on U.

(ii) for any $h \in \mathcal{H}_+(U)$ there exists a unique representing measure μ_h on ∂U with

$$h(y) = \int p(y, x) \mu_h(dx) \qquad (y \in U)$$

(iii) $\lim_{y \to x} p(y, x_0) = 0 \quad (x_0, x \in \partial U, x_0 \neq x)$

Combining [35], Theorem 3.3 with Remark 1.1 we get

Theorem 2.1. *If U is an NTA-domain and $Y_0 \in U$ then there exists a Poisson kernel which satisfies also*

(iv) $\mu_y^U(A) = \int_A p(y, x) \mu_{y_0}^U(dx) \quad$ ($y \in U$, $A \subset \partial U$ *measurable*)

(v) $p(y_0, x) = 1 \quad (x \in \partial U)$

Let now U be a relatively compact, resolutive domain which possesses a Poisson kernel p. We write $p_x(y) := p(y, x)$. It follows from (ii) that $\{\lambda p_x: \lambda > 0, x \in \partial U\}$ coincides with the set of minimal harmonic functions on U ($h \neq 0$, $h \in \mathcal{H}_+(U)$ is *minimally harmonic* if $u \leq h$ implies $u = \alpha h$, $\alpha \in \boldsymbol{R}_+$ for any $u \in H_+(U)$). A subset $A \subset U$ is said to be (*minimally*) *thin* at x if $R_{p_x}^A \neq p_x$ where

$$R_s^A := \inf\{t \in \mathcal{S}_+(U) : s \leq t \text{ on } A\}$$

is just the usual reduite of a function $s \in \mathcal{S}_+(U)$. Because of the preceding remark this notion does not depend on the special Poisson kernel. For any $g: U \to \bar{\boldsymbol{R}}$ and $x \in \partial U$ we define

$$f\text{—}\limsup_{y \to x} g(y) = \inf\{\sup_{y \in A} g(y): A \subset U, U \backslash A \text{ thin at } x\}$$

If we have even

$$f\text{—}\limsup_{y \to x} g(y) = f\text{—}\liminf_{y \to x} g(y) := -\limsup_{y \to x} -g(y)$$

then we say that g has a *fine limit* at x. We now have

Theorem 2.2 (Naïm-Doob). *Let u, $h \in \mathcal{H}_+(U)$, $h > 0$ with representing measures μ_u, μ_h be given. Then for μ_h-almost all $x \in \partial U$ we have*

$$f\text{—}\lim_{y \to x} \frac{u(y)}{h(y)} = \frac{d\mu_u}{d\mu_h}(x)$$

where the function on the right side is the Radon-Nikodym derivative of the absolutely continuous part of μ_u with respect to μ_h.

A simple proof of Theorem 2.2 is given in Bliedtner, Loeb [5].

3. The main result

In this section U will be an NTA-domain and p will be a Poisson kernel of U.

A subset $A \subset U$ is said to be *non-tangential* at $x \in \partial U$ if $x \in \bar{A}$ and if there exists $\alpha > 0$ such that

$$d(y, x\backslash U)\geq \alpha d(y, x) \qquad (y\in A).$$

By the interior corkscrew condition,

$$NT(x, \alpha):=\{y\in U: d(y, X\backslash U)\geq \alpha d(x, y)\}$$

is non-tangential at x and conversely any set A which is non-tangential at x is contained in $NT(x, \alpha)$ if α is sufficiently small. We say that $g: U\rightarrow \bar{\boldsymbol{R}}$ converges *non-tangentially* at $x\in\partial U$ to $a\in\bar{\boldsymbol{R}}$ if

$$\lim_{A\ni y\to x} g(y)=a$$

for any set $A\subset U$ which is non-tangential at x and we write

$$NT\text{—}\lim_{y\to x} g(y)=a$$

Theorem 3.1. *Let $u, h\in\mathcal{H}_+(U)$, $h>o$ and let μ_u, μ_h be the corresponding representing measures. Then for μ_h-almost all $x\in\partial U$ we have*

$$NT\text{—}\lim_{y\to x}\frac{u(y)}{h(y)}=\frac{d\mu_u}{d\mu_h}(x)$$

where the function on the right side is as in Theorem 2.2.

Corollary 3.2. *For any harmonic function $u\geq 0$, $y_0\in U$*

$$NT\text{—}\lim_{y\to x} u(y)$$

exists for $\mu_{y_0}^U$-almost all $x\in\partial U$.

Proof of the Corollary. Let p be the Poisson kernel of Theorem 2.1 and

$$h(y):=\int p(y, x)\mu_{y_0}^U(dx)$$

Then $\mu_{x_0}^U$ is the representing measure of h. The assertion follows now from Theorem 3.1 and (U is regular!)

$$\lim_{y\to x} h(y)=1 \qquad (x\in\partial U).$$

For the proof of the theorem we need two fundamental estimates. An estimate of the first kind appears in every modern paper dealing with our problem and goes back to Carleson [10]. The second seems to be quite obvious from the Markov process point of view but its proof is extremely delicate. Recalling Remark 1.1 both estimates follow from [35], Corollary 1.3 and [35], Theorem 2.1.

Lemma 3.3. (a) *For any $0<\alpha<1$ there exists a constant C_α depending* *only on U, C_H and α such that for any $x_0\in\partial U$, $0<r\leq R_H$ and $u\in\mathcal{H}_+(U)$ with*

$$\lim_{y\to x} u(y)=0 \qquad \left(x\in\partial U\backslash B\left(x_0, \frac{1}{2}r\right)\right)$$

we have

$$u(y)\leq C_\alpha u(y_0) \quad (y\in U\backslash B(x_0, r), y_0\in NT(x_0, \alpha, r):=NT(x_0, \alpha)\cap\partial B(x_0, r)).$$

(b) *If we denote $U(x, r):=U\backslash\bar{B}(x, r)$ $(x\in\partial U, r>0)$ then there exists $\varepsilon_U>0$, $0<\gamma_U<1$ depending only on U and C_H such that for any $x\in\partial U$, $y\in U$ there exists $R(x, y)>0$ with $y\in U(x, R(x, y))$ and*

$$\varepsilon_U\mu_x^{U(x,r)}(U)\leq\mu_x^{U(x,r)}(NT(x, \gamma_U, r)) \qquad (0<r\leq R(x, y)).$$

Lemma 3.4. *Let $x\in\in\partial U$ and $r_n\geq r_{n+1}>0$, $n\in\boldsymbol{N}$, $\lim_{n\in N} r_n=0$. Then the set*

$$A:=\bigcup_{n\to N} NT(x, \gamma_U, r_n)$$

is not minimally thin at x.

Proof. Denote $A_n:=\bar{B}(x, r_n)\cap A$ and $T(x, \alpha, r):=(U\backslash NT(x, \alpha))\cap\partial B(x, r)$. Then any $s\in\mathcal{S}_+(U)$ with $p_x\leq s$ on A_n satisfies

$$\liminf_{z\to x'} s(z)\geq p_x(x') \qquad (x'\in NT(x, \gamma_U, r_n))$$

and therefore

$$(1) \qquad \int_{NT(x,\gamma_U,r_n)} p_x(x')\mu_x^{U(x,r_n)}(dx')\leq s(y) \qquad (y\in U(x_0, r_n)).$$

On the other hand, using property (iii) of the Poisscn kernel and Lemma 3.3, we have

$$\begin{aligned} p_x(y) &= \int_U p_{x_0}(x')\mu_y^{U(x,r_n)}(dx') = \int_{T(x,\gamma_U,r_n)} + \int_{NT(x,\gamma_U,r_n)} \\ &\leq\mu_y^{U(x,r_n)}(U)C\gamma_U \inf\{p_x(x'): x\in NT(x, \gamma_U, r_n)\} + \int_{NT(x,\gamma_U,r_n)} \\ &\leq(\varepsilon_U^{-1}C\gamma_U \int_{NT(x,\gamma_U,r_n)})+\int_{NT(x,\gamma_U,r_n)} \end{aligned}$$

if $r_n\leq R(x, y)$. Together with (1) this entails

$$p_x(y)\leq(1+\varepsilon_U^{-1}C\gamma_U)s(y) \qquad (r_n\leq R(x, y))$$

Taking the infimum over all such s we get

$$p_x(y)\leq CR_{p_x}^{A_n}(y) \qquad (r_n\geq R(x, y))$$

But now $p_x \leq CR^A_{p_x}$ shows that A is not minimally thin at x. Otherwise $p_x \leq CR^A_{p_x}$ would be a potential (cf. [16], p. 313)

Proposition 3.5. *For any $u, h \in \mathcal{H}_+(U)$, $h>0$, $0<\alpha<1$ and $x \in \partial U$ with $x \in \overline{NT(x, \alpha)}$ we have*

$$\limsup_{NT(x,\alpha)\ni y\to x} \frac{u(y)}{h(y)} \leq B_\alpha \left(f\text{—}\limsup_{y\to x} \frac{u(y)}{h(y)}\right)$$

where B_α depends only on α, U and C_H.

Proof. By the Harnack chain condition there exists B_α depending only on α, C_U, C_H such that

$$(1) \qquad \sup\left\{\frac{u(y)}{h(y)} : y \in NT(x, \alpha, r)\right\} \leq B_\alpha \inf\left\{\frac{u(y)}{h(y)} : y \in NT(x, \gamma_U, r)\right\} \qquad (x \in \partial U, r>0)$$

Assume now that there exist $x \in \partial U$, $\varepsilon>0$ and a sequence (y_n) in $NT(x, \alpha)$ with

$$\lim_{n\to\infty} y_n = x, \ \frac{u(y_n)}{h(y_n)} \geq (1+\varepsilon)B_\alpha b, \ b := f\text{—}\limsup_{y\to x} \frac{u(y)}{h(y)} < \infty$$

Then, by (1), we have

$$\frac{u(y)}{h(y)} \geq (1+\varepsilon)b \qquad (y \in NT(x, \gamma_U, r_n),\ r_n := d(x, y_n))$$

and therefore, by the definition of f-lim sup,

$$\bigcup_{n\in N} NT(x, \gamma_U, r_n)$$

must be minimally thin at x contradicting Lemma 3.4.

Proof of Theorem 3.1. Let $h \in \mathcal{H}_+(U)$, $h>0$ with representing measure μ_h. Let $A \subset \partial U$ be a measurable set. By Theorem 2.2 there exists a μ_h-null set $N_A \subset \partial U$ with

$$f\text{—}\limsup_{y\to x} \frac{h_A(A)}{h(y)} = 0 \quad (x \in \partial U \setminus (A \cup N_A))$$

$$f\text{—}\limsup_{y\to x} \frac{h'_A(y)}{h(y)} = 0 \quad (x \in A \setminus N_A)$$

where

$$h_A(y) := \int_A p(y, x)\mu_h(dx), \ h'_A(y) = \int_{\partial U\setminus A} p(y, x)\mu_h(dx)$$

By Proposition 3.5 we have

$$\limsup_{NT(x,\alpha)\ni y\to x} \frac{h_A(y)}{h(y)} = 0 \qquad (x\in U\backslash(A\cup N_A),\ 0<\alpha<1,\ x\in\overline{NT(x,\alpha)})$$

$$\limsup_{NT(x,\alpha)\ni y\to x} \frac{h'_A(y)}{h(y)} = 0 \qquad (x\in A\backslash N_{A'},\ 0<\alpha<1,\ x\in\overline{NT(x,\alpha)})$$

Since also $h_A+h'_A=h$ we obtain

$$\lim_{NT(x,\alpha)\ni y\to x} \frac{h_A(y)}{h(y)} = 1_A(x) \qquad (x\in\partial U\backslash N_A,\ 0<\alpha<1,\ x\in\overline{NT(x,\alpha)})$$

Then for any measurable step function f on ∂U there exists a μ_h-null set N_f such that

$$(1) \qquad NT\text{—}\lim_{y\to x} \frac{h_f(y)}{h(y)} = f(x) \qquad (x\in\partial U\backslash N_f)$$

where

$$h_f(y) := \int p(y,x)f(y)\mu_h(dy)\,.$$

Let now $f\in L^1(\mu)$, $f\geq 0$ be arbitrary. Then there exist measurable step functions $f_n\uparrow f$.
We set

$$M := \bigcup_{n=1}^{\infty} N_{f_n},\ \tilde{M}_{n,\varepsilon} := \left\{x\in\partial U: f\text{—}\limsup_{y\to x} \frac{h_{f-f_n}(y)}{h(y)} \geq \varepsilon\right\}$$

By Theorem 2.2.

$$\mu_h(\tilde{M}_{n,\varepsilon}) = \mu_h\{f-f_n\geq\varepsilon\}$$

and therefore

$$(2) \qquad \mu_h(\tilde{M}) = 0,\ \tilde{M} := \bigcup_{\varepsilon>0} \bigcap_{n\in N} \tilde{M}_{n,\varepsilon}$$

By (1) and Proposition 3.5 we have for any $0<\alpha<1$, $x\in\partial U\backslash(M\cup\tilde{M}_{n,\varepsilon})$, $x\in\overline{NT(x,\alpha)}$

$$\limsup_{NT(x,\alpha)\ni y\to x} \frac{h_f(y)}{h(y)} \leq f_n(x)+B_\alpha f\text{—}\limsup_{y\to x} \frac{h_{f-f_n}(y)}{h(y)} \leq f_n(x)+B_\alpha\varepsilon$$

and therefore

$$\limsup_{NT(x,\alpha)\ni y\to x} \frac{h_f(y)}{h(y)} \leq \sup_{n\in N} f_n(x) = f(x)$$

for any $0<\alpha<1$, $x\in\partial U\backslash(M\cup\tilde{M})$, $x\in\overline{NT(x,\alpha)}$.
Since, by (1), obviously also

$$\liminf_{NT(x,\alpha)\ni y\to x} \frac{h_f(y)}{h(y)} \geq \sup_{n\in N} f_n(x)$$

for any $0<\alpha<1$, $x\in\partial U\backslash M$, $x\in\overline{NT(x,\alpha)}$ we get

$$NT\text{—}\lim_{x\to y} \frac{h_f(y)}{h(y)} = f(x) \qquad (x\in\partial U\backslash N_f) \tag{3}$$

where $N_f := M\cup\tilde{M}$ is a μ_h-null set, by (2).

Let now $u\in\mathcal{H}_+(U)$ be arbitrary. Setting $f := \frac{d\mu_u}{d\mu_h}$ we have the decomposition

$$u = h_f + v$$

where the representing measure μ_v is singular with respect to μ_h. Hence by Theorem 2.2

$$f\text{—}\limsup_{y\to x} \frac{v(y)}{h(y)} = 0 \qquad \mu_h\text{-a.e.}$$

and by Proposition 3.5 also

$$NT\text{—}\lim_{y\to x} \frac{v(y)}{h(y)} = 0 \qquad \mu_h\text{-a.e.}$$

Together with (3) the assertion follows.

4. Examples

In the sequel $C^2(U)$ (resp. $C(U)$) will be the space of twice continuously differentiable (resp. continuous) functions on an open set $U\subset\boldsymbol{R}^n$.

4.1. Let (Y, d) be an n-dimensional Euclidean space and

$$L := \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + c$$

be a differential operator on an open set $X\subset Y$ such that

(i) the functions a_{ij}, b_i, c are bounded and Hölder-continuous on X and $c\leq 0$

(ii) $$C_L^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \qquad (\xi\in\boldsymbol{R}^n,\ x\in X)$$

where C_L is a constant independent of x and $|\xi|$ is the Euclidean norm of ξ. Then the sheaf

$$\mathcal{H}(U) := \{h\in C^2(U) : Lh = 0\}$$

forms a Brelot harmonic space. By Serrin [31], Theorem 4 the uniform Harnack inequality holds. Moreover an inspection of the proof of this theorem shows that for any $x\in X$, $0<r\leq R_H$, $\bar{B}(x, r)\subset X$ the harmonic measure μ_x^r satisfies

$$\text{(iii)} \qquad C^{-1}\sigma_x^r(A)\leq\mu_x^r(A)\leq C\sigma_x^r(A) \qquad (A\subset\partial B(x, r) \text{ measurable})$$

where σ_x^r is the normalized surface measure on $\partial B(x, r)$ and C is a constant independent of x, r.

Clearly (iii) implies condition (HB) and therefore any geometric NTA-domain is an NTA-domain. In particular, the non-tangential limit theorem holds for Zygmund and Lipschitz domains. In the case $L=\Delta$ and U a Lipschitz domain Corollary 3.2 (resp. Theorem 3.1) was shown by Hunt, Wheeden [20] (resp. [21]). A "soft analysis" proof of Corollary 3.2 when $L=\Delta$ and U Lipschitz was given in Wittmann [34]. For L as above and U Lipschitz their results were shown by Ancona [2]. For $L=\Delta$ and U a geometric NTA-domain Corollary 3.2 was shown by Jerison, Kenig [22]. Their method works also for general L but it requires a difficult geometric localization lemma of P.W. Jones [24]. Finally Taylor [32] proved Theorem 3.1 for $L=\Delta$ and U a geometric NTA-domain based on [22].

The assumption $c\leq 0$ can be dropped by using arguments similar to Remark 1.1.

For the next two examples we need Sobolev spaces. For $U\subset\boldsymbol{R}^n$ open $C_0^\infty(U)$ will be the space of all infinitely differentiable functions on U with compact carrier. We denote by $W^{2,p}(\boldsymbol{R}^n)$ (resp. $W^{1,p}(\boldsymbol{R}^n)$) the completion in L^p of $C_0^\infty(\boldsymbol{R}^n)$ with respect to the norm

$$||f||_{2,p}:=\left(\int\left(|(f(x)|^p+\sum_{i=1}^n\left|\frac{\partial}{\partial x_i}f(x)\right|^p+\sum_{i,j=1}^n\left|\frac{\partial^2}{\partial x_i\partial x_j}f(x)\right|^p\right)dx\right)^{1/p}$$

$$\left(\text{resp. } ||f||_{1,p}:=\left(\int\left(|f(x)|^p+\sum_{i=1}^n\left|\frac{\partial}{\partial x_i}f(x)\right|^p\right)dx\right)^{1/p}\right)$$

Finally for any open U let $W_{\text{loc}}^{i,p}(U)$ ($i=1, 2$) be the space of all functions f on U such that for any relatively compact $V\subset\bar{V}\subset U$ there exists $g\in W^{i,p}(\boldsymbol{R}^n)$ with $f|_V=g|_V$ a.e..

Of course $\dfrac{\partial}{\partial x_i}$ $\left(\text{resp. } \dfrac{\partial^2}{\partial x_i\partial x_j}\right)$ extend continuously to continuous operators from $W_{\text{loc}}^{1,p}(U)$ (resp. $W_{\text{loc}}^{2,p}(U)$) to $L^p(U)$.

4.2. Let (Y, d) be Euclidean space and $(a_{i,i})_{1\leq i,j\leq n}$ be a symmetric matrix of bounded measurable functions on an open set $X\subset Y$ satisfying 4.1 (ii).

For any open set $U\subset X$ let $\mathcal{H}(U)$ be the space of all continuous functions $h\in W_{\text{loc}}^{1,2}(V)$ with

$$\sum_{i,j=1}^n\int_U a_{i\,j}(x)\frac{\partial}{\partial x_i}h(x)\frac{\partial}{\partial x_j}\varphi(x)dx=0 \qquad (\varphi\in C_0^\infty(U))$$

Then the sheaf $\mathcal{H}(U)$ forms a Brelot harmonic space (cf. R.M. Hervé [19]) which satisfies the uniform Harnack inequality (cf. Moser [29]). Applying [9], Lemma 2.1 to balls and using the uniform Harnack inequality we see that condition (*HB*) also holds though 4.1 (iii) fails in general. For Lipschitz domains U and the above class of operators Corollary 3.2 was shown by Cafarelli, Fabes, Mortola, Salsa [9]. By using the method of Grüter, Widman [17] one can drop the symmetry assumption on $(a_{i,j})$.

4.3. Let (Y, d) be Euclidean space and

$$L: = \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c$$

be a differential operator on an open set $X \subset Y$ such that 4.1 (ii) and
(i) the functions $(a_{i,j})$ are bounded and uniformly continuous. b_i and c are bounded measurable and $c \leq 0$.
Let $\mathcal{H}(U)$ be the space of all continuous functions f on U with $f \in W^{2;p}_{\mathrm{loc}}(U)$ $(1 < p < \infty)$ and $Lf = 0$. Then $(X, \mathcal{H})$ is a harmonic space (this follows from the L^p-Schauder estimates in [1]). The uniform Harnack inequality was proved by Krylov, Safanov [26] and Trudinger [33], but also the ingenious method of Landis [27] can be adapted to the present case. Condition (*HB*) follows from Miller's [28] uniform barrier if $c = 0$. In the general case (*HB*) may be proved by elementary comparison arguments.

Unlike 4.1 and 4.2 the Green function may behave very badly in the present case and Ancona's method [2] has to be modified strongly. This was done by Bauman [4]. But non-tangential limit theorems are only stated and not proved in this paper.

If Y is two dimensional then it is enough to assume that the $a_{i,j}$ are only measurable instead of being uniformly continuous.

4.4. Let (Y, d), X and L be as in 4.1. But now $\mathcal{H}(U)$ will be the space of all continuous function h such that

$$\int_U h(x) L\varphi(x) dx = 0 \qquad (\varphi \in C_0^\infty(U))$$

Then $(X, \mathcal{H})$ is an harmonic space adjoint to the harmonic space of 4.1 in the sense of R.M. Hervé [18], p. 537. By Ancona [3], Proposition 10 the uniform Harnack inequality and condition (*HB*) hold.

4.5. (generalized Schrödinger equation). Let (Y, d) be n-dimensional Euclidean space $X = Y$ with $n \geq 3$. We denote by $G(x, y)$ the fundamental solution of Δ on Y, i.e.

$$G(x, y) = c_n |x - y|^{2-n}$$

Let μ be a signed Radon measures on $\boldsymbol{R}^n$ with total variation $|\mu|$ such that

$$\lim_{r\to 0} \sup_{x,y\in X} \int_{B(x,r)} G(y, z)|\mu|(dz) = 0$$

For any $U\subset X$ open let $\mathcal{H}(U)$ be the space of all continuous functions h on U satisfying

$$\int_U f(x)\Delta\varphi(x)dx - \int_U f\varphi d\mu = 0 \qquad (\varphi\in C_0^\infty(U))$$

Then, by Boukricha, Hansen, Hueber [7], (X, H) is a Brelot harmonic space and by [7], Theorem 7.7 the uniform Harnack inequality as well as condition 4.1 (iii) are satisfied. Hence the same results as in 4.1 hold. The above results hold also for $n=2$ if some standard changes are made.

4.6. Let $\boldsymbol{H}_n := \{(x, y, t)\colon x\in\boldsymbol{R}^n, y\in\boldsymbol{R}^n, t\in\boldsymbol{R}\}$ be the Heisenberg group with multiplication

$$(x, y, t)\,(x', y', t') := (x+x', y+y', t+t'+2\sum_{i=1}^n x_i y_i' + x_i' y_i)$$

and $(\delta_r)_{r>0}$ be the group automorphisms defined by

$$\delta_r(x, y, t) := (rx, ry, r^2 t)$$

The left translation invariant metric

$$d((x, y, t), (x', y', t')) := \rho((x, y, t)^{-1}(x', y', t'))\,,$$
$$\rho(x, y, t) := ((|x|^2+|y|^2)^2+t^2)^{1/4}$$

is naturally associated with the left translation invariant differential operator

$$L := \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}\right)^2$$

by its fundamental solution (cf. Folland [15]). It is well known that L is hypoelliptic. Moreover, by the work of Bony [6], p. 302

$$\mathcal{H}(U) := \{h\in C^2(U)\colon Lu = 0\} \qquad (U\subset X := Y := \boldsymbol{H}_n \text{ open})$$

is a Brelot harmonic space.
Since

(i) $$d(\delta_r(x, y, t), \delta_r(x', y', t')) = rd((x, y, t), (x', y', t'))$$

and

$$h\circ\delta_r \in \mathcal{H}(\delta_r^{-1}(U)) \qquad (h\in\mathcal{H}(U))$$

the uniform Harnack inequality holds. Condition (HB) may also be shown by an elementary comparison argument involving the known fundamental solution [15] (cf. Wittmann [36] for more details). It is an interesting open question whether the unit ball (with respect to the above metric) is an NTA-domain. On the other hand there exist other left invariant metrics on $\boldsymbol{H}^n$ satisfying 4.6 (i) and having balls for which the NTA conditions are easy to verify (cf. the "infinitesimal" metric in Koranyi [25], sect. 4). Although the Green function is very nice in the present case the method of Jerison, Kenig [22] seems to be hardly applicable here because a proof of Jones localization which is already quite difficult for $\boldsymbol{R}^n$ looks hopeless for the above metric space.

4.7. Let L be a degenerate elliptic differential operator on a compact manifold $X=Y$ as in Sanchez-Calle [30]. Let further d be the metric on Y naturally associated with L (cf. [30], p. 150 and also Fefferman, Phong [14]). Then it will be shown in Wittmann [36] that the uniform Harnack inequality as well as condition (HB) hold for the harmonic space defined by

$$\mathcal{H}(U) := \{h \in C^2(U) : Lu = 0\}$$

(cf. Bony [6], p. 302).
A non-elliptic generalization of example 4.2 containing the uniform Harnack inequality of 4.7 as a very special case is given in Franchi-Laconelli [37].

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