Shigekawa, I.
Osaka J. Math.
24 (1987), 37-59

EXISTENCE OF INVARIANT MEASURES OF DIFFUSIONS ON AN ABSTRACT WIENER SPACE

ICHIRO SHIGEKAWA

(Received November 25, 1985)

1. Introduction

In this paper, we consider diffusions on an *abstract Wiener space* (B, H, μ), B is a separable (real) Banach space with a norm $\|\cdot\|_B$, H is a separable (real) Hilbert space that is densely and continuously imbedded in B with an inner product $\langle\cdot, \cdot\rangle_H$ and a norm $|\cdot|_H=\sqrt{\langle\cdot, \cdot\rangle_H}$ and μ is the Wiener measure, i.e., Borel probability measure with the characteristic function $\hat{\mu}$ given by

$$\hat{\mu}(l) = \int_B e^{\sqrt{-1}(x,l)}\,\mu(dx) = \exp\{-\frac{1}{2}|l|_H^2\}, \qquad l\in B^*$$

where B^* is the dual space of B, $(\ ,\ )$ is the natural bilinear form on $B\times B^*$ and we regard B^* as a subspace of H: $B^*\subseteq H^*=H$.

Typical example of a diffusion on the abstract Wiener space is the *Ornstein-Uhlenbeck process*. We denote its generator by $\frac{1}{2}L$ and call L the Ornstein-Uhlenbeck operator. We consider diffusions generated by operators of the form $A=\frac{1}{2}L+b$ where b is an H-valued bounded function on B and we regard b as a vector field on B. Our main aim is to show the *existence of invariant measures* of these diffusions.

By the way, as is well-known, such a diffusion is obtained by the transformation of the drift for the Ornstein-Uhlenbeck process. Hence our diffusions are closely related to the Ornstein-Uhlenbeck process. But, a calculus for the Ornstein-Uhlenbeck process, sometimes called *Malliavin's calculus*, was developed by many authors. So our discussion is based on Malliavin's calculus, especially on the theories of Ornstein-Uhlenbeck semigroup and Sobolev spaces over the abstract Wiener space which were studied by P.A. Meyer and H. Sugita. In this paper, we mainly follow Sugita [10].

Our strategy to prove the existence of an invariant measure is to solve the equation $A^*\rho=0$ where A^* is the dual operator of A. First we solve this equation in finite dimensional case by using the stability of the index. Secondly we solve it in infinite dimensional case by limiting procedure. In the second step, Gross' *logarithmic Sobolev inequality* (see [2]) plays an essential role.

Furthermore we discuss the *symmetry* of the semigroup with respect to the invariant measure and, denoting the invariant measure by ν, study under what condition $\nu=\mu$ holds.

Connected to the above problems, E. Nelson [7] and A.N. Kolmogorov [5] considered the diffusions on a Riemannian manifold. They studied the diffusion generated by $\frac{1}{2}\Delta+b$ where Δ is the Laplace-Beltrami operator and b is a C^∞ vector field and obtained the necessary and sufficient condition for the symmetry of the semigroup and for the equality of the invariant measure and the Riemannian volume. In their studies, de Rham-Hodge-Kodaira's decomposition of the space of 1-forms is crusial. In [9], the author obtained de Rham-Hodge-Kodaira's decomposition on the abstract Wiener space. Hence in our case, parallel argument can be done.

This paper is organized as follows. In the section 1, we construct the diffusion by the transformation of the drift and define the associated semigroup on $L^2(B, \mu)$. Moreover we decide the domain of the generator. It is important in order to characterize invariant measures as solutions of $A^*\rho=0$. We prove the existence of an invariant measure in the section 3. We prove it by two steps; first in finite dimensional case and secondly in infinite dimensional case. In the section 4, we discuss the symmetry of the semigroup.

2. Construction of the diffusion

Let (B, H, μ) be an abstract Wiener space and L be the Ornstein-Uhlenbeck operator. Let b be an H-valued measurable function on B and we assume that b is bounded:

$$(B.1) \qquad ||b||_\infty = \sup_{x\in B} |b(x)|_H < \infty .$$

In this section we construct a diffusion on B generated by an operator $A=\frac{1}{2}L+b$. Here we regard b as a vector field on B:

$$bf(x) = \langle b(x), Df(x)\rangle_H$$

where Df is the H-derivative of f.

We characterize this diffusion by a martingale formulation. To do this, we first prepare a space of testing functions as follows; let $\mathcal{D}$ be a set of all functions u: $B\to\boldsymbol{R}$ represented as

$$u(x) = f((x, \phi_1), (x, \phi_2), \cdots, (x, \phi_n))$$

for some $n\in\boldsymbol{N}$, $f\in C_0^\infty(\boldsymbol{R}^n)$ and $\phi_1, \phi_2, \cdots, \phi_n\in B^*$ where B^* is the dual space of B and $(,)$ is the natural bilinear form on $B\times B^*$. Then for $u\in\mathcal{D}$, Au

is a bounded function on B. Secondly let $W(B)$ be a set of all continuous paths w: $[0, \infty)\to B$. We define a metric ρ on $W(B)$ by

$$\rho(w, v)=\sum_{n=1}^{\infty}\frac{1}{2^n}\sup_{0\leqq t\leqq n}(\|w_t-v_t\|_B\wedge 1).$$

Then $W(B)$ is a separable complete metric space. We denote by $\mathcal{B}(W(B))$ the topological σ-algebra and $\mathcal{B}_t(W(B))$, $t\geqq 0$ the sub σ-algebra generated by w_s, $s\leqq t$.

DEFINITION 2.1. A diffusion generated by an operator $A=\frac{1}{2}L+b$ is a family of probability measures $\{Q_x\}_{x\in B}$ on $W(B)$ satisfying:
(i) for $u\in\mathcal{D}$

$$u(w_t)-u(w_0)-\int_0^t Au(w_s)ds$$

is $(\mathcal{B}_t(W(B)))$-martingale under Q_x for all $x\in B$,
(ii) $Q_x(w_0=x)=1$,
(iii) $x\mapsto Q_x(E)$ is measurable for $E\in\mathcal{B}(W(B))$,
(iv) $\{Q_x\}$ is a strongly Markovian system.

Typical example is the Ornstein-Uhlenbeck process. We can give it by solving a stochastic differential equation as follows. Let $(W_t)_{t\geqq 0}$ be a B-valued Wiener process on an auxiliary probability space $(\Omega, P, \mathcal{F}, (\mathcal{F}_t))$ with the mean 0 and the covariance

$$E^P[(W_t, \phi)(W_s, \psi)]=(t\wedge s)\langle\phi, \psi\rangle_H \qquad \text{for } \phi, \psi\in B^*\subseteq H. \tag{2.1}$$[1)]

We may assume that (W_t) is canonically realized on $W_0(B)=\{w\in W(B)\,|\,w_0=0\}$, i.e., $\Omega=W_0(B)$, $\mathcal{F}=\mathcal{B}(W_0(B))$, $\mathcal{F}_t=\mathcal{B}_t(W(B))$ and $W_t(w)=w_t$ where $\mathcal{B}(W_0(B))$ and $\mathcal{B}_t(W_0(B))$ are restrictions of $\mathcal{B}(W(B))$ and $\mathcal{B}_t(W(B))$ to $W_0(B)$ respectively. We consider the following stochastic differential equation:

$$\begin{cases} dX_t=dW_t-\frac{1}{2}X_t dt \\ X_0=x. \end{cases} \tag{2.2}$$

Then (2.2) has a unique solution, which we denote by $(X(t, x))_{t\geqq 0}$. Let P_x be a law of $(X(t, x))$ on $W(B)$. Then $\{P_x\}_{x\in B}$ is the diffusion generated by $\frac{1}{2}L$ called the Ornstein-Uhlenbeck process.

Next we consider the general case $A=\frac{1}{2}L+b$. For $x\in B$ define a process $(M_t^x)_{t\geqq 0}$ by

1) E^P stands for the expectation with respect to P. In the sequel, we use this convention without mentioning.

$$M_t^x = \exp\{\int_0^t (b(X(s, x)), dW_s) - \frac{1}{2}\int_0^t |b(X(s, x))|_H^2 ds\}$$

where $X(s, x)$ is the solution of (2.2) and the first term of the exponent in the right hand side is the stochastic integral (see, e.g., [6]). Under the assumption *(B.1)*, (M_t^x) is a martingale and then there exists a probability measure $\tilde{P}_x$ on $\Omega = W_0(B)$ such that

$$\tilde{P}_x | \mathcal{F}_t = M_t^x P | \mathcal{F}_t \qquad \text{for all } t \geqq 0$$

where $|\mathcal{F}_t$ stands for the restriction to $\mathcal{F}_t$. Let Q_x be a law of $(X(t, x))$ under $\tilde{P}_x$. Now the following proposition can be obtained by a standard argument.

Proposition 2.1. *$\{Q_x\}_{x\in B}$ is a unique diffusion generated by $A = \frac{1}{2}L + b$.*

Let us define the semigroup $\{T_t\}_{t\geqq 0}$ associated with A as follows:

$$T_t u(x) = E^{Q_x}[u(w_t)] = E^P[u(X(t, x))M_t^x] \tag{2.3}$$

for $u \in \mathcal{B}_b(B)$ where $\mathcal{B}_b(B)$ is a set of all bounded Borel measurable functions on B. It is well-known that $\{T_t\}$ is actually a semigroup on $\mathcal{B}_b(B)$ but we have to extend it to $L^2(B, \mu)$.

Proposition 2.2. *T_t can be extended to a bounded linear operator on $L^2(B, \mu)$. Moreover, writing this extension by T_t also, $\{T_t\}_{t\geqq 0}$ forms a strongly continuous semigroup on $L^2(B, \mu)$.*

Proof. We denote the L^2-norm by $|\cdot|_2$. Then for $u \in \mathcal{B}_b(B)$ we have by the Schwarz inequality and (B.1)

$$
\begin{aligned}
& |T_t u|_2^2 \\
&= \int_B T_t u(x)^2 \mu(dx) \\
&= \int_B E^P[u(X(t, x))\exp\{\int_0^t (b(X(s, x)), dW_s) - \frac{1}{2}\int_0^t |b(X(s, x))|_H^2 ds\}]^2 \mu(dx) \\
&= \int_B E^P[u(X(t, x))\exp\{\frac{1}{2}\int_0^t |b(X(s, x))|_H^2 ds\} \\
&\qquad \times \exp\{\int_0^t (b(X(s, x)), dW_s) - \int_0^t |b(X(s, x))|_H^2 ds\}]^2 \mu(dx) \\
&\leqq \int_B E^P[u(X(t, x))^2 \exp\{\int_0^t |b(X(s, x))|_H^2 ds\}] \\
&\qquad \times E^P[\exp\{\int_0^t (2b(X(s, x)), dW_s) - \frac{1}{2}\int_0^t |2b(X(s, x))|_H^2 ds\}]\mu(dx) \\
&\leqq e^{t\|b\|_\infty^2} \int_B E^P[u(X(t, x))^2]
\end{aligned}
$$

$$\times E^P[\exp\{\int_0^t (2b(X(s, x)), dW_s) - \frac{1}{2}\int_0^t |2b(X(s, x))|_H^2 ds\}]\mu(dx) .$$

Noting that μ is the invariant measure of the Ornstein-Uhlenbeck process and

$$\exp\{\int_0^t (2b(X(s, x)), dW_s) - \frac{1}{2}\int_0^t |2b(X(s, x))|_H^2 ds\} \tag{2.4}$$

is a martingale, we have

$$|T_t u|_2^2 \leqq e^{t\|b\|_\infty^2}\int_B u(x)^2\mu(dx) = e^{t\|b\|_\infty^2}|u|_2^2 . \tag{2.5}$$

Hence T_t can be extended to $L^2(B, \mu)$.

Next we show the strong continuity of T_t. Since a set of all bounded continuous functions on B is dense in $L^2(B, \mu)$, it is enough to show that $T_t u \to u$ in $L^2(B, \mu)$ as $t \to 0$ for a bounded continuous function u. But $T_t u$ converges to u pointwise. Hence $T_t u \to u$ in $L^2(B, \mu)$ by Lebesgue's dominated convergence theorem. □

Hereafter, to the end of the section 3, we consider $\{T_t\}$ as a strongly continuous semigroup on $L^2(B, \mu)$. Let $\hat{A}$ be an infinitesimal generator of the semigroup $\{T_t\}$ in operator theoretical sense. We denote the domain of $\hat{A}$ by $D(\hat{A})$. Then we have;

Proposition 2.3. *$\mathcal{D} \subseteq D(\hat{A})$ and $\hat{A} = A$ on $\mathcal{D}$.*

Proof. By the Itô formula, we have for $u \in \mathcal{D}$,

$$\begin{aligned} T_t u(x) - u(x) &= E^P[u(X(t, x))M_t^x] - u(x) \\ &= E^P[\int_0^t Au(X(s, x))M_s^x ds] \\ &= \int_0^t T_s Au(x) ds . \end{aligned}$$

Hence by the Schwarz inequality, we get

$$\begin{aligned} |\frac{1}{t}(T_t u - u) - Au|_2^2 &= \int_B |\frac{1}{t}\int_0^t T_s Au(x) - Au(x)|^2 \mu(dx) \\ &\leqq \frac{1}{t}\int_0^t \int_B |T_s Au(x) - Au(x)|^2 \mu(dx) ds \\ &= \frac{1}{t}\int_0^t |T_s Au - Au|_2^2 ds . \end{aligned}$$

Now the rest is easy by the strong continuity of the semigroup. □

Next we will get the concrete expression of $\hat{A}$. To do this, let us review Sobolev spaces on an abstract Wiener space. (See, e.g., [10] for details. But we use different notations.)

For $n\in\boldsymbol{Z}_+$, define a norm $|\cdot|_{n,2}$ by

$$|u|_{n,2}^2=|u|_2^2+|Du|_2^2+\cdots+|D^nu|_2^2, \quad u\in\mathcal{D}$$

where $Du, \cdots, D^nu$ are H-derivatives of u and, for example, $|Du|_2$ is the norm of $Du\colon B\to H$ in $L^2(B, \mu; H)$, the space of all square-integrable H-valued functions on B. We denote by $W^{n,2}$ the completion of $\mathcal{D}$ by the norm $|\cdot|_{n,2}$. Moreover for any separable Hilbert space K, we can similarly define a Sobolev space of K-valued functions and we denote it by $W^{n,2}(K)$. We also denote the dual space of $W^{n,2}(K)$ by $W^{-n,2}(K)$ and its norm by $|\cdot|_{-n,2}$.

For $u\in W^{2,2}$, Lu and Du are well-defined and belong to $L^2(B, \mu)$ and $L^2(B, \mu; H)$ respectively. Hence $Au(x)=\frac{1}{2}Lu(x)+\langle b(x), Du(x)\rangle_H$ is well-defined as an element of $L^2(B, \mu)$. We extend A to $W^{2,2}$ in this manner. Now by Proposition 2.3, we easily have

Proposition 2.4. *$W^{2,2}\subseteq D(\hat{A})$ and $\hat{A}=A$ on $W^{2,2}$.*

Remainder of this section is devoted to the proof of $D(\hat{A})=W^{2,2}$. Before proceeding, we need some results on the Ornstein-Uhlenbeck process. Let $\{T_t^{O-U}\}_{t\geq 0}$ be the Ornstein-Uhlenbeck semigroup on $L^2(B, \mu)$ and J_0 be the projection operator to the space of constant functions:

$$J_0u(x)=\int_B u(x)\mu(dx)\,. \tag{2.6}$$

Then we can define the potential operator G as follows:

$$G=\int_0^\infty (T_t^{O-U}-J_0)dt\,. \tag{2.7}$$

Moreover G is the bounded linear operator from $W^{n,2}$ into $W^{n+2,2}$ and

$$LG=-\mathrm{id}+J_0 \qquad \text{on } W^{n,2} \tag{2.8}$$

for $n\in\boldsymbol{Z}_+$ where id is the identity mapping.

To show $D(A)=W^{2,2}$, we need the following Proposition.

Proposition 2.5. *The operator A on $L^2(B, \mu)$ with the domain $W^{2,2}$ is a closed operator and moreover $\lambda>\|b\|_\infty^2/2$ is in the resolvent set of A.*

Proof. Take $\lambda>\|b\|_\infty^2/2$. We first note that $\lambda-A$ is a bounded linear operator from $W^{2,2}$ into $W^{0,2}=L^2(B, \mu)$. We shall show that $\lambda-A$ is bijective as a mapping from $W^{2,2}$ into $W^{0,2}$. To do so, define a bilinear form Φ_λ on $W^{1,2}\times W^{1,2}$ by

$$\Phi_\lambda(u, v)=\frac{1}{2}\int_B\langle Du(x), Dv(x)\rangle_H\mu(dx) \tag{2.9}$$

$$-\int_B \langle b(x), Du(x)\rangle_H v(x)\mu(dx)+\lambda\int_B u(x)v(x)\mu(dx)\,.$$

Note that if $u\in W^{2,2}$ and $v\in W^{1,2}$ then

$$\Phi_\lambda(u, v)=\int_B (\lambda-A)u(x)v(x)\mu(dx)\,.$$

It is easy to see that Φ_λ is continuous; there exists a positive constant c such that

$$\Phi_\lambda(u, v)\leqq c|u|_{1,2}|v|_{1,2}\,. \tag{2.10}$$

From the assumption, we can take $0<\alpha<1$ so that

$$\lambda>\|b\|_\infty^2/(2\alpha)\,.$$

Then we have

$$\begin{aligned}\Phi_\lambda(u, u) &= \frac{1}{2}|Du|_2^2-\int_B \langle b(x), Du(x)\rangle_H u(x)\mu(dx)+\lambda|u|_2^2 \\ &\geqq \frac{1}{2}|Du|_2^2-\|b\|_\infty|Du|_2|u|_2+\lambda|u|_2^2 \\ &= \frac{1}{2}(1-\alpha)|Du|_2^2+(\lambda-\|b\|_\infty^2/(2\alpha))|u|_2^2 \\ &\quad +\left(\sqrt{\frac{\alpha}{2}}|Du|_2-\frac{\|b\|_\infty}{\sqrt{2\alpha}}|u|_2\right)^2 \\ &\geqq \frac{1}{2}(1-\alpha)|Du|_2^2+(\lambda-\|b\|_\infty^2/(2\alpha))|u|_2^2\,.\end{aligned} \tag{2.11}$$

Take any $v\in L^2(B, \mu)$. Note that the linear functional $w\mapsto\int_B w(x)v(x)\mu(dx)$ on $W^{1,2}$ is bounded since

$$\left|\int_B w(x)v(x)\mu(dx)\right|\leqq|w|_2|v|_2\leqq|w|_{1,2}|v|_2\,.$$

By (2.10) and (2.11), we can use the Lax-Milgram theorem (see e.g., [11]) and obtain that there exists $u\in W^{1,2}$ such that

$$\Phi_\lambda(u, w)=\int_B v(x)w(x)\mu(dx),\ w\in W^{1,2}\,. \tag{2.12}$$

Then, for $w\in L^2(B, \mu)=W^{0,2}$, we have that $Gw\in W^{2,2}$ and hence

$$\begin{aligned}&\int_B v(x)Gw(x)\mu(dx) \\ &=\Phi_\lambda(u, Gw) \\ &=\frac{1}{2}\int_B \langle Du(x), DGw(x)\rangle_H\mu(dx)-\int_B \langle b(x), Du(x)\rangle_H Gw(x)\mu(dx)\end{aligned}$$

$$
\begin{aligned}
&+\lambda\int_B u(x)Gw(x)\mu(dx)\\
&=-\frac{1}{2}\int_B u(x)LGw(x)\mu(dx)-\int_B\langle b(x),\,Du(x)\rangle_H Gw(x)\mu(dx)\\
&\quad+\lambda\int_B u(x)Gw(x)\mu(dx)\\
&=\frac{1}{2}\int_B u(x)w(x)\mu(dx)-\frac{1}{2}\int_B u(x)J_0w(x)\mu(dx)\\
&\quad-\int_B\langle b(x),\,Du(x)\rangle_H Gw(x)\mu(dx)+\lambda\int_B u(x)Gw(x)\mu(dx)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(2.13)\qquad &\int_B u(x)w(x)\mu(dx)\\
&\leqq \left|\int_B (2v(x)-2\lambda u(x))Gw(x)\mu(dx)\right|+\left|\int_B u(x)J_0w(x)\mu(dx)\right|\\
&\quad+\left|\int_B\langle b(x),\,Du(x)\rangle_H Gw(x)\mu(dx)\right|\\
&\leqq |2v-2\lambda u|_2|Gw|_2+|u|_2|J_0w|_2+\|b\|_\infty|Du|_2|Gw|_2\\
&\leqq (|2v-2\lambda u|_2\|G\|_{W^{-2,2},W^{0,2}}+|u|_2\|J_0\|_{W^{-2,2},W^{0,2}}\\
&\quad+\|b\|_\infty|Du|_2\|G\|_{W^{-2,2},W^{0,2}})|w|_{-2,2}
\end{aligned}
$$

where $\|\cdot\|_{W^{-2,2},W^{0,2}}$ is the operator norm from $W^{-2,2}$ into $W^{0,2}$. Thus we have $u\in W^{2,2}$ and $(\lambda-A)u=v$ which implies that $\lambda-A$ is surjective. Moreover it is easy to see that $\lambda-A$ is injective from (2.11) and hence $\lambda-A$ is bijective.

Now, by Banach's closed graph theorem, $(\lambda-A)^{-1}$ is a bounded linear operator from $W^{0,2}$ into $W^{2,2}$. Noting that the inclusion $W^{2,2}\hookrightarrow W^{0,2}$ is continuous, we have that $(\lambda-A)^{-1}$ is a bounded linear operator from $W^{0,2}$ into $W^{0,2}$. This implies that λ is in the resolvent set of A. Further $\lambda-A$ is closed as a linear operator from $W^{0,2}$ into $W^{0,2}$ and hence A is closed. □

Now we can get a main theorem in this section. We denote the dual operator of A by A^* and its domain by $D(A^*)$.

Theorem 2.1. *$D(\hat{A})=W^{2,2}$ and hence $\hat{A}=A$. Moreover $D(A^*)\subseteq W^{1,2}$.*

Proof. Take $\lambda>\|b\|_\infty^2/2$. Then λ is in the resolvent set of A by Proposition 2.4 and also is in the resolvent set of $\hat{A}$ by (2.5). On the other hand, from Proposition 2.3, we have $A\subseteq\hat{A}$ and hence $(\lambda-A)^{-1}\subseteq(\lambda-\hat{A})^{-1}$. But $(\lambda-A)^{-1}$ and $(\lambda-\hat{A})^{-1}$ are defined everywhere on $L^2(B,\,\mu)$. Therefore we have $(\lambda-A)^{-1}=(\lambda-\hat{A})^{-1}$ and hence $\hat{A}=A$.

Next we show the second assertion. Take any $u\in D(A^*)$ and set $v=A^*u$. Then for $w\in W^{2,2}$

$$\langle w, v\rangle_2 = \langle Aw, u\rangle_2$$

where $\langle\ ,\ \rangle_2$ is the inner product of $L^2(B, \mu)$. Then, substituting Gw for w, we have

$$\langle Gw, v\rangle_2 = \int_B (\frac{1}{2} LGw(x) + \langle b(x), DGw(x)\rangle_H) u(x)\mu(dx)$$
$$= \int_B (\frac{1}{2} J_0 w(x) - \frac{1}{2} w(x) + \langle b(x), DGw(x)\rangle_H) u(x)\mu(dx) .$$

Hence

$$\begin{aligned}
&|\langle w, u\rangle_2| \\
&\leqq |J_0 w|_2 |u|_2 + 2\|b\|_\infty |DGw|_2 |u|_2 + 2|Gw|_2 |v|_2 \\
&\leqq \|J_0\|_{W^{-2,2},W^{0,2}} |w|_{-2,2} |u|_2 + 2\|b\|_\infty \|G\|_{W^{-1,2},W^{1,2}} |w|_{-1,2} |u|_2 \\
&\quad + 2\|G\|_{W^{-2,2},W^{0,2}} |w|_{-2,2} |v|_2 \\
&\leqq (\|J_0\|_{W^{-2,2},W^{0,2}} |u|_2 + 2\|b\|_\infty \|G\|_{W^{-1,2},W^{1,2}} |u|_2 \\
&\quad + 2\|G\|_{W^{-2,2},W^{0,2}} |v|_2) |w|_{-1,2} .
\end{aligned}$$

Now it is easy to see that $u \in W^{1,2}$. This completes the proof. □

3. Existence of the invariant measure

In the previous section, we constructed a diffusion $\{Q_x\}_{x\in B}$ generated by $A = \frac{1}{2}L + b$. Hence associated transition probabilites are defined by

$$q(t, x, dy) = Q_x(w_t \in dy) . \tag{3.1}$$

Especially, for their importance in our discussion, we denote the transition probabilites of the Ornstein-Uhlenbeck process by $p(t, x, dy)$:

$$p(t, x, dy) = P_x(w_t \in dy) . \tag{3.2}$$

An invariant measure of the diffusion $\{Q_x\}_{x\in B}$ is a signed measure ν satisfying

$$\int_B T_t f(x)\nu(dx) = \int_B f(y) q(t, x, dy)\nu(dx) = \int_B f(x)\nu(dx)$$

for any $f \in \mathscr{B}_b(B)$. Throughout the paper, we always assume that signed measures are of finite total variation. As is well-known, the Wiener measure μ is the unique invariant measure of the Ornstein-Uhlenbeck process. In this section we shall show the existence of an invariant measure of $\{Q_x\}$. First of all, we prepare some results on invariant measures.

Lemma 3.1. *Let ν be an invariant measure and ν_+, ν_- be positive part and negative part of ν respectively. Then ν_+, ν_- are both invariant measures.*

Proof. By the Hahn decomposition, there exist Borel sets B_+ and B_- such that $B=B_+\cup B_-$, $B_+\cap B_-=\phi$, $\nu_+(\cdot)=\nu(\cdot\cap B_+)$ and $\nu_-(\cdot)=-\nu(\cdot\cap B_-)$. Then we have

$$\begin{aligned}\nu(B_+) &= \int_B q(t, x, B_+)\nu(dx) \leqq \int_B q(t, x, B_+)\nu_+(dx)\\ &\leqq \int_B \nu_+(dx) = \nu_+(B) = \nu(B_+)\,.\end{aligned}$$

Hence we have

$$\int_B q(t, x, B_+)\nu_+(dx) = \int_B \nu_+(dx)\,.$$

Noting that $q(t, x, B_+)\leqq 1$, we have

$$q(t, x, B_+) = 1 \quad \nu_+\text{-a.e.}$$

and hence

$$q(t, x, B_-) = 0 \quad \nu_+\text{-a.e.}$$

Therefore, for any Borel set E,

$$\begin{aligned}\int_B q(t, x, E)\nu_+(dx) &= \int_B \{q(t, x, E\cap B_+)+q(t, x, E\cap B_-)\}\nu_+(dx)\\ &= \int_B q(t, x, E\cap B_+)\nu_+(dx)\\ &\geqq \int_B q(t, x, E\cap B_+)\nu(dx)\\ &= \nu(E\cap B_+)\\ &= \nu_+(E)\,.\end{aligned}$$

Similarly we have

$$\int_B q(t, x, B\backslash E)\nu_+(dx) \geqq \nu_+(B\backslash E)\,.$$

On the other hand, it holds that

$$\begin{aligned}\int_B q(t, x, E)\nu_+(dx)+\int_B q(t, x, B\backslash E)\nu_+(dx) &= \nu_+(B)\\ &= \nu_+(E)+\nu_+(B\backslash E)\,.\end{aligned}$$

Hence we have

$$\int_B q(t, x, E)\nu_+(dx) = \nu_+(E)$$

which implies that ν_+ is an invariant measure. ν_- is similar. □

By the above lemma, it is enough to consider only probability measures as invariant measures. First we discuss the uniqueness.

Proposition 3.1. *An invariant probability measure that is absolutely continuous with respect to* μ*, if it exists, is unique. Moreover it is mutually absolutely continuous with respect to* μ*.*

Proof. Let ν be such an invariant measure. We denote the Radon-Nikodym derivative by $\rho=\frac{d\nu}{d\mu}$. Set

$$E=\{x\in B;\ \rho(x)>0\}\,.$$

Then we have

$$1=\nu(E)=\int_B q(t,x,E)\nu(dx)\leqq\int_B \nu(dx)=\nu(B)=1\,.$$

Hence

$$q(t,x,E)=1 \qquad \nu\text{-a.e.}$$

Since μ is absolutely continuous with respect to ν on E, we have

$$q(t,x,E)=1 \qquad \mu\text{-a.e.} \quad \text{on } E\,.$$

By the way, from the construction of $\{Q_x\}$, $q(t, x, dy)$ and $p(t, x, dy)$ are mutually absolutely continuous. Therefore

$$p(t,x,E)=1 \qquad \mu\text{-a.e.} \quad \text{on } E\,.$$

Hence we have

$$\begin{aligned}\mu(E)&=\int_B p(t,x,E)\mu(dx)\geqq\int_E p(t,x,E)\mu(dx)=\int_E \mu(dx)\\&=\mu(E)\,.\end{aligned}$$

Thus we have

$$p(t,x,E)=1_E(x) \qquad \mu\text{-a.e.}$$

which implies $T_t^{O-U}1_E=1_E$ where T_t^{O-U} is the Ornstein-Uhlenbeck semigroup. This is equivalent to $L1_E=0$. By the way, the kernel of L is the space of all constant functions. Hence we have $1_E=1$ μ-a.e., i.e., $\mu(E)=1$. Thus ν and μ are mutually absolutely continuous.

The first assertion follows from the above fact. In fact, assume that ν_1 and ν_2 be two invariant probability measures. Then, by Lemma 3.1, $(\nu_1-\nu_2)_+$ and $(\nu_1-\nu_2)_-$ are both invariant measures and each of them, if it is not equal to 0, is mutually absolutely continuous with respect to μ. But $(\nu_1-\nu_2)_+$ and $(\nu_1-\nu_2)_-$ are mutually singular. Hence one of them is equal to 0 which leads to $\nu_1=\nu_2$. □

REMARK. The above proposition show that the uniqueness holds if we restrict ourselves to probability measures that are absolutely continuous with respect to μ. Hence in finite dimensional case, we can show the uniqueness. But in infinite dimensional case, we could not exclude the possibility of the existence of singular invariant measures. Difficulty lies on the fact that transition probabilities are singular with respect to μ. We can only say that assuming the smoothness of b, the uniqueness holds in the scope of generalized Wiener functionals by the hypoellipticity (see e.g., [9]). But probability measures do not belong to the space of generalized Wiener functionals in general.

Now we proceed to the existence of an invariant measure. To do this, we characterize invariant measures as follows.

Proposition 3.2. *Let ν be a signed measure that is absolutely continuous with respect to μ. We assume that the Radon-Nikodym derivative $\rho=\frac{d\nu}{d\mu}$ belongs to $L^2(B, \mu)$. Then, the following three conditions are equivalent:*

(i) *ν is an invariant measure,*

(ii) $\langle A\phi, \rho\rangle_2=\int_B A\phi(x)\rho(x)\mu(dx)=0, \qquad \phi\in\mathcal{D}$,

(iii) *$\rho\in D(A^*)$ and $A^*\rho=0$.*

Proof. Equivalence of (ii) and (iii) is clear. We first show the implication (i)$\Rightarrow$(ii). Assume that ν is an invariant measure. Then for any $\phi\in\mathcal{D}$, we have

$$\int_B T_t\phi(x)\rho(x)\mu(dx) = \int_B \phi(x)\rho(x)\mu(dx) .$$

By noting that

$$T_t\phi-\phi = \int_0^t T_sA\phi ds ,$$

we have

$$\int_0^t \langle T_sA\phi, \rho\rangle_2 ds = 0 .$$

Hence

$$\langle A\phi, \rho\rangle_2 = \lim_{t\to 0}\frac{1}{t}\int_0^t \langle T_sA\phi, \rho\rangle_2 ds = 0 .$$

Secondly we show the implication (ii)$\Rightarrow$(i). Under the assumption (ii), we have for any $u\in W^{2,2}$,

$$\int_B Au(x)\rho(x)\mu(dx) = 0 .$$

Hence for $\phi\in\mathcal{D}$

$$\int_B T_t\phi(x)\nu(dx)-\int_B \phi(x)\nu(dx) = \langle T_t\phi, \rho\rangle_2-\langle\phi, \rho\rangle_2$$
$$= \int_0^t \langle AT_s\phi, \rho\rangle_2 ds = 0 .$$

This implies that ν is an invariant measure. □

In the remainder of this section, we will establish the existence of the solution to (iii) in Proposition 3.2. First we consider the finite dimensional case. Assume $B=\boldsymbol{R}^n$ and

$$\mu_n(dx) = \left(\frac{1}{2\pi}\right)^{n/2} e^{-|x|^2/2}dx .$$

Then $(\boldsymbol{R}^n, \mu_n)$ is a finite dimensional abstract Wiener space. In this case, c.o.n.s. in $L^2(R^n, \mu_n)$ is constructed as follows. Let H_k, $k\in\boldsymbol{Z}_+$ be Hermite polynomials:

$$H_k(\xi) = \frac{(-1)^k}{k!} e^{\xi^2/2} \frac{d^k}{d\xi^k} e^{-\xi^2/2} \qquad \xi\in\boldsymbol{R} . \tag{3.3}$$

For a multi-index $a=(a_1, a_2, \cdots, a_n)\in\boldsymbol{Z}_+^n$, we define H_a: $\boldsymbol{R}^n\to\boldsymbol{R}$ by

$$H_a(x) = H_{a_1}(x^1)H_{a_2}(x^2) \cdots H_{a_n}(x^n), \quad x = (x^1, x^2, \cdots, x^n)\in\boldsymbol{R}^n . \tag{3.4}$$

Then $\{\sqrt{a!}H_a;\ a\in\boldsymbol{Z}_+^n\}$[2] forms a c.o.n.s. in $L^2(\boldsymbol{R}^n, \mu_n)$. Setting $h_a=\sqrt{a!}H_a$, it holds that

$$\langle u, v\rangle_2 = \sum_{a\in\boldsymbol{Z}_+^n} \langle u, h_a\rangle_2\langle v, h_a\rangle_2 . \tag{3.5}$$

On the other hand, it is well-known that h_a is an eigenfunction of L for an eigenvalue $-|a|=-(a_1+a_2+\cdots+a_n)$ and an inner product in $W^{1,2}$ is given by

$$\langle u, v\rangle_{1,2} = \sum_{a\in\boldsymbol{Z}_+^n} \langle u, h_a\rangle_2\langle v, h_a\rangle_2+\sum_{a\in\boldsymbol{Z}_+^n} |a|\langle u, h_a\rangle_2\langle v, h_a\rangle_2 . \tag{3.6}$$

Of course, this inner product defines a norm $|\cdot|_{1,2}$:

$$|u|_{1,2}^2 = |u|_2^2+|Du|_2^2 = \langle u, u\rangle_{1,2} \tag{3.7}$$

(see e.g., [8]).

Proposition 3.3. *Assume that $B=\boldsymbol{R}^n$ and*

$$\mu_n(dx) = \left(\frac{1}{2\pi}\right)^{n/2} e^{-|x|^2/2}dx . \tag{3.8}$$

Then there exists a non-trivial element $\rho\in D(A^*)$ such that $A^*\rho=0$.

2) $a!=a_1!a_2!\cdots a_n!$

Proof. Take $l\in\boldsymbol{N}$ so that $l\geqq 8\|b\|_\infty^2+1$ and set $K=\frac{1}{8}+\|b\|_\infty^2$. Define a bounded operator $\tilde{A}$: $W^{2,2}\to L^2(\boldsymbol{R}^n, \mu_n)$ by

$$\tilde{A}u(x)=\frac{1}{2}Lu(x)+\langle b(x), Du(x)\rangle_H-\sum_{\substack{a\in Z_+^n\\|a|\leqq l}}K\langle u, h_a\rangle h_a(x) \tag{3.9}$$

and a bilinear form Φ: $W^{1,2}\times W^{1,2}\to\boldsymbol{R}$ by

$$\begin{aligned}\Phi(u, v)=&\frac{1}{2}\int_B\langle Du(x), Dv(x)\rangle_H\mu(dx)-\int_B\langle b(x), Du(x)\rangle_H v(x)\mu(dx)\\&+\sum_{\substack{a\in Z_+^n\\|a|\leqq l}}K\langle u, h_a\rangle_2\langle v, h_a\rangle_2,\quad u, v\in W^{1,2}.\end{aligned} \tag{3.10}$$

Then, for $u\in W^{2,2}$, $v\in W^{1,2}$, it holds that

$$\Phi(u, v)=-\langle\tilde{A}u, v\rangle_2.$$

It is easy to see that there exists a constant $c>0$ such that

$$|\Phi(u, v)|\leqq c|u|_{1,2}|v|_{1,2}. \tag{3.11}$$

Moreover we have

$$\begin{aligned}&\Phi(u, u)\\
&\geqq\frac{1}{2}\int_B\langle Du(x), Du(x)\rangle_H\mu(dx)-\int_B|b(x)|_H|Du(x)|_H|u(x)|\mu(dx)\\
&\quad+\sum_{\substack{a\in Z_+^n\\|a|\leqq l}}K\langle u, h_a\rangle_2^2\\
&\geqq\frac{1}{2}|Du|_2^2-\frac{1}{2}\int_B\Big(\frac{1}{2}|Du(x)|_H^2+2\|b\|_\infty^2|u(x)|^2\Big)\mu(dx)\\
&\quad+\sum_{\substack{a\in Z_+^n\\|a|\leqq l}}K\langle u, h_a\rangle_2^2\\
&=\frac{1}{4}\sum_{a\in Z_+^n}|a|\langle u, h_a\rangle_2^2-\|b\|_\infty^2\sum_{a\in Z_+^n}\langle u, h_a\rangle_2^2+\sum_{\substack{a\in Z_+^n\\|a|\leqq l}}K\langle u, h_a\rangle_2^2\\
&=\frac{1}{8}\sum_{a\in Z_+^n}|a|\langle u, h_a\rangle_2^2+\sum_{\substack{a\in Z_+^n\\|a|\leqq l}}\Big(\frac{1}{8}|a|+K-\|b\|_\infty^2\Big)\langle u, h_a\rangle_2^2\\
&\quad+\sum_{\substack{a\in Z_+^n\\|a|\leqq l}}\Big(\frac{1}{8}|a|-\|b\|_\infty^2\Big)\langle u, h_a\rangle_2^2\\
&\geqq\frac{1}{8}\sum_{a\in Z_+^n}|a|\langle u, h_a\rangle_2^2+\frac{1}{8}\sum_{a\in Z_+^n}\langle u, h_a\rangle_2^2\\
&=\frac{1}{8}|u|_{1,2}^2.\end{aligned} \tag{3.12}$$

By the way, for any $v \in L^2(\boldsymbol{R}^n, \mu_n)$, a linear functional $u \mapsto -\langle v, u\rangle_2$ on $W^{1,2}$ is bounded and hence, by the Lax-Milgram theorem, there exists $w \in W^{1,2}$ so that

$$\Phi(w, u) = -\langle v, u\rangle_2, \qquad u \in W^{1,2}.$$

Therefore, by the same argument as in the proof of Proposition 2.5, we have that $w \in W^{2,2}$ and $\tilde{A}w = v$. Hence $\tilde{A}$ is surjective and moreover injective by (3.12). Thus $\tilde{A}$ is a Fredholm operator with the index 0.

Note that

$$u \mapsto \sum_{\substack{\alpha \in Z_+^n \\ |\alpha| \leq l}} K\langle u, h_\alpha\rangle_2 h_\alpha$$

is a compact operator. Hence, by the stability of the index for Fredholm operators, we have that A is a Fredholm operator with the index 0. But dim Ker$(A) \geqq 1$ because $A\mathbf{1} = 0$ where $\mathbf{1}$ is the function identically equal to 1. Hence codim Im(A) = dim Ker$(A) \geqq 1$ which implies the existence of ρ such that $A^*\rho = 0$. □

Now we proceed to the infinite dimensional case. We will show the existence in this case by limitting procedure. To this end, take a sequence $\{e_i\}_{i=1}^\infty \subseteq B^* \subseteq H$ such that $\{e_i\}$ forms a c.o.n.s. in H. Set

$$f_i(x) = \langle b(x), e_i\rangle_H$$

then we have

$$b(x) = \sum_{i=1}^\infty f_i(x) e_i \qquad \text{in } H.$$

For $n \in \boldsymbol{N}$, let H_n be a linear span of $\{e_1, e_2, \cdots, e_n\}$, $H_n^\perp$ be its orthogonal complement in H and B_n be a closure of $H_n^\perp$ in B. Then

$$B = H_n \oplus B_n \qquad \text{(direct sum)}.$$

We denote the projection to H_n and B_n by π_n and $\pi_n^\perp$ respectively. By the above decomposition, writing as

$$x = \sum_{i=1}^n \xi_i e_i + y, \qquad \xi_i = (x, e_i), \ y \in B_n$$

it holds that

$$\mu(dx) = \left(\frac{1}{2\pi}\right)^{n/2} e^{-|\xi|^2/2} d\xi \times \mu_n(dy), \ \xi = (\xi_1, \xi_2, \cdots, \xi_n)$$

where $\mu_u^\perp$ is the image measure $\mu \circ (\pi_n^\perp)^{-1}$. Clearly $(B_n, H_n^\perp, \mu_n^\perp)$ is an abstract Wiener space.

Further we define projection operator π_n: $L^2(B, \mu) \to L^2(B, \mu)$ by

$$(\pi_n u)(x) = \int_{B_n} u\Big(\sum_{i=1}^n (x, e_i) e_i + y\Big) \mu_n(dy).$$

It is easy to see that π_n is actually a projection operator and $\pi_n \to \mathrm{id}$ strongly as $n\to\infty$. Define

$$b_n(x)=\sum_{i=1}^{n}\pi_n f_i(x)e_i$$

and

$$A_n=\frac{1}{2}L+b_n\,.$$

Then $||b_n||_\infty \leqq ||b||_\infty$, $b_n\to b$ in $L^2(B,\mu;H)$ as $n\to\infty$ and moreover for $u\in W^{2,2}$, $A_n u\to Au$ in $L^2(B,\mu)$ as $n\to\infty$.

For the proof in infinite dimensional case, the following Gross' logarithmic Sobolev inequality is essential (see L. Gross [2]). For $u\in W^{1,2}$,

$$\int_B u(x)^2\log|u(x)|\mu(dx)\leqq\int_B |Du(x)|_H^2\mu(dx)+|u|_2^2\log|u|_2\,. \tag{3.13}$$

Using this inequality, we can prove the following proposition.

Proposition 3.4. *Let $\{u_n\}_{n=1}^\infty$ be a sequence in $W^{1,2}$ such that $\{Du_n\}$ is a bounded sequence in $L^2(B,\mu;H)$ and u_n converges in probability to u where u is a Borel measurable function which is finite a.e. Then $u\in L^2(B,\mu)$ and $u_n\to u$ in $L^2(B,\mu)$ as $n\to\infty$.*

Proof. Taking a subsequence if necessary, we may assume that $u_n\to u$ a.e. It is well-known (see, e.g., [8], [10]) that

$$\int_B\{u_n(x)-\int_B u_n(y)\mu(dy)\}^2\mu(dx)\leqq\int_B|Du_n(x)|_H^2\mu(dx)\,. \tag{3.14}$$

We shall show that $\{\int_B u_n(y)\mu(dy)\}_{n=1}^\infty$ is bounded. Otherwise, there exists a subsequence $\{u_{n'}\}$ such that

$$|\int_B u_{n'}(y)\mu(dy)|\to\infty \qquad \text{as } n'\to\infty\,.$$

Hence by Fatou's lemma, we have

$$\begin{aligned}\infty &= \int_B \lim_{n'\to\infty}\{u_{n'}(x)-\int_B u_{n'}(y)\mu(dx)\}^2\mu(dx)\\ &\leqq \varliminf_{n'\to\infty}\int_B\{u_{n'}(x)-\int_B u_{n'}(y)\mu(dy)\}^2\mu(dx)\end{aligned}$$

which contradicts (3.14). Thus $\{\int_B u_n(y)\mu(dy)\}$ is bounded and hence $\{u_n\}$ is bounded in $L^2(B,\mu)$. Then by Gross' logarithmic Sobolev inequality, we have

$$\int_B u_n(x)^2\log|u_n(x)|\mu(dx)\leqq\int_B|Du_n(x)|_H^2\mu(dx)+|u_n|_2^2\log|u_n|_2\,.$$

Since the right hand side is bounded in n, $\{u_n^2\}$ is uniformly integrable. Now it is easy to see that $u_n \to u$ in $L^2(B, \mu)$. □

Now we can prove the infinite dimensional case.

Proposition 3.5. *There exists a non-trivial element* $\rho \in D(A^*) \subseteq W^{1,2}$ *such that* $A^*\rho = 0$.

Proof. Let b_n, A_n be as above. Then, by Proposition 3.3, there exists $\rho_n \in D(A_n^*)$ such that $A_n^*\rho_n = 0$. By Proposition 3.1 and Theorem 2.1 we may assume that $\rho_n \geqq 0$, $\rho_n \in W^{1,2}$ and $\int_B \rho_n(x)^2 \mu(dx) = 1$. Moreover, since $A_n^*\rho_n = 0$ we get

$$\begin{aligned} 0 &= -\langle \rho_n, A_n^*\rho_n \rangle_2 \\ &= \frac{1}{2}\int_B \langle D\rho_n(x), D\rho_n(x) \rangle_H \mu(dx) - \int_B \langle b_n(x), D\rho_n(x) \rangle_H \rho_n(x) \mu(dx) . \end{aligned}$$

Hence

$$\begin{aligned} |D\rho_n|_2^2 &= \int_B \langle D\rho_n(x), D\rho_n(x) \rangle_H \mu(dx) \\ &= 2\int_B \langle b_n(x), D\rho_n(x) \rangle_H \rho_n(x) \mu(dx) \\ &\leqq 2\|b_n\|_\infty |D\rho_n|_2 |\rho_n|_2 \leqq 2\|b\|_\infty |D\rho_n|_2 |\rho_n|_2 . \end{aligned}$$

Thus we have

$$|D\rho_n|_2 \leqq 2\|b\|_\infty |\rho_n|_2 = 2\|b\|_\infty . \tag{3.15}$$

Furthermore, since $\{\rho_n\}$ is bounded in $L^2(B, \mu)$, there exists a subsequence $\{\rho_{n'}\}$ which converges weakly in $L^2(B, \mu)$. Denoting a weak limit of $\{\rho_{n'}\}$ by ρ, we shall show that $\rho \neq 0$. In fact, if $\rho = 0$, then

$$0 = \lim_{n' \to \infty} \langle \rho_{n'}, 1_B \rangle = \lim_{n' \to \infty} \int_B \rho_{n'}(x) \mu(dx)$$

and noting that $\rho_n \geqq 0$, we have $\rho_n \to 0$ in $L^1(B, \mu)$. Hence $\rho_n \to 0$ in probability. Combining this with (3.15), we can use Proposition 3.4 and have that $\rho_n \to 0$ strongly in $L^2(B, \mu)$ which contradicts $|\rho_n|_2 = 1$. Therefore we have that $\rho \neq 0$. Moreover, for any $u \in W^{2,2}$

$$\langle Au, \rho \rangle_2 = \lim_{n' \to \infty} \langle A_{n'}u, \rho_{n'} \rangle_2 = 0$$

since $A_{n'}u \to Au$ strongly in $L^2(B, \mu)$. Thus $A^*\rho = 0$.

The inclusion $D(A^*) \subseteq W^{1,2}$ is proved in Theorem 2.1. □

Now we have established the following theorem.

Theorem 3.1. *There exists a unique invariant probability measure ν which is absolutely continuous with respect to μ. Moreover, ν and μ are mutually absolutely continuous and the Radon-Nikodym derivative $\rho=\frac{d\nu}{d\mu}$ belongs to $W^{1,2}$.*

4. Symmetry of the semigroup

In this section, we discuss the symmetry of the semigroup. Let notations be same as before. We denote by ν an invariant measure of the diffusion $\{Q_x\}$ which was guaranteed in Theorem 3.1. We also denote by $\rho=\frac{d\nu}{d\mu}$ the Radon-Nikodym derivative. Then the semigroup is called symmetric with respect to ν if

$$(4.1)\qquad \int_B T_t f(x)g(x)\nu(dx) = \int_B f(x)T_t g(x)\nu(dx) \qquad \text{for } f, g\in \mathcal{B}_b(B)\,.$$

We want to know whether the semigroup $\{T_t\}$ is symmetric with respect to ν or not.

Before answering this question, we consider another diffusion $\{Q'_x\}_{x\in B}$ generated by $A'=\frac{1}{2}L+b'$, b' being a bounded vector field. We denote by $\{T'_t\}$, ν' and ρ' the semigroup, the invariant measure and its Radon-Nikodym derivative associated with the diffusion $\{Q'_x\}$ respectively.

Also, let us review differential forms on an abstract Wiener space (see [9] for details). Let $\mathcal{AL}^n_{(2)}(H; \boldsymbol{R})$ be a set of all n-linear functionals on $\underbrace{H\times\cdots\times H}_{n}$ which are alternative and of Hilbert-Schmidt class. We regard an element of $W^{1,2}(\mathcal{AL}^n_{(2)}(H; \boldsymbol{R}))$ as an n-form on B and we denote the exterior derivative and its dual operator by d and d^* respectively. Then we have the following.

Theorem 4.1.

(i) *$\nu=\nu'$ if and only if there exists $\beta\in W^{1,2}(\mathcal{AL}^2_{(2)}(H; \boldsymbol{R}))$ such that*

$$(4.2)\qquad \rho(b-b') = d^*\beta\,.$$

(ii) *It holds that for $f, g\in \mathcal{B}_b(B)$*

$$(4.3)\qquad \int_B T_t f(x)g(x)\nu(dx) = \int_B f(x)T'_t g(x)\nu(dx)$$

if and only if $\log\rho\in W^{1,2}$ and $b+b'=D\log\rho$.

Proof. First we show (i). Assume $\nu=\nu'$. Then for $u\in W^{2,2}$, we have

$$0 = \int_B Au(x)\rho(x)\mu(dx) = \int_B A'u(x)\rho(x)\mu(dx)$$

and

$$\int_B \left(\frac{1}{2}Lu(x)+\langle b(x), Du(x)\rangle_H\right)\rho(x)\mu(dx)$$
$$=\int_B \left(\frac{1}{2}Lu(x)+\langle b'(x), Du(x)\rangle_H\right)\rho(x)\mu(dx) .$$

Hence, since $D=d$ in this case, we obtain

$$\int_B \langle \rho(x)\,(b(x)-b'(x)),\, du(x)\rangle_H \mu(dx) = 0 .$$

This implies that $d^*(\rho(b-b'))=0$. Therefore, by de Rham-Hodge-Kodaira's decomposition (see [9]), there exists $\beta\in W^{1,2}(\mathcal{AL}^2_{(2)}(H;\boldsymbol{R}))$ such that

$$\rho(b-b') = d^*\beta .$$

Conversely, if $\rho(b-b')=d^*\beta$ for some $\beta\in W^{1,2}(\mathcal{AL}^n_{(2)}(H;\boldsymbol{R}))$, then

$$D^*\rho(b-b') = D^*d^*\beta = d^*d^*\beta = 0 .$$

Hence for $u\in W^{2,2}$, we have

$$\int_B \left(\frac{1}{2}Lu(x)+\langle b'(x), Du(x)\rangle_H\right)\rho(x)\mu(dx)$$
$$=\int_B \left(\frac{1}{2}Lu(x)+\langle b(x), Du(x)\rangle_H\right)\rho(x)\mu(dx)$$
$$-\int_B \langle \rho(x)\,(b(x)-b'(x)),\, Du(x)\rangle_H \mu(dx)$$
$$=\int_B D^*\rho(b-b')\,(x)u(x)\mu(dx)$$
$$=0 .$$

Hence $(A')^*\rho=0$ and by the uniquencess of the invariant measure, we have $\rho=\rho'$.

Next we show (ii). Assume (4.3). Then for $u, v\in\mathcal{D}$, we have

$$\langle T_t u-u,\, \rho v\rangle_2 = \langle \rho u,\, T'_t v-v\rangle_2$$

and

$$\langle \frac{1}{t}(T_t u-u),\, \rho v\rangle_2 = \langle \rho u,\, \frac{1}{t}(T'_t v-v)\rangle_2 .$$

Letting $t\to 0$, we get

$$\langle Au,\, \rho v\rangle_2 = \langle \rho u,\, A'v\rangle_2 . \tag{4.4}$$

On the other hand, since $A^*\rho=0$, we have

$$\langle A(uv),\, \rho\rangle_2 = 0 .$$

But it holds that

$$A(uv)(x) = Au(x)v(x)+u(x)Av(x)+\langle Du(x), Dv(x)\rangle_H .$$

Hence we have

$$\langle Au, \rho v\rangle_2+\langle \rho u, Av\rangle_2+\langle \rho, \langle Du, Dv\rangle_H\rangle_2 = 0 .$$

Combining this with (4.4), we have

$$\begin{aligned} 0 &= \langle \rho u, A'v\rangle_2+\langle \rho u, Av\rangle_2+\langle \rho, \langle Du, Dv\rangle_H\rangle_2 \\ &= \langle \rho u, Lv\rangle_2+\langle \rho u, \langle b+b', Dv\rangle_H\rangle_2+\langle \rho, \langle Du, Dv\rangle_H\rangle_2 \\ &= -\langle D(\rho u), Dv\rangle_2+\langle \rho u, \langle b+b', Dv\rangle_H\rangle_2+\langle \rho, \langle Du, Dv\rangle_H\rangle_2 \\ &= -\langle u, \langle D\rho, Dv\rangle_H\rangle_2-\langle \rho, \langle Du, Dv\rangle_H\rangle_2+\langle \rho u, \langle b+b', Dv\rangle_H\rangle_2 \\ &\quad +\langle \rho, \langle Du, Dv\rangle_H\rangle_2 \\ &= \langle u, \rho\langle b+b', Dv\rangle_H-\langle D\rho, Dv\rangle_H\rangle_2 . \end{aligned}$$

Since $\mathcal{D}$ is dense in $L^2(B, \mu)$, we get

$$\langle D\rho, Dv\rangle_H = \rho\langle b+b', Dv\rangle_H .$$

Taking $v(x)=(x, e)$ for $e\in B^*\subseteq H$, we have

$$\langle D\rho(x), e\rangle_H = \rho(x)\langle b(x)+b'(x), e\rangle_H \quad \text{a.e.}$$

Hence

$$D\rho = \rho(b+b') .$$

Now for $n\in \boldsymbol{N}$, set

$$f_n = \log(\rho+\frac{1}{n}) .$$

Then

$$Df_n = \frac{D\rho}{\rho+\frac{1}{n}} = \frac{\rho}{\rho+\frac{1}{n}}(b+b') .$$

Hence $|Df_n|_2\leqq\|b+b'\|_\infty$ and $f_n\to\log\rho$ a.e. as $n\to\infty$. By Proposition 3.4, we have that $\log\rho\in W^{1,2}$ and $D\log\rho=b+b'$.

Next we show the converse. By pursueing above argument conversely, we get, for $u, v\in\mathcal{D}$,

$$\langle Au, \rho v\rangle_2 = \langle \rho u, A'v\rangle_2 . \tag{4.5}$$

But it is easy to see that (4.5) holds for $u\in W^{2,2}$ and $v\in\mathcal{D}$. Hence, noting that $T_t u\in W^{2,2}$ for $u\in\mathcal{D}$, we have for $u, v\in\mathcal{D}$,

$$\langle AT_t u, \rho v\rangle_2 = \langle \rho T_t u, A'v\rangle_2 . \tag{4.6}$$

Similarly (4.6) holds for $u \in \mathcal{D}$ and $v \in W^{2,2}$. Therefore for $u, v \in \mathcal{D}$, it holds that

$$\langle AT_t u, \rho T'_s v\rangle_2 = \langle \rho T_t u, A' T'_s v\rangle_2 .$$

Now for $u, v \in \mathcal{D}$, define

$$g(s) = \langle T_{t-s} u, \rho T'_s v\rangle_2 \qquad 0 \leqq s \leqq t .$$

Differentiating with respect to s, we have

$$g'(s) = -\langle AT_{t-s} u, \rho T'_s v\rangle_2 + \langle T_{t-s} u, \rho A' T'_s v\rangle_2 = 0 .$$

Hence we get $g(t) = g(0)$, i.e.,

$$\langle T_t u, \rho v\rangle_2 = \langle \rho u, T'_t v\rangle_2 .$$

This completes the proof. □

Now we can answer the problem of symmetry. We consider a diffusion $\{Q_x\}_{x \in B}$ generated by $A = \frac{1}{2}L + b$. As before, $\nu(dx) = \rho(x)\mu(dx)$ denotes the invariant measure.

Theorem 4.2.
(i) *$\nu = \mu$ if and only if there exists $\beta \in W^{1,2}(\mathcal{AL}^n_{(2)}(H; \boldsymbol{R}))$ such that*

(4.7) $$b = d^*\beta .$$

(ii) *$\{T_t\}$ is symmetric with respect to ν if and only if there exists $f \in W^{1,2}$ such that*

$$b = Df .$$

Moreover, in that case, it holds that

$$\rho = ce^{2f}$$

where c is a normalizing constant (f has an ambiguity up to constant).

For the proof, we need the following lemma.

Lemma 4.1. *Let f be an element of $W^{1,2}$ such that Df is essentially bounded. Then $e^f \in W^{1,2}$ and $De^f = e^f Df$.*

Proof. Define a C^∞ function $\phi: \boldsymbol{R} \to \boldsymbol{R}$ so that

$$\phi(\xi) = \begin{cases} \xi & \xi \leqq 0 \\ 1 & \xi \geqq 1 \end{cases}$$

and for $n \in \boldsymbol{N}$, $\phi_n(\xi) = n + \phi(\xi - n)$. Set

$$f_n(x) = \phi_n(f(x)).$$

Then $f_n \in W^{1,2}$ and

$$Df_n = \phi_n'(f(x))Df(x).$$

Hence Df_n, $n \in \boldsymbol{N}$ are essentially bounded and

$$\|Df_n\|_\infty \leqq \|\phi_n'\|_\infty \|Df\|_\infty = \|\phi'\|_\infty \|Df\|_\infty .$$

Moreover set

$$u_n = c_n e^{f_n}$$

where c_n is chosen so that $|u_n|_2 = 1$. Then $u_n \in W^{1,2}$ and

$$Du_n = u_n Df_n .$$

Hence

$$|Du_n|_2 \leqq |u_n|_2 \|Df_n\|_\infty \leqq \|\phi'\|_\infty \|Df\|_\infty .$$

On the other hand, since c_n is clearly non-increasing, setting $c = \lim_{n \to \infty} c_n$, we have

$$u_n(x) \to c e^{f(x)} \quad \text{a.e.}$$

Thus we can use Proposition 3.4 and obtain that $ce^f \in L^2(B, \mu)$ and $u_n \to ce^f$ in $L^2(B, \mu)$. We will show $c \neq 0$. If not, then $u_n \to 0$ in $L^2(B, \mu)$. But this contradict $|u_n|_2 = 1$. Thus $c \neq 0$ and $e^f \in L^2(B, \mu)$. Moreover, it is not difficult to see that $e^f \in W^{1,2}$ and $De^f = e^f Df$. □

Proof of Theorem 4.2. (i) is easily obtained from Theorem 3.1 (i). We will show (ii). The sufficiency is also easily obtained from Theorem 3.1 (ii). To show the necessity, set

$$\rho = e^{2f} .$$

Then, by lemma 4.1, $\rho \in W^{1,2}$ and

$$D\rho = 2e^{2f} Df = 2\rho b .$$

Hence, for $u \in W^{2,2}$,

$$\begin{aligned}
\langle Au, \rho \rangle_2 &= \frac{1}{2} \int_B Lu(x)\rho(x)\mu(dx) + \int_B \langle b(x), Du(x) \rangle_H \rho(x)\mu(dx) \\
&= -\frac{1}{2} \int_B \langle Du(x), D\rho(x) \rangle_H \mu(dx) + \int_B \langle b(x), Du(x) \rangle_H \rho(x)\mu(dx) \\
&= -\int_B \langle Du(x), \rho(x)b(x) \rangle_H \mu(dx) + \int_B \langle b(x), Du(x) \rangle_H \rho(x)\mu(dx)
\end{aligned}$$

$$= 0 .$$

Thus $A^*\hat{\rho}=0$. Hence, by the uniqueness of the invariant measure, we have $\rho=c\hat{\rho}$ for some constant $c>0$. Now the rest is easy. □

References

[1] L. Gross: *Abstract Wiener spaces*, Proc. Fifth Berkeley Symp. Math Statist. Prob. II, Part 1, 31–41, Univ. Calif. Press, Berkeley, 1965.

[2] L. Gross: *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), 1061–1083.

[3] N. Ikeda, S. Watanabe: Stochastic differential equations and diffusion processes, Kodansha/North-Holland, Tokyo/Amsterdam, 1981.

[4] N. Ikeda, S. Watanabe: *An introduction to Malliavin's calculus*, Proc. Taniguchi Intern. Symp. on Stochastic Analysis, Katata and Kyoto, 1982, ed. by K. Itô, Kinokuniya, 1984.

[5] A.N. Kolmogoroff, *Zur Umkehrbarkeit der statistischen Naturgesetze*, Math. Ann. **113** (1937), 766–772.

[6] H.-H. Kuo: Gaussian measures in Banach spaces, Lecture Notes in Math. vol. 463, Springer, 1973.

[7] E. Nelson: *The adjoint Markoff process*, Duke Math. J. **25** (1958), 671–690

[8] I. Shigekawa: *Derivatives of Wiener functionals and absolute continuity of induced measures*, J. Math. Kyoto Univ. **20** (1980), 263–289.

[9] I. Shigekawa: *De Rham-Hodge-Kodaira's decomposition on an abstract Wiener space*, J. Math. Kyoto Univ. **26** (1986), 191–202.

[10] H. Sugita: *Sobolev spaces of Wiener functionals and Malliavin's calculus*, J. Math. Kyoto Univ. **25** (1985), 31–48.

[11] K. Yosida: Functional analysis, Springer, New York, 1971.

Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka 560 Japan