Morisugi, K.
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HOMOTOPY GROUPS OF SYMPLECTIC GROUPS AND THE QUATERNIONIC JAMES NUMBERS

KAORU MORISUGI

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0. Introduction

Let $Sp(m)$ $(0\leqq m\leqq\infty)$ be the m-th symplectic group. For convenience we denote $Sp(\infty)$ by Sp. Our purpose is to determine the homotopy groups of $Sp(m)$, $\pi_i(Sp(m))$. If $i<4m+2$ then $\pi_i(Sp(m))$ is isomorphic to $\pi_i(Sp)$. And $\pi_i(Sp)$ is well-known by Bott periodicity. Suppose $i\geqq 4m+2$, then it is not difficult to see that if $i=0$, 1, 3 or 7 mod 8 then $\pi_i(Sp(m))$ is isomorphic to $\pi_{i+1}(Sp/Sp(m))$ and if $i=4$ or 5 mod 8 then $\pi_i(Sp(m))$ is isomorphic to $\pi_{i+1}(Sp/Sp(m))+Z/2$, (direct sum), where $Sp/Sp(m)$ is the factor space of Sp by the subgroup $Sp(m)$. Thus if $i\neq 2$ mod 4 then the calculation of π_i $(Sp(m))$ can be reduced to that of $\pi_{i+1}(Sp/Sp(m))$. In the meta-stable range of i, $4m+2\leqq i\leqq 8m+4$, $\pi_i(Sp/Sp(m))$ is isomorphic to $\pi_i^s(Q_{n,n-m})$ for sufficiently large n, where $Q_{n,n-m}$ is the stunted quaternionic quasi-projective space [8]. And when the value of i-$4m$ is small we can calculate the group $\pi_i^s(Q_{n,n-m})$ (see [15]). On the other hand in the case that $i=2$ mod 4, even if we know the group $\pi_{i+1}(Sp/Sp(m))$ this is not sufficient to determine $\pi_i(Sp(m))$. Let $i=4n-2$. There are two steps in the computation of $\pi_{4n-2}(Sp(m))$; one is to determine the quaternionic James number and the other to solve a certain group extension problem. Let us explain these. Let $X_{n,k}$ be the quaternionic Stiefel manifold of all symplectic k-frames in H^n (n-dimensional vector space over the quaternions H) and let $p\colon X_{n,k}\to S^{4n-1}$ be the bundle projection which associates with each frame its last vector. Then the quaternionic James number $X\{n,k\}$ is defined as the index of $p_*\pi_{4n-1}(X_{n,k})$. Thus $X\{n,1\}=1$, $X\{n,l\}$ divides $X\{n,k\}$ if $l<k$ and, by the classical work of Bott [3], $X\{n,n\}=a(n-1)\cdot(2n-1)!$, where $a(\mathrm{i})=2$ if i is odd and $=1$ if i is even. It is well-known that $X_{n,k}$ is homeomorphic to $Sp(n)/Sp(n-k)$. Let $d(n,m)=X\{n,n\}/X\{n,n-m\}$. Then there exists a short exact sequence (*):

$$0\to \mathrm{Tor}(\pi_{4n-1}(X_{n,n-m}))\xrightarrow{\Delta}\pi_{4n-2}(Sp(m))\xrightarrow{i_*}Z/d(n,m)\to 0\,,$$

where Δ is the restriction to $\mathrm{Tor}(\pi_{4n-1}(X_{n,n-m}))$ (the torsion subgroup of π_{4n-1} $(X_{n,n-m})$) of the boundary homomorphism $\Delta'\colon\pi_{4n-1}(X_{n,n-m})\to\pi_{4n-2}(Sp(m))$ associated with the bundle $Sp(m)\to Sp(n)\to X_{n\,n-m}$, and $i_*\colon \pi_{4n-2}(Sp(m))\to\pi_{4n-2}(Sp(n-1))\cong Z/X\{n,n\}$ ([3][12]) is induced by the inclusion. Note that $\pi_{4n-1}(X_{n,n-m})\cong\pi_{4n-1}(Sp/Sp(m))$. Thus in the case that $i=4n-2(n\geqq m+1)$, in order to reduce the calculation of $\pi_{4n-2}(Sp(m))$ to that of $\pi_{4n-1}(Sp/Sp(m))$ we must determine the number $d(n,m)$ and solve the extension problem for the above short eaxct sequence (*).

Concerning the number $d(n,m)$, there is an upper bound $d^A(n,m)$ of $d(n,m)$, that is, $d(n,m)$ divides $d^A(n,m)$. This upper bound $d^A(n,m)$ is obtained by using KO-theory. Explicitly $d^A(n,m)$ is given by:

$$d^A(n,m)=\underset{s\geqq m+1}{\text{g.c.d.}}\left\{\frac{a(n-1)}{a(n-s)}(2s-1)!M(n,s)\right\},$$

where $M(n,s)$ is an integer defined by the equation:

$$(e^t+e^{-t}-2)^s=\sum_{n\geqq 1}\frac{(2s)!}{(2n)!}M(n,s)t^{2n}.$$

For small values of l, it is known that $d(n,n-l)=d^A(n,n-l)$ [16], [17], [18]. So it seems possible that for all $n\geqq m+1$, $d(n,m)=d^A(n,m)$.

Concerning the extension problem, let $j\colon Sp\to Sp/Sp(m)$ be the projection and $\alpha_n\in\pi_{4n-1}(Sp)\cong Z$ be a generator. Then we see that the sequence (*) splits if and only if $j_*(\alpha_n)$ is divisible by $d(n,m)$ in $\pi_{4n-1}(Sp/Sp(m))$.

Thus if we can show that $j_*(\alpha_n)=d^A(n,m)\beta$ for some $\beta\in\pi_{4n-1}(Sp/Sp(m))$, then we see that $d(n,m)=d^A(n,m)$ and that the sequence (*) splits. For this reason we want to know the divisibility of $j_*(\alpha_n)$ in $\pi_{4n-1}(Sp/Sp(m))$. For this purpose it is convenient to look at p-primary components for each prime p separately. In this paper we are concerned only with 2-primary component.

Let $\nu_2(k)$ be the index of 2 in the prime decomposition of an integer k. For convenience we denote $\nu_2(X\{n,k\})$, $\nu_2(d(n,m))$ and $\nu_2(d^A(n,m))$ by $X_2\{n,k\}$, $d_2(n,m)$ and $d_2^A(n,m)$ respectively. For a space X, $\pi_*(X;2)$ means the 2-primary component of $\pi_*(X)$. Then our main results are as follows.

Theorem I. *Let* $1\leqq m\leqq 3$ *and* $n\geqq m+1$. *Then, the following hold.*

0) $d_2^A(n,1)=2$ *if n is even and*$=0$ *if n is odd.*

$d_2^A(n,2)=3$ *if n is odd,*$=4$ *if* $n=0$ *mod* 4 *and*$=5$ *if* $n=2$ *mod* 4.

$d_2^A(n,3)=4$ *if* $n=1$ *mod* 4,$=5$ *if* $n=3$ *mod* 3 *or if n is even,*$=6$ *if* $n=15$ *mod* 16 *and*$=7$ *if* $n=7$ *mod* 16.

1) $X_2\{n,n-m\}=\nu_2(a(n-1)\cdot(2n-1)!)-d_2^A(n,m)$.

2) $\pi_{4n-2}(Sp(m);2)\cong Tor(\pi_{4n-1}(Sp/Sp(m);2))+Z/2^{d_2^A(n,m)}$ *(direct sum).*

3) *There exists a periodic family* $\alpha_{n,m}\in\pi_{4n-2}(Sp(m);2)$ *such that* $|\alpha_{n,m}|=|i_*\alpha_{n,m}|=2^{d_2^A(n,m)}$, *where* $i\colon Sp(m)\to Sp(n-1)$ *is the inclusion and* $|x|$ *means the order of x.*

The above family $\alpha_{n,m}$ seems to coincide with those studied by Barratt

[2], Mori [13] and Oda [19]. Theorem I is, in a sense, an unstable version of the results in [6], but the methods used in this paper are different from those in our previous papers [6], [7].

Theorem II. *Let* $1\leqq m\leqq 5$ *and* $n\geqq m+1$. *Then,*

1) $X_2\{n,m\}=\nu_2(a(n-1)\cdot(2n-1)!)-d_2^A(n,n-m)$,

2) $\pi_{4n-2}(Sp(n-m);2)\cong \mathrm{Tor}(\pi_{4n-1}(Sp/Sp(n-m);2))+Z/2^{d_2^A(n,n-m)}$ *(direct sum),*

3) *There exists a periodic family* $\alpha_{n,n-m}\in\pi_{4n-2}(Sp(n-m);2)$ *such that* $|\alpha_{n,n-m}|=|i_*\alpha_{n,n-m}|=2^{d_2^A(n,n-m)}$, *where* $i\colon Sp(n-m)\to Sp(n-1)$ *is the inclusion.*

In Theorem II the assertion 1) was already obtained with a few exceptions by Ōshima [16], [17], [18].
This paper is organized as follows. In §1 we investigate some properties of $d(n,m)$ and $d^A(n,m)$. In §2 we study the relation between $\pi_{4n-2}(Sp(m))$ and James numbers. §3 is devoted to the proof of Theorem I. In §4 we prove Theorem II.

This paper was motivated by the works of Walker [22] and Crabb-Knapp [4]. In this respect the author thanks them. The author also thanks M. Imaoka and H. Ōshima for valuable discussions with them.

1. The James numbers

Let $X\{n,k\}$ be the quaternionic James number (see §0) and for $n\geqq m+1$, $d(n,m)=X\{n,n\}/X\{n,n-m\}$. As mentioned in §0, there exists an upper bound $d^A(n,m)$ for the number $d(n,m)$. In this section we give some properties of $d^A(n,m)$ and $d(n,m)$ which are needed in later sections. The contents of this section are very similar to [5]. See [5].

DEFINITION 1.1 [22]. Let $s\geqq 1$ and $n\geqq 1$. Define a number $M(n,s)$ by the following equation:

$$(e^t+e^{-t}-2)^s=\sum_{n\geqq 1}\frac{(2s)!}{(2n)!}M(n,s)t^{2n}.$$

Lemma 1.2 [22]. 1) *The following recursive formula holds:*

$$M(n,s)=M(n-1,s-1)+s^2M(n-1,s).$$

In particular $M(n,s)$ *is an integer,* $M(n,1)=1$, $M(n,n)=1$ *and* $M(n,s)=0$ *if* $n<s$.

2) $$(2s-1)!M(n,s)=s^{2s-1}+\sum_{i=1}^{s-1}(-1)^i\left\{\binom{2s-1}{i}-\binom{2s-1}{i-1}\right\}(s-i)^{2n-1}.$$

DEFINITION 1.3.

1) $$d^A(n,m)=\underset{s\geqq m+1}{\mathrm{g.c.d.}}\left\{\frac{a(n-1)}{a(n-s)}(2s-1)!M(n,s)\right\},$$

where $a(i)=1$ if i is even and $=2$ if i is odd.
2) $d_2^A(n,m)$ is the index of 2 in the prime decomposition of the integer $d^A(n,m)$.

The following propoistion easily follows from [22] or [7].

Propositton 1.4 ([22], [7]). *Let $n\geqq m+1$. Then the integer $d(n,m)$ is a divisor of the integer $d^A(n,m)$.*

Proposition 1.5.
1) $d^A(n,n-1)=a(n-1)\cdot(2n-1)!$ *if* $n>1$.
2) $d^A(n,n-2)=a(n-1)\cdot(2n-1)!(n,24)/24$ *if* $n>2$, *where* $(n,24)$ *is the greatest common divisor of* n *and* 24.
3) *If* $n\geqq m+1$ *and* $(n,m)\neq(2,1)$ *then* $d_2^A(n,m)\leqq 2n-3$ $(d_2^A(2,1)=2)$.
4) *Let* $n\geqq m+1$. *If* $d_2^A(n,m)\geqq b$, *then for any* $k\geqq 1$ $d_2^A(n+k\cdot t(b), m)\geqq b$. *If* $d_2^A(n,m)=b$, *then for any* $k\geqq 1$ $d_2^A(n+k\cdot t(b+1), m)=b$. *Here* $t(b)=\max\{2,2^{b-3}\}$.

Proof. 1) is obvious from Definition 1.3 and Lemma 1.2. 2) follows from the fact that $M(n,n-1)=n(n-1)(2n-1)/6$. Since, by definition, $d_2^A(n,m)\leqq d_2^A(n,n-2)$ if $n\geqq m+2$, 3) follows easily from 2). Now we shall prove 4). It is enough to show that for any $s\geqq m+1$,

$$\frac{a(n-1)}{a(n-s)}(2s-1)!M(n,s)=\frac{a(n+t(b)-1)}{a(n+t(b)-s)}(2s-1)!M(n+t(b), s) \mod 2^b .$$

Note that $\frac{a(n-1)}{a(n-s)}=\frac{a(n+t(b)-1)}{a(n+t(b)-s)}$ because $t(b)$ is even. Unless n is odd and s is even then, $\frac{a(n-1)}{a(n-s)}$ and $\frac{a(n+t(b)-1)}{a(n+t(b)-s)}$ are integers. Thus using the formula 2) of Lemma 1.2 and the fact that for any odd integer l, $l^{2t(b)}=1 \mod 2^b$, we have the desired result. If n is odd and s is even, then $\frac{a(n-1)}{a(n-s)}=1/2$. But in this case the number $(1/2)\left\{\binom{2s-1}{i}-\binom{2s-1}{i-1}\right\}$ is an integer. Since $b\leqq 2n-2$, by a similar argument, we have the desired results. q.e.d.

The proof of the next proposition is long and we shall omit the details. An outline is given the last remark of this section.

Proposition 1.6. *If* $n\geqq m+1$ *then,*

$$d_2^A(n, m)\leqq m(m+1) ,$$

As an immediate corollary we have

Corollary 1.7. 1) *If* $d^A(n,m)=0$ *mod* 2^b *then for any* $k\geqq 1$ $d^A(n+k\cdot t(b),m)=0$ *mod* 2^b.

2) *For a fixed integer m if we regard the integer* $d_2^A(n,m)$ *as a function of* $n(n\geqq m+1)$, *then the function* $d_2^A(n,m)$ *is periodic.*

Proof. 1) is clear from 4) of Proposition 1.5. By Proposition 1.6, the function $d_2^A(n,m)$ has a maximum, say, d. Then using 4) of Proposition 1.5, it is easy to see that $t(d)$ is a period of the function $d_2^A(n,m)$. q.e.d.

Now we shall investigate the number $d(n,m)$. Let $b\geqq 1$. We denote the Moore space $S^1\cup_f e^2$ by M_b, where $f\colon S^1\to S^1$ is a map of degree 2^b. Let π_0: $M_b\to S^2$ be the projection. The following theorem is essentially due to M. Mahowald [9] [10] (See also [5]).

Theorem 1.8. *Let* $\alpha_n\in\pi_{4n-1}(Sp)\cong Z$ *be a generator. If* $b\leqq 2n-3$ *or* $b=n=2$ *then for each* $k\geqq 1$ *there exists a map*

$$^kA_b\colon \Sigma^{4(n+k\cdot t(b))-3}M_b\to\Sigma^{4n-3}M_b,$$

such that the following diagram is homotopy commutative up to units:

$$\begin{array}{ccc}
\Sigma^{4(n+k\cdot t(b))-3}M_b & \xrightarrow{\ ^kA_b\ } & \Sigma^{4n-3}M_b \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\
S^{4(n+k\cdot t(b))-1} & & S^{4n-1} \\
{\scriptstyle \alpha_{n+k\cdot t(b)}}\searrow & & \swarrow{\scriptstyle \alpha_n} \\
& Sp\,. &
\end{array}$$

Corollary 1.9. *Let* $n\geqq m+1$ *and* $m\geqq 1$. *Let* $j\colon Sp\to Sp/Sp(m)$ *be the projection and* $\alpha_n\in\pi_{4n-1}(Sp)$ *be a generator. If* $j_*\alpha_n=2^b\beta$ *for some* $\beta\in\pi_{4n-1}(Sp/Sp(m))$ *then for each* $k\geqq 1$, $j_*\alpha_{n+k\cdot t(b)}=2^b\beta'$ *for some* $\beta'\in\pi_{4(n+k\cdot t(b))-1}(Sp/Sp(m))$.

Proof. The assumption that $j_*\alpha_n=2^b\beta$ for some $\beta\in\pi_{4n-1}(Sp/Sp(m))$ implies that $b\leqq d_2(n,m)$. Since $d_2(n,m)\leqq d_2^A(n,m)\leqq 2n-3$, the rest of the proof is obvious from Theorem 1.8. q.e.d.

Corollary 1.9 is, in a sense, a geometrical realization of 1) of Corollary 1.7.

REMARK. There is a stable version of the James number, that is, the stable James number $X^s\{n,k\}$ can be defined as the index of $p_*\pi_{4n-1}^s(X_{n,k})$ in $\pi_{4n-1}^s(S^{4n-1})$ [16], where $\pi_*^s(\ )$ is the stable homotopy group. The number $X^s\{n,k\}$ is a divisor of $X\{n,k\}$ and is equal to the stable order of the attaching map of the top cell in the stunted quaternionic quasi-projective space $Q_{n,k}$ [16] [8]. Using KO-theory and the Pontrjagin character, we can obtain a lower bound $X^A\{n,k\}$, say, for $X^s\{n,k\}$ ([22], [7]). Then our $d^A(n,m)$ is equal to $X^A\{n,n\}/X^A\{n,n-m\}$. Since $X\{n,n\}=X^s\{n,n\}=X^A\{n,n\}$ ([3], [8], [14]), clearly $d^A(n,m)$ is an upper bound of $d(n,m)=X\{n,n\}/X\{n,n-m\}$.

There is a (-1)-connected spectrum A (Cf. [4]) which is essentially the fiber spectrum of the Adams operation ψ^3-1: $KO(\ )_{(2)}\to KO(\ )_{(2)}$, where $KO(\ )_{(2)}$ is KO-theory localized at 2. Using A-theory, $d^A(n,m)$ can be interpreted as the modulo torsion index of $j_*A_{4n-1}(Q_{n,n})$ in $A_{4n-1}(Q_{n,n-m})$. Then from information about the stable self maps of the infinite quaternionic qusai-projective space [14] we can show that for any $n\geqq m+1$ $d_2^A(n,m)-d_2^A(n,m-1)\leqq 2m$. (Cf. [5]). Thus we have $d_2^A(n,m)\leqq m(m+1)$.

Using a technique similar to that in [5] or [4], it can be shown that if n is sufficiently large, compared with m, then $d_2^A(n,m)=\nu_2(X^s\{n,n\}/X^s\{n,n-m\})$. However, in the unstable case, it is not clear whether $d_2^A(n,m)=d_2(n,m)$ or not.

2. The relation between $d(n,m)$ and $\pi_{4n-2}(Sp(m))$

Let $Sp(k)$ be the k-th symplectic group. Recall that the quaternionic Stiefel manifold $X_{n,k}$ is defined as $Sp(n)/Sp(n-k)$. Throughout this section we always assume that $0<m\leqq n-1$. Consider the commutative diagram:

$$\begin{array}{ccccc} Sp(m) & \longrightarrow & Sp(n) & \xrightarrow{j} & X_{n,n-m} \\ \downarrow{\scriptstyle i} & & \| & & \downarrow{\scriptstyle p} \\ Sp(n-1) & \longrightarrow & Sp(n) & \xrightarrow{p_1} & S^{4n-1}, \end{array}$$

where j,p and p_1 are bundle projections and i and unlabeled maps are inclusions. Applying $\pi_*(\ )$ we have the commutative diagram:

Diagram (**)

$$\begin{array}{ccccccc} & & \pi_l(X_{n-1,n-m-1}) & = & \pi_l(X_{n-1,n-m-1}) & & \\ & & \downarrow & & \downarrow & & \\ \pi_l(Sp(n)) & \xrightarrow{j_*} & \pi_l(X_{n,n-m}) & \xrightarrow{\Delta'} & \pi_{l-1}(Sp(m)) & \longrightarrow & \pi_{l-1}(Sp(n)) \\ \| & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle i_*} & & \| \\ \pi_l(Sp(n)) & \xrightarrow{p_{1*}} & \pi_l(S^{4n-1}) & \xrightarrow{\partial} & \pi_{l-1}Sp(n-1)) & \longrightarrow & \pi_{l-1}(Sp(n)), \end{array}$$

where $l=4n-1$, Δ' and ∂ are the boundary homomorphisms induced by the bundles $Sp(m)\to Sp(n)\xrightarrow{j}X_{n,n-m}$ and $Sp(n-1)\to Sp(n)\xrightarrow{p_1}S^{4n-1}$ respectively and all straight lines are exact. Note that $\pi_{l-1}(Sp(n))\cong 0$, $\pi_l(Sp(n))\cong Z$, $\pi_l(S^{4n-1})\cong Z$ and $\pi_l(X_{n,n-m})\cong Z+\text{Torsion}$ for $l=4n-1$. Since by definition the index of $p_{1*}\pi_l(Sp(n))$ in $\pi_l(S^{4n-1})$ is $X\{n,n\}$ and $X\{n,n\}$ is non-zero, it follows that j_* is a monomorphism. Recall that the James number $X\{n,n-m\}$ is defined as the index of p_* in the above sequence; also recall that $d(n,m)$ is defined as $X\{n,n\}/$

$X\{n,n-m\}$. Then we have

Proposition 2.1. *There exists a short exact sequence:*

$$0 \to \mathrm{Tor}(\pi_{4n-1}(X_{n,n-m})) \overset{\Delta}{\to} \pi_{4n-2}(Sp(m)) \to \mathrm{Im}\, i_* \to 0\,, \tag{2.1}$$

where Δ is the restriction of Δ' to the torsion subgroup of $\pi_{4n-1}(X_{n,n-m})$. And $\mathrm{Im}\, i_*$, *the image of i_*: $\pi_{4n-2}(Sp(m)) \to \pi_{4n-2}(Sp(n-1))$, is isomorphic to the cyclic group $Z/d(n,m)$.*

Proof. The proof follows easily by chasing Diagram (**). q.e.d.

Concerning the splitting of the above short exact sequence we have

Proposition 2.2. *The following are equivalent:*

1) (2.1) *is split.*
2) *There exists an element $\alpha_{n,m} \in \pi_{4n-2}(Sp(m))$ such that $|\alpha_{n,m}| = |i_*\alpha_{n,m}| = d(n,m)$, where $|x|$ is the order of x.*
3) *Let $\alpha_n \in \pi_{4n-1}(Sp(n)) \cong Z$ be a generator. Then $j_*\alpha_n = d(n,m)\beta$ for some $\beta \in \pi_{4n-1}(X_{n,n-m})$.*

It should be noticed that the 2-primary version of the above proposition still holds. This follows easily from the proof of Proposition 2.2 below.

The proof that 1) is equivalent to 2) is clear. For the proof that 2) is equivalent to 3), we need

Lemma 2.3. *Let a be an integer. Then the following are equivalent:*

1) *$j_*\alpha_n = a \cdot \beta$ for some $\beta \in \pi_{4n-1}(X_{n,n-m})$.*
2) *There exists an element α such that $|\alpha| = |i_*\alpha| = a$.*

Proof. Let $g \in \pi_{4n-1}(S^{4n-1})$ be a generator and y be a generator of the free part of $\pi_{4n-1}(X_{n,n-m}) \cong Z + \mathrm{Torsion}$. If $j_*\alpha_n = a \cdot \beta$ for some $\beta \in \pi_{4n-1}(X_{n,n-m})$ then $a\, p_*\beta = \pm X\{n,n\}\, g$. Thus $X\{n,n\}/a$ is an integer and $p_*\beta = \pm \frac{X\{n,n\}}{a} g$. Put $\alpha = \Delta'(\beta)$. Then clearly α is of order a. Since $i_*\alpha = i_*\Delta'(\beta) = \partial(p_*\beta) = \pm \frac{X\{n,n\}}{a} \partial(g)$ and since $\partial(g)$ is a generator of the cyclic group $\pi_{4n-2}(Sp(n-1) \cong Z/X\{n,n\}$, therefore $i_*\alpha$ is also of order a. Thus 1) implies 2). Suppose that $j_*\alpha_n = d(n,m)y + \gamma$ for y (a generator of the free part) and γ (a torsion element). Then if the statement 2) holds, clearly a is a divisor of $d(n,m)$. Note that $p_*y = \pm X\{n,n-m\}\, g$. By chasing the Diagram (**) it is easy to see that $i_*(\frac{d(n,m)}{a} \Delta' y)$ is of order a. Thus if the statement 2) holds then there exists an element α' of order a such that $i_*\alpha' = i_*(\frac{d(n,m)}{a} \Delta' y)$. Therefore by the exact sequence (2.1) we have that $\frac{d(n,m)}{a}\Delta' y=\alpha'+\Delta'\gamma'$ for some $\gamma'\in \mathrm{Tor}(\pi_{4n-1}(X_{n,n-m}))$. On the other hand since $-\Delta'\gamma=a(\frac{d(n,m)}{a}\Delta' y)=a(\alpha'+\Delta'\gamma')=\Delta'(a\gamma')$ and Δ' is monomorphic on the torsion subgroup, $\mathrm{Tor}(\pi_{4n-1}(X_{n,n-m}))$, thus we see that $\gamma=-a\gamma'$. Therefore 1) holds. q.e.d.

Proposition 2.4. *Suppose $i\geqq 4m+2$. If $i=0,1,3$ or 7 mod 8 then $\pi_i(Sp(m))$ is isomorphic to $\pi_{i+1}(Sp/Sp(m))$ and if $i=4$ or 5 then $\pi_i(Sp(m))$ is isomorphic to $\pi_{i+1}(Sp/Sp(m))+Z/2$ (direct sum).*

Proof. Consider the exact sequence;

$$\cdots \to \pi_{i+1}(Sp) \overset{j_*}{\to} \pi_{i+1}(Sp/Sp(m)) \overset{\Delta}{\to} \pi_i(Sp(m)) \to \pi_i(Sp) \to \cdots$$

If $i=4$ or 5 mod 8 then $\pi_i(Sp)\cong Z/2$. Note that $Sp(1)$ is homeomorphic to S^3. As is well-known there exist periodic families $\mu_k\in\pi_{8k+4}(S^3)$ and $\mu_k\eta\in\pi_{8k+5}(S^3)$ which are of order 2 and detected by d-invariant of KO-theory [1]. This implies that if $i=4$ or 5 mod 8 then $\pi_i(Sp)\cong Z/2$ is a direct summand of $\pi_i(Sp(m))$. Thus if $i=4$ mod 8 then Δ is monomorphic. If $i=5$ mod 8 then, since $\pi_{i+1}(Sp)=0$, Δ is monomorphic. Thus if $i=4$ or 5 mod 8 then we have the desired result. The other cases follow from the well-known structure of $\pi_i(Sp)$ and the cases $i=4$ or 5 mod 8. q.e.d.

Combining Corollary 1.9 and a 2-primary version of Proposition 2.2 we have the following theorem.

Theorem 2.5. *Let $n\geqq m+1$ and $b=d_2^A(n,m)$. If $j_*\alpha_n$ is divisible by 2^b in $\pi_{4n-1}(X_{n,n-m})$ or if there exists an element $\alpha\in\pi_{4n-2}(Sp(m))$ such that $|\alpha|=|i_*\alpha|=2^b$, and if $d_2^A(n+k\cdot t(b),m)=b$ for some k, then $d_2(n+k\cdot t(nb),m)=b$ and $\pi_{4(n+k\cdot t(b))-2}(Sp(m);2)$ is isomorphic to the direct sum of* Tor $(\pi_{4(n+kt(b))-1}(Sp/Sp(m);2))$ *and $Z/2^b$.*

3. Proof of Theorem I

This section is devoted to the proof of Theorem I in §0. Throughout this section we use the following notation. α_k is a generator of $\pi_{4k-1}(Sp)$ and j_m: $Sp\to Sp/Sp(m)$ is the projection. $a(k)=1$ if k is even and $=2$ if k is odd. Let $t(b)=\max\{2, 2^{b-3}\}$. We denote the greatest common divisor of integers k and l by (k,l).

The following proposition is well-known (for example, [12]).

Proposition 3.1. *Let $n\geqq 1$. Then,*

1) $j_{n*}(\alpha_{n+1})=d^A(n+1,n)\cdot g_n$, *where g_n is a generator of $\pi_{4n+3}(Sp/Sp(n))\cong Z$ and* $d^A(n+1,n)=a(n)\cdot(2n+1)!$,

2) $j_{n*}(\alpha_{n+2})=d^A(n+2,n)\cdot g'$, *where* g' *is a generator of* $\pi_{4n+7}(Sp/Sp(n))\cong Z$ *and* $d^A(n+2,n)=a(n+1)\cdot(2n+3)!(n+2,24)/24$.

Lemma 3.2.

$$d_2^A(n,1)=\begin{cases}0 & \text{if } n \text{ is odd,}\\ 2 & \text{if } n \text{ is even.}\end{cases}$$

Proof. From the recursive formula of Lemma 1.2 it is easy to see that $M(n,2)$ is odd if $n\geqq 2$. Thus $\nu_2(\frac{a(n-1)}{a(n-2)}3!M(n,2))=2$ if n is even and $=0$ if n is odd. On the other hand, since $M(n,s)$ is an integer, $\nu_2(\frac{a(n-1)}{a(n-s)}(2s-1)!M(n,s))$ $\geqq\nu_2(\frac{a(n-1)}{a(n-s)}(2s-1)!)$. It is clear that if $s\geqq 3$ then $\nu_2(\frac{a(n-1)}{a(n-s)}(2s-1)!)\geqq 3$. Therefore we have

$$\begin{aligned}d_2^A(n,1)&=\min_{s\geqq 2}\{\nu_2(\frac{a(n-1)}{a(n-s)}(2s-1)!M(n,s))\}\\ &=\nu_2(\frac{a(n-1)}{a(n-2)}3!M(n,2))\\ &=\begin{cases}0 & \text{if } n \text{ is odd,}\\ 2 & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

q.e.d.

Theorem 3.3. $j_{1*}\alpha_n=2^{d_2^A(n,1)}\beta$ *for some* $\beta\in\pi_{4n-1}(Sp/Sp(1))$.

Proof. From 1) of Proposition 3.1, it is obvious that $j_{1*}\alpha_2$ is divisible by 4. Then, by Corollary 1.9, for any $k\geqq 1$, $j_{1*}\alpha_{2+2k}$ is divisible by 4. Thus from Lemma 3.2 we have the desired result. q.e.d.

Lemma 3.4. *Let* $n\geqq 3$. *Then,*

$$d_2^A(n,2)=\begin{cases}3 & \text{if } n \text{ is odd,}\\ 4 & \text{if } n=0 \bmod 4,\\ 5 & \text{if } n=2 \bmod 4.\end{cases}$$

Proof. By the use of the formula 2) of Lemma 1.2 and by simple arithmetic (I used a computer.), we see that $d_2^A(3,2)=3$, $d_2^A(4,2)=4$, and $d_2^A(6,2)=d_2^A(10,2)$ $=5$. Then, by 4) of Proposition 1.5, it follows that for any $k\geqq 1$, $d_2^A(3+k\cdot t(4),2)$ $=d_2^A(3,2)$. Thus we obtain that $d_2^A(3+2k,2)=3$. Similarly, since $d_2^A(4+k\cdot t(5),2)$ $=d_2^A(4,2)$, we obtain that $d_2^A(4+4k,2)=4$. Since $d_2^A(6,2)=5$, it follows that $d_2^A(6+8k,2)=5$ for any $k\geqq 1$. Since $d_2^A(10,2)=5$, $d_2^A(10+8k,2)=5$. Therefore, for any $k\geqq 1$, $d_2^A(6+4k,2)=5$. q.e.d.

Lemma 3.5. *There exists an element* $\beta\in\pi_{22}(Sp(2);2)$ *such that both* β *and* $i_*\beta$ *are of order* 2^5, *where* $i:Sp(2)\to Sp(4)$ *is the inclusion.*

Proof. Consider the homotopy exact sequence associated with the bunlde $Sp(1) \xrightarrow{i'} Sp(2) \xrightarrow{p} S^7$, where i' is the inclusion and p is the projection:

$$\pi_{22}(Sp(1)) \xrightarrow{i'_*} \pi_{22}(Sp(2)) \xrightarrow{p_*} \pi_{22}(S^7).$$

According to [11], there exists an element $[\rho'']\in\pi_{22}(Sp(2); 2)$ such that $8[\rho'']=i'_*\bar{\mu}'$, $\bar{\mu}'$ is of order 4 and $e_R(\bar{\mu}')=1/4$ in $Q/Z_{(2)}$, where e_R is the Adams e-invariant, Q is the rational number field and $Z_{(2)}$ is the integers localized at (2). Thus clearly $[\rho'']$ is of order 2^5. In order to show that $i_*[\rho'']$ is still of order 2^5, consider the commutative diagram:

$$\begin{array}{ccccc} \pi_{22}(Sp(1)) & \xrightarrow{i'_*} & \pi_{22}(Sp(2)) & \xrightarrow{i_*} & \pi_{22}(Sp(4)) \\ \downarrow\theta & & \downarrow\theta & & \downarrow\theta \\ \pi^s_{22}(Q_1) & \xrightarrow{i'_*} & \pi^s_{22}(Q_2) & \xrightarrow{i_*} & \pi^s_{22}(Q_4), \end{array}$$

where Q_n is the quaternionic quasi-projective space of dim $4n-1$ and $\theta: Sp(n)\to \Omega^\infty\Sigma^\infty Q_n$ is James's stable splitting [8]. Let $\tilde{x}\in\widetilde{KO}^3(Q_n)$ be the element which corresponds to $1\in KO^0(HP^{n-1})$ under the Thom isomorphism. (Q_n in the Thom space of a certain Spin(3) bundle.) Let f: $S^{4(n+k)-2}\to Q_n$ be a stable map for some $k\geqq 1$. Then the e-invariant $e(f)$: $KO^{4s-1}(Q_n)\to Q/Z$ can be defined as the functional Pontrjagin character ([1], [21], [22]). Then $e_R(\bar{\mu}')=1/4$ implies that $e(\theta\bar{\mu}')(\tilde{x})=1/4$. By naturality and additivity of the e-invariant, it follows that $e(\theta[\rho''])(\tilde{x})=1/32$. Thus, by naturality again, we see that $e(\theta i_*[\rho''])(\tilde{x})=1/32$. Thus $i_*[\rho'']$ is still of order 2^5. q.e.d.

Theorem 3.6. *$j_{2*}\alpha_n=2^{d_2^A(n,2)}\beta$ for some $\beta\in\pi_{4n-1}(Sp/Sp(2))$.*

Proof. From 1) Proposition 3.1 it is clear that $j_{2*}\alpha_3$ is divisible by 5. In particular, $j_{2*}\alpha_3$ is divisible by 2^3. Thus, by Corollary 1.9, for any $k\geqq 1$ $j_{2*}\alpha_{3+2k}$ is divisible by 2^3. Therefore in the case that n is odd, by Lemma 3.4 we have the desired result. From 2) of Proposition 3.1, since $\nu_2(d^A(4,2))=4$, $j_{2*}\alpha_4$ is divisible by 2^4. Since $t(4)=2$, using Corollary 1.9, we see that for any $k\geqq 1$ $j_{2*}\alpha_{4+2k}$ is divisible by 2^4. Therefore if $n=0$ mod 4 then we have the desired result. For the case that $n=2$ mod 4, using Lemmas 3.5 and 2.3, we see that $j_{2*}\alpha_6$ is divisible by 2^5. Since $t(5)=4$, using Corollary 1.9 we have the result. q.e.d.

Lemma 3.7.

$$d_2^A(n,3)=\begin{cases} 4 & \textit{if } n=1 \textit{ mod } 4,\\ 5 & \textit{if } n=3 \textit{ mod } 8 \textit{ or } n \textit{ is even},\\ 6 & \textit{if } n=15 \textit{ mod } 16,\\ 7 & \textit{if } n=7 \textit{ mod } 16.\end{cases}$$

Proof. By the use of the recursive formula 1) of Lemma 1.2 it is easy to see that if n is even and $\geqq 4$ then $M(n, 4)$ is odd. Thus if n is even then $\nu_2(\frac{a(n-1)}{a(n-4)}7!M(n,4))=\nu_2(2\cdot 7!)=5$. If $s\geqq 5$, then $\nu_2(\frac{a(n-1)}{a(n-s)}(2s-1)!M(n, s))\geqq \nu_2(\frac{a(n-1)}{a(n-s)}(2s-1)!)\geqq 6$. Thus, if n is even then $d_2^A(n,3)=5$. By using techniques like those in the proof of Lemma 3.4, other cases follow from the facts that $d_2^A(5, 3)=4$, $d_2^A(7, 3)=d_2^A(23, 7)=7$, $d_2^A(11, 3)=5$ and $d_2^A(15, 3)=6$. These facts were verified by computer. Details are omitted. q.e.d.

Lemma 3.8. *$j_{3*}\alpha_7=2^7\beta$ for some $\beta\in\pi_{27}(Sp/Sp(3))$.*

Proof. For convenience, in this proof we localize all spaces at (2). So homotopy groups should be considered as 2-local groups. Let β_k be a generator of the free part of $\pi_{27}(Sp/Sp(k))\cong Z_{(2)}+2$-Torsion. Remark that there are several choices of β_k. Let j': $Sp/Sp(2)\to Sp/Sp(3)$ be the canonical projection. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_{27}(Sp) & = & \pi_{27}(Sp) & & \\
\downarrow{\scriptstyle j_{2*}} & & \downarrow{\scriptstyle j_{3*}} & & \\
\pi_{27}(Sp/Sp(2)) & \xrightarrow{j'_*} & \pi_{27}(Sp/Sp(3)) & \xrightarrow{\partial} & \pi_{26}(S^{11}) \\
\downarrow{\scriptstyle \Delta'} & & \downarrow & & \| \\
\pi_{26}(Sp(2)) & \longrightarrow & \pi_{26}(Sp(3)) & \longrightarrow & \pi_{26}(S^{11}),
\end{array}$$

where all straight sequences are exact. Since $2^{d_2(7,3)}$ is the modulo torsion index of j_{3*}, from the fact [16] that $d_2(7,3)=7$ and $d_2(7,2)=3$, it follows that for any choice of β_2 there is a choice of β_3 such that

$$j'_*(\beta_2) = 2^4\beta_3+t\,,$$

where t is a 2-torsion element of $\pi_{27}(Sp/Sp(3))$. Since $2^4\pi_{26}(S^{11})=0$[20], it follows that $\partial(t)=0$. Thus, there exists an element $t'\in\mathrm{Tor}(\pi_{27}(Sp/Sp(2))$ such that $j'_*t'=t$. On the other hand it is known that $2^3\pi_{26}(Sp(2))=0$ [19]. Since Δ: $\mathrm{Tor}(\pi_{27}(Sp/Sp(2))\to\pi_{26}(Sp(2))$ is monomorphic, it follows that $2^3t=2^3j'_*t'=j'_*(2^3t')=0$. Therefore, from Theorem 3.6, $j_3\alpha_7=j'_*(j_{2*}\alpha_7)=j'_*(2^3\beta_2)=2^3(2^4\beta_3+t)=2^7\beta_3$. This completes the proof. q.e.d.

Theorem 3.9. *$j_{3*}\alpha_n=2^{d_2^A(n,3)}\beta$ for some $\beta\in\pi_{4n-1}(Sp/Sp(3))$.*

Proof. Since $\nu_2(d^A(4, 3))=5$ and $t(5)=4$, using Proposition 3.1 and Corollary 1.9 we see that $j_{3*}\alpha_{4+4k}$ is divisible by 2^5 for any $k\geqq 1$. Since $j_{2*}\alpha_6$ is divisible by 2^5 in $\pi_{23}(Sp/Sp(2))$, it is clear that $j_{3*}\alpha_6$ is divisible by 2^5 in $\pi_{23}(Sp/Sp(3))$. This implies that if n is even then $j_{3*}\alpha_n$ is divisible by 2^5 in $\pi_{4n-1}(Sp/Sp(3))$. By Lemma 3.8 it is obvious that $j_{3*}\alpha_7$ is divisible by 2^l for $5\leqq l\leqq 7$. Thus $j_{3*}\alpha_{7+k\cdot t(l)}$ is divisible by 2^l for any $k\geqq 1$. Since $t(5)=4$, $t(6)=8$ and $t(7)=16$, we see that if $n=3$ mod 8 then $j_{3*}\alpha_n$ is divisible by 2^5, if $n=15$ mod 16 then $j_{3*}\alpha_n$ is divisible by 2^6 and if $n=7$ mod 16 then $j_{3*}\alpha_n$ is divisible by 2^7. Similarly we see that if $n=1$ mod 4 then $j_{3*}\alpha_n$ is divisible by 2^4 in $\pi_{4n-1}(Sp/Sp(3))$. Thus, using Lemma 3.7 we have completed the proof. q.e.d.

Now the proof of Theorem I is clear. By Theorems 3.3, 3.6 and 3.9, $d_2(n,m)=d_2^A(n,m)$ for $m=1,2$ or 3. Thus, using Proposition 2.2, we have the desired results.

4. The proof of Theorem II

This section is devoted to the proof of Theorem II. The next theorem follows from [16], [17] and [18]. The ambiguity for $m=5$, mentioned in [16], is removed. Details will appear in the forthcoming paper [15].

Theorem 4.1. *Let* $1\leqq m\leqq 5$ *and* $n\geqq m+1$. *Then,*

$$d_2(n, n-m) = d_2^A(n, n-m)\,.$$

Proposition 4.2. *Let* $1\leqq m\leqq 5$. *Then for any* $n\geqq m+1$,

$$2^3\mathrm{Tor}(\pi_{4n-1}(Sp/Sp(n-m); 2) = 0\,.$$

Proof. If $m=1$ or 2 then $\pi_{4n-1}(Sp/Sp(n-m))$ is torsion free. Thus the assertion is clear. Now consider the exact sequence:

$$\pi_{4n-1}(S^{4(n-m)+3}) \to \pi_{4n-1}(Sp/Sp(n-m)) \to \pi_{4n-1}(Sp/Sp(n-m+1))\,.$$

Let $m=3$. Then $\pi_{4n-1}(S^{4n-9}; 2)$ is isomorphic to $Z/2+Z/2+Z/2$ if $n=4$ and to $Z/2+Z/2$ if $n\geqq 5$ [20]. Thus, from the above exact sequence, it follows that $2\mathrm{Tor}(\pi_{4n-1}(Sp/Sp(n-3); 2))=0$. Now let $m=4$. Since $\pi_{4n-1}(S^{4n-13}; 2)$ is isomorphic to $Z/2$ if $n=6$ and to zero if $n\geqq 5$ and $n\neq 6$ [20], using the exact sequence for $m=4$ we obtain that $2^2\mathrm{Tor}(\pi_{4n-1}(Sp/Sp(n-4); 2))=0$. Similarly using the structure of $\pi_{4n-1}(S^{4n-17}; 2)$[20], we see that $2^3\mathrm{Tor}(\pi_{4n-1}(Sp/Sp(n-5); 2))=0$. q.e.d.

Proof of Theorem II. For convenience we localize all spaces at (2). Thus homotopy groups are always (2)-local groups. By Theorem 3.6, $j_{2*}\alpha_n=2^k\beta_2$, where $k=d_2^A(n,2)$, β_2 is a generator of the free part of $\pi_{4n-1}(Sp/Sp(2))\cong Z_{(2)}+2$-torsion. Let $j'\colon Sp/Sp(2)\to Sp/Sp(n-m)$ be the canonical projection. Then clearly if $n\geqq m+2$ then $j_{n-m}=j'\circ j_2$. Let $k'=d_2^A(n,n-m)$ and $k=d_2^A(n,2)$. Since $2^{d_2(n,n-m)}$ is the modulo torsion index of $j_{n-m*}\colon \pi_{4n-1}(Sp)\to\pi_{4n-1}(Sp/Sp(n-m))$, Theorem 4.1 implies that $j'_*\beta_2=2^{k'-k}\beta_{n-m}+t$, where t is a 2-torsion element of

$\pi_{4n-1}(Sp/Sp(n-m))$ and β_{n-m} is a generator of the free part of $\pi_{4n-1}(Sp/Sp(n-m)) \cong Z_{(2)}+2$-torsion. Then $j_{n-m*}\alpha_n=j'_*(j_{2*}\alpha_n))=j'_*(2^k\beta_2)=2^{k'}\beta_{n-m}+2^k t$. Since $k \geqq 3$, by Proposition 4.2 $2^k t=0$. Therefore we have $j_{n-m*}\alpha_n=2^{k'}\beta_{n-m}$. The rest of the proof is similar to that of Theorem I. This completes the proof of Theorem II.

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Department of Mathematics
University of Wakayama
Wakayama 640, Japan