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ON MAXIMAL SUBMODULES OF A FINITE DIRECT SUM OF HOLLOW MODULES V

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Throughout this note, R is a ring with identity, J the Jacobson radical of R, and all R-modules are unitary right R-modules. Further we let e be a local idempotent of R and U_R a submodule of eJ such that eR/U is of finite length. Put $D:=eRe/eJe$ and $D(U):=\{x+eJe\in D\,|\,x\in eRe,\ xU\leq U\}$. Then $D(U)$ is a division subring of D. In [3]–[6], we have given a relationship between the dimension $[D: D(U)]_r$ of D as a *right* $D(U)$-vector space and the property (**, n) of maximal submodules of $(eR/U)^{(n)}$ defined there. (The dual result had been obtained in [1, Proposition 2.1] from another point of view.)

In this short note, we shall study the dimension $[D: D(U)]_l$ of D as a *left* $D(U)$-vector space and give it a meaning. Originally our considerations had been restricted to the case of uniform modules of Loewy length 2 over an artinian ring with proofs along the line of Sumioka [8, Lemma 5.3], and later by different proofs we generalized and dualized to get the present form. Hence it should be noted that by dualizing the arguments all the parallel results hold for uniform modules if we assume that $[D: D(U)]_l<\infty$.

In what follows, we denote by $|M|$ and by $\sharp I$ the composition length of each R-module M and the cardinality of each set I, respectively. Let L be a submodule of an R-module M. Then we say that L is a *characteristic* submodule of M if $fL\leq L$ for every endomorphism f of M. By $M^{(I)}$ we denote the direct sum of $\sharp I$ copies of M for each R-module M and each set I. We regard $Me=\mathrm{Hom}_R(eR, M)$ for every R-module M by identifying each $t\in Me$ with the map $eR\to M$ defined via $x\mapsto tx$ for each $x\in eR$. So in particular for each $t\in eRe=\mathrm{End}_R(eR)$, $t^{-1}U$ means the inverse image $\{x\in eR\,|\,tx\in U\}$ of U under t.

Now the canonical epimorphism π: $eR\to eR/U$ induces a monomorphism $\mathrm{End}_R(eR/U)\to\mathrm{Hom}_R(eR, eR/U)$ by which we regard $\mathrm{End}_R(eR/U)=\{f\in\mathrm{Hom}_R(eR, eR/U)\,|\,f(U)=0\}\leq\mathrm{Hom}_R(eR, eR/U)$. Consider the epimorphism

$$\delta\colon \mathrm{Hom}_R(eR, eR/U)\to\mathrm{Hom}_R(eR, eR/eJ)=eRe/eJe$$

induced from the canonical epimorphism $eR/Ue\to R/eJ$ by the exact functor

$\mathrm{Hom}_R(eR, -)$. Then δ induces a ring homomorphism $\delta_U:=\delta|_{\mathrm{End}_R(eR/U)}$: $\mathrm{End}_R(eR/U)\to eRe/eJe$ and a ring isomorphism $\mathrm{Im}\,\delta_U\cong\mathrm{End}_R(eR/U)/K_U$ by which we identify these rings, where $K_U:=\mathrm{Ker}\ \delta_U=\{f\in\mathrm{End}_R(eR/U)\,|\,\mathrm{Im} f\leq eJ/U\}$ is the Jacobson radical of the local ring $\mathrm{End}_R(eR/U)$. Hence $\mathrm{Im}\,\delta_U$ is a division subring of eRe/eJe. Putting $D:=eRe/eJe$ and $D(U):=\mathrm{Im}\,\delta_U$, we examine the dimension $[D\colon D(U)]_l$ of D as a *left* $D(U)$-vector space. Note that this definition of $D(U)$ coincides with that defined in the introduction. Observe that the left module structure of $\mathrm{Hom}_R(eR, eR/U)$ over $\mathrm{End}_R(eR/U)$ induces that of $\overline{\mathrm{Hom}}_R(eR, eR/U):=\mathrm{Hom}_R(eR, eR/U)/K$ over $\mathrm{End}_R(eR/U)/K_U=D(U)$ where $K:=\mathrm{Ker}\ \delta=\{f\in\mathrm{Hom}_R(eR, eR/U)\,|\,\mathrm{Im}\, f\leq eJ/U\}$. Then δ induces an isomorphism

$$\sharp)\quad {}_{D(U)}\overline{\mathrm{Hom}}_R(eR, eR/U)\cong{}_{D(U)}D$$

which plays a basic role in our study. By $\bar{f}$ we denote the coset of f in $\overline{\mathrm{Hom}}_R(eR, eR/U)$ for each $f\in\mathrm{Hom}_R(eR, eR/U)$. Now the isomorphism $\sharp$) tells us that $[D\colon D(U)]_l=n$ iff there exists a map $f=(f_i)^T_{i\in I}\colon eR\to(eR/U)^{(I)}$ satisfying the following condition ($\sharp$) in case $\sharp I=n$ but not in case $\sharp I=k$ for any $k<n$.

($\sharp$) For each $g\colon eR\to eR/U$, there exists $h=(h_i)_{i\in I}\colon (eR/U)^{(I)}\to eR$ such that $\bar{g}=\sum_{i\in I}\bar{h}_i\bar{f}_i$, i.e. $(g-hf)(eR)\leq eJ/U$.

For, ($\sharp$) is equivalent to saying that the set $\{\bar{f}_i\}_{i\in I}$ generates ${}_{D(U)}\overline{\mathrm{Hom}}_R(eR, eR/U)$. We shall relate this condition with an inner structure of eR under suitable assumptions.

Lemma 1. *Let I be a set, $t_i\in eRe$ for each $i\in I$ and $t\in eRe$. Put $s_i:=\pi t_i$ and $s:=\pi t\colon eR\to eR/U$. Consider the following conditions.*

(a) $\bar{s}\in\sum_{i\in I}D(U)\,\bar{s}_i$.

(b) $\bigcap_{i\in I}t_i^{-1}U\leq t^{-1}U$.

Then

(1) *If $(eJe)U\leq U$ and $t_l\notin eJe$ for some $l\in I$, then* (a) *implies* (b).

(2) *If eR/U is quasi-injective, then* (b) *implies that $s\in\sum_{i\in I}\mathrm{End}_R(eR/U)\,s_i$ whence also implies* (a).

Proof. (1) By (a), $s=\sum_{i\in I}r_i s_i+k$ for some $(r_i)_{i\in I}\colon (eR/U)^{(I)}\to eR/U$ and some $k\colon eR\to eJ/U$. By the projectivity of eR, $k=\pi j$ for some $j\in eJe$. So $t^{-1}U=\mathrm{Ker}\, s\geq\bigcap_{i\in I}\mathrm{Ker}\, s_i\cap\mathrm{Ker}\, k=(\bigcap_{i\in I}t_i^{-1}U)\cap j^{-1}U$. Here since $jt_l^{-1}\in eJe$, $(jt_l^{-1})U\leq U$ by assumption. Thus $t_l^{-1}U\leq j^{-1}U$. Hence $t^{-1}U\geq\bigcap_{i\in I}t_i^{-1}U$.

(2). Put $s_I:=(s_i)^T_{i\in I}\colon eR\to(eR/U)^{(I)}$. It follows from (b) that $\bigcap_{i\in I}\mathrm{Ker}\, s_i\leq\mathrm{Ker}\, s$. So there is some $r\colon \mathrm{Im}\, s_I\to eR/U$ which makes the diagram (without the broken arrow)

$$\begin{tikzcd} eR \arrow[r, "s_I"] \arrow[d, "s"'] & \operatorname{Im} s_I \arrow[r, hook] \arrow[dl, "r"] & (eR/U)^{(I)} \arrow[dll, dashed, "q"] \\ eR/U & & \end{tikzcd}$$

commutative. Since eR/U is quasi-injective, eR/U is $(eR/U)^{(I)}$-injective by Azumaya, Mbuntum and Varadarajan [2, Proposition 1.16 (2)]. Hence we have a homomorphism $q=(q_i)_{i\in I}$: $(eR/U)^{(I)}\to eR/U$ which completes the commutative diagram above. Thus $s=\sum_{i\in I} q_i s_i\in\sum_{i\in I}\operatorname{End}_R(eR/U)\, s_i$.

We put $U^*=\bigcap_{t\in eRe} t^{-1}U$. Then as easily seen, U^* is the largest characteristic submodule of eR contained in U.

The following is a direct consequence of Lemma 1.

Proposition. *Let I, t_i and s_i be as in* Lemma 1.

(1) *Assume that $(eJe)U\leq U$ and $t_i\notin eJe$ for all $i\in I$. Then*

(i) *If $\{\bar{s}_i\}_{i\in I}$ generates ${}_{D(U)}\overline{\operatorname{Hom}}_R(eR, eR/U)$, then $U^*=\bigcap_{i\in I} t_i^{-1}U$.*

(ii) *If the intersection $\bigcap_{i\in I} t_i^{-1}U$ is irredundant, then $\{\bar{s}_i\}_{i\in I}$ is linearly independent in ${}_{D(U)}\overline{\operatorname{Hom}}_R(eR, eR/U)$.*

(2) *Assume that eR/U is quasi-injective. Then the converse assertions of* (i) *and* (ii) *above hold.*

Theorem. *Assume that $(eJe)U\leq U$ and eR/U is quasi-injective. Then the following cardinal numbers are equal.*

(1) $n:=[D: D(U)]_l$.

(2) $k:=\min\{\sharp K\mid U^*=\bigcap_{i\in K} t_i^{-1}U$ *for some* $t_i\in eRe\}$.

(3) $l:=\sharp I$ *for a set I such that there is a $t_i\in eRe/eJe$ for each $i\in I$ and $U^*=\bigcap_{i\in I} t_i^{-1}U$ is an irredundant intersection.*

Proof. Let $\{\bar{s}_i\}_{i\in N}$ be a basis of ${}_{D(U)}\overline{\operatorname{Hom}}_R(eR, eR/U)$ where $s_i\in\operatorname{Hom}_R(eR, eR/U)$ for each $i\in N$ and $n=\sharp N$. Since eR is projective, $s_i=\pi t_i$ for some $t_i\in eRe$ for each i. Noting that $t_i\notin eJe$ since $\bar{s}_i\neq 0$ for each i, we have $U^*=\bigcap_{i\in N} t_i^{-1}U$ by Proposition (1) (i). Thus $k\leq n$. Also by Proposition (2) (i), $n\leq k$, i.e. $n=k$. So since the intersection $U^*=\bigcap_{i\in N} t_i^{-1}U$ above is irredundant, l exists. Clearly $k\leq l$ for every I in (3). It follows immediately from Proposition (1) (ii) that $l\leq n$ for every I. Hence $l=k=n$.

Lemma 2. *Let $U^*=\bigcap_{i\in I} t_i^{-1}U$ be an irredundant intersection with each $t_i\in eRe\backslash eJe$. If there is a characteristic submodule T of eR containing U with $|T/U|=1$, then $\sharp I=|T/U^*|$.*

Proof. For each $i \in I$, since $|T/t_i^{-1}U|=1$ and $\bigcap_{j \neq i} t_j^{-1}U \nleq t_i^{-1}U$, we have $T=t_i^{-1}U+\bigcap_{j \neq i} t_j^{-1}U$. Hence $|T/U^*|=|\bigoplus_{i \in I} T/t_i^{-1}U|=\sharp I$.

By Lemma 2, Proposition (1) (i) and (2) (ii), we obtain the following.

Corollary. *Assume that* $[D: D(U)]_l<\infty$ *and that there is a characteristic submodule* T *of* eR *containing* U *with* $|T/U|=1$ (*e.g.* $T=eJ$ *and* U *is a maximal submodule of* eJ). *Then*

(1) *If* $(eJe)U \leq U$, *then* $|T/U^*| \leq [D: D(U)]_l$.

(2) *If* eR/U *is quasi-injective, then* $[D: D(U)]_l \leq |T/U^*|$.

REMARK. (1) If Ker $\delta=0$ (e.g. $eJe=0$), then ${}_{D(U)}D \cong {}_{\mathrm{End}_R(eR/U)}\mathrm{Hom}_R(eR, eR/U)$.

(2) Assume that R is a finite dimensional algebra over a field. If U is a maximal submodule of eJ (i.e. eR/U is a uniserial module of length 2), then $(eJe)U \leq U$ since $eJ^2 \leq U \leq eJ$. Put $\mathcal{C}:=\{U \mid U$ is a maximal submodule of $eJ\}$, and let $\{eR/U_i\}_{i=1}^n$ be a complete set of representatives of isomorphism classes of the class $\{eR/U \mid U \in \mathcal{C}\}$. Then since $eJ^2 \leq U^* \leq U$ for each $U \in \mathcal{C}$, and $eR/U \cong eR/V$ implies $U^*=V^*$ for any U and V in $\mathcal{C}$, we have $eJ^2 \leq \bigcap_{U \in \mathcal{C}} U^* = \bigcap_{i=1}^n U_i^* \leq \bigcap_{U \in \mathcal{C}} U=(eJ)J$ whence $eJ^2=\bigcap_{i=1}^n U_i^*$. So $|eJ/eJ^2| \leq |\bigoplus_{i=1}^n eJ/U_i^*| \leq \sum_{i=1}^n [D: D(U_i)]_l$ by Corollary (1). On the other hand, by [3, Lemma 3 and Proposition 6 i)], [5, Condition I] implies $\sum_{i=1}^n [D: D(U_i)]_l=\sum_{i=1}^n [D: D(U_i)]_r \leq 2$. Hence in this case [5, Condition I] implies $|eJ/eJ^2| \leq 2$, i.e. [5, Condition II]. The dual argument also works in [7] and [9].

(3) By using [2, Proposition 1.16 (1)] in the proof of the dual version of Lemma 1 (2), we see that all the results dual to the above hold if we assume that $[D: D(U)]_l<\infty$. The details are left to the reader.

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