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ASYMPTOTIC DIRICHLET PROBLEM FOR A COMPLEX MONGE-AMPÈRE OPERATOR

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1. Introduction

Let M be a complex manifold of dimension n and $P(M)$ denote the set of plurisubharmonic functions on M. For $u \in P(M) \cap C^2(M)$, we write $(dd^c u)^k$ for $\underbrace{dd^c u \wedge dd^c u \wedge \cdots dd^c u}_{k \text{ times}}$ where $d^c = \sqrt{-1}(\bar{\partial} - \partial)$. In the case $k=n$, the operator $u \rightarrow (dd^c u)^n$ is called a complex Monge-Ampère operator. In general, let u be a locally bounded plurisubharmonic function on M. In [5], [6], Bedford and Taylor defined a positive (k, k) current $(dd^c u)^k$ inductively by

$$\int \psi \wedge (dd^c u)^k = \int u \cdot dd^c \psi \wedge (dd^c u)^{k-1}$$

for any smooth $(n-k, n-k)$ form ψ with compact support on M. In the same paper they studied the Dirichlet problem for the complex Monge-Ampère operator on strongly pseudoconvex bounded domains in $\boldsymbol{C}^n$.

In this paper we shall consider the Dirichlet problem at *infinity* on certain negatively curved Kähler manifolds. Before stating our main theorem, we recall some definitions in [10]: Let M be a simply connected complete Riemannian manifold of nonpositive curvature. Two geodesic rays γ_1, γ_2 parametrized by arc length are called *asymptotic* if the distance $d(\gamma_1(t), \gamma_2(t))$ is bounded for $t \geqq 0$. The equivalence classes of geodesic rays are called *asymptotic classes*, the set of which will be denoted by $M(\infty)$. Then $\bar{M} = M \cup M(\infty)$ equipped with the "cone topology" is a compact topological space homeomorphic to a cell.

Theorem. *Let M be a simply connected complete Kähler manifold whose sectional curvature K satisfies*

$$(1) \qquad -a^2 \leqq K \leqq -1 \qquad (a \geqq 1).$$

We denote by ω the Kähler form on M and by $r(x)$ the distance function relative to a fixed point $o \in M$. Then for any continuous function f on $M(\infty)$ and for any *nonnegative continuous function* ρ *on* $\bar{M}=M\cup M(\infty)$ *which satisfies*

$$0\leqq\rho(x)\leqq C\exp(-2n\cdot r(x)) \tag{2}$$

for some constant $C>0$, *there exists a unique continuous plurisubharmonic function* u *on* M *such that*

$$\begin{cases}(dd^c u)^n=\rho\cdot\omega^n/n! & \text{in}\quad M\\ u=f & \text{on}\quad M(\infty)\,.\end{cases} \tag{3}$$

By applying the argument of Cegrell [8], we get more generally

Corollary. *Under the same assumption as in Theorem, let* $H(t, x)$ *be a Lebesgue measurable nonnegative function on* $(-\infty, \sup f)\times M$ *with*

$$0\leqq H(t, x)\leqq C\cdot\exp(-2n\cdot r(x)) \tag{2'}$$

for some constant $C>0$. *If* $H(t, x)$ *is a continuous function in* t, *then the Dirichlet problem*

$$\begin{cases}(dd^c u)^n=H(t, x)\omega^n/n! & \text{in}\quad M\\ \lim_{x\to\xi} u(x)=f(\xi) & \text{for any}\quad \xi\in M(\infty)\end{cases} \tag{3'}$$

has a solution $u\in P(M)\cap L^\infty(M, \mathrm{loc})$, *where* $P(M)\cap L^\infty(M, \mathrm{loc})$ *denotes the set of locally bounded plurisubharmonic functions on* M.

We mention here some works previous to ours. In [20], H. Wu proposed, among other things, the following question: Is a simply connected complete Kahler manifold with nonpositive Riemannian curvature and with negative holomorphic sectional curvature bounded away from zero biholomorphic to a bounded domain in C^n? (See also Aomoto [2].) Around this problem, a number of interesting results has been obtained (cf. e.g. [17]). In particular, in [11], Greene and Wu showed a geometric method of constructing suitable bounded plurisubharmonic functions on a Kähler manifold M as in theorem, and applying the $L^2-\bar{\partial}$ theory to M, they proved that M possesses the Bergman metric. As a Riemannian counterpart to the above problem, Choi [9] and Kasue [13] considered the Dirichlet problem at infinity for Laplace operator on a simply connected complete Riemannian manifold satisfying (1). Then Anderson [1] showed that such a Riemannian manifold possesses abundant global convex subsets, which allows to solve the Dirichlet problem for Laplace operator (cf. [9], [13]). We depend essentially on Anderson's result; in fact, we can construct so called barrier functions, by making use of his result.

REMARK. The decay conditions (2) and (2′) would be reasonable for our situation. In the case of complex n ball with Bergman metric, these conditions correspond to the boundedness of density function measured by the usual Lebesgue measure. After finishing this work, H. Kaneko treats our problem from the probabilistic standpoint and showed that the decay condition can be weakened ([14]).

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2. Proof of the Theorem

In what follows, we preserve the notations introduced in Theorem. Let us denote by $B(\rho, f)$ the class of subsolutions to the Dirichlet problem (3), i.e. the set of functions $v \in P(M) \cap L^\infty(M, \text{loc})$ satisfying

$$(4) \qquad (dd^c v)^n \geqq \rho \cdot \omega^n/n! \qquad \text{(in the sense of Bedford-Taylor [5], [6])}$$
$$\limsup_{x \to \xi} v(x) \geqq f(\xi) \qquad \text{for any} \quad \xi \in M(\infty).$$

The upper envelope of the class $B(\rho, f)$ is by definition the function

$$(5) \qquad u(x) = \sup \{v(x): v \in B(\rho, f)\} \qquad (x \in M).$$

We first show the following

Lemma 1.
(1) *$B(\rho, f)$ is not empty.*
(2) *The upper regularization u^* of the upper envelope u belongs to the class $B(\rho, f)$ and satisfies*

$$\lim_{x \to \xi} u^*(x) = f(\xi)$$

for any $\xi \in M(\infty)$. In particular, $u = u^*$.

Proof. Set

$$(6) \qquad \beta(x) = \exp 2 \int_1^{r(x)} (\sinh t)^{-1} dt,$$

where $r(x)$ stands for the distance function between a point x and a fixed point o of M. By the Hessian comparison theorem (cf. [11]: Theorem A.)

$$(7) \qquad dd^c \beta \geqq 2\beta (\sinh r)^{-2} \cdot \omega \qquad \text{on} \quad M - \{o\}.$$

Here ω denotes the Kähler form on M. It follows from (7) that

$$(8) \qquad (dd^c \beta)^n \geqq (2\beta)^n (\sinh r)^{-2n} \cdot \omega^n$$

as positive currents on M. This shows that for some constants $C_1 > 0$ and C_2, $C_1 \beta + C_2$ belongs to $B(\rho, f)$, because of the assumption (2). Now we fix a point $\xi \in M(\infty)$. By a theorem of Anderson ([1: Theorem 3.1]), for any positive number ε there exists an open neighborhood $U'_{\xi, \varepsilon}$ of ξ in $\bar{M}$ such that

$$|f(\eta)-f(\xi)|<\varepsilon \qquad \eta\in M(\infty)\cap U'_{\xi,\varepsilon},$$
$$M-U'_{\xi,\varepsilon} \text{ is a totally convex domain in } M.$$

By the approximation theorem, we may assume that the boundary of $M-U'_{\xi,\varepsilon}$ is smooth (cf. [12], [13: Corollary (2.5)]), so that the distance function $r_{\xi,\varepsilon}(x)$ to the boundary of $M-U'_{\xi,\varepsilon}$ is smooth on $U'_{\xi,\varepsilon}\cap M$ (cf. [$B-O$]). Now we set

$$\beta_{\xi,\varepsilon}(x)=\exp 2\int_1^{r_{\xi,\varepsilon}(x)}(\cosh t)^{-1}dt\,. \tag{9}$$

By the Hessian comparison theorem for hypersurfaces (cf. [13: Theorem 2.49])

$$dd^c\beta_{\xi,\varepsilon}\geqq 2(\cosh r_{\xi,\varepsilon})^{-2}\min\{1,\sinh r_{\xi,\varepsilon}\}\cdot\omega \tag{10}$$

on $M\cap U'_{\xi,\varepsilon}$. We take two constants

$$A=\exp(-2\int_0^1(\cosh t)^{-1}dt) \qquad B=\exp(2\int_1^\infty(\cosh t)^{-1}dt)\,,$$

and extend $\beta_{\xi,\varepsilon}$ to a plurisubharmonic function on M by setting $\beta_{\xi,\varepsilon}=A$ on $M-U'_{\xi,\varepsilon}$. We set

$$\underline{\beta}_{\xi,\varepsilon}(x)=f(\xi)-2\varepsilon+C_3(\beta_{\xi,\varepsilon}(x)-B)+C_4(\beta(x)-C)\,, \tag{11}$$

where $C=\exp 2\int_1^\infty(\sinh t)^{-1}dt$, $C_3>0$ and $C_4<0$.
It follows from (8), (10), (11) that $\underline{\beta}_{\xi,\varepsilon}$ belongs to $B(\rho, f)$ and satisfies

$$\underline{\beta}_{\xi,\varepsilon}\geqq f(\xi)-3\varepsilon \qquad \text{on} \quad U_{\xi,\varepsilon}\cap M$$

for a small neighborhood $U_{\xi,\varepsilon}\subset U'_{\xi,\varepsilon}$ of ξ in $\bar{M}$. Then we have

$$u^*(x)\leqq\underline{\beta}_{\xi,\varepsilon}(x)\leqq f(\xi)-3\varepsilon \qquad (x\in U_{\xi,\varepsilon}\cap M)\,. \tag{12}$$

Set

$$\bar{\beta}_{\xi,\varepsilon}(x)=(f(\xi)+2\varepsilon)-C_5(\beta_{\xi,\varepsilon}(x)-B)\,, \tag{13}$$

where C_5 is a positive constant. Then, for a sufficiently small neighborhood $U_{\xi,\varepsilon}$ of ξ, we have

$$\bar{\beta}_{\xi,\varepsilon}\leqq f(\xi)+3\varepsilon \qquad \text{on} \quad U_{\xi,\varepsilon}\cap M\,. \tag{14}$$

Since the function $\bar{\beta}_{\xi,\varepsilon}$ is plurisuperharmonic, we have

$$u^*\leqq\bar{\beta}_{\xi,\varepsilon} \qquad \text{on} \quad U_{\xi,\varepsilon}\cap M\,. \tag{15}$$

Since ε is an arbitrary small positive constant, it follows from (12), (14), (15) that

$$\limsup_{x\to\xi} u^*(x) = f(\xi)\,.$$

This proves the last statement of Lemma 1. Now we shall complete the proof of the Lemma. We may choose an increasing sequence of functions $u_j \in B(\rho, f)$ such that $u^* = (\lim_{j\to\infty} u_j)^*$. Then we see that

$$(dd^c u^*)^n = \lim_{j\to\infty} (dd^c u_j)^n \leqq \rho\cdot\omega^n/n!$$

([6: Theorem 7.4]). This implies $u^* \in B(\rho, f)$, which proves the Lemma 1.

Let $\{Z_i\}_{1\leqq i\leqq n}$ be a frame of holomorphic vector fields on M. Here we remark that the holomorphic tangent bundle is holomorphically trivial. Let Ω be a relatively compact domain of M. For sufficiently small positive constant δ, we can define a smooth map Φ: $\Delta_\delta^n \times \Omega \to M$ ($\Delta_\delta = \{z\in C: |z|<\delta\}$) by

$$\Phi(w, x) = \mathrm{Exp}(\mathrm{Re}\sum_{i=1}^n w_i Z_i)(x) \qquad (w=(w_1, \cdots, w_n)\in \Delta_\delta^n)\,. \tag{16}$$

For $w \in \Delta_\delta^n$, we denote the holomorphic map $x \to \Phi(w, x)$ by Φ_w. Now we shall prove the following

Lemma 2. *The function u defined by* (5) *is a continuous function on* $\bar{M}$.

Proof. Given $\varepsilon>0$ and $\xi\in M(\infty)$, we choose a neighborhood $U_{\xi,\varepsilon}$ of ξ in $\bar{M}$ as in Lemma 1. For sufficiently large $R>0$, M is covered by $\{U_{\xi,\varepsilon}\}_{\xi\in M(\infty)}$ and the geodesic ball $B(o, R) = \{x\in M: r(x)<R\}$. For sufficiently small $\delta>0$, we may assume that

$$|u(x) - u(\Phi_w(x))| < \varepsilon \qquad (w\in\Delta_\delta^n,\ x\in\partial B(o, R))$$

because of (12), (14), (15). Now we define a plurisubharmonic function $U(x)$ on M by

$$U(x) = \begin{cases} u(x) & \text{if } x\in M-B(o, R) \\ \max\{u(x),\ u\circ\Phi_w(x) - 2\varepsilon + C_6(\beta(x) - C)\} & \text{if } x\in B(o, R) \end{cases} \tag{17}$$

where C is a positive constant and $w \in \Delta_\delta^n$. Then for any $\eta>0$, it follows from (8) that on $B(o, R)$,

$$\begin{aligned} [dd^c(u\circ\Phi_w + C_6\beta)]^n &\geqq [dd^c(u\circ\Phi_w)]^n + C_6^n(dd^c\beta^n) \\ &\geqq (\rho-\eta)\omega^n + 2^n C_6^n \beta^n (\sinh r)^{-n}\cdot\omega^n/n!\,. \end{aligned} \tag{18}$$

Now we see that $U\in B(\rho, f)$, by setting

$$C = \eta^{1/n} \sup_{B(o,R)} \frac{\sinh r}{\beta}$$

In particular,

$$u\circ\Phi_w - 2\varepsilon + C_6(\beta - C) \leqq u \qquad \text{on} \quad B(o, R)(w \in \Delta_\delta^n)\,.$$

That is,

$$u\circ\Phi_w(x) - u(x) \leqq 2\varepsilon + C_6 C\,.$$

By taking $\eta(<\varepsilon)$ sufficiently small, we obtain

$$u\circ\Phi_w - u \leqq \varepsilon \qquad \text{on } B(o, R) \quad (w \in \Delta_\delta^n)\,.$$

This shows the continuity of u.

Now lemma 1, lemma 2 and the original argument of Bedford-Taylor ([5: Theorem 8.3]) show that

$$(dd^c u)^n = \rho\cdot\omega^n/n! \qquad \text{on} \quad M$$
$$u = f \qquad \text{on} \quad M(\infty)\,.$$

Thus our theorem has been proved.

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