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ON EXTREMAL QUASICONFORMAL MAPPINGS WITH VARYING DILATATION BOUNDS

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1. Introduction

Let a homeomorphism of the boundary of the unit disk $D=\{w\,|\,|w|<1\}$ onto itself be given which can be extended to a quasiconformal mapping of the whole disk onto itself. Then one asks for extremal extensions, i.e. quasiconformal extensions with the smallest possible maximal dilatation. This problem has been posed by Teichmüller ([21], p. 184) and main contributions have been made by Strebel ([17], [18]). Later Hamilton [7] and Reich-Strebel [14] have derived necessary and sufficient conditions for extremality of a mapping, the Hamilton-condition.

Due to a remark of Teichmüller ([21], p. 15, lines 16–20) this problem had been generalized first by Kühnau [9] in certain cases and then by Reich [12] in the following way. In addition to the fixed given boundary values the competing quasiconformal mappings are supposed to have their dilatations pointwise bounded by a prescribed function on a certain given subset E of D. Then their maximal dilatations in the remaining part of the disk have to be minimized.

In the analogous way as the Hamilton-condition is derived in the former case $E=\emptyset$, Reich has treated this problem in [12] and for a complete description of the solution we refer to section 2. Since our paper contains direct developments of Reich's work, we hope readers to be well acquainted with his paper. Mainly our contributions deal with the question how extremal mappings and certain quadratic differentials related to them depend on the dilatation bounds when these vary. To make this clear let us now explain the situation precisely.

Given are a quasisymmetric boundary mapping h: $\partial D\to\partial D$ and a closed set σ on ∂D, which is always assumed to contain at least four points. In addition a (possibly empty) measurable subset E of D such that $D\backslash E$ has positive measure is given and the dilatation bound b on E. The latter is a measurable function $b(w)\geqq 0$ on E with $||b||_\infty:=\operatorname{ess\,sup}_{w\in E} b(w)<1$. We consider the set $Q=Q(h, \sigma, E, b)$ of all qc (quasiconformal) mappings F: $D\to D$ with $F|_\sigma=h|_\sigma$ and with complex dilatation κ_F that satisfies

$$(1.1) \qquad |\kappa_F(w)| \leqq b(w) \quad \text{a.e. in } E\,.$$

An element $F \in Q(h, \sigma, E, b)$ is called an extremal mapping with dilatation bound b or simply extremal in $Q(h, \sigma, E, b)$ if the number

$$\operatorname*{ess\,sup}_{w \in D \setminus E} |\kappa_F(w)|$$

is minimal. It is called uniquely extremal in Q if it is the only extremal element in Q.

If E is relatively compact in D, then obviously Q is never empty. In fact, if we choose r, $0<r<1$, such that $\bar{E} \subset \{w \mid |w|<r\}$, then, as is well-known, there exists a qc selfmapping of the annulus $\{w \mid r<|w|<1\}$ with boundary values h on $|w|=1$ and identity on $|w|=r$ (see e.g. [8]). Extending the mapping by the identity inside will clearly yield an element in Q. And if Q is not empty, then extremal elements exist by a normality reasoning and Strebel's Satz on p. 469 in [19] which asserts that a local uniform limit of functions in $Q(h, \sigma, E, b)$ satisfies (1.1) too. It will be our general assumption throughout this paper that σ is finite or E is relatively compact in $\bar{D} \setminus \sigma$.

In our treatment we will also include the case where the dilatation bound is replaced by the constant one. By this we mean that we consider all qc extensions of $h|_\sigma$ which need only to be defined in $D \setminus E$ and minimize their maximal dilatations. In this case the general assumption on E shall be strengthened in the natural way that E is supposed to be compact in $\bar{D} \setminus \sigma$ and $D \setminus E$ to be a domain. To be precise, $Q(h, \sigma, E, 1)$ is the set of all qc mappings F: $D \setminus E \to F(D \setminus E) \subset D$ where F is continuously extensible on $\partial D \cap (\bar{D} \setminus E)$ with $F(\partial D \cap (\bar{D} \setminus E)) \subset \partial D$ and $F|_\sigma = h|_\sigma$. In the image-domain we then have free boundary components (except $F(\partial D \cap (\bar{D} \setminus E))$) and we call an extremal element absolutely extremal as it has been done in [2] where this problem had been introduced. Again by normality it easily follows that absolutely extremal mappings exist.

In section 2 we collect results on extremal elements in $Q(h, \sigma, E, b)$ and give a brief explanation how they are proved. Our main interest will be directed to the "non-degenerate" case when there is no substantial boundary point, i.e. there is no point ζ on the boundary ∂D where the local dilatation H_ζ^σ of h with respect to σ is equal to the maximal dilatation of the extremal mapping in $D \setminus E$. (For the definition of H_ζ^σ we refer to [11], p. 392 or [1], p. 571.) Based on the main result in [1] it then follows that the inverse of the extremal mapping in Q is of "generalized" Teichmüller type with an associated quadratic differential of finite norm. Moreover then this mapping is uniquely extremal in Q.

In section 3 we then investigate the dependence of extremal dilatation, extremal qc mapping and quadratic differential on varying dilatation bounds. Under the assumption that the area-measure of ∂E is zero we will prove: If a sequence of dilatation bounds b_n tends to a continuous dilatation bound b_∞ uniformly in E, then the corresponding extremal dilatations converge to the extremal dilatation for $Q(h, \sigma, E, b_\infty)$. And if the extremal mapping with dilatation bound b_∞ does not have a substantial boundary point, then the corresponding sequences of extremal mappings and of quadratic differentials converge to the extremal mapping and associated quadratic differential in $Q(h, \sigma, E, b_\infty)$. In the special case where E consists of finitely many closed quasidisks in D we will also prove: If a sequence of dilatation bounds b_n tends to 1 in the sense that $\operatorname*{ess\,inf}_{w\in E} b_n(w)\to 1$ for $n\to\infty$, then corresponding extremals contain a subsequence which tends to an absolutely extremal mapping, and if there is no substantial boundary point, then the sequence itself converges. For arbitrary closed sets E, however, such questions remain open.

Finally we would like to thank the referee for several valuable suggestions.

2. Extremal mappings in $Q(h, \sigma, E, b)$

We denote by L_∞^{ad} the set of admissible dilatation bounds on E, this is the subset of $L_\infty(E)$ consisting of all real-valued non-negative measurable functions b on E with L_∞-norm $||b||_\infty<1$. $L_\infty^{ad}\cup\{1\}$ is a partially ordered set by defining $b_1\leqq b_2$ iff $b_1(w)\leqq b_2(w)$ a.e. in E. Here we furthermore introduce the notations $D_F(w)$ to denote the dilatation of a qc mapping F at the point w, κ_F to denote the complex dilatation of F and we put $K[F]=\operatorname{ess\,sup} D_F(w)$.

Since h is quasisymmetric there is a $b_0\in L_\infty^{ad}$ such that $Q(h, \sigma, E, b_0)\neq\emptyset$. For a function $b\geqq b_0$, $b\in L_\infty^{ad}$, a necessary and sufficient condition for an element $F\in Q(h, \sigma, E, b)$ to be extremal can be expressed by the complex dilatation κ_f of the inverse mapping $f=F^{-1}$. We write $z=F(w)$, $w=f(z)$, $k_b=\operatorname*{ess\,sup}_{w\in D\setminus E}|\kappa_F(w)|$ and $E_0=\{w\in E\,|\,b(w)=0\}$. Then we put

$$\hat{\kappa}(z)=\begin{cases}\kappa_f(z) & z\in F(D\setminus E)\\ k_b\kappa_f(z)/b(f(z)) & z\in F(E\setminus E_0)\\ 0 & z\in F(E_0)\end{cases}$$

and this condition reads

$$\sup_{\substack{\phi\in\beta_{\sigma'}\\ ||\phi||_{F(D\setminus E_0)}=1}} \operatorname{Re}\iint_D \hat{\kappa}(z)\phi(z)dxdy = k_b \tag{2.1}$$

where $\sigma'=h(\sigma)$, $\beta_{\sigma'}=\{\phi\in L_1(D)\,|\,\phi$ analytic in D, continuously extensible on $\partial D\setminus\sigma'$ and ϕdz^2 real on $\partial D\setminus\sigma'\}$ and $||\phi||_{F(D\setminus E_0)}=\iint_{F(D\setminus E_0)}|\phi(z)|dxdy$. In the case where E is empty, this is the Hamilton-condition [7]. In case that there is a number $\varepsilon>0$ such that $b(w)\geqq\varepsilon$ in E, this has been proved by Reich [12].

In case $b(w)=0$ in E it has been proved for finite σ in [6] and for arbitrary σ in [4]. The generalization to the arbitrary case is due to Sakan [16].

From this condition it follows that two cases are possible. First, this supremum can be attained. Then there is a $\phi\in\beta_{\sigma'}$, $\phi\neq 0$, such that $\hat{\kappa}=k_b\bar{\phi}/|\phi|$ a.e. in D. We then normalize ϕ by $\|\phi\|=1$, and if $k_b\neq 0$ we have

$$\kappa_f(z)=\begin{cases} k_b\bar{\phi}(z)/|\phi(z)| & \text{a.e. in } D\backslash F(E) \\ b(f(z))\bar{\phi}(z)/|\phi(z)| & \text{a.e. in } F(E)\end{cases} \tag{2.2}$$

and so f is of "generalized" Teichmüller type. If σ is finite, then the supremum is attained and this case is hence treated.

Second, and we remark here that both cases can occur simultaneously, there is a sequence $\phi_n\in\beta_{\sigma'}$, $\|\phi_n\|_{F(D\backslash E_0)}=1$ with $\operatorname{Re}\iint_D \hat{\kappa}\phi_n dxdy \overset{n\to\infty}{\to} k_b$ and $\phi_n \overset{n\to\infty}{\to} 0$ locally uniformly in $\bar{D}\backslash\sigma'$. From the relative compactness of E_0 in $\bar{D}\backslash\sigma$ it follows that $\|\phi_n\|_{F(E_0)} \overset{n\to\infty}{\to} 0$. Hence $\|\phi_n\| \overset{n\to\infty}{\to} 1$ and if we put $\hat{\phi}_n:=\phi_n/\|\phi_n\|$ we get a degenerating Hamilton sequence for the complex dilatation $\hat{\kappa}$, this is defined to be a sequence $\hat{\phi}_n\in\beta_{\sigma',1}:=\{\phi\in\beta_{\sigma'}\,|\,\|\phi\|=1\}$ where $\operatorname{Re}\iint_D \hat{\kappa}\hat{\phi}_n dxdy \overset{n\to\infty}{\to} \|\hat{\kappa}\|_\infty=k_b$ and $\hat{\phi}_n \overset{n\to\infty}{\to} 0$ locally uniformly in $\bar{D}\backslash\sigma'$. We denote by $\hat{f}$ a qc self-mapping of D with complex dilatation $\hat{\kappa}$. We conclude first that $\hat{f}$ is extremal for its boundary values on σ', i.e. in $Q(\hat{f}|_{\partial D}, \sigma', \emptyset, 0)$, and by Satz 5.2 in [1] (or Theorem 1.1 in [5]) there exists a substantial boundary point on σ'. Since $\hat{f}\circ F$ is conformal in the set $D\backslash E$ which contains a neighborhood of σ and since local dilatations of the boundary mapping are preserved under conformal mapping we conclude that F has a substantial boundary point on σ, i.e. $K_b=\operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma$ where $K_b=(1+k_b)/(1-k_b)$.

Furthermore, if κ_f fulfills the equation (2.2), it has been proved in [12] (see also Remark 3 in [15]) that then F is uniquely extremal in $Q(h, \sigma, E, b)$. We have the

Corollary 2.1. *If σ is finite or if E is relatively compact in $\bar{D}\backslash\sigma$, then the assumption $K_b>\operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma$ implies that there is a unique extremal element F in $Q(h, \sigma, E, b)$ and a unique $\phi\in\beta_{\sigma',1}$ such that* (2.2) *holds.*

Next we assume that E is a compact set in $\bar{D}\backslash\sigma$ and that $D\backslash E$ is a domain. An analogous necessary condition for a mapping F in $Q(h, \sigma, E, 1)$ to be absolutely extremal has been given in Theorem 1 in [2]. Again we put $f=F^{-1}$ in $F(D\backslash E)$, $k_1=\operatorname{ess\,sup}_{w\in D\backslash E} |\kappa_F(w)|$ and

$$\hat{\kappa}(z)=\begin{cases} \kappa_f(z) & z\in F(D\backslash E) \\ 0 & z\in D\backslash F(D\backslash E)\,. \end{cases}$$

Then this necessary condition reads

$$(2.3) \qquad \sup_{\phi\in\beta_{\sigma',1}} \mathrm{Re}\iint_D \hat{\kappa}(z)\phi(z)dxdy = k_1 .$$

Here we give a simplified proof of it which is directly based on the Hamilton-condition.

Proof. Let F be absolutely extremal and as above $\hat{f}$ denotes a qc self-mapping of D with complex dilatation $\hat{\kappa}$. We claim that $\hat{f}$ is extremal in the class of all qc selfmappings of D with boundary values $\hat{f}$ on σ', i.e. extremal in $Q(\hat{f}|_{\partial D}, \sigma', \emptyset, 0)$. From this then (2.3) follows by the usual Hamilton-condition since $k_1=||\hat{\kappa}||_\infty$. So let us assume that there is a qc mapping g: $D\to D$ with $g^{-1}\circ\hat{f}=\mathrm{id}$ on σ' and $||\kappa_g||_\infty<k_1$. Then the qc mapping $g^{-1}\circ\hat{f}\circ F$ is defined in $D\backslash E$ and belongs to $Q(h, \sigma, E, 1)$. Moreover, since $\hat{f}\circ F$ is conformal in $D\backslash E$, the complex dilatation is bounded by $||\kappa_g||_\infty$ in $D\backslash E$ in contradiction to the absolute extremality of F. We conclude that $\hat{f}$ is extremal in $Q(\hat{f}|_{\partial D}, \sigma', \emptyset, 0)$.

We remark here that this consideration even shows that for every component Λ of $D\backslash F(D\backslash E)$ the qc mapping $\hat{f}|_{D\backslash\Lambda}$ is extremal in the class of all qc mappings from $D\backslash\Lambda$ onto $D\backslash\hat{f}(\Lambda)$ with boundary values $\hat{f}$ on σ'. As above it is also clear that $\hat{f}$ contains a substantial boundary point iff F contains such a point, i.e. iff $K_1=\operatorname{Max}_{\zeta\in\sigma} H^\sigma_\zeta$ where $K_1=(1+k_1)/(1-k_1)$.

The supremum in condition (2.3) can be attained or not. This gives two possibilities again: First, the absolute extremal mapping F satisfies

$$(2.4) \qquad \kappa_f(z) = k_1\bar{\phi}(z)/|\phi(z)| \qquad \text{a.e. in } F(D\backslash E)$$

with $\phi\in\beta_{\sigma',1}$ and if $k_1\neq 0$ then evidently the area-measure of $D\backslash F(D\backslash E)$ has to be zero. (In [2], p. 341 it is further shown, using the remark above, that the lack of a substantial boundary point on σ for F then implies that every component of $D\backslash F(D\backslash E)$ is a subarc of a vertical trajectory or a connected subset of the vertical critical graph of the quadratic differential ϕ.) And then second, as before it follows that there is a substantial boundary point, i.e. $K_1=\operatorname{Max}_{\zeta\in\sigma} H^\sigma_\zeta$.

In this problem, unlike condition (2.1), the condition (2.3) is only necessary but not sufficient for F to be absolutely extremal. In case that the compact set E is contained in D necessary and sufficient conditions have been given in [2] for finitely connected domains $D\backslash E$ and in [3] for arbitrary domains $D\backslash E$. Later we will only use the result that if $K_1>\operatorname{Max}_{\zeta\in\sigma} H^\sigma_\zeta$, then the absolutely extremal mapping is uniquely extremal. We hence have

Corollary 2.2. *If E is compact in D and $D\backslash E$ is a domain, then the assump-*

tion $K_1>\operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma$ *implies that there is a unique extremal element* F *in* $Q(h, \sigma, E, 1)$ *and a unique* $\phi\in\beta_{\sigma',1}$ *such that* (2.4) *holds. Moreover the area-measure of* $D\backslash F(D\backslash E)$ *is zero.*

3. Varying dilatation bounds

Let $b_0\in L_\infty^{ad}$ be fixed such that $Q(h, \sigma, E, b_0)\neq\emptyset$. We then define

$$L(b_0) = \{b\in L_\infty^{ad}\cup\{1\}\,|\,b\geqq b_0\}$$
$$C(b_0) = \{b\in L(b_0)\,|\,b \text{ continuous on } E\}.$$

For every $b\in L(b_0)$ we denote by F_b an extremal mapping in $Q(h, \sigma, E, b)$ and its inverse by $f_b=F_b^{-1}$. Furthermore we put $K_b=\operatorname{ess\,sup}_{w\in D\backslash E} D_{F_b}(w)$ and $k_b=(K_b-1)/(K_b+1)$. Then the function $b\to K_b$ is well-defined on $L(b_0)$ with values bounded by $\operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma\leqq K_b\leqq K_{b_0}$.

We then consider

$$L_0(b_0) = \{b\in L(b_0)\,|\,K_b>\operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma\}$$
$$C_0(b_0) = \{b\in C(b_0)\,|\,K_b>\operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma\}\,.$$

For every $b\in L_0(b_0)$ and the corresponding assumption on E we conclude by Corollary 2.1 and 2.2 that the extremal F_b is uniquely determined and there is a unique $\phi_b\in\beta_{\sigma',1}$ with

$$\kappa_{f_b} = \begin{cases} k_b\overline{\phi_b}/|\phi_b| & \text{a.e. in } F_b(D\backslash E) \\ (b\circ f_b)\overline{\phi_b}/|\phi_b| & \text{a.e. in } F_b(E) \end{cases} \tag{3.1}$$

if $b\neq 1$ and

$$\kappa_{f_1} = k_1\overline{\phi_1}/|\phi_1| \qquad \text{a.e. in } F_1(D\backslash E) \tag{3.2}$$

if $b = 1$.

We hence have two functions on $L_0(b_0)$, $b\to F_b$ and $b\to\phi_b$, one of them with values in $\bigcup_{b\in L_0(b_0)} Q(h, \sigma, E, b)$ and one of them with values in $\beta_{\sigma',1}$. These two functions as well as $b\to K_b$ shall now be investigated.

A sequence b_n in $L(b_0)$ is called bounded, if there is a $q<1$ such that $\|b_n\|_\infty\leqq q$ for every n. We call it bounded convergent to $b\in L(b_0)$, if the sequence is bounded and $b_n(w)\overset{n\to\infty}{\to} b(w)$ pointwise a.e. in E, and we call it locally uniformly bounded convergent to $b\in L(b_0)$, if the sequence is bounded and $b_n\overset{n\to\infty}{\to} b$ locally uniformly in E. If a sequence b_n in $L(b_0)$ is considered which is not bounded, then we always assume that E is compact in D and $D\backslash E$ is a domain. Such a sequence is called convergent to 1 if $\operatorname{ess\,inf}_{w\in E} b_n(w)\overset{n\to\infty}{\to} 1$.

If $b_\infty \in C(b_0)$, $b_\infty \neq 1$, then we call $b \to K_b$ continuous at b_∞ in $C(b_0)$ if $\lim_{n\to\infty} K_{b_n} = K_{b_\infty}$ for every sequence $b_n \in L(b_0)$ which is locally uniformly bounded convergent to b_∞. And this function is called continuous at 1 if $\lim_{n\to\infty} K_{b_n} = K_1$ for every sequence $b_n \in L(b_0)$ which is convergent to 1.

In the same way we define continuity of $b \to \phi_b$ and $b \to F_b$ at $b_\infty \in C_0(b_0)$. $\phi_{b_\infty} = \lim_{n\to\infty} \phi_{b_n}$ then means local uniform convergence in $\bar{D}\backslash\sigma'$ and $F_{b_\infty} = \lim_{n\to\infty} F_{b_n}$ means local uniform convergence in D if $b_\infty \neq 1$, otherwise only in $D\backslash E$.

Let b_n be a bounded sequence in $L(b_0)$. Then it has the property P1: Any sequence of corresponding extremals F_{b_n} contains a subsequence $F_{b_{n_i}}$ where $K_{b_{n_i}} \overset{i\to\infty}{\to} \underline{\lim}_{n\to\infty} K_{b_n}$ and $F_{b_{n_i}} \overset{i\to\infty}{\to} F$ locally uniformly in D where $F|_\sigma = h|_\sigma$ and $\operatorname{ess\,sup}_{w\in D\backslash E} D_F(w) \leqq \underline{\lim}_{n\to\infty} K_{b_n}$. Furthermore, if b_n is bounded convergent to b_∞, then $F \in Q(h, \sigma, E, b_\infty)$ and $K_{b_\infty} \leqq \underline{\lim}_{n\to\infty} K_{b_n}$.

This follows by normality of the sequence F_{b_n}, since $K[F_{b_n}] \leqq \operatorname{Max}\{K_{b_0}, (1+q)/(1-q)\}$ for every n and σ contains more than three points, and by Strebel's Satz in [19] which gives $|\kappa_F(w)| \leqq \overline{\lim}_{n\to\infty} |\kappa_{F_{b_n}}(w)| \leqq \lim_{n\to\infty} b_n(w) = b_\infty(w)$ a.e. in E.

Let b_n be a sequence in $L(b_0)$ which is not bounded. Then it has property P2: Any sequence of corresponding extremals F_{b_n} contains a subsequence $F_{b_{n_i}}$ where $K_{b_{n_i}} \overset{i\to\infty}{\to} \underline{\lim}_{n\to\infty} K_{b_n}$ and $F_{b_{n_i}} \overset{i\to\infty}{\to} F$ locally uniformly in $D\backslash E$ where $\operatorname{ess\,sup}_{w\in D\backslash E} D_F(w) \leqq \underline{\lim}_{n\to\infty} K_{b_n}$. Evidently $F \in Q(h, \sigma, E, 1)$ and hence $K_1 \leqq \underline{\lim}_{n\to\infty} K_{b_n}$.

For this reason the function $b \to K_b$ is called lower semicontinuous on $L(b_0)$.

Lemma 3.1. *Let b_n be a bounded sequence in $L_0(b_0)$ or (if E is compact in D and $D\backslash E$ a domain) an unbounded sequence in $L_0(b_0)$. If $\underline{\lim}_{n\to\infty} K_{b_n} > \operatorname{Max}_{\zeta\in\sigma} H^\sigma_\zeta$, then the uniquely determined $\phi_{b_n} \in \beta_{\sigma',1}$ contain a subsequence which converges locally uniformly in $\bar{D}\backslash\sigma'$ to a limit $\phi \in \beta_{\sigma',1}$.*

Proof. First we introduce the abbreviations $H = \operatorname{Max}_{\zeta\in\sigma} H^\sigma_\zeta$, $C = \underline{\lim}_{n\to\infty} K_{b_n}$, $c = (C-1)/(C+1)$ and we write F_n, K_n, ϕ_n, f_n instead of F_{b_n}, K_{b_n}, etc. Then we also remark, that if $b_n = 1$ for infinitely many n (which of course can only occur if b_n is an unbounded sequence) then the statement is trivial because then $K_1 = \underline{\lim}_{n\to\infty} K_n > H$ and so we then just choose this subsequence. Hence we may assume that all $b_n \neq 1$ and the extremals F_n are defined in all of D.

By normality of $\beta_{\sigma',1}$ we can find a subsequence of ϕ_n, which we call ϕ_n again, such that $\phi_n \overset{n\to\infty}{\to} \phi \in \beta_{\sigma'}$ locally uniformly in $\bar{D}\backslash\sigma'$ and evidently $||\phi||$

$\leqq 1$. We now have to prove that $\|\phi\|=1$, but we remark here that for most of the following statements it would be sufficient to know that $\phi\neq 0$, which can be proved more easily by Strebel's frame mapping criterion [20]. We use the more complicated Lemma 4.1 in [3] to prove this stronger result.

If σ is finite, then $\|\phi\|=1$ anyway. So we may assume that $\bar{E}$ is compact in $\bar{D}\backslash\sigma$. By property P1 for a bounded sequence or by property P2 for an unbounded sequence we can pass to a further subsequence and have $\lim_{n\to\infty} K_n = C$ and $F_n \overset{n\to\infty}{\to} F$ locally uniformly in D or in $D\backslash E$. In both cases we have $F|_\sigma = h|_\sigma$ and $\operatorname{ess\,sup}_{w\in D\backslash \bar{E}} D_F(w) \leqq C$. We can also assume that $H<K_n\leqq C+1$ for all n and hence by (3.1) all f_n have complex dilatations

$$\kappa_{f_n}(z) = k_n\bar{\phi}_n(z)/|\phi_n(z)| \qquad \text{a.e. in } F_n(D\backslash E)\,. \tag{3.3}$$

Let $0<\varepsilon<1$. We choose an open set A in $\bar{D}$ with $\bar{E}\subset A$, $\bar{A}\subset\bar{D}\backslash\sigma$ and

$$\iint_{D\backslash F(D\backslash A)} |\phi(z)|\,dxdy \geqq \|\phi\| - \varepsilon\,.$$

If $\bar{E}\subset D$ we choose A to be a disc $\{w|\ |w|<r\}$, $r<1$. Otherwise we choose it to be bounded by finitely many analytic arcs in D and subarcs of ∂D such that a neighborhood of σ is cut off from $\bar{D}$.

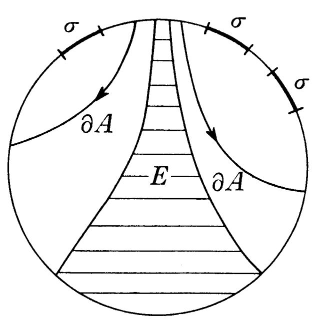

Fig. 1

By the local uniform convergence of F_n at least in $D\backslash E$ we have for every neighborhood U of σ in $\bar{D}\backslash\bar{A}$ $\iint_{D\backslash F_n(U)} |\phi_n|\,dxdy \overset{n\to\infty}{\to} \iint_{D\backslash F(U)} |\phi|\,dxdy$ and therefore $\iint_{F_n(U)} |\phi_n|\,dxdy = 1 - \iint_{D\backslash F_n(U)} |\phi_n|\,dxdy \overset{n\to\infty}{\to} 1 - \iint_{D\backslash F(U)} |\phi|\,dxdy \leqq 1 - \|\phi\| + \varepsilon$. Since $\iint_{F_n(D\backslash U\backslash A)} |\phi_n|\,dxdy \overset{n\to\infty}{\to} \iint_{F(D\backslash U\backslash A)} |\phi|\,dxdy \leqq \iint_{F(D\backslash A)} |\phi|\,dxdy \leqq \varepsilon$ we can choose some n_0 such that for $n\geqq n_0$

$$\iint_{F_n(D\backslash U\backslash A)} |\phi_n|\,dxdy \leqq 2\varepsilon \text{ and } \iint_{F_n(U)} |\phi_n|\,dxdy \leqq 1 - \|\phi\| + 2\varepsilon\,. \tag{3.4}$$

By Lemma 4.4 of [3] there is a $(H+\varepsilon)$-qc extension of $h|_\sigma$ into a neighborhood of ∂D. We apply Lemma 4.1 (4.1′ if $\sigma \neq \partial D$) and the following remark from [3] on this qc extension and $F_n|_A$. Here it is needed to say that in case that $\bar{E}$ is compact in D, i.e. A is a disk $\{|w|<r\}$, then this lemma in [3] can be applied as it is formulated there. (We take the opportunity here to remark that in the proof of that lemma in [3], evidently all qc extensions into annuli $D_{rr'}$ have to be chosen in the given homotopy class by F). If $\bar{E}$ is compact in $\bar{D}\backslash\sigma$ only, it can be modified in an obvious way. We conclude that for a fixed number $M>\mathrm{Max}\,\{C+1,\ H+1\}=C+1$ there is a qc mapping $g\colon D\to D$ with $g=F_n$ in A, $g|_\sigma=h|_\sigma$ and g is $(H+\varepsilon)$-qc in a certain neighborhood U of σ in D and g is M-qc in $D\backslash E$.

The Main Inequality ([14] or p. 110 in [13]) applied to $g\circ f_n$ and ϕ_n reads

$$1\leqq\iint_D|\phi_n|\,|1-\kappa_{g\circ f_n}\phi_n/|\phi_n|\,|^2/(1-|\kappa_{g\circ f_n}|^2)dxdy\,.$$

Since $g\circ f_n=\mathrm{id}$ in $F_n(A)$ we estimate as usual

$$1\leqq\iint_{F_n(A)}|\phi_n|dxdy+\iint_{D\backslash F_n(A)}|\phi_n|(|1-\kappa_{f_n}\phi_n/|\phi_n|\,|^2/(1-|\kappa_{f_n}|^2))D_g(f_n(z))dxdy\,.$$

Because of (3.3) the second integral can be estimated by

$$(H+\varepsilon)\iint_{F_n(U)}|\phi_n|dxdy/K_n+M\iint_{F_n(D\backslash U\backslash A)}|\phi_n|dxdy/K_n$$

and by (3.4) we get

$$1\leqq\iint_{F_n(A)}|\phi_n|dxdy+(H+\varepsilon)\,(1-||\phi||+2\varepsilon)/K_n+2\varepsilon M/K_n\,.$$

Letting n tend to ∞ and then $\varepsilon\to0$ gives

$$1\leqq||\phi||+H(1-||\phi||)/C$$

or

$$C(1-||\phi||)\leqq H(1-||\phi||)\,.$$

From $H<C$ we hence infer that $||\phi||=1$ and this finishes the proof.

Let m_2 denote 2-dimensional Lebesgue-measure. Then we have

Theorem 3.1.

(a) *If σ is finite or $\bar{E}$ compact in $\bar{D}\backslash\sigma$, then $m_2(\partial E)=0$ implies that $b\to K_b$ is continuous on $C(b_0)\backslash\{1\}$ and so are the functions $b\to\phi_b$, $b\to F_b$ on $C_0(b_0)\backslash\{1\}$.*

(b) *If E is compact in D and $D\backslash E$ a domain, then property* (A) *of E below implies that $b\to K_b$ is also continuous at* 1 *and if in addition $1\in C_0(b_0)$ then so are the functions $b\to\phi_b$ and $b\to F_b$.*

A set E is called to have property (A) if it is a union of finitely many measurable sets $E_1, E_2, \cdots, E_p$ such that: For every $\varepsilon>0$ there are a number M, open quasidisks B_ν and closed quasidisks T_ν with $T_\nu\subset E_\nu\subset B_\nu$, $\overline{B_\nu}\subset D$ for $\nu\leqq p$ such that all $\overline{B_\nu}$ are disjoint and to each ν there is a M-qc mapping H^ν_ε: $B_\nu\to B_\nu$ with

$$H^\nu_\varepsilon=\mathrm{id}\quad \text{on } \partial B_\nu\,,$$
$$\operatorname*{ess\,sup}_{w\in B_\nu\setminus \overline{E}_\nu} D_{H^\nu_\varepsilon}(w)\leqq 1+\varepsilon \quad\text{and}$$
$$\overline{E_\nu}\subset \operatorname{int} H^\nu_\varepsilon(T_\nu)\,.$$

Lemma 3.2. *If E is a disjoint union of finitely many closed quasidisks in D, then E has property* (A).

Proof of Theorem 3.1. Let $b_\infty\in C(b_0)$, $b_\infty\neq 1$ and $b_n\in L(b_0)$ be a sequence which is locally uniformly bounded convergent to b_∞. We first assume that $C=\varliminf_{n\to\infty} K_{b_n}>\operatorname*{Max}_{\zeta\in\sigma} H^\sigma_\zeta$. W. l. o. g. all b_n belong to $L_0(b_0)$ and by Lemma 3.1 we may choose a subsequence, which we call b_n again, such that the uniquely determined $\phi_{b_n}\in\beta_{\sigma',1}$ tend to a $\phi\in\beta_{\sigma',1}$ locally uniformly in $\bar{D}\setminus\sigma'$. By property P1 of the sequence b_n we can also get that $C=\lim_{n\to\infty} K_{b_n}$ and $F_{b_n}\overset{n\to\infty}{\to}F$ locally uniformly in D where $F\in Q(h, \sigma, E, b_\infty)$. By the continuity of b_∞ and the local uniform continuity of b_n and f_{b_n} we have the pointwise limit

$$\kappa_{f_{b_n}}(z)\overset{n\to\infty}{\to}\begin{cases} c\bar{\phi}(z)/|\phi(z)| & \text{a.e. in } F(D\setminus\bar{E})\\ b_\infty(f(z))\bar{\phi}(z)/|\phi(z)| & \text{a.e. in } F(\operatorname{int} E)\end{cases}$$

where $c=(C-1)/(C+1)$ and $f=F^{-1}$. By the assumption $m_2(\partial E)=0$ we hence have a sequence of complex dilatations which converges a.e. in D and hence by Theorem 5.2 in [10], p. 187 we have

$$\kappa_f=\begin{cases} c\bar{\phi}/|\phi| & \text{a.e. in } F(D\setminus E)\\ (b_\infty\circ f)\bar{\phi}/|\phi| & \text{a.e. in } F(E)\,.\end{cases}$$

By [12] we conclude that F is uniquely extremal in $Q(h, \sigma, E, b_\infty)$ and hence $k_{b_\infty}=c$ (and therefore $b_\infty\in C_0(b_0)$), $\phi=\phi_{b_\infty}$ and $F=F_{b_\infty}$. By the same reason we know that every subsequence of the original subsequence b_n must have a subsequence where $K_{b_n}\to K_{b_\infty}$, $\phi_{b_n}\to\phi_{b_\infty}$ and $F_{b_n}\to F_{b_\infty}$ for $n\to\infty$. But then the original sequences themselves must have these limits.

Then we assume that $C=\varliminf_{n\to\infty} K_{b_n}=\operatorname*{Max}_{\zeta\in\sigma} H^\sigma_\zeta$. If $\varlimsup_{n\to\infty} K_{b_n}=\operatorname*{Max}_{\zeta\in\sigma} H^\sigma_\zeta$, then we have by the lower semicontinuity of $b\to K_b$ the inequality $\operatorname*{Max}_{\zeta\in\sigma} H^\sigma_\zeta\leqq K_{b_\infty}\leqq \varliminf_{n\to\infty} K_{b_n}\leqq\varlimsup_{n\to\infty} K_{b_n}=\operatorname*{Max}_{\zeta\in\sigma} H^\sigma_\zeta$ and hence $K_{b_\infty}=\lim_{n\to\infty} K_{b_n}$ and $b_\infty\notin C_0(b_0)$, i.e., the required continuity at b_∞ is proved. If $\varlimsup_{n\to\infty} K_{b_n}>\operatorname*{Max}_{\zeta\in\sigma} H^\sigma_\zeta$ then the former reasoning applied to a subsequence b_{n_k} where $\varliminf_{k\to\infty} K_{b_{n_k}} > \operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma$ then gives $K_{b_\infty} = \varliminf_{k\to\infty} K_{b_{n_k}} > \operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma$ in contradiction to $K_{b_\infty} \leqq \varliminf_{n\to\infty} K_{b_n} = \operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma$, hence this case can not occur and therefore $b_\infty \in C_0(b_0)$, $b_\infty \neq 1$ implies that we are in the first case where all three sequences converge as required. Statement (a) is hence proved.

Let now b_n be a sequence in $L(b_0)$, convergent to 1. Again we first assume that $C > \operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma$ and $b_n \in L_0(b_0)$. By Lemma 3.1 we again pass to a subsequence where $\phi_{b_n} \overset{n\to\infty}{\to} \phi \in \beta_{\sigma',1}$ locally uniformly in $\bar{D}\backslash\sigma'$. By property P2 we also get $C = \lim_{n\to\infty} K_{b_n}$ and $F_{b_n} \overset{n\to\infty}{\to} F$ locally uniformly in the domain $D\backslash E$. The complex dilatations $\kappa_{f_{b_n}}$ then tend to

$$c\bar{\phi}(z)/|\phi(z)| \quad \text{a.e. in } F(D\backslash E)$$

and hence as before we have for $f = F^{-1}$

$$\kappa_f = c\bar{\phi}/|\phi| \quad \text{a.e. in } F(D\backslash E)\,.$$

We only know that $k_1 \leqq c$, since these properties of F are in general not sufficient for absolute extremality. If $k_1 = c$, then F would be absolutely extremal, i.e., $F = F_1$ and $K_1 = C > \operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma$ implies that $1 \in C_0(b_0)$ and by Corollary 2.2 F_1 would be uniquely determined as well as $\phi_1 = \phi \in \beta_{\sigma',1}$. Again the original sequences themselves would have to converge to ϕ_1, F_1 and K_1, respectively. Furthermore, if $k_1 = c$ the same reasoning as above would also treat the case $C = \operatorname{Max}_{\zeta\in\sigma} H_\zeta^\sigma$, since $K_1 \leqq \varliminf_{n\to\infty} K_{b_n} = C$. Then $1 \notin C_0(b_0)$ and $b \to K_b$ is also continuous there.

The crucial part is to show that we even have equality in $K_1 \leqq C$. We are not able to do this for arbitrary sets E but for those with property (A). Then we can define the following variation:

Let F_1 be absolutely extremal, i.e., $K_1 = \operatorname{ess\,sup}_{w\in D\backslash E} D_{F_1}(w)$. Since ∂B_ν and $H_\varepsilon^\nu(\partial T_\nu)$ are quasicircles and $H_\varepsilon^\nu(\partial T_\nu) \subset D\backslash E$, we can extend $F_1|_{H_\varepsilon^\nu(\partial T_\nu)}$ quasiconformally in $H_\varepsilon^\nu(T_\nu)$. We call this extension F_ε^ν.

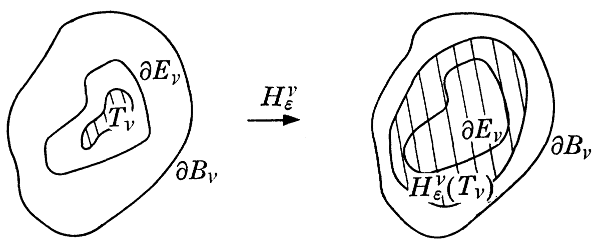

Fig. 2

Then we define

$$G:=\begin{cases} F_1 & \text{in } D\setminus\bigcup_{\nu=1}^{p}\overline{B_\nu} \\ F_1\circ H_\varepsilon^\nu & \text{in } \overline{B_\nu}\setminus T_\nu \\ F_\varepsilon^\nu\circ H_\varepsilon^\nu & \text{in } T_\nu \end{cases}$$

which is a qc mapping in D. Its maximal dilatation in $D\setminus E$ is bounded by $(1+\varepsilon)K_1$ and for $\operatorname{ess\,inf}_{w\in E} b_n(w)$ close to 1 we evidently have $G\in Q(h, \sigma, E, b_n)$, hence

$$(1+\varepsilon)K_1\geqq\varlimsup_{n\to\infty} K_{b_n}\geqq C\,.$$

This holds for every $\varepsilon>0$ and so $C\leqq K_1$, which proves $C=K_1$.

To have some concrete examples for sets with property (A) we finally prove Lemma 3.2:

Let $E=\bigcup_{\nu=1}^{p} E_\nu$ where the E_ν are disjoint closed quasidisks in D. We may choose $T_\nu=E_\nu$ and verify property (A). For every E_ν let f be a conformal mapping of the outside of the unit disk D onto the outside of E_ν which is quasiconformally extended in D. For every ν we choose B_ν to be the image of $D_r=\{z|\ |z|<r\}$ for an $r>1$ such that all $\overline{B_\nu}$ are disjoint. Then for the given $\varepsilon>0$ we define $g_\varepsilon\colon D_r\to D_r$ by

$$g_\varepsilon(z)=\begin{cases} \rho z & |z|\leqq 1 \\ \rho z|z|^{-\log\rho/\log r} & 1\leqq|z|\leqq r \end{cases}$$

where ρ, $1<\rho<r$, is chosen such that g_ε is $(1+\varepsilon)$-qc in $1\leqq|z|\leqq r$, which is possible since with ρ close to one its dilatation $\log r/(\log r/\rho)$ is close to one. Then $H_\varepsilon^\nu:=f\circ g_\varepsilon\circ f^{-1}$ in B_ν has the desired properties.

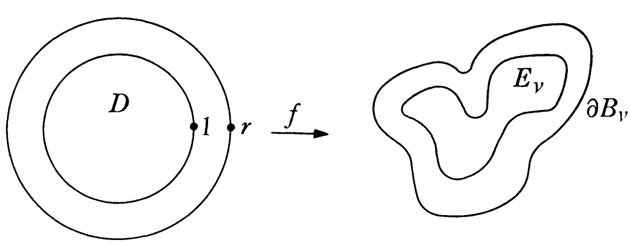

Fig. 3

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