Sakane, Y.
Osaka J. Math.
23 (1986), 585–616

EXAMPLES OF COMPACT EINSTEIN KÄHLER MANIFOLDS WITH POSITIVE RICCI TENSOR

YUSUKE SAKANE[1)]

(Received May 7, 1985)

Let (P, J, g) be a compact Kähler manifold. If (P, J, g) is Einstein Kähler, the first Chern class $c_1(P)$ of P is positive, zero or negative. It has been proved by Aubin [1] and Yau [20] that if (P, J) is a compact complex manifold with $c_1(P)<0$ there exists a unique Einstein Kähler metric on (P, J), and by Yau [20] that if (P, J) is a compact Kähler manifold with $c_1(P)=0$ there exists an Einstein Kähler metric on (P, J). In the case of $c_1(P)>0$ it is known that there exist compact Kähler manifolds which do not admit any Einstein Kähler metric (cf. [6], [8], [19]). Up to now known obstructions to the existence of Einstein Kähler metrics on compact Kähler manifolds with positive first Chern class are (1) Matsushima's theorem ([10], [12]), that is, if (P, J, g) is an Einstein Kähler manifold, the Lie algebra of all Killing vector fields on P is a real form of the Lie algebra of all holomorphic vector fields on P and (2) Futaki invariant [6].

The purpose of this note is to give some examples of compact Einstein Kähler manifolds with positive first Chern class which are not homogeneous. We give a necessary and sufficient condition to the existence of Einstein Kähler metrics on $P^1(\boldsymbol{C})$-bundles over hermitian symmetric spaces of compact type. In the category of Riemannian manifolds, compact Einstein manifolds of cohomogeneity one have been studied by Bérard Bergery [2]. In our case the $P^1(\boldsymbol{C})$-bundle P is of cohomogeneity one with respect to a maximal compact subgroup of the complex Lie group of all holomorphic transformations on P and to prove our Main Theorem we use the similar method used by Bérard Bergery in [2]. We also remark that our Corollary 2 (2) to our Main Theorem generalizes the example given in Futaki [6].

The author would like to express his thanks to professors Tadashi Nagano and Norihito Koiso for their useful suggestions given during the preparation of this paper.

1 Main Theorem

Let M be an irreducible hermitian symmetric space of compact type.

---

1) This work was supported by Grant-in-Aid for Scientific Research (No. 59460001), The Ministry of Education, Science and Culture.

We denote by $H^1(M, \theta^*)$ the isomorphism classes of all holomorphic line bundles over M. It is known that $H^1(M, \theta^*)$ is isomorphic to the second cohomology group $H^2(M, \boldsymbol{Z})\simeq\boldsymbol{Z}$ ([5]). Take a generator L of $H^1(M, \theta^*)$ which has a positive Chern class $c_1(L)>0$. Then the first Chern class $c_1(M)$ of M is given by $c_1(M)=\kappa c_1(L)$ where κ is an integer: $2\leqq\kappa\leqq\dim_{\boldsymbol{C}}M+1$ (cf. [5]).

Consider a product X of two irreducible hermitian symmetric spaces of compact type M_1 and M_2 and a holomorphic vector bundle $p_1^*L_1^a\oplus p_2^*L_2^b$ over X where p_i: $X\rightarrow M_i$ $(i=1, 2)$ are projections, L_i $(i=1, 2)$ are the generators of $H^1(M_i, \theta^*)$ and a, b are positive integers. We denote by P the $P^1(\boldsymbol{C})$-bundle $P(p_1^*L_1^a\oplus p_2^*L_2^b)$ over X. It is not difficult to see that the first Chern class $c_1(P)$ of P is positive if $a<\kappa_1$ and $b<\kappa_2$ where κ_i $(i=1, 2)$ are positive integers given by $c_1(M_i)=\kappa_i c_1(L_i)$ (cf. [15] proof of theorem (5.56)).

Main Theorem. *For irreducible hermitian symmetric spaces of compact type M_1 of complex m-dimension and M_2 of complex n-dimension, and positive integers a, b with $a<\kappa_1$ and $b<\kappa_2$, there exists an Einstein Kähler metric on the compact complex manifold P if and only if*

$$\int_{-1}^{1}(\kappa_1-ax)^m(\kappa_2+bx)^n x dx = 0\,.$$

Corollary 1. *For irreducible hermitian symmetric spaces of compact type $M=M_1=M_2$ and a positive integer $a=b$ with $a<\kappa$, there exists an Einstein Kähler metric on the $P^1(\boldsymbol{C})$-bundle P over $M\times M$.*

Corollary 2.

(1) *For $M=M_1=M_2$ and positive integers a, b such that a, $b<\kappa$ and $a\neq b$, the $P^1(\boldsymbol{C})$-bundle P over $M\times M$ has the first positive Chern class but P does not admit any Einstein Kähler metric.*

(2) *For $M_1=P^1(\boldsymbol{C})$, $M_2\neq P^1(\boldsymbol{C})$ and a positive integer b with $b<\kappa_2$, the $P^1(\boldsymbol{C})$-bundle P over $P^1(\boldsymbol{C})\times M_2$ has the positive first Chern class but P does not admit any Einstein Kähler metric.*

2 Orbits on $P^1(C)$-bundles over a Kähler C-space

Let X be a Kähler C-space, that is, a simply connected compact complex homogeneous space with a Kähler metric. By a result of H.C. Wang [18], X can be written as $X=G/U$ where G is a simply connected complex semisimple Lie group and U is a parabolic subgroup of G. Let ρ: $U\rightarrow\boldsymbol{C}^*$ be a holomorphic representation of U and ξ_ρ the homogeneous holomorphic line bundle on X associated to ρ, that is, ξ_ρ is obtained from the product $G\times\boldsymbol{C}^*$ by identifying (gu, w) with $(g, \rho^{-1}(u)w)$ where $g\in G$, $u\in U$ and $w\in\boldsymbol{C}^*$. It is known that every holomorphic line bundle on a Kähler C-space X is homogeneous (cf. Ise [7]).

For a holomorphic line bundle ξ on X, we consider a $P^1(\boldsymbol{C})$-bundle $P(1\oplus\xi)$ over X, where 1 denotes the trivial line bundle on X. Then G acts on $P(1\oplus\xi)$ in the natural way.

Proposition 2.1. *If ξ is a non-trivial holomorphic line bundle on X, the $P^1(\boldsymbol{C})$-bundle $P=P(1\oplus\xi)$ is a disjoint union of three G-orbits One of orbits is open in P and it is isomorphic to the principal $\boldsymbol{C}^*$-bundle associated to ξ. The other two orbits are isomorphic to X*

Proof The equivalence class of $(g, (w_1, w_2))\in G\times\boldsymbol{C}^2$ is denoted by $[g, (w_1, w_2)]\in 1\oplus\xi$. Let p: $1\oplus\xi-(0\text{-section})\to P$ denote the canonical projection Consider the G-orbit of the point $p[e, (1,1)]$ where e is the identity of G. We shall show that the orbit $G\cdot p[e, (1,1)]$ is isomorphic to the principal $\boldsymbol{C}^*$-bundle associated to the line bundle ξ. Let ρ: $U\to\boldsymbol{C}^*$ denote the holomorphic representation such that $\xi=\xi_\rho$. Then the principal $\boldsymbol{C}^*$-bundle associated to ξ is obtained from the product $G\times\boldsymbol{C}^*$ by identifying (gu, w) with $(g, \rho^{-1}(u)w)$ where $g\in G$, $u\in U$ and $w\in\boldsymbol{C}^*$, and the principal $\boldsymbol{C}^*$-bundle is denoted by $G\times_\rho\boldsymbol{C}^*$. The equivalence class of $(g, w)\in G\times\boldsymbol{C}^*$ is denoted by $[g, w]\in G\times_\rho\boldsymbol{C}^*$. We define a map φ: $G\cdot p[e, (1,1)]\to G\times_\rho\boldsymbol{C}^*$ by $\varphi(gp[e, (1,1)])=[g, 1]$. It is not difficult to see that φ is an injective holomorphic map. Since ρ is not trivial, ρ: $U\to\boldsymbol{C}^*$ is surjective and thus we see that φ is surjective. Moreover for each element $p[g, (w_1, w_2)]$ $(w_1\neq 0, w_2\neq 0)$ there is an element $u\in U$ such that $\rho(u)=w_1^{-1}w_2\in\boldsymbol{C}^*$. Thus $p[g, (w_1, w_2)]=p[gu, (1,1)]$. By the same way we see that the orbits $G\cdot p[e, (1,0)]$ and $G\cdot p[e, (0,1)]$ are isomorphic to $X=G/U$. Thus the orbit $G\cdot p[e, (1,1)]$ is open in $P(1\oplus\xi)$. q.e.d.

For a holomorphic line bundle $\xi=\xi_\rho$ on X let $\tilde{U}$ be the isotropy subgroup of G at $p[e, (1,1)]\in P(1\oplus\xi)$. Then $\tilde{U}=\{g\in U\,|\,\rho(g)=1\}$ and $\dim_{\boldsymbol{C}}\tilde{U}=\dim_{\boldsymbol{C}}U-1$ if ξ is non-trivial. The natural $\boldsymbol{C}^*\times\boldsymbol{C}^*$-action on $1\oplus\xi$ induces a $\boldsymbol{C}^*$-action on $P(1\oplus\xi)$. Note that $G\times\boldsymbol{C}^*$-orbits in $P(1\oplus\xi)$ coincide with G-orbit and that the $\boldsymbol{C}^*$-action on the orbit $G\cdot p[e, (1,1)]$ corresponds to the right $\boldsymbol{C}^*\simeq U/\tilde{U}$-action on the principal fiber bundle $G/\tilde{U}$ over X.

Let G_u denote a maximal compact subgroup of G and $V=G_u\cap U$. Then G_u/V is diffeomorphic to G/U. Put $\tilde{V}=\{g\in V\,|\,\rho(g)=1\}$. If ρ: $U\to\boldsymbol{C}^*$ is non-trivial, $\dim_{\boldsymbol{R}}\tilde{V}=\dim_{\boldsymbol{R}}V-1$.

Proposition 2.2. *Let ρ: $U\to\boldsymbol{C}^*$ be non-trivial. Then the principal $\boldsymbol{C}^*$-bundle $G\times_\rho\boldsymbol{C}^*$ over X is $G_u\times S^1$-equivariantly diffeomorphic to $G_u/\tilde{V}\times\boldsymbol{R}_+$ where $G_u\times S^1$ acts on $\boldsymbol{R}_+$ trivially.*

Proof. For $g\in G$, there exist elements $k\in G_u$ and $u\in U$ such that $g=ku$, since $G_u/V=G/U$. Since each element of $G\times_\rho\boldsymbol{C}^*$ may written as $[g, 1]\in G\times_\rho\boldsymbol{C}^*$, we have $[g, 1]=[k, \rho(u)]$. Let $G_u\times_\rho\boldsymbol{C}^*$ denote the space obtained from the product $G_u \times \boldsymbol{C}^*$ by identifying (kv, w) with $(k, \rho^{-1}(v)w)$ where $k \in G_u$, $v \in V$ and $w \in \boldsymbol{C}^*$. The equivalence class of $(k, w) \in G_u \times \boldsymbol{C}^*$ is also denoted by $[k, w]$. Then the map $[g, 1] \mapsto [k, \rho(g)]$: $G \times_\rho \boldsymbol{C}^* \to G_u \times_\rho \boldsymbol{C}^*$ is a $G_u \times S^1$-equivariantly diffeomorphism. Put $\rho(u) = re^{i\theta}$ $(r \in \boldsymbol{R}_+)$. Then r is uniquely detei mined by the class $[g, 1] \in G \times_\rho \boldsymbol{C}^*$. In fact, if $g = ku = k_1 u_1$ $(k, k_1 \in G_u, u, u_1 \in U)$, $k^{-1}k_1 = uu_1^{-1} \in G_u \cap U = V$. Since $\rho(uu_1^{-1}) \in S^1 = \{e^{i\theta} \mid \theta \in \boldsymbol{R}\}$, $\rho(u_1) = \rho(u_1 u^{-1})\rho(u) = re^{i\theta_1}$ for some $\theta_1 \in \boldsymbol{R}$. Define a map ψ: $G_u \times_\rho \boldsymbol{C}^* \to G_u / \tilde{V} \times \boldsymbol{R}_+$ by $\psi([k, w]) = (kv\tilde{V}, r)$ where $w = re^{i\theta}$ and $\rho(v) = e^{i\theta}$ $(v \in V)$. Then it is easy to see that ψ is a $G_u \times S^1$-equivariantly diffeomorphism. q.e.d.

For a compact complex manifold Y let $\mathrm{Aut}_0(Y)$ denote the connected component of the identity of the group of all holomorphic automorphisms of Y.

Proposition 2.3. *Let ξ be a non-trivial holomorphic line bundle on a Kähler C-space $X = G/U$. Then the complex Lie group* $\mathrm{Aut}_0(P(1 \oplus \xi))$ *is reductive if and only if $H^0(X, \xi) = H^0(X, \xi^{-1}) = (0)$. Moreover in this case the Lie algebra of* $\mathrm{Aut}_0(P(1 \oplus \xi))$ *coincides with the Lie algebra of* $\mathrm{Aut}_0(X) \times \boldsymbol{C}^*$.

Proof. Let π: $P(1 \oplus \xi) \to X$ be the natural projection. By a theorem of Blanchard [4], the projection π induces a Lie group homomorphism, denoted also by π,

$$\pi \colon \mathrm{Aut}_0(P(1 \oplus \xi)) \to \mathrm{Aut}_0(X) .$$

It is known that the Lie algebra of $\mathrm{Ker}\,\pi$ is isomorphic to $H^0(X, \mathrm{End}(1 \oplus \xi))$ and thus it is isomorphic to

$$\left\{ \begin{pmatrix} w_1 & s_1 \\ s_2 & w_2 \end{pmatrix} \mid s_1 \in H^0(X, \xi),\ s_2 \in H^0(X, \xi^{-1}),\ w_1, w_2 \in \boldsymbol{C} \right\} \Big/ \left\{ \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \mid w \in \boldsymbol{C} \right\}$$

(cf. [8]). By a Borel-Weil theorem (cf. for example [7]), for a non-trivial holomorphic line bundle ξ, if $H^0(X, \xi) \neq 0$, $H^0(X, \xi^{-1}) = 0$. Thus if one of $H^0(X, \xi)$, $H^0(X, \xi^{-1})$ is non-zero, $\mathrm{Aut}_0(P(1 \oplus \xi))$ is not reductive. Conversely, if $H^0(X, \xi) = H^0(X, \xi^{-1}) = (0)$, $\dim_{\boldsymbol{C}} \mathrm{Ker}\,\pi = 1$. Note also that π: $\mathrm{Aut}_0(P(1 \oplus \xi)) \to \mathrm{Aut}_0(X)$ is surjective. The Lie algebra of $\mathrm{Aut}_0(P(1 \oplus \xi))$ always contains the Lie algebra of $\mathrm{Aut}_0(X) \times \boldsymbol{C}^*$. Thus the Lie algebra of $\mathrm{Aut}_0(P(1 \oplus \xi))$ coincides with the Lie algebra of $\mathrm{Aut}_0(X) \times \boldsymbol{C}^*$, which is reductive, since $\mathrm{Aut}_0(X)$ is a complex semi-simple Lie group. q.e.d.

Corollary 2.4. *Let ξ be a non-trivial holomorphic line bundle on a Kähler C-space. Then $P(1 \oplus \xi)$ is almost homogeneous but not homogeneous.*

Proof. By proposition 2.1, $P(1 \oplus \xi)$ is almost homogeneous. If $\mathrm{Aut}_0(P(1 \oplus \xi))$ acts transitively on the simply connected compact projective manifold $P(1 \oplus \xi)$, the Lie group $\mathrm{Aut}_0(P(1\oplus\xi))$ is a semi-simple complex Lie group (cf. Takeuchi [16] p. 174). Since $\mathrm{Aut}_0(P(1\oplus\xi))$ is not semi-simple by Proposition 2.3, this is a contradiction. q.e.d.

3 $G_u\times S^1$-invariant Kähler metrics on the open orbit

We consider a $G_u\times S^1$-invariant Kähler metric on the open orbit $G\cdot p[e, (1,1)]\cong G\times_\rho C^*$ in $P(1\oplus\xi)$. Let $\mathfrak{g}_u$, $\mathfrak{v}$, $\tilde{\mathfrak{v}}$ be the Lie algebra of G_u, V, $\tilde{V}$ respectively. Since G_u is a compact semi-simple Lie group, the Killing form of $\mathfrak{g}_u$ is negative definite. Let $\langle\ ,\ \rangle$ denote the $\mathrm{Ad}(G_u)$-invariant inner product on $\mathfrak{g}_u$ induced from the Killing form and let $\mathfrak{m}\subset\mathfrak{g}_u$ be the orthogonal complement of $\mathfrak{v}$ with respect to the inner product $\langle\ ,\ \rangle$. Then $\mathfrak{g}_u=\mathfrak{v}+\mathfrak{m}$ and $[\mathfrak{v}, \mathfrak{m}]\subset\mathfrak{m}$. Let $\mathfrak{c}_\mathfrak{p}$ be the orthogonal complement of $\tilde{\mathfrak{v}}$ in $\mathfrak{v}$ with respect to the inner product $\langle\ ,\ \rangle$. Then we have

$$[\mathfrak{c}_\mathfrak{p}, \tilde{\mathfrak{v}}]=(0)\,. \tag{3.1}$$

In fact, we can write $\mathfrak{v}=\mathfrak{c}+\mathfrak{v}_s$ where $\mathfrak{c}$ is the center of $\mathfrak{v}$ and $\mathfrak{v}_s$ is the semi-simple part of $\mathfrak{v}$. Note that $\langle\mathfrak{c}, \mathfrak{v}_s\rangle=(0)$ and $\tilde{\mathfrak{v}}\supset\mathfrak{v}_s$. Thus $\mathfrak{c}_\mathfrak{p}\subset\mathfrak{c}$ and hence $[\mathfrak{c}_\mathfrak{p}, \tilde{\mathfrak{v}}]=(0)$. Moreover if the holomorphic representation $\rho\colon U\to C^*$ corresponds to the weight Λ, then $\sqrt{-1}\Lambda$ generates $\mathfrak{c}_\mathfrak{p}$ and thus $\mathfrak{c}_\mathfrak{p}$ generates a closed subgroup of G_u, that is, a circle group S^1.

Put $\mathfrak{p}=\mathfrak{c}_\mathfrak{p}+\mathfrak{m}$. Then we have orthogonal decompositions of $\mathfrak{g}_u$, $\mathfrak{p}$ and $\mathfrak{v}$ with respect to $\langle\ ,\ \rangle$:

$$\mathfrak{g}_u=\tilde{\mathfrak{v}}+\mathfrak{p},\ \mathfrak{p}=\mathfrak{c}_\mathfrak{p}+\mathfrak{m},\ \mathfrak{v}=\tilde{\mathfrak{v}}+\mathfrak{c}_\mathfrak{p}\,. \tag{3.2}$$

Moreover we have

$$[\mathfrak{v}, \mathfrak{c}_\mathfrak{p}]=(0),\ [\mathfrak{v}, \mathfrak{m}]\subset\mathfrak{m}\,. \tag{3.3}$$

Let $\boldsymbol{R}_+$ be the subgroup of C^* defined by $\{r>0\,|\,re^{i\theta}\in C^*\}$. Since the open orbit $G\times_\rho C^*$ in $P(1\oplus\xi)$ is also a $G\times C^*$-orbit in $P(1\oplus\xi)$ and $G\times_\rho C^*$ is diffeomorphic to $G_u/\tilde{V}\times\boldsymbol{R}_+$, the Lie subgroup $G_u\times\boldsymbol{R}_+$ of $G\times C^*$ also acts on $G\times_\rho C^*$ transitively. Take a basis $\{\tilde{H}\}$ of the Lie algebra of $\boldsymbol{R}_+$. Then $\mathfrak{g}_u+\boldsymbol{R}\tilde{H}=\tilde{\mathfrak{v}}+\mathfrak{p}+\boldsymbol{R}\tilde{H}$ and $\mathrm{Ad}(\tilde{V})(\mathfrak{p}+\boldsymbol{R}\tilde{H})\subset\mathfrak{p}+\boldsymbol{R}\tilde{H}$. We identify $\mathfrak{p}+\boldsymbol{R}\tilde{H}$ with the tangent space $T_0(G\times_\rho C^*)$ at the origin $o=[e, 1]$ of $G\times_\rho C^*$. Since the complex structure J on $G\times_\rho C^*$ is invariant by the action of $G\times C^*$, it induces a linear isomorphism $I\colon\mathfrak{p}+\boldsymbol{R}\tilde{H}\to\mathfrak{p}+\boldsymbol{R}\tilde{H}$ which satisfies $I^2=-id$ and $I\circ\mathrm{Ad}(g)=\mathrm{Ad}(g)\circ I$ for every $g\in\tilde{V}$. Note that at the origin o of $G\times_\rho C^*$ the orbit of the right S^1-action coincides with the orbit of the left S^1-action defined by $\mathfrak{c}_\mathfrak{p}$ and that the complex structure of the fiber C^* is induced from the natural complex structure of C. Therefore we have

$$I\mathfrak{c}_\mathfrak{p}=\boldsymbol{R}\tilde{H}\,. \tag{3.4}$$

Moreover, since the complex structure on $P(1\oplus\xi)$ is compatible with the invariant complex structure on $G/U=G_u/V$,

$$(3.5)\qquad I\mathfrak{m}=\mathfrak{m}\,.$$

To investigate a $G_u\times S^1$-invariant hermitian metric on the open orbit $G\times_\rho \boldsymbol{C}^*$, we consider a $G_u\times\boldsymbol{R}_+$-invariant hermitian metric on $G\times_\rho\boldsymbol{C}^*=G_u/\tilde{V}\times\boldsymbol{R}_+$ for the moment. Note that there is a natural one-to-one correspondence between $G_u\times\boldsymbol{R}_+$-invariant hermitian metrics on $G_u/\tilde{V}\times\boldsymbol{R}_+$ and the $\mathrm{Ad}(\tilde{V})$-invariant hermitian inner products on $\mathfrak{p}+\boldsymbol{R}\tilde{H}$ (cf. [11]).

From now on we assume that

$$(3.6)\qquad [\tilde{\mathfrak{v}},\mathfrak{m}]=\mathfrak{m}\,.$$

Let B be an $\mathrm{Ad}(\tilde{V})$-invariant hermitian inner product on $\mathfrak{p}+\boldsymbol{R}\tilde{H}$. Then B has the following properties:

$$(3.7)\qquad \text{(a)}\ \ B(\mathfrak{c}_\mathfrak{p},\tilde{H})=(0)\qquad \text{(b)}\ \ B(\mathfrak{c}_\mathfrak{p},\mathfrak{m})=(0)$$
$$\text{(c)}\ \ B(\tilde{H},\mathfrak{m})=(0)\,.$$

In fact, (a) follows from (3.4). To see (b), $B(\mathfrak{c}_\mathfrak{p},\mathfrak{m})=B(\mathfrak{c}_\mathfrak{p},[\tilde{\mathfrak{v}},\mathfrak{m}])=B([\tilde{\mathfrak{v}},\mathfrak{c}_\mathfrak{p}],\mathfrak{m})=(0)$ by (3.1). Now (c) follows from (b) and (3.5).

We decompose $\tilde{\mathfrak{v}}$-module $\mathfrak{m}$ into irreducible component $\mathfrak{m}_j$; $\mathfrak{m}=\sum_j\mathfrak{m}_j$. By (3.6) we have

$$(3.8)\qquad [\tilde{\mathfrak{v}},\mathfrak{m}_j]=\mathfrak{m}_j\qquad\text{for every } j\,.$$

From now on we also assume that

$$(3.9)\qquad [\mathfrak{v},\mathfrak{m}_j]=\mathfrak{m}_j\qquad\text{for every } j\,,$$

$$(3.10)\qquad I\mathfrak{m}_j=\mathfrak{m}_j\qquad\text{for every } j \text{ and}$$

(3.11) each multiplicity of irreducible components of $\mathfrak{m}$ as $\tilde{\mathfrak{v}}$-module is 1.

Now the hermitian inner product B can be written uniquely as

$$(3.12)\qquad B=d(\langle\ ,\ \rangle|_{\mathfrak{c}_\mathfrak{p}}+\langle I\circ,I\circ\rangle|_{\boldsymbol{R}\tilde{H}})+\sum_j c_j\langle\ ,\ \rangle|_{\mathfrak{m}_j}$$

where d, c_j are positive real numbers, $\langle\ ,\ \rangle|\mathfrak{c}_\mathfrak{p}$ and $\langle\ ,\ \rangle|\mathfrak{m}_j$ denote the inner products on $\mathfrak{c}_\mathfrak{p}$ and $\mathfrak{m}_j$ induced from $\langle\ ,\ \rangle$ respectively, and $\langle I\circ,I\circ\rangle_{\boldsymbol{R}\tilde{H}}$ denotes the inner product on $\boldsymbol{R}\tilde{H}$ defined by $\langle IX,IY\rangle$ for $X, Y\in\boldsymbol{R}\tilde{H}$. Note that $\langle\ ,\ \rangle|\mathfrak{c}_\mathfrak{p}$, $\langle I\circ,I\circ\rangle|_{\boldsymbol{R}\tilde{H}}$ and $\langle\ ,\ \rangle|\mathfrak{m}_j$ are $\mathrm{Ad}(\tilde{V})$-invariant symmetric bilinear form on $\mathfrak{p}+\boldsymbol{R}\tilde{H}$. Let β_0, β_1, α_j be the $G_u\times\boldsymbol{R}_+$-invariant symmetric tensors on $G_u/\tilde{V}\times\boldsymbol{R}_+$ corresponding to $\langle\ ,\ \rangle|\mathfrak{c}_\mathfrak{p}$, $\langle I\circ,I\circ\rangle|_{\boldsymbol{R}\tilde{H}}$, $\langle\ ,\ \rangle_{\mathfrak{m}_j}$ respectively. Then the $G_u\times\boldsymbol{R}_+$-invariant hermitian metric g_B corresponding to B is given by

$$g_B = d(\beta_0+\beta_1)+\sum_j c_j\alpha_j\,.$$

Lemma 3.1. *The $G_u\times \boldsymbol{R}_+$-invariant symmetric tensors β_0, β_1 on $G_u/\tilde{V}\times\boldsymbol{R}_+$ are invariant by the right* S^1-action.

Proof. (cf. [9] §2) Let $\tilde{\gamma}$ be $\mathfrak{c}_{\mathfrak{p}}$-valued left invariant 1-form on G_u, defined by

$\tilde{\gamma}(Y)$=the $\mathfrak{c}_{\mathfrak{p}}$-component of $Y\in\mathfrak{g}_u$ with respect to the decomposition $\mathfrak{g}_u=\tilde{\mathfrak{v}}+\mathfrak{c}_{\mathfrak{p}}+\mathfrak{m}$.
Then there is a unique G_u-invariant connection, called the canonical connection, on the principal S^1-bundle $G_u/\tilde{V}$ over G_u/V such that the connection form γ is given by $\pi_1{}^*\gamma=\tilde{\gamma}$ where $\pi_1\colon G_u\to G_u/\tilde{V}$ is the canonical projection. Using the connection form γ, the symmetric tensor β_0 on $G_u/\tilde{V}\times\boldsymbol{R}_+$ can be written as $\beta_0=\langle\gamma,\gamma\rangle$, that is, $\beta_0(X,Y)=\langle\gamma(X),\gamma(Y)\rangle$ for $X, Y\in T_p(G_u/\tilde{V}\times\boldsymbol{R}_+), p\in G_u/\tilde{V}\times\boldsymbol{R}_+$. In particular, β_0 is invariant by the right S^1-action. We also have $\beta_1=\langle\gamma\circ J,\gamma\circ J\rangle$. Since the right S^1-action is holomorphic, β_1 is also invariant by the right S^1-action. q.e.d.

Let $\tilde{\alpha}_j$ denote the G_u-invariant symmetric tensor on $X=G_u/V$ corresponding to Ad(V)-invariant symmetric bilinear form $\langle\ ,\ \rangle|_{\mathfrak{m}_j}$ on $\mathfrak{m}$. Let $\pi\colon G\times_\rho \boldsymbol{C}^*\to G_u/V$ denote the canonical projection. Then we have $\alpha_j=\pi^*\tilde{\alpha}_j$. In particular, α_j is also invariant by the right S^1-action.

We now consider a $G_u\times S^1$-invariant hermitian metric g on $G\times_\rho\boldsymbol{C}^*\simeq G_u/\tilde{V}\times\boldsymbol{R}_+$. Let $\tilde{X}$ denote the vector field on $G_u/\tilde{V}\times\boldsymbol{R}_+$ induced by $X\in\mathfrak{g}_u$.

Proposition 3.2. *A $G_u\times S^1$-invariant hermitian metric g on $G\times_\rho\boldsymbol{C}^*$ can be written as*

$$g = F^2(\beta_0+\beta_1)+\sum_j H_j^2\alpha_j \tag{3.13}$$

where F, H_j are $G_u\times S^1$-invariant positive valued C^∞ functions on $G\times_\rho\boldsymbol{C}^*$.

Proof. We denote by $\tilde{o}$ the origin of $G_u/\tilde{V}$ and identify the tangent space $T_{(\tilde{o},r)}(G_u/\tilde{V}\times\boldsymbol{R}_+)$ at $(\tilde{o}, r)$ with $\mathfrak{c}_{\mathfrak{p}}+\mathfrak{m}+\boldsymbol{R}\frac{\partial}{\partial r}$. Then

$$g_{(\tilde{o},r)}\left(u,\frac{\partial}{\partial r}\right) = 0 \qquad \text{for } u\in T_{\tilde{o}}(G_u/\tilde{V})\,. \tag{3.14}$$

In fact, if $u\in\mathfrak{m}$, then $u=\sum_i[\tilde{X}_i,\tilde{Y}_i]_{\tilde{o}}$ for some $X_i\in\tilde{\mathfrak{v}}$, $Y_i\in\mathfrak{m}$ by our assumption (3.6). Since $(\tilde{X}_i)_{\tilde{o}}=0$ and $[\tilde{X}_j,\frac{\partial}{\partial r}]=0$, we have $g_{(\tilde{o},r)}(u,\frac{\partial}{\partial r})=\sum_i g_{(\tilde{o},r)}([\tilde{X}_i,\tilde{Y}_i]_{\tilde{o}},\frac{\partial}{\partial r})=-\sum_i g_{(\tilde{o},r)}(Y_i,[\tilde{X}_i,\frac{\partial}{\partial r}]_{(\tilde{o},r)})=0$. Since the orbits of the left and right S^1-actions at the point $(\tilde{o},r)\in G_u/\tilde{V}\times\boldsymbol{R}_+$ coincide, we have $I\mathfrak{c}_{\mathfrak{p}}=\boldsymbol{R}\frac{\partial}{\partial r}$.

Therefore $g_{(\tilde{o},r)}(u, \frac{\partial}{\partial r})=0$ if $u\in\mathfrak{c}_{\mathfrak{p}}$.

Since G_u acts on $\boldsymbol{R}_+$ trivially, for each point $(p, r)\in G_u/\tilde{V}\times\boldsymbol{R}_+$

$$g_{(p,r)}(u, \frac{\partial}{\partial r}) = 0 \qquad \text{for } u\in T_p(G_u/\tilde{V})\,. \tag{3.15}$$

Now it is easy to see that g can be written as

$$g = F_0^2\beta_0+F_1^2\beta_1+\sum_j H_j^2\alpha_j$$

where F_0, F_1 and H_j are positive valued C^∞-functions on $G_u/\tilde{V}\times\boldsymbol{R}_+$. Since g, β_0, β_1 and α_j are $G_u\times S^1$-invariant, so are F_0, F_1 and H_j. Moreover we have $F_0=F_1$, since $\beta_1(X, Y)=\beta_0(JX, JY)$ and g is hermitian. q.e.d.

Now we consider conditions that a $G_u\times S^1$-invariant hermitian metric g on $G\times_\rho\boldsymbol{C}^*$ of the form (3.13) to be Kähler. For $X\in\mathfrak{c}_{\mathfrak{p}}$ let X^* denote the vector field on $G_u/\tilde{V}\times\boldsymbol{R}_+$ induced by the right action of $S^1=\{\exp tX\,|\,t\in\boldsymbol{R}\}$. For a fixed non-zero $X\in\mathfrak{c}_{\mathfrak{p}}$, define 1-forms θ_0 and θ_1 on $G_u/\tilde{V}\times\boldsymbol{R}_+$ by

$$\theta_0(A) = \beta_0(X^*, A) \tag{3.16}$$

$$\theta_1(A) = -\beta_1(JX^*, A) \tag{3.17}$$

where A is a C^∞-vector field on $G_u/\tilde{V}\times\boldsymbol{R}_+$. Then θ_0 and θ_1 are $G_u\times S^1$-invariant forms.

Lemma 3.3. *At the origin $o\in G\times_\rho\boldsymbol{C}^*$, we have*

(1) $d\theta_1=0$

(2) $d\theta_0(Y, Z)=\begin{cases} -\langle X, [Y, Z]\rangle & \textit{if } Y, Z\in\mathfrak{m} \\ 0 & \textit{otherwise}\,. \end{cases}$

Proof. Since θ_0 and θ_1 are G_u-invariant, $L_{\tilde{Y}}\theta_0=L_{\tilde{Y}}\theta_1=0$ for $Y\in\mathfrak{p}$. For $Y, Z\in\mathfrak{p}$, $(d\theta_i)(\tilde{Y}, \tilde{Z})=\tilde{Y}\theta_i(\tilde{Z})-\tilde{Z}\theta_i(\tilde{Y})-\theta_i([\tilde{Y}, \tilde{Z}])=-\theta_i([\tilde{Z}, \tilde{Y}])=\theta_i(\widetilde{[Z, Y]})$, $i=0, 1$. Thus $d\theta_1(Y, Z)=0$ and $d\theta_0(Y, Z)=-\langle X, [Y, Z]\rangle$. For $Y\in\mathfrak{p}$, $d\theta_i(\tilde{Y}, \frac{\partial}{\partial r})=\tilde{Y}\theta_i(\frac{\partial}{\partial r})-\frac{\partial}{\partial r}\theta_i(\tilde{Y})-\theta_i([\tilde{Y}, \frac{\partial}{\partial r}])=-\frac{\partial}{\partial r}\theta_i(\tilde{Y})=-\theta_i([\frac{\partial}{\partial r}, \tilde{Y}])=0$. Therefore $d\theta_i(Y, \tilde{H})=0$ for $Y\in\mathfrak{p}$. q.e.d.

Let ω be the Kähler form on $G\times_\rho\boldsymbol{C}^*$ of a hermitian metric g, that is, $\omega(A, B)=g(A, JB)$, and let ω_j be the 2-form on $G_\rho\times\boldsymbol{C}^*$ corresponding to the J-invariant symmetric forms α_j. The Kähler form ω on $G\times_\rho\boldsymbol{C}^*$ corresponding to the hermitian metric g of the form (3.13) is given by

$$\omega = \frac{F^2}{\beta_0(X^*, X^*)}\theta_0\wedge\theta_1+\sum_j H_j^2\omega_j\,. \tag{3.18}$$

Now we define a vector field H on $G\times_\rho \boldsymbol{C}^*$ by

$$H=-\frac{1}{g(X^*, X^*)^{1/2}}JX^* . \tag{3.19}$$

Proposition 3.4. *Assume that every 2-form ω_j is d-closed. Then a hermitian metric g on $G\times_\rho \boldsymbol{C}^*$ of the form (3.13) is Kähler if and only if*

$$-\frac{F}{\langle X, X\rangle^{1/2}}\langle X, [A, IB]\rangle+\sum_j d(H_j^2)(H)\langle A, B\rangle|_{\mathfrak{m}_j}=0 \tag{3.20}$$

where $A, B\in\mathfrak{m}$, $0\neq X\in\mathfrak{c}_\mathfrak{p}$.

Proof. Since $dF=-(JX^*)F\dfrac{1}{\beta_0(X^*, X^*)}\theta_1$, $d\theta_1=0$ and $d\omega_j=0$, $d\omega=\dfrac{F^2}{\beta_0(X^*, X^*)}d\theta_0\wedge\theta_1+\sum_j d(H_j^2)\wedge\omega_j$. For A, $C\in\mathfrak{m}$, $(d\theta_0\wedge\theta_1)(\tilde{A}, \tilde{C}, JX^*)=-\theta_0(\widetilde{[A, C]})\beta_0(X^*, X^*)$. Note also that $(d\theta_0\wedge\theta_1)(\tilde{A}, \tilde{B}, \tilde{C})=(\theta_1\wedge\omega_j)(\tilde{A}, \tilde{B}, \tilde{C})=0$ for A, B, $C\in\mathfrak{m}$, $(d\theta_0\wedge\theta_1)(\tilde{A}, X^*, JX^*)=(\theta_1\wedge\omega_j)(\tilde{A}, X^*, JX^*)=0$ for $A\in\mathfrak{m}$, $X\in\mathfrak{c}_\mathfrak{p}$ and $(d\theta_0\wedge\theta_1)(\tilde{A}, \tilde{B}, X^*)=(\theta_1\wedge\omega_j)(\tilde{A}, \tilde{B}, X^*)=0$ for A, $B\in\mathfrak{m}$, $X\in\mathfrak{c}_\mathfrak{p}$. Thus we have $d\omega=0$ if and only if, at $(\tilde{o}, r)\in G_u/\tilde{V}\times\boldsymbol{R}_+$,

$$d\omega(\tilde{A}, \tilde{C}, JX^*)=0 \qquad \text{for } A, C\in\mathfrak{m} \text{ and } X\in\mathfrak{c}_\mathfrak{p} . \tag{3.21}$$

Since

$$\begin{aligned} d\omega(\tilde{A}, \tilde{C}, JX^*) &= -F^2\theta_0(\widetilde{[A, C]})+\sum_j d(H_j^2)(JX^*)\omega_j(\tilde{A}, \tilde{C}) \\ &= -F^2\beta_0(X^*, \widetilde{[A, C]})-g(X^*, X^*)^{1/2}\sum_j d(H_j^2)(H)\omega_j(\tilde{A}, \tilde{C}) \\ &= -F^2\beta_0(X^*, \widetilde{[A, C]})-F\beta_0(X^*, X^*)^{1/2}\sum_j d(H_j^2)(H)\omega_j(\tilde{A}, \tilde{C}) , \end{aligned}$$

we see that (3.21) holds if and only if

$$F\beta_0(X^*, \widetilde{[A, C]})/(\beta_0(X^*, X^*)^{1/2})+\sum_j d(H_j^2)(H)\alpha_j(\tilde{A}, J\tilde{C})=0$$

for A, $C\in\mathfrak{m}$ and $X\in\mathfrak{c}_\mathfrak{p}$. Therefore $d\omega=0$ if and only if

$$F\langle X, [A, C]\rangle/(\langle X, X\rangle^{1/2})+\sum_j d(H_j^2)(H)\langle A, IC\rangle|_{\mathfrak{m}_j}=0$$

for A, $C\in\mathfrak{m}$, $X\in\mathfrak{c}_\mathfrak{p}$. Since $I\mathfrak{m}_j=\mathfrak{m}_j$, we get our claim by putting $B=IC$. q.e.d.

4 Extensive conditions of a $G_u\times S^1$-invariant metric

Now we consider conditions of a $G_u\times S^1$-invariant Kähler metric on the open orbit $G\times_\rho \boldsymbol{C}^*$ which can be extended to a Kähler metric on $P(1\oplus\xi)$. For a Kähler manifold (Y, J, g) let ∇ denote the Riemannian connection.

Lemma 4.1. *For a holomorphic Killing vector field X on Y and a Killing vector field A on Y such that $[A, X]=0$, we have $g(\nabla_{JX}JX, A)=0$.*

Proof. Since A is a Killing vector field, $Ag(X, X)=2g([A, X], X)=0$. Thus $g(\nabla_A X, X)=\frac{1}{2}Ag(X, X)=0$. Since X is also Killing, $g(\nabla_X X, A)+g(X, \nabla_A X)=0$. Therefore $g(\nabla_X X, A)=0$. Since g is a Kähler metric and X is holomorphic, $\nabla_{JX}JX=J\nabla_{JX}X=J\nabla_X JX=-\nabla_X X$, and hence we get $g(\nabla_{JX}JX, A)=0$. q.e.d.

Now we consider a $G_u\times S^1$-invariant Kähler metric g on the open orbit $G\times_\rho \boldsymbol{C}^*$ of the form (3.13). Let H be the vector field on $G\times_\rho \boldsymbol{C}^*$ defined by (3.19).

Lemma 4.2. *On the open orbit $G\times_\rho \boldsymbol{C}^*$, we have*

$$\nabla_H H=0\,. \tag{4.1}$$

Proof. By Lemma 4.1, we have $g(\nabla_{JK^*}JX^*, \tilde{A})=0$ for a Killing vector field $\tilde{A}$ on $G\times_\rho \boldsymbol{C}^*$ where $A\in\mathfrak{g}_u$. Since

$$\nabla_H H=\frac{1}{g(X^*, X^*)}\nabla_{JX^*}JX^*+\frac{1}{g(X^*, X^*)^{1/2}}(JX^*)\,(g(X^*, X^*)^{1/2})JX^*$$

and $g(JX^*, \tilde{A})=0$, we have $g(\nabla_H H, \tilde{A})=0$. Since $g(H, H)=1$, $g(\nabla_H H, H)=0$. Therefore we have $\nabla_H H=0$, q.e.d.

Let $\rho\colon U\to\boldsymbol{C}^*$ be the holomorphic representation corresponding to the weight Λ and identify $\sqrt{-1}\Lambda$ with an element of $\mathfrak{c}_\mathfrak{p}$. From now on denote by X_0 the element of $\mathfrak{c}_\mathfrak{p}$ defined by $\Lambda(X_0)=\sqrt{-1}$. Then the right S^1-action $\{\exp tX_0\,|\,t\in\boldsymbol{R}\}$ on $P(1\oplus\xi_\rho)$ corresponds to the natural S^1-action on $P(1\oplus\xi_\rho)$ induced by the S^1-action on each fiber $P^1(\boldsymbol{C})$. We also define a symmetric tensor β_0 on $G_u/\tilde{V}\times\boldsymbol{R}_+$ by $\tilde{\beta}_0=(1/\langle X_0, X_0\rangle)\beta_0$ and a function $\tilde{F}$ on $G_u/\tilde{V}\times\boldsymbol{R}_+$ by $\tilde{F}=\langle X_0, X_0\rangle^{1/2}F$ for a C^∞ function F on $G_u/\tilde{V}\times\boldsymbol{R}_+$. Then $\tilde{F}^2\tilde{\beta}_0=F^2\beta_0$. Let r be the canonical coordinate of $\boldsymbol{R}_+$ as before. Thus we have $JX_0^*=-r(\partial/\partial r)$ on $G_u/\tilde{V}\times\boldsymbol{R}_+$. Thus a $G_u\times S^1$-invariant hermitian metric g on $G_u/\tilde{V}\times\boldsymbol{R}_+$ of the form (3.13) can be written as

$$g=(\tilde{F}/r)^2dr^2+\tilde{F}^2\tilde{\beta}_0+\sum_j H_j^2\alpha_j\,. \tag{4.2}$$

Now we consider a $G_u\times S^1$-invariant Kähler metric g_0 on $P(1\oplus\xi_\rho)$. We know that there is a $G_u\times S^1$-invariant Kähler metric on $P(1\oplus\xi_\rho)$, since $P(1\oplus\xi_\rho)$ is a Kähler manifold and the compact Lie group $G_u\times S^1$ acts on $P(1\oplus\xi_\rho)$ as a holomorphic transformation group. Note that the functions $\tilde{F}$ and H_j can be regarded as functions on $\boldsymbol{R}_+$, since they are $G_u\times S^1$-invariant.

Lemma 4.3. *For a $G_u\times S^1$-invariant Kähler metric g_0 on $P(1\oplus\xi)$, let its restriction g_0 to the open orbit $G_u/\tilde{V}\times\boldsymbol{R}_+$ be of the form* (4.2). *Then the function $\tilde{F}$ extends to a C^∞-function $\tilde{F}\colon[0, \infty)\to\boldsymbol{R}$ such that $\tilde{F}(0)=0$, $\tilde{F}'(0)>0$ and*

$\tilde{F}(r)$ is an odd function at $r=0$, that is, $\tilde{F}(r)=-\tilde{F}(-r)$, and the functions H_j extend to C^∞ functions H_j: $[0, \infty)\to \boldsymbol{R}_+$ such that $H_j(0)>0$ and H_j are even functions at $r=0$.

Proof. Note that the intersection of the open orbit $G_u/\tilde{V}\times \boldsymbol{R}_+$ and a fiber $P^1(\boldsymbol{C})$ is identified with $\boldsymbol{C}^*$ and that the right S^1-action on $G_u/\tilde{V}\times\boldsymbol{R}_+$ induces a natural S^1-action on $\boldsymbol{C}^*$. On the intersection $\boldsymbol{C}^*$, the metric g_0 is given by

$$g_{0|P^1(\boldsymbol{C})} = (\tilde{F}(r)/r)^2 dr^2 + \tilde{F}(r)^2 d\theta^2 \tag{4.3}$$

by using polar coordinates (r, θ) on $\boldsymbol{C}^*$, and thus it is written as

$$g_{0|P^1(\boldsymbol{C})} = (\tilde{F}(r)/r)^2(dx^2+dy^2) \qquad \text{on } \boldsymbol{C}^*$$

by using a canonical coordinate $z=x+\sqrt{-1}y$ on $\boldsymbol{C}$. Therefore a metric $(\tilde{F}(r)/r)^2dr^2+\tilde{F}(r)^2d\theta^2$ extends to a metric on $\boldsymbol{C}$ if and only if $\tilde{F}$ extends to a C^∞ function $\tilde{F}$: $[0, \infty)\to\boldsymbol{R}$ such that $\tilde{F}(0)=0$, $\tilde{F}'(0)>0$ and $\tilde{F}$ is an odd function at $r=0$ (cf. [3] Proposition 4.6). By the same way we see that H_j extend to C^∞ functions H_j: $[0, \infty)\to\boldsymbol{R}_+$ such that $H_j(0)>0$ and H_j are even functions at $r=0$. q.e.d.

We now consider a geodesic $c(t)$ of the compact Kähler manifold $(P(1\oplus\xi), g_0)$ through the origin $c(t_0)=(\tilde{o}, 1)\in G_u/\tilde{V}\times\boldsymbol{R}_+$ with $\dot{c}(t_0)=H_{c(t_0)}$, parametrized by arc length. Since $\nabla_H H=0$, $c(t)$ is the integral curve of H through $(\tilde{o}, 1)$, that is,

$$\dot{c}(t) = H_{c(t)}\,. \tag{4.4}$$

Note also that

$$H = -(1/\tilde{F}(r))JX_0^* = (r/\tilde{F}(r))\,(\partial/\partial r)\,. \tag{4.5}$$

We set $\dot{c}(t)=(dr/dt)\,(\partial/\partial r)$. Then $c(t)$ satisfies an ordinary differential equation

$$dr/dt = r/\tilde{F}(r)\,. \tag{4.6}$$

By Lemma 4.3, the function $\tilde{F}(r)/r$ extends to a C^∞ function $\tilde{f}(r)$: $[0, \infty)\to\boldsymbol{R}_+$ such that $\tilde{f}(r)$ is even at $r=0$. Thus $p_0(r)=\int_0^r\tilde{f}(u)du$: $[0, \infty)\to\boldsymbol{R}^\infty$ is a monotone increasing C^∞ function and is odd at $r=0$, and we have $t=p_0(r)$.

Let L_0 denote the length of the geodesic $c(t)$ of $P(1\oplus\xi)$ between two singular orbits of $G_u\times S^1$. By taking the inverse function $r=q_0(t)$ of $t=p_0(r)$, we define C^∞ functions f_0, h_j^0: $(0, L_0)\to\boldsymbol{R}_+$ by

$$\begin{cases} f_0(t) = \tilde{F}(q_0(t)) \\ h_j^0(t) = H_j(q_0(t))\,. \end{cases} \tag{4.7}$$

By using a similar argument for a neighborhood of $c(L_0)$, we see that the functions f_0, h_j^0 extend to C^∞ functions f_0, h_j^0: $[0, L_0]\to \boldsymbol{R}$ which satisfy $f_0(0)=f_0(L_0)=0, f_0'(0)=1=-f_0'(L_0), f_0^{(2k)}(0)=f_0^{(2j)}(L_0)=0$ for each positive integer k, $h_j^0(0)>0$, $h_j^0(L_0)>0$ and $(h_j^0)^{(2k-1)}(0)=(h_j^0)^{(2k-1)}(L_0)=0$ for each positive integer k. Therefore we get the first part of the following theorem.

Theorem 4.4 (cf. [2] Section 4).

(1) *Let g_0 be a $G_u\times S^1$-invariant Kähler metric on $P(1\oplus\xi)$. Then the metric g_0 is given by*

$$g_0 = dt^2+f_0^2(t)\beta_0+\sum_j h_j^0(t)^2\alpha_j$$

on the open orbit $G\times_\rho \boldsymbol{C}^*$, where f_0, h_j^0 are C^∞ functions on $[0, L_0]$ such that

$$(4.8)\quad \begin{cases} f_0, h_j^0 \text{ are positive valued on } (0, L_0), f_0(0)=f_0(L_0)=0, \\ f_0'(0)=1=-f_0'(L_0), f_0^{(2k)}(0)=f_0^{(2k)}(L_0)=0 \text{ for each} \\ \text{positive integer } k,\ h_j^0(0)>0,\ h_j^0(L_0)>0 \text{ and } (h_j^0)^{(2k-1)}(0) \\ =(h_j^0)^{(2k-1)}(L_0)=0 \text{ for each positive integer } k\,. \end{cases}$$

(2) *Conversely let $f(s)$, $h_j(s)$ be C^∞ functions on $[0, L]$ which satisfy the properties* (4.8). *Then the metric*

$$g = ds^2+f(s)^2\beta_0+\sum_j h_j(s)^2\alpha_j$$

is defined on the open orbit $G\times_\rho \boldsymbol{C}^*$ and extends to a C^∞ metric on $P(1\oplus\xi)$.

Proof. We prove the second part. At first we consider the ordinary differential equation

$$(4.9)\qquad dr/ds=(1/f(s))r\,.$$

A solution of (4.9) is given by

$$r=q(s)=\exp\int_{s_0}^{s}(1/f(u))du$$

where $s_0\in(0, L)$ is the point corresponding to $r=1$. By our assumption on $f(s)$ at $s=0$, $f(s)=s(1+s^2f_1(s))$ where $f_1(s)$ is a C^∞ function on $[0, L)$ and $f_1^{(2k-1)}(0)=0$ for every positive integer k. Since

$$\exp\int_{s_0}^{s}(1/f(u))du=\frac{s}{s_0}\exp\left(-\int_{s_0}^{s}\frac{uf_1(u)}{1+u^2f_1(u)}du\right),$$

the solution $r=sq_1(s)$ of the equation (4.9) extends to a C^∞ function on $[0, L)$ such that $q_1(0)>0$ and $q_1^{(2k-1)}(0)=0$ for each positive integer k. Note also that $r=sq_1(s)$ is a monotone increasing function. If we put $r_1=1-r$, the equation (4.9) is written as

$$dr_1/ds = -(1-f(s))r_1,$$

and, from our assumption on $f(s)$ at $s=L$, we see that the solution r_1 of the equation is of the form

$$r_1 = (L-s)\tilde{q}_1(s)$$

where $\tilde{q}_1(s)$ is a C^∞ function on $(0, L]$ such that $\tilde{q}_1(L)>0$ and $\tilde{q}_1^{(2j-1)}(L)=0$ for each positive integer k. Let $s=p(r)$: $[0, \infty)\to[0, L)$ be the inverse function of $r=q(s)$. Then the metric g can be written in the form (4.2). Moreover, since $s=p(r)$ and $t=p_0(r)$ are monotone increasing C^∞ functions on $[0, \infty)$, s is a C^∞ function of t defined on $[0, L_0)$ such that $s(0)=0$, $(ds/dt)(0)>0$ and $d^{2k-1}s/dt^{2k-1}(0)=0$ for each positive integer k. Similarly we see that s is a C^∞ function of t on $(0, L_0]$, and hence $s=s(t)$: $[0, L_0]\to[0, L]$ is an onto diffeomorpishm which satisfies

$$ds/dt = f(s)/f_0(t) \qquad \text{and}$$
$$d^{2k}s/dt^{2k}(0) = d^{2k}s/dt^{2k}(L_0) = 0 \qquad \text{for each positive integer } k.$$

Thus $h_j(s)=h_j(s(t))$ satisfies $d^{2k-1}h_j/dt^{2k-1}(0)=d^{2k-1}h_j/dt^{2k-1}(L_0)=0$ for each integer k, and hence it is C^∞ at neighborhoods of singular orbits, since the square of the distance from a point on a Riemannian manifold is C^∞ at a neighborhood of the point. Now the metric g can be written as

$$\begin{aligned} g &= (ds/dt)^2dt^2+(f(s)/f_0(t))^2f_0(t)^2\tilde{\beta}_0+\sum_j h_j(s)^2\alpha_j \\ &= (ds/dt)^2(dt^2+f_0(t)^2)\tilde{\beta}_0+\sum_j h_j(s(t))^2\alpha_j \\ &= (ds/dt)^2(g_0-\sum_j h_j^0(t)^2\alpha_j)+\sum_j h_j(s(t))^2\alpha_j\,. \end{aligned}$$

Since ds/dt is an even function at $t=0$ and $t=L_0$, $ds/dt(0)>0$ and $ds/dt(L_0)>0$, we see that g extends to a C^∞ Riemannian metric g on $P(1\oplus\xi)$. q.e.d.

REMARK. If the metric g on the open orbit $G\times_\rho \boldsymbol{C}^*$ is Kähler, so is the extended metric g on $P(1\oplus\xi)$.

5 Computations of Ricci curvature

We now compute the Ricci tensor of a $G_u\times S^1$-invariant Kähler metric g on the open orbit $G\times_\rho \boldsymbol{C}^*$ in the projective bundle $P(1\oplus\xi)$. We assume that the metric g is of the form

$$g = ds^2+g_s = ds^2+f(s)^2\tilde{\beta}_0+\sum_j h_j(s)^2\alpha_j\,. \tag{5.1}$$

To calculate the curvature of the metric $g=ds^2+g_s$ on $G_u/\tilde{V}\times(0, L)$ we use the notion of a Riemannian submersion according to Bérard Bergery [2]. Note that the vector field H is given by the vector field $\partial/\partial s$. Let ∇ be the

Riemannian connection of g as before and $\hat{\nabla}$ that of g_s in each fiber of the Riemannian submersion $G_u/\tilde{V}\times(0, L)\to(0, L)$. We recall that, by definition, $T_X Y$ is the horizontal part of $\nabla_X Y$ for vertical vector fields X and Y, $T_X H$ is the vertical part of $\nabla_X H$ and if we put $T_H H=T_H X=0$, we obtain a tensor T of type (1, 2) on $G_u/\tilde{V}\times(0, L)$. Now the formulas of O'Neill is given by

$$(5.2)\qquad \begin{cases} \nabla_X Y = \hat{\nabla}_X Y+T_X Y \\ \nabla_X H = T_X H \\ \nabla_H X \text{ and } \nabla_X H \quad \text{are vertical} \\ \nabla_H H = 0 \end{cases}$$

for vertical vector fields X and Y. Note that the tensor A of O'Neill [14] is zero, since the base space $(0, L)$ of the Riemannian submersion is 1-dimensional. Note also that

$$(5.3)\quad g(T_X Y, H) = -g(T_X H, Y),\ T_X Y = T_Y X,\ g(T_X H, Y) = g(T_Y H, X)\,.$$

If X and Y are vertical vector fields which commute with H, that is, $[X, H]=[Y, H]=0$, we have

$$(5.4)\qquad g(T_X Y, H) = -\frac{1}{2}Hg(X, Y) = -g(T_X H, Y)\,.$$

By the formulas of O'Neill if X, Y, Z, V are vertical vectors and $\hat{R}$ is the curvature tensor of the metric g_s on $G_u/\tilde{V}$, we obtain the followings for the curvature R of $g=ds^2+g_s$:

$$(5.5)\quad \begin{cases} g(R(X, Y)Z, V) = g(\hat{R}(X, Y)Z, V)-g(T_X Z, T_Y V)+g(T_X V, T_Y Z) \\ g(R(X, Y)Z, H) = g((\nabla_Y T)_X Z, H)-g((\nabla_X T)_Y Z, H) \\ g(R(X, H)Y, H) = g((\nabla_H T)_X Y, H)-g(T_X H, T_Y H)\,. \end{cases}$$

To calculate the Ricci tensor r of the metric $g=ds^2+g_s$, we take an orthonormal basis $(X_i)_{i=1,\cdots,n-1}$ of the tangent space of an orbit $G_u/\tilde{V}$ with respect to g_s and introduce the following notations:

the principal normal vector $N=\sum_i T_{X_i}X_i$,
the norm $||T||$ of T, $||T||^2=\sum_i g(T_{X_i}H, T_{X_i}H)$ and
$\hat{\delta}T(X) = -\sum_i (\nabla_{X_i}T)_{X_i}X$ for a vertical vector X.

(Note that all these notations are independent of the choice of the basis.) We also denote by $\hat{r}$ the Ricci tensor of the metric g_s on each orbit. Then the Ricci tensor r of the metric g is given by the following formulas.

Proposition 5.1 (Bérard Bergery [2]). *If X and Y are vertical,*

$$(5.6) \qquad r(X, Y) = \hat{r}(X, Y) - g(N, T_X Y) + g((\nabla_H T)_X Y, H)$$

$$(5.7) \qquad r(X, H) = g(\hat{\delta} T(X), H)$$

$$(5.8) \qquad r(H, H) = Hg(N, H) - \|T\|^2 .$$

Lemma 5.2 (cf. [2] Proposition 3.18). *For a $G_u \times S^1$-invariant Kähler metric g on the open orbit $G \times_\rho C^*$ of the form* (5.1), *we have*

$$(5.9) \qquad r(X, H) = 0 \qquad \text{for all vertical vectors } X .$$

Proof. Since the Ricci tensor r is invariant by the complex structure J on $G \times_\rho C^*$ and by the action of $G_u \times S^1$, we get our claim by the same way as the proof of Proposition 3.2. q.e.d.

Lemma 5.3. *If vertical vector fields X, Y commute with H, we have*

$$(5.10) \quad g((\nabla_H T)_X Y, H) = -\frac{1}{2} H \cdot H \cdot g(X, Y) + 2g(T_X H, T_Y H) .$$

Proof. $g(\nabla_H T)_X Y, H) = g(\nabla_H (T_X Y), H) - g(T_{\nabla_H X} Y, H) - g(T_X(\nabla_H Y), H)$

$$= Hg(T_X Y, H) - g(T_Y(\nabla_H X), H) - g(T_X(\nabla_H Y), H)$$

$$= -\frac{1}{2} H \cdot H \cdot g(X, Y) + g(\nabla_H X, T_Y H) + g(\nabla_H Y, T_X H) \quad \text{by (5.3), (5.4)}$$

$$= -\frac{1}{2} H \cdot H \cdot g(X, Y) + 2g(T_X H, T_Y H), \text{ since } [X, H] = [Y, H] = 0 .$$

q.e.d.

From now on we assume that the Kähler C-space X is a product of two irreducible hermitian symmetric spaces of compact type M_1 and M_2 and that the projective bundle $P(1 \oplus \xi)$ is induced from a vector bundle $1 \oplus \xi$ where ξ is a line bundle given by $p_1^* L_1^{-a} \otimes p_2^* L_2^b$ for some positive integers a and b. Then our assumptions (3.6), (3.9), (3.10) and (3.11) are satisfied by taking canonical decompositions of symmetric spaces: $(\mathfrak{g}_i)_u = \mathfrak{b}_i + \mathfrak{m}_i$ $(i=1, 2)$. Thus a $G_u \times S^1$-invariant hermitian metric g on the open orbit $G \times_\rho C^*$ is given by the form

$$(5.11) \qquad g = ds^2 + f(s)^2 \tilde{\beta}_0 + h_1(s)^2 \alpha_1 + h_2(s)^2 \alpha_2$$

where α_i $(i=1, 2)$ are symmetric tensors induced from the invariant metrics on M_i corresponding to the inner product $\langle\ ,\ \rangle = -$Killing form.

As in section 4 let $X_0 \in \mathfrak{c}_\mathfrak{p}$ be the element defined by $\Lambda(X_0) = \sqrt{-1}$. Then $\tilde{\beta}_0(X_0, X_0) = 1$. We put $m = \dim_C M_1$ and $n = \dim_C M_2$. Take an orthonormal basis $\{B_1, \cdots, B_{2m}, C_1, \cdots, C_{2n}\}$ of $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ with respect to the inner product $\langle\ ,\ \rangle$ such that $B_j \in \mathfrak{m}_1$ and $C_j \in \mathfrak{m}_2$.

Proposition 5.4. *For an orthonormal basis* $\left\{H, \frac{1}{f} X_0, \frac{1}{h_1} B_1, \cdots, \frac{1}{h_1} B_{2m}, \frac{1}{h_2} C_1,\right.$

$\cdots, \dfrac{1}{h_2} C_{2n}\Big\}$, *we have*

$$r(H, H) = -\left(\frac{f''}{f} + 2m\frac{h_1''}{h_1} + 2n\frac{h_2''}{h_2}\right)$$

$$r\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) = \hat{r}\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) - \frac{f'}{f}\left(2m\frac{h_1'}{h_1} + 2n\frac{h_2'}{h_2}\right) - \frac{f''}{f}$$

$$r\left(\frac{1}{h_1}B_i, \frac{1}{h_1}B_i\right) = \hat{r}\left(\frac{1}{h_1}B_i, \frac{1}{h_1}B_i\right) - \frac{f'h_1'}{fh_1} - \frac{h_1''}{h_1} - (2m-1)\left(\frac{h_1'}{h_1}\right)^2 - 2n\frac{h_1'h_2'}{h_1 h_2}$$

$$r\left(\frac{1}{h_2}C_i, \frac{1}{h_2}C_i\right) = \hat{r}\left(\frac{1}{h_2}C_i, \frac{1}{h_2}C_i\right) - \frac{f'h_2'}{fh_2} - \frac{h_2''}{h_2} - (2n-1)\left(\frac{h_2'}{h_2}\right)^2 - 2m\frac{h_1'h_2'}{h_1 h_2}$$

$$r\left(\frac{1}{f}X_0, \frac{1}{h_1}B_i\right) = r\left(\frac{1}{f}X_0, \frac{1}{h_2}C_i\right) = r\left(\frac{1}{h_1}B_i, \frac{1}{h_1}B_j\right) = r\left(\frac{1}{h_2}C_i, \frac{1}{h_2}C_j\right) = 0$$

for $i \neq j$ *and*

$$r\left(\frac{1}{h_1}B_i, \frac{1}{h_2}C_j\right) = 0 \quad \textit{for each} \quad (i, j).$$

Proof. Note that $[\tilde{Y}, H]=0$ for $Y \in \mathfrak{p}$. Since $g(N,H)=g(T_{(1/f)X_0}(1/f)X_0, H) + \sum_i g(T_{(1/h_1)B_i}(1/h_1)B_i, H) + \sum_j g(T_{(1/h_2)C_j}(1/h_2)C_j, H) = (1/f^2)g(T_{\tilde{X}_0}\tilde{X}_0, H) + (1/h_1^2) \sum_i g(T_{\tilde{B}_i}\tilde{B}_i, H) + (1/h_2^2)\sum_j g(T_{\tilde{C}_j}\tilde{C}_j, H) = -\frac{1}{2}\{(1/f^2)Hg(\tilde{X}_0, \tilde{X}_0) + (1/h_1^2) \sum_i Hg(\tilde{B}_i, \tilde{B}_i) + (1/h_2^2)\sum_i Hg(\tilde{C}_i, \tilde{C}_i)\} = -(f'/f)\beta_0(\tilde{X}_0, \tilde{X}_0) - (h_1'/h_1)\sum_i \alpha_1(\tilde{B}_i, \tilde{B}_i) - (h_2'/h_2)\sum_i \alpha_2(\tilde{C}_i, \tilde{C}_i) = -(f'/f) - 2m(h_1'/h_1) - 2n(h_2'/h_2)$ by (5.4), we have

$$Hg(N, H) = -\frac{f''f-(f')^2}{f^2} - 2m\frac{h_1''h_1-(h_1')^2}{h_1^2} - 2n\frac{h_2''h_2-(h_2')^2}{h_2^2}.$$

Note that, for $Y \in \mathfrak{p}$, $g(T_Y H, T_Y H) = \sum_k g(T_Y H, X_k)^2$ where $\{X_k\}$ is an orthonormal basis of a tangent space of an orbit $G_u/\tilde{V}$. Thus $g(T_{X_0}H, T_{X_0}H) = (f')^2$, $g(T_{B_i}H, T_{B_i}H) = (h_1')^2$ and $g(T_{C_i}H, T_{C_i}H) = (h_2')^2$. Therefore $\|T\|^2 = \sum_k \|T_{X_k}H\|^2 = (f'/f)^2 + 2m(h_1'/h_1)^2 + 2n(h_2'/h_2)^2$ and hence $r(H,H) = -(f''/f) - 2m(h_1''/h_1) - 2n(h_2''/h_2)$ by (5.8).

Since $g((\nabla_H T)_{(1/f)X_0}(1/f)X_0, H) = (1/f^2)g((\nabla_H T)_{X_0}X_0, H) = (1/f^2)\{-\frac{1}{2}H\cdot H\cdot g(\tilde{X}_0, \tilde{X}_0) + 2g(T_{\tilde{X}_0}H, T_{\tilde{X}_0}H)\} = (-f''f + (f')^2)/f^2$, we have, by (5.6)

$$r\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) = \hat{r}\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) - (f'/f)\left(2m\frac{h_1'}{h_1} + 2n\frac{h_2'}{h_2}\right) - \frac{f''}{f}.$$

By the same way we get two other formulas for Ricci tensor r. Since

Ricci tensor r is invariant by the complex structure J and by the action $G_u \times S^1$, we get last claims by the same way as in proof of Proposition 3.2. q.e.d.

Now to compute Ricci tensor $\hat{r}$ we recall known facts on a hermitian symmetric space M of compact type. We write $M=G/K$ where G is the identity component of the group of all isometris of M. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K respectively and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{n}$ be a canonical decomposition. By identifying $\mathfrak{n}$ with the tangent space of G/K at the origin, let I be the complex structure on $\mathfrak{n}$ induced by the invaiiant complex structure J on M. By extending I to the complexification $\mathfrak{n}^C$ of $\mathfrak{n}$, we have the decomposition $\mathfrak{n}^C=\mathfrak{n}^+ + \mathfrak{n}^-$, $\mathfrak{n}^+ \cap \mathfrak{n}^- = (0)$, $\bar{\mathfrak{n}}^+ = \mathfrak{n}^-$, where the bar denotes complex conjugation with respect to $\mathfrak{n}$. It is known that there exists an element Z in the center $\mathfrak{c}$ of $\mathfrak{k}$ such that $\mathrm{ad}(Z)=I$. Moreover it is also known that $\dim \mathfrak{c}=1$ if M is irreducible. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing Z. Then the centralizer of Z coincides with $\mathfrak{k}$. We denote by Σ the root system of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$ and $\mathfrak{g}_\alpha$ the eigenspace of the root α. Note that $\bar{\mathfrak{g}}_\alpha = \mathfrak{g}_{-\alpha}$ where the bar denotes complex conjugation with ıespect to $\mathfrak{g}$. By setting $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha(Z) = \sqrt{-1}\}$, we have

$$\mathfrak{n}^+ = \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha,\ \mathfrak{n}^- = \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha \,.$$

We denote by $\mathfrak{h}_0$ the real subspace $\sqrt{-1}\mathfrak{h}$ of $\mathfrak{h}^C$ and introduce a lexicographical order in the dual space $\mathfrak{h}_0^*$ by taking a basis $\{H_1, \cdots, H_l\}$ of $\mathfrak{h}_0$ such that $H_1 = -\sqrt{-1}Z$. We denote by Σ_0^+ the set of positive roots not belonging to $\Sigma_\mathfrak{n}^+$. Then

$$\Sigma_0^+ = \{\alpha \in \Sigma \mid \alpha > 0,\ \alpha(Z) = 0\}$$

and

$$\mathfrak{k}^C = \mathfrak{h}^C + \sum_{\alpha \in \Sigma_0^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

We also identify a linear form $\lambda \in \mathfrak{h}_0^*$ with an element $H_\lambda \in \mathfrak{h}_0$ by means of the Killing form φ on $\mathfrak{g}^C$,

$$\lambda(H) = \varphi(H, H_\lambda) \qquad \text{for all } H \in \mathfrak{h}_0 \,.$$

It is also known that if M is an irreducible hermitian symmetric space there is a unique simple root α_1 belonging to $\Sigma_\mathfrak{n}^+$. We denote by $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ the set of all simple roots and by $\{\Lambda_\alpha\}_{\alpha\in\Pi}$ the fundamental weights of $\mathfrak{g}^C$ corresponding to Π. Then Σ_0^+ is spanned by $\{\alpha_2, \cdots, \alpha_l\}$ and thus the center $\mathfrak{c}$ of $\mathfrak{k}$ is given by $\sqrt{-1}\boldsymbol{R}\Lambda_{\alpha_1}$.

Let $\langle\ ,\ \rangle$ denote the inner product of $\mathfrak{h}_0$ induced from the Killing form φ on $\mathfrak{g}^C$ as before. If M is an irreducible hermitian symmetric space, the element $Z \in \mathfrak{c}$ such that $\mathrm{ad}(Z)=I$ is given by

(5.12) $$Z=\frac{2\sqrt{-1}}{\langle\alpha_1,\alpha_1\rangle}\Lambda_{\alpha_1}.$$

Lemma 5.5. *Put* $\delta_{\mathfrak{n}}=\frac{1}{2}\sum_{\alpha\in\Sigma_{\mathfrak{n}}^+}\alpha$. *Then* $\delta_{\mathfrak{n}}$ *belongs to the center of* $\mathfrak{k}^C$ *and* $\langle\delta_{\mathfrak{n}},\alpha\rangle=1/4$ *for* $\alpha\in\Sigma_{\mathfrak{n}}^+$.

Proof. See Murakami [13] Part II Lemma 1.1 and Corollary of Lemma 5.1, or Takeuchi [16].

It is also known that if M is irreducible there is a canonical isomorphism $\boldsymbol{Z}\Lambda_{\alpha_1}\to H^2(M,\boldsymbol{Z})$ and the first Chern class $c_1(M)$ of M corresponds to $\kappa\Lambda_{\alpha_1}$ where κ is an integer given by

(5.13) $$\kappa=\frac{2\langle 2\delta_{\mathfrak{n}},\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle}.$$

Therefore we have

(5.14) $$Z=2\sqrt{-1}\kappa\Lambda_{\alpha_1}.$$

Now we choose $E_\alpha\in\mathfrak{g}_\alpha$ for $\alpha\in\Sigma$ with the following properties:

$$[E_\alpha, E_{-\alpha}]=-\alpha,\ \varphi(E_\alpha,E_{-\alpha})=-1,\ \bar{E}_\alpha=E_{-\alpha}.$$

Put $B_\alpha=\frac{1}{\sqrt{2}}(E_\alpha+E_{-\alpha})$ for $\alpha\in\Sigma_{\mathfrak{n}}^+$. Then $B_\alpha\in\mathfrak{n}$, $IB_\alpha=\frac{\sqrt{-1}}{\sqrt{2}}(E_\alpha-E_{-\alpha})$ and $\{B_\alpha, IB_\alpha\mid\alpha\in\Sigma_{\mathfrak{n}}^+\}$ is an orthonormal basis of $\mathfrak{n}$ with respect to the inner product $\langle\ ,\ \rangle$ induced from the Killing form. Note that $[B_\alpha, IB_\alpha]=\sqrt{-1}\alpha$ for $\alpha\in\Sigma_{\mathfrak{n}}^+$,

(5.15) $$\langle[B_\alpha, IB_\alpha],\sqrt{-1}\Lambda_{\alpha_1}\rangle=1/2\kappa\qquad\text{for }\alpha\in\Sigma_{\mathfrak{n}}^+$$

by (5.14) and $\alpha(Z)=\sqrt{-1}$.

Now consider a product X of two irreducible hermitian symmetric spaces of compact type M_1 and M_2 and a projective bundle $P(1\oplus p_1^*L_1^{-a}\otimes p_2^*L_2^b)$ where L_1 and L_2 are generators of the group of all holomorphic line bundles $H^1(M_1,\theta^*)$ and $H^1(M_2,\theta^*)$ respectively and a, b are positive integers. Let $\Lambda^{(1)}$ and $\Lambda^{(2)}$ be the fundamental weights corresponding to L_1 and L_2 respectively. Then the weight Λ corresponding to the holomorphic line bundle $p_1^*L_1^{-a}\otimes p_2^*L_2^b$ over $X=M_1\times M_2$ is given by $\Lambda=-a\Lambda^{(1)}+b\Lambda^{(2)}$.

Now we take an orthonormal basis of $\mathfrak{m}$ such that $\{B_1,\cdots,B_m, IB_1,\cdots, IB_m\}$ is a basis of $\mathfrak{m}_1$ and $\{C_1,\cdots,C_n, IC_1,\cdots, IC_n\}$ is a basis of $\mathfrak{m}_2$ which satisfy (5.15). Let κ_i be the positive integers corresponding to the first Chern class $c_1(M_i)$ of M_i as before.

Lemma 5.6.

(1) (5.16) $\begin{cases} \langle\sqrt{-1}\Lambda, [B_i, IB_i]\rangle = -a/2\kappa_1 & \textit{for each } i. \\ \langle\sqrt{-1}\Lambda, [C_i, IC_i]\rangle = b/2\kappa_2 & \textit{for each } i. \end{cases}$

(2) *A $G_u\times S^1$-invariant hermitian metric g on the open orbit $G\times_\rho \boldsymbol{C}^*$ of the form* (5.11) *is Kähler if and only if*

$$(5.17)\qquad \begin{cases} (a/2\kappa_1)f+2h_1h_1' = 0. \\ (-b/2\kappa_2)f+2h_2h_2' = 0. \end{cases}$$

Proof. At first (5.16) follows from (5.15). Since M_1 and M_2 are hermitian symmetric spaces of compact type, the assumption of Proposition 3.4 is satisfied. The condition (3.20) can be written as

$$-(f(s)/\langle X_0, X_0\rangle^{1/2}\cdot\langle X, X\rangle^{1/2})\langle X, [A, IB]\rangle+\sum_{j=1}^{2}(d(h_j^2)/ds)\langle A, B\rangle_{|\mathfrak{m}_j} = 0$$

for $A, B\in\mathfrak{m}$, $0\neq X\in\mathfrak{c}_\mathfrak{p}$. Since $X_0\in\mathfrak{c}_\mathfrak{p}$ is given by $\Lambda(X_0)=\sqrt{-1}$, $X_0=\sqrt{-1}\,\Lambda/\langle\Lambda, \Lambda\rangle$ and thus $X_0=\langle X_0, X_0\rangle\sqrt{-1}\Lambda$. Now by taking an orthonormal basis of $\mathfrak{m}$ as before, we see that the condition (3.20) is equivalent to (5.17). q.e.d.

Now we compute Ricci tensor $\hat{r}$ of a metric $g_s=f(s)^2\beta_0+h_1(s)^2\alpha_1+h_2(s)^2\alpha_2$ on $G_u/\tilde{V}$. Let $\mathfrak{g}_u=\tilde{\mathfrak{v}}+\mathfrak{p}$ be the decomposition as before. Then

$$\mathfrak{p} = \mathfrak{c}_\mathfrak{p}+\mathfrak{m}_1+\mathfrak{m}_2,\ [\mathfrak{m}_i, \mathfrak{m}_i]\subset\tilde{\mathfrak{v}}+\mathfrak{c}_\mathfrak{p}\ (i = 1, 2)$$

and $[\mathfrak{c}_\mathfrak{p}, \mathfrak{m}_i]\subset\mathfrak{m}_i\ (i=1, 2)$. We denote by $\hat{R}$ the curvature tensor of $(G_u/\tilde{V}, g_s)$. Note also that the metric g_s corresponds to an inner product

$$(5.18)\quad \langle\ ,\ \rangle_s = (f(s)^2/\langle X_0, X_0\rangle)\langle\ ,\ \rangle_{\mathfrak{c}_\mathfrak{p}}+h_1(s)^2\langle\ ,\ \rangle|_{\mathfrak{m}_1}+h_2(s)^2\langle\ ,\ \rangle|_{\mathfrak{m}_2}$$

on $\mathfrak{p}$.

Lemma 5.7. *For $X, Y\in\mathfrak{p}$, we have*

(5.19) $\langle\hat{R}(X, Y)Y, X\rangle_s = -(3/4)\langle[X, Y]_\mathfrak{p}, [X, Y]_\mathfrak{p}\rangle_s -\langle[[X, Y]_{\tilde{\mathfrak{v}}}, Y], X\rangle_s$
$-(1/2)\langle Y, [X, [X, Y]_\mathfrak{p}]_\mathfrak{p}\rangle_s-(1/2)\langle X, [Y, [Y, X]_\mathfrak{p}]_\mathfrak{p}\rangle_s+\langle U(X, Y), U(X, Y)\rangle_s$
$+\langle U(X, X), U(Y, Y)\rangle_s$

where $Z_{\tilde{\mathfrak{v}}}, Z_\mathfrak{p}$ denote $\tilde{\mathfrak{v}}$-component, $\mathfrak{p}$-component of $Z\in\mathfrak{g}_u$ respectively, and U: $\mathfrak{p}\times\mathfrak{p}\to\mathfrak{p}$ is a symmetric bilinear form defined by

$$\langle U(X, Y), Z\rangle_s = \frac{1}{2}\{\langle[Z, X]_\mathfrak{p}, Y\rangle_s+[Z, Y]_\mathfrak{p}, X\rangle\rangle_s\}$$

for $X, Y, Z\in\mathfrak{p}$.

Proof. See [17] Lemma 7.1.

Proposition 5.8. *For an orthonormal basis* $\left\{\frac{1}{f}X_0, \frac{1}{h_1}B_1, \cdots, \frac{1}{h_1}B_m, \frac{1}{h_1}IB_1,\right.$

$\cdots, \frac{1}{h_1}IB_m, \frac{1}{h_2}C_1, \cdots, \frac{1}{h_2}C_n, \frac{1}{h_2}IC_1, \cdots, \frac{1}{h_2}IC_n\}$ *of* $\mathfrak{p}$, *we have*

$$\hat{r}\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) = 2m\left(\frac{a}{2\kappa_1}\right)^2\frac{f^2}{4h_1^4} + 2n\left(\frac{b}{2\kappa_2}\right)^2\frac{f^2}{4h_2^4} \tag{5.20}$$

$$\hat{r}\left(\frac{1}{h_1}B_i, \frac{1}{h_1}B_i\right) = \hat{r}\left(\frac{1}{h_1}IB_i, \frac{1}{h_1}IB_i\right) = \frac{1}{2h_1^2} - \left(\frac{a}{2\kappa_1}\right)^2\frac{f^2}{2h_1^4} \tag{5.21}$$

$$\hat{r}\left(\frac{1}{h_2}C_j, \frac{1}{h_2}C_j\right) = \hat{r}\left(\frac{1}{h_2}IC_j, \frac{1}{h_2}IC_j\right) = \frac{1}{2h_2^2} - \left(\frac{b}{2\kappa_2}\right)^2\frac{f^2}{2h_2^4} \tag{5.22}$$

for $i=1, \cdots, m, j=1, \cdots, n$.

Proof. For simplicity we put $B'_i=B_i$, $B'_{i+m}=IB_i$ for $i=1, \cdots, m$ and $C'_j=C_j$, $C'_{j+n}=IC_j$ for $j=1, \cdots, n$. Note that $[X_0, Y]=-(a/2\kappa_1)\langle X_0, X_0\rangle IY$ for $Y\in\mathfrak{m}_1$ and $[X_0, Y]=(b/2\kappa_2)\langle X_0, X_0\rangle IY$ for $Y\in\mathfrak{m}_2$. By straightforward computations, we have

$$-\frac{3}{4}\langle\left[\frac{1}{f}X_0, \frac{1}{h_1}B'_i\right]_{\mathfrak{p}}, \left[\frac{1}{f}X_0, \frac{1}{h_1}B'_i\right]_{\mathfrak{p}}\rangle_s = -\frac{3}{4}\frac{1}{f^2}\left(\frac{a}{2\kappa_1}\langle X_0, X_0\rangle\right)^2,$$

$$-\frac{1}{2}\langle\frac{1}{h_1}B', \left[\frac{1}{f}X_0, \left[\frac{1}{f}X_0, \frac{1}{h_1}B'_i\right]_{\mathfrak{p}}\right]_{\mathfrak{p}}\rangle_s = \frac{1}{2}\frac{1}{f^2}\left(\frac{a}{2\kappa_1}\langle X_0, X_0\rangle\right)^2,$$

$$-\frac{1}{2}\langle\frac{1}{f}X_0, \left[\frac{1}{h_1}B'_i, \left[\frac{1}{h_1}B'_i, \frac{1}{f}X_0\right]_{\mathfrak{p}}\right]_{\mathfrak{p}}\rangle_s = \frac{1}{2}\frac{1}{h_1^2}\left(\frac{a}{2\kappa_1}\right)^2\langle X_0, X_0\rangle,$$

$$\langle U\left(\frac{1}{f}X_0, \frac{1}{h_1}B'_i\right), U\left(\frac{1}{f}X_0, \frac{1}{h_1}B'_i\right)\rangle_s = \frac{1}{4}\frac{1}{f^2h_1^2}\left(\frac{a}{2\kappa_1}\right)^2\left\{h_1\langle X_0, X_0\rangle - \frac{f^2}{h_1}\right\}^2$$

and

$$\langle U\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right), U\frac{1}{h_1}B'_i, \frac{1}{h_1}B'_i)\rangle_s = 0.$$ Note also that

$[X_0, B'_i]=0$. Thus by Lemma 5.6, we get

$$\langle\hat{R}\left(\frac{1}{f}X_0, \frac{1}{h_1}B'_i\right)\frac{1}{h_1}B'_i, \frac{1}{f}X_0\rangle_s = \frac{1}{4}\left(\frac{a}{2\kappa_1}\right)^2\frac{f^2}{h_1^4}.$$

By the same way we get

$$\langle\hat{R}\left(\frac{1}{f}X_0, \frac{1}{h_2}C'_j\right)\frac{1}{h_2}C'_j, \frac{1}{f}X_0\rangle_s = \frac{1}{4}\left(\frac{b}{2\kappa_2}\right)^2\frac{f^2}{h_2^4}.$$

Since $\hat{r}\left(\frac{1}{f}X_0, \frac{1}{f}X_0\right) = \sum_{i=1}^{2m}\langle\hat{R}\left(\frac{1}{f}X_0, \frac{1}{h_1}B'_i\right)\frac{1}{h_1}B'_i, \frac{1}{f}X_0\rangle_s$
$+\sum_{j=1}^{2n}\langle\hat{R}\left(\frac{1}{f}X_0, \frac{1}{h_2}C'_j\right)\frac{1}{h_2}C'_j, \frac{1}{f}X_0\rangle_s$, we get (5.20).

Note that $[B_i, B_j]_{\mathfrak{p}}=0$, $[IB_i, IB_j]_{\mathfrak{p}}=0$ and $[B_i, IB_j]_{\mathfrak{p}}=[B_i, IB_j]_{\mathfrak{c}_{\mathfrak{p}}}=\delta_{ij}\frac{-a}{2\kappa_1}X_0$, and $[\mathfrak{c}_{\mathfrak{p}}, \mathfrak{m}_i]\subset\mathfrak{m}_i$ $(i=1, 2)$. By straightforward computations, we have

$$\langle\hat{R}\Big(\frac{1}{h_1}B_i, \frac{1}{h_1}IB_i\Big)\frac{1}{h_1}IB_i, \frac{1}{h_1}B_i\rangle_s = -\frac{3}{4}\Big(\frac{a}{2\kappa_1}\Big)^2\frac{f^2}{h_1^4}-\frac{1}{h_1^2}\langle[[B_i, IB_i], IB_i], B_i\rangle$$

and

$$\langle\hat{R}\Big(\frac{1}{h_1}B_i', \frac{1}{h_1}B_j'\Big)\frac{1}{h_1}B_i', \frac{1}{h_1}B_j'\rangle_s = -\frac{1}{h_1^2}\langle[[B_i', B_j'], B_j'], B_i'\rangle$$

otherwise.

We note that if $\bar{R}_1$ is the curvature tensor of the hermitian symmetric space M_1 with the metric induced from the Killing form then

$$\langle\bar{R}_1(B_i', B_j')B_j', B_i'\rangle = -\langle[[B_i', B_j'], B_j''], B_i'\rangle\,.$$

Moreover it is known that the Ricci tensor $\bar{r}_1$ of a hermitian symmetric space M_1 is given by

$$\bar{r}_1(X, Y) = \frac{1}{2}\langle X, Y\rangle \qquad \text{for } X, Y\in\mathfrak{m}_1$$

(see [11] Proposition 9.7). Obviously we have

$$\langle\hat{R}\Big(\frac{1}{h_1}B_i', \frac{1}{h_2}C_j'\Big)\frac{1}{h_2}C_j', \frac{1}{h_1}B_i'\rangle_s = 0 \qquad \text{for each } (i, j)\,.$$

Therefore we get

$$\begin{aligned}\hat{r}\Big(\frac{1}{h_1}B_i', \frac{1}{h_1}B_i'\Big) &= \langle\hat{R}\Big(\frac{1}{h_1}B_i', \frac{1}{f}X_0\Big)\frac{1}{f}X_0, \frac{1}{h_1}B_i'\rangle \\ &+\sum_{j=1}^{2m}\langle\hat{R}\Big(\frac{1}{h_1}B_i', \frac{1}{h_1}B_j'\Big)\frac{1}{h_1}B_j', \frac{1}{h_1}B_i'\rangle \\ &= -\frac{1}{2}\Big(\frac{a}{2\kappa_1}\Big)^2\frac{f^2}{h_1^4}+\frac{1}{2h_1^2}\,.\end{aligned}$$

By the same way we also get (5.22). q.e.d.

By Proposition 5.4, Lemma 5.6 and Proposition 5.8, we get following theorem.

Theorem 5.9. *Let X be a product of two irreducible hermitian symmetric spaces of compact type M_1 and M_2 and let $P(1\oplus\xi_\rho)$ be a projective bundle on X such that $\xi_\rho=p_1^*L_1^{-a}\otimes p_2^*L_2^b$ where a, b are positive integers. Then a $G_u\times S^1$-invariant hermitian metric g on the open orbit $G\times_\rho C^*$ of the form* (5.11) *is Einstein Kähler if and only if f, h_1 and h_2 satisfy the following ordinary differential equations:*

$$
(5.23)\quad \begin{cases}
(1) & \dfrac{a}{2\kappa_1}f+2h_1h_1' = 0 \\
(2) & -\dfrac{b}{2\kappa_2}f+2h_2h_2' = 0 \\
(3) & -\left(\dfrac{f''}{f}+2m\dfrac{h_1''}{h_1}+2n\dfrac{h_2''}{h_2}\right) = \lambda \\
(4) & -\dfrac{f''}{f}-\dfrac{f'}{f}\left(2m\dfrac{h_1'}{h_1}+2n\dfrac{h_2'}{h_2}\right)+2m\left(\dfrac{a}{2\kappa_1}\right)^2\dfrac{f^2}{4h_1^4}+2n\left(\dfrac{b}{2\kappa_2}\right)^2\dfrac{f^2}{4h_2^4} = \lambda \\
(5) & -\dfrac{h_1''}{h_1}-\dfrac{f'h_1'}{fh_1}-(2m-1)\left(\dfrac{h_1'}{h_1}\right)^2-2n\left(\dfrac{h_1'h_2'}{h_1h_2}\right)+\dfrac{1}{2h_1^2}-\left(\dfrac{a}{2\kappa_1}\right)^2\dfrac{f^2}{2h_1^4} = \lambda \\
(6) & -\dfrac{h_2''}{h_2}-\dfrac{f'h_2'}{fh_2}-(2n-1)\left(\dfrac{h_2'}{h_2}\right)^2-2m\left(\dfrac{h_1'h_2'}{h_1h_2}\right)+\dfrac{1}{2h_2^2}-\left(\dfrac{b}{2\kappa_2}\right)^2\dfrac{f^2}{2h_2^4} = \lambda
\end{cases}
$$

for some constant $\lambda>0$.

6 A proof of Main Theorem

At first we shall solve the system of ordinary differential equations (5.23). We consider a solution such that f, h_1 and h_2 are positive valued functions on an open interval. By (5.23) (2) we see that $h_2'>0$. From (5.23) (1) and (2) we have

$$
(6.1)\qquad \frac{f'}{f} = \frac{h_1''}{h_1'}+\frac{h_1'}{h_1} = \frac{h_2''}{h_2'}+\frac{h_2'}{h_2}
$$

and

$$
(6.2)\qquad \begin{cases}
\dfrac{f'}{f}\dfrac{h_1'}{h_1} = \dfrac{h_1''}{h_1}+\left(\dfrac{h_1'}{h_1}\right)^2 = \dfrac{h_1''}{h_1}+\left(\dfrac{a}{2\kappa_1}\right)^2\dfrac{f^2}{4h_1^4} \\
\dfrac{f'}{f}\dfrac{h_2'}{h_2} = \dfrac{h_2''}{h_2}+\left(\dfrac{h_2'}{h_2}\right)^2 = \dfrac{h_2''}{h_2}+\left(\dfrac{b}{2\kappa_2}\right)^2\dfrac{f^2}{4h_2^4}\,.
\end{cases}
$$

Thus under the equations (5.23) (1) and (2), the equations (5.23) (3) and (4) are identical.

From (5.23) (1) and (2) we also get

$$
(6.3)\qquad a\kappa_2h_2'h_2+b\kappa_1h_1'h_1 = 0\,,
$$

and we introduce a constant $\delta>0$ by

$$
(6.4)\qquad \delta^2 = a\kappa_2h_2^2+b\kappa_1h_1^2\,.
$$

Now we introduce a new variable $y=y(h_2)$ by

$$
(6.5)\qquad h_2' = \sqrt{y(h_2)}\,.
$$

Then we have

$$\frac{d^2h_2}{ds^2}=\frac{1}{2}\frac{dy}{dh_2} \text{ and } \frac{d^3h_2}{ds^3}=\frac{1}{2}\frac{d^2y}{dh_2^2}\frac{dh_2}{ds}. \tag{6.6}$$

By (6.1), (6.3) and (5.23) (2), the equation (5.23) (6) is written as

$$-2\frac{h_2''}{h_2}-(2n+2)\left(\frac{h_2'}{h_2}\right)^2+2m\frac{a\kappa_2}{b\kappa_1}\frac{1}{h_1^2}(h_2')^2+\frac{1}{2h_2^2}=\lambda .$$

Thus by (6.5) and (6.6) we get

$$\frac{dy}{dh_2}+2\left(\frac{n+1}{h_2}-m\frac{a\kappa_2}{b\kappa_1}\frac{h_2}{h_1^2}\right)y=\frac{1}{2h_2}-\lambda h_2 . \tag{6.7}$$

Similarly, by (6.2), the equation (5.23) (5) is written as

$$-2\frac{h_1''}{h_1}-(2m+2)\left(\frac{h_1'}{h_1}\right)^2-2n\frac{h_1'h_2'}{h_1h_2}+\frac{1}{2h_1^2}=\lambda . \tag{6.8}$$

From (6.3), (6.4), (6.5) and (6.6) we obtain

$$\left(\frac{h_1'}{h_1}\right)^2=\left(\frac{a\kappa_2}{b\kappa_1}\right)^2\frac{h_2^2}{h_1^4}y \tag{6.9}$$

and

$$\frac{h_1''}{h_1}=-\frac{1}{2}\frac{a\kappa_2}{b\kappa_1}\frac{h_2}{h_1^2}\frac{dy}{dh_2}-\frac{a\kappa_2}{(b\kappa_1)^2}\frac{\delta^2}{h_1^4}y . \tag{6.10}$$

Therefore the equation (6.8) is written as

$$\frac{dy}{dh_2}+2\left(\frac{n+1}{h_2}-m\frac{a\kappa_2}{b\kappa_1}\frac{h_2}{h_1^2}\right)y=\frac{b\kappa_1}{a\kappa_2}\frac{h_1^2}{h_2}\lambda-\frac{1}{2}\frac{b\kappa_1}{a\kappa_2}\frac{1}{h_2} . \tag{6.11}$$

From the equations (6.7), (6.11) and (6.4), we obtain a relation

$$a\kappa_2+b\kappa_1=2\lambda\delta^2 . \tag{6.12}$$

Now by (5.23) (2) and (6.6), we have

$$\frac{f''}{f}=3\frac{h_2''}{h_2}+\frac{h_2'''}{h_2'}=\frac{3}{2}\frac{1}{h_2}\frac{dy}{dh_2}+\frac{1}{2}\frac{d^2y}{dh_2^2} . \tag{6.13}$$

Thus the equation (5.23) (3) is written as

$$\frac{d^2y}{dh_2^2}+\left(\frac{2n+3}{h_2}-\frac{2ma\kappa_2h_2}{b\kappa_1h_1^2}\right)\frac{dy}{dh_2}-\frac{4ma\kappa_2\delta^2}{(b\kappa_1)^2h_1^4}y=-2\lambda . \tag{6.14}$$

Now it is easy to see that the equation (6.14) is obtained from the equation (6.7) by differentiation and (6.4). Hence we get the following lemma.

Lemma 6.1. *The system of differential equations* (5.23) *is equivalent to the following system of equations*:

$$(6.15)\qquad \begin{cases} \dfrac{a}{2\kappa_1}f+2h_1h_1'=0, \ -\dfrac{b}{2\kappa_2}f+2h_2h_2'=0 \\ h_2'=\sqrt{y(h_2)}\,, \quad 2\lambda(a\kappa_2h_2^2+b\kappa_1h_1^2)=a\kappa_2+b\kappa_1 \\ \dfrac{dy}{dh_2}+2\Big(\dfrac{n+1}{h_2}-m\dfrac{a\kappa_2}{b\kappa_1}\dfrac{h_2}{h_1^2}\Big)y=\dfrac{1}{2h_2}-\lambda h_2\,. \end{cases}$$

Now we consider the first order linear differential equation (6.7). Since an integral factor μ is given by

$$(6.16)\qquad \mu=h_2^{2(n+1)}(\delta^2-a\kappa_2h_2^2)^m=h_2^{2(n+1)}(b\kappa_1h_1^2)^m\,,$$

a solution y of the equation (6.16) is given by

$$(6.17)\qquad y=\frac{1}{2h_2^{2(n+1)}(b\kappa_1h_1^2)^m}\Big\{\int h_2^{2n+1}(b\kappa_1h_1^2)^m(1-2\lambda h_2^2)dh_2+C\Big\}$$

where C is a constant and $a\kappa_2h_2^2+b\kappa_1h_1^2=\delta^2$.

Now we recall the following theorem on a compact Einstein Kähler manifold.

Theorem 6.2 (Matsushima [12]). *Let* (P, J, g) *be a compact Einstein Kähler manifold with positive Ricci tensor. Then the Lie algebra* $\mathfrak{k}(P, g)$ *of all Killing vector fields on* P *is a real form of the Lie algebra* $\mathfrak{g}(P, J)$ *of all holomorphic vector fields on* P.

Let $P(1\oplus\xi_\rho)$ be the projective bundle on X as in Theorem 5.9 and assume that g is an Einstein Kähler metric on $P(1\oplus\xi_\rho)$. Then we assume that g is invariant by the maximal compact Lie group $G_u\times S^1$ by Theorem 6.2, and hence g is of the form (5.11) on the open orbit $G\times_\rho \boldsymbol{C}^*$, and f, h_1, h_2 satisfy the equations (5.23) and conditions of Theorem 4.4 at the boundaries 0 and L. By (5.23) (1) and (2), we obtain

$$(6.18)\qquad \begin{cases} \dfrac{a}{2\kappa_1}f'+2h_1h_1''+2(h_1')^2=0\,, \\ -\dfrac{b}{2\kappa_2}f'+2h_2h_2''+2(h_2')^2=0 \end{cases}$$

Since $f'(0)=1$, $f'(L)=-1$, $h_1'(0)=h_1'(L)=h_2'(0)=h_2'(L)=0$, we have

$$(6.19)\qquad \begin{cases} \dfrac{a}{2\kappa_1}+2h_1(0)h_1''(0)=0, \ -\dfrac{a}{2\kappa_1}+2h_1(L)h_1''(L)=0\,, \\ -\dfrac{b}{2\kappa_2}+2h_2(0)h_2''(0)=0, \ \dfrac{b}{2\kappa_2}+2h_2(L)h_2''(L)=0\,. \end{cases}$$

By (6.7) and (6.8) we have

$$(6.20)\qquad -4h_i''(0)h_i(0) = 2\lambda h_i^2(0)-1,\ -4h_i''(L)h_i(L) = 2\lambda h_i^2(L)-1$$

for $i=1, 2$. Thus by (6.19) and (6.20), we get

$$(6.21)\qquad \begin{cases} 2\lambda h_1^2(0) = 1+(a/\kappa_1),\ 2\lambda h_1^2(L) = 1-(a/\kappa_1)\,, \\ 2\lambda h_2^2(0) = 1-(b/\kappa_2),\ 2\lambda h_2^2(L) = 1+(b/\kappa_2)\,. \end{cases}$$

In particular, we obtain conditions $a<\kappa_1$ and $b<\kappa_2$, which are known as the conditions for the first Chern class of $P(1\oplus\xi_p)$ being positive. Now, since $y(h_2(0))=(h_2'(0))^2=0$, $y(h_2)$ is given by

$$(6.22)\qquad y(h_2) = \frac{1}{2h_2^{2(n+1)}(b\kappa_1 h_1^2)^m}\int_{h_2(0)}^{h_2} h_2^{2n+1}(b\kappa_1 h_1^2)^m(1-2\lambda h_2^2)dh_2\,.$$

Since $y(h_2(L))=0$, we have

$$y(h_2(L)) = \frac{1}{2h_2^{2(n+1)}(L)\,(b\kappa_1 h_1^2(L))^m}\int_{h_2(0)}^{h_2(L)} h_2^{2n+1}(b\kappa_1 h_1^2)^m(1-2\lambda h_2^2)dh_2 = 0\,.$$

Hence, if g is an Einstein Kähler metric on $P(1\oplus\xi_p)$, we have

$$(6.23)\qquad \int_{\sqrt{(1-(b/\kappa_2))/2\lambda}}^{\sqrt{(1+(b/\kappa_2))/2\lambda}} h_2^{2n+1}(b\kappa_1 h_1^2)^m(1-2\lambda h^2)dh_2 = 0$$

where $2\lambda(a\kappa_2 h_2^2+b\kappa_1 h_1^2)=a\kappa_2+b\kappa_1$. Now we put $u=2\lambda h_2^2-1$. Then (6.23) can be written as

$$\int_{-b/\kappa_2}^{b/\kappa_2}(u+1)^n(b\kappa_1-a\kappa_2 u)^m u\,du = 0\,,$$

since $2\lambda(a\kappa_2 h_2^2+b\kappa_1 h_1^2)=a\kappa_2+b\kappa_1$.

Thus by setting $x=(\kappa_2/b)u$, we see that (6.23) is given by

$$\int_{-1}^{1}(\kappa_2+bx)^n(\kappa_1-ax)^m x\,dx = 0\,.$$

Conversely, assume that (6.23) is satisfied. We define $y(h_2)$ on a neighborhood of $[\sqrt{(1-(b/\kappa_2))/2\lambda},\ \sqrt{(1+(b/\kappa_2))/2\lambda}]$ by

$$y(h_2) = \frac{1}{2h_2^{2(n+1)}(b\kappa_1 h_1^2)^m}\int_{\sqrt{(1-(b/\kappa_2))/2\lambda}}^{h_2} h_2^{2n+1}(b\kappa_1 h_1^2)^m(1-2\lambda h_2^2)dh_2\,.$$

For simplicity, we put $h^0=\sqrt{(1-(b/\kappa_2))/2\lambda}$, $h^1=\sqrt{(1+(b/\kappa_2))/2\lambda}$. Then $y(h^0)=y(h^1)=0$ and $y(h_2)>0$ for $h^0<h_2<h^1$. Note also that $dy/dh_2(h^0)>0$ and $dy/dh_2(h^1)>0$. Define a function $\tilde{t}(h_2)$ on (h^0, h^1) by

$$(6.24)\qquad \tilde{t}(h_2) = 1\int_{\sqrt{1/2\lambda}}^{h_2}\frac{1}{\sqrt{y(h_2)}}\,dh_2\,.$$

Since $h_2=h^0$, h^1 are simple roots of $y(h_2)=0$, $\lim_{h_2\to h^0+}\tilde{t}(h_2)$ and $\lim_{h_2\to h^1-}\tilde{t}(h_2)$ exist. We put

$$\tilde{t}_0=\lim_{h_2\to h^0+}\tilde{t}(h_2) \quad \text{and} \quad \tilde{t}_1=\lim_{h_2\to h^1-}\tilde{t}(h_2)\,.$$

We also define a function $t(h_2)$ on $[h^0, h^1]$ by

$$t(h_2)=\tilde{t}(h_2)-\tilde{t}_0,\ t(h^0)=0 \ \text{and} \ t(h^1)=\tilde{t}_1-\tilde{t}_0$$

and we put $L=t(h^1)$. Then $t(h_2)\colon [h^0, h^1]\to[0, L]$ is a monotone increasing continuous function which is C^∞ on (h^0, h^1).

Now let $h_2(t)$ be the inverse function of $t(h_2)$. Then $dh_2/dt=\sqrt{y(h_2)}$ on $(0, L)$. We claim that $h_2(t)$ can be extended to a C^∞ function $h_2(t)\colon [0, L]\to\boldsymbol{R}_+$ such that $h_2^{(2k-1)}(0)=h_2^{(2k-1)}(L)=0$ for each positive integer k. For a sufficient small $\varepsilon>0$, we extend $h_2(t)$ to a function $h_2(t)\colon (-\varepsilon, L+\varepsilon)\to\boldsymbol{R}_+$ by $h_2(t)=h_2(-t)$ for $-\varepsilon<t<0$ and $h_2(t+L)=h_2(L-t)$ for $0<t<\varepsilon$. Then we see that $h_2(t)\colon (-\varepsilon, L+\varepsilon)\to\boldsymbol{R}$ is continuous and is a C^∞ function except $t=0$ and $t=L$. Since $dh_2/dt=\sqrt{y(h_2)}$ on $(0, L)$, $dh_2/dt=-\sqrt{y(h_2)}$ on $(-\varepsilon, 0)$ and $\lim_{t\to 0}\frac{dh_2}{dt}=0$, we see that $dh_2/dt(0)$ exists and $dh_2/dt(0)=0$. Similarly we have $dh_2/dt(L)=0$. Thus we see that $h_2(t)\colon (-\varepsilon, L+\varepsilon)\to\boldsymbol{R}_+$ is a function of class C^1. By $dh_2/dt=\sqrt{y(h_2)}$ on $(0, L)$, we have

$$\frac{d^2h_2}{dt^2}=\frac{1}{2}\frac{dy}{dh_2}(h_2(t)) \quad \text{on } (0, L)\,.$$

By $dh_2/dt=-\sqrt{y(h_2)}$ on $(-\varepsilon, 0)$, we also have

$$\frac{d^2h_2}{dt^2}=\frac{1}{2}\frac{dy}{dh_2}(h_2(t)) \quad \text{on } (-\varepsilon, 0)\,.$$

Thus we see that $\lim_{t\to 0} d^2h_2/dt^2$ exists and

$$\frac{d^2h_2}{dt^2}(0)=\frac{1}{2}\frac{dy}{dh_2}(h^0)=\frac{1}{2}\left(\frac{1}{2h^0}-\lambda h^0\right).$$

Similarly we see that $\lim_{t\to L} d^2h_2/dt^2$ exists and

$$\frac{d^2h_2}{dt^2}(L)=\frac{1}{2}\frac{dy}{dh_2}(h^1)=\frac{1}{2}\left(\frac{1}{2h^1}-\lambda h^1\right).$$

Therefore $h_2(t)\colon (-\varepsilon, L+\varepsilon)\to\boldsymbol{R}_+$ is of class C^2. Now we put $\varphi(h_2)=\frac{1}{2}\frac{dy}{dh_2}$. Then $\varphi(h_2)$ is a C^∞ function on a neighborhood of $[h^0, h^1]$ and

$$\frac{d^2h_2}{dt^2}=\varphi(h_2(t)) \quad \text{on } (0, L)\,. \tag{6.25}$$

Lemma 6.3. *On* $(0, L)$, *we have, for each positive integer* k,

$$\text{(6.26)}\quad \frac{dh_2^{2k+1}}{dt^{2k+1}} = \frac{d^{2k-1}\varphi}{dh_2^{2k+1}}\left(\frac{dh_2}{dt}\right)^{2k-1} + \sum_{j=1}^{k-1}\Phi_{2(k-j)-1}^{2k+1}\left(\varphi, \frac{d\varphi}{dh_2}, \cdots, \frac{d^{2k-1-j}\varphi}{dh_2^{2k-1-j}}\right)\left(\frac{dh_2}{dt}\right)^{2(k-j)-1}$$

$$\text{(6.27)}\quad \frac{dh_2^{2k}}{dt^{2k}} = \frac{d^{2k-2}\varphi}{dh_2^{2k-2}}\left(\frac{dh_2}{dt}\right)^{2k-2} + \sum_{j=1}^{k-1}\Phi_{2(k-j)-2}^{2k}\left(\varphi, \frac{d\varphi}{dh_2}, \cdots, \frac{d^{2k-2-j}\varphi}{dh_2^{2k-2-j}}\right)\left(\frac{dh_2}{dt}\right)^{2(k-j)-2}$$

where $\Phi_{l-1-2j}^{l}\left(\varphi, \frac{d\varphi}{dh_2}, \cdots, \frac{d^{l-1-j}\varphi}{dh_2^{l-1-j}}\right)$ *are polynomials of* $\varphi, \frac{d\varphi}{dh_2}, \cdots, \frac{d^{l-1-j}\varphi}{dh_2^{l-1-j}}$.

Proof. By routine computations using induction.

In particular, we see that

$$\lim_{t\to 0}\frac{dh_2^{2k+1}}{dt^{2k+1}} = \lim_{t\to L}\frac{dh_2^{2k+1}}{dt^{2k+1}} = 0\,,$$

$$\lim_{t\to 0}\frac{dh_2^{2k}}{dt^{2k}} = \Phi_0^{2k}\left(\varphi(h^0), \frac{d\varphi}{dh_2}(h^0), \cdots, \frac{d^{k-1}\varphi}{dh_2^{k-1}}(h^0)\right)$$

$$\lim_{t\to L}\frac{dh_2^{2k}}{dt^{2k}} = \Phi_0^{2k}\left(\varphi(h^1), \frac{d\varphi}{dh_2}(h^1), \cdots, \frac{d^{k-1}\varphi}{dh_2^{k-1}}(h^1)\right),$$

and hence $h_2(t)\colon (-\varepsilon, L+\varepsilon)\to \boldsymbol{R}_+$ is a C^∞ function such that $h_2^{(2k-1)}(0)=h_2^{(2k-1)}(L)=0$ for each positive integer k. We define a function f by

$$f = (4\kappa_2/b)h_2h_2'$$

and a function $h_1>0$ by

$$2\lambda(a\kappa_2h_2^2+b\kappa_1h_1^2) = a\kappa_2+b\kappa_1\,.$$

Then f is a C^∞ function on $[0, L]$ such that $f(0)=f(L)=0, f'(0)=-f'(L)=1$ and $f^{(2k)}(0)=f^{(2k)}(L)=0$ for each positive integer k, and f, h_1, h_2 satisfy the equation (5.23). Therefore a metric $g=dt^2+f(t)^2\tilde{\beta}_0+h_1(t)^2\alpha_1+h_2(t)\alpha_2$ is an Einstein Kähler metric on $P(1\oplus\xi_\rho)$ by Theorem 4.4 and Theorem 5.9. This proves our Main Theorem.

Proof of Corollary 1. Since $\int_{-1}^{1}(\kappa-ax)^m(\kappa+ax)^m x\,dx=0$, we see that there exists an Einstein Kähler metric on P by our Main Theorem.

Proof of Corollary 2 (1). By our Main Theorem it is enough to see that

$$\int_{-1}^{1}(\kappa+bx)^m(\kappa-ax)^m x\,dx\neq 0 \qquad \text{for } a\neq b\,.$$

We may assume that $b>a$.

$$\int_{-1}^{1}(\kappa+bx)^m(\kappa-ax)^m xdx=\int_{-1}^{1}(\kappa^2+(b-a)x-abx^2)^m xdx$$
$$=\sum_{j=a}^{m}\int_{-1}^{1}\binom{m}{j}(\kappa^2-abx^2)^{m-j}((b-a)x)^j xdx$$
$$=\sum_{k\geqq 1}\int_{-1}^{1}\binom{m}{2k-1}(\kappa^2-abx^2)^{m-2k+1}(b-a)^{2k-1}x^{2k}dx$$
$$=2\sum_{k\geqq 1}\int_{0}^{1}\binom{m}{2k-1}(\kappa^2-abx^2)^{m-2k+1}(b-a)^{2k-1}x^{2k}dx>0\,. \qquad \text{q.e.d.}$$

Proof of Corollary 2 (2). Since $\kappa_1=2$ and $a=1$, we have to show that

$$(6.28)\qquad \int_{-1}^{1}(2-x)(\kappa_2+bx)^n xdx\neq 0 \qquad \text{for } n\geqq 2\,.$$

Put $y=\kappa_2+bx$. Then the integral (6.28) is given by

$$\int_{\kappa_2-b}^{\kappa_2+b}\frac{1}{b^3}(2b+\kappa_2-y)(y-\kappa_2)y^n dy\,.$$

Now we have

$$(6.29)\qquad \int_{\kappa_2-b}^{\kappa_2+b}(2b+\kappa_2-y)(y-\kappa_2)y^n dy$$
$$=\frac{1}{(n+1)(n+2)(n+3)}[(\kappa_2-b)^{n+1}(2\kappa_2^2+(2n+4)2b\kappa_2+(n+1)(3n+8)b^2)$$
$$-(\kappa_2+b)^{n+1}(2(\kappa_2^2+2b\kappa_2)-b^2(n^2+5n+4))]\,.$$

Case 1. $b\geqq 2$.

Since $b<\kappa_2\leqq n+1$,

$$b^2(n^2+5n+4)-2(\kappa_2^2+2b\kappa_2)\geqq b^2(n^2+5n+4)-2(n+1)(n+1+2b)$$
$$=(b^2-2)n^2+(5b^2-2b-2)n+(4b^2-4b-2)>0 \qquad \text{if } b\geqq 2\,.$$

Thus the integration (6.29) is positive.

Case 2. $b=1$.

We use a classification of irreducible hermitian symmetric spaces. It is also known that the integer κ of an irreducible hermitian symmetric space of compact type M is given as follows (cf. [5]):

I $M=U(p+q)/(U(p)\times U(q))$	$\kappa=p+q$	$\dim_C M=pq$
II $M=SO(2q)/U(q)$ $(q\geqq 5)$	$\kappa=2q-2$	$\dim_C M=q(q-1)/2$
III $M=Sp(q)/U(q)$ $(q\geqq 3)$	$\kappa=q+1$	$\dim_C M=q(q+1)/2$
IV $M=SO(q+2)/(SO(2)\times SO(q))$ $(q\geqq 3)$	$\kappa=q$	$\dim_C M=q$

V $M = E_6/(Spin(10)\times T^1)$ $\quad \kappa = 12 \quad \dim_C M = 16$

VI $M = E_7/(E_6\times T^1)$ $\quad \kappa = 18 \quad \dim_C M = 27$.

Now, since $b=1$, (6.29) is given by

$$(6.30)\quad \int_{\kappa_2-1}^{\kappa_2+1}(2+\kappa_2-y)(y-\kappa_2)y^n dy$$
$$= \frac{1}{(n+3)(n+2)(n+1)}[(\kappa_2-1)^{n+1}(2\kappa_2^2+2(2n+4)\kappa_2+(n+1)(3n+8))$$
$$-(\kappa_2+1)^{n+1}(2(\kappa_2^2+2\kappa_2)-(n^2+5n+4))].$$

Case 2.1.

If $M=U(p+q)/(U(p)\times U(q))$ and $p, q\geqq 2$,

$$n^2+5n+4-2(\kappa_2^2+2\kappa_2) = (pq)^2+5pq+4-2(p+q)^2-4(p+q)$$
$$= (p^2-2)(q^2-2)+pq-4p-4q\geqq 2(p^2-2)+q(p-4)-4p.$$

If $p\geqq 4$, $2(p^2-2)+q(p-4)-4p\geqq 2(p^2-2)+2(p-4)-4p$
$= 2(p-3)(p+2)\geqq 0$.

We may also assume that $p\geqq q$. If $p=3\geqq q\geqq 2$,

$$n^2+5n+4-2(\kappa_2^2+2\kappa_2) = 7(q^2-2)+3q-12-4q = 7q^2-q-26>0.$$

Note that if $p=q=2$ then M is a quadric $Q^4(\boldsymbol{C})$.

Case 2.2.

If $M=SO(2q)/U(q)$ $(q\geqq 5)$,

$$n^2+5n+4-2(\kappa_2^2+2\kappa_2) = (q(q-1)/2)^2+5(q(q-1)/2)+4$$
$$-2(2q-2)^2-4(2q-2).$$

Since $n=q(q-1)/2$, $n^2+5n+4-2(\kappa_2^2+2\kappa_2)=n^2-11n+4>0$ if $q\geqq 6$, that is, $n\geqq 15$.

For $q=5$, we have $n=10$ and thus (6.30) becomes

$$\frac{1}{13\times 12\times 11}(7^{11}(2^9+11\times 38)-9^{11}\times 6)\neq 0$$

Case 2.3.

If $M=Sp(q)/U(q)$ $(q\geqq 3)$,

$$n^2+5n+4-2(\kappa_2^2+2\kappa_2) = (q(q+1)/2)^2+5q(q+1)/2+4-2((q+1)^2+2(q+1))$$

Put $p(x)=(x(x+1)/2)^2+5x(x+1)/2+4-2((x+1)^2+2(x+1))$.

Then $p(3)=22$ and $p'(x)>0$ for $x>3$ and hence $n^2+5n+4-2(\kappa_2^2+2\kappa_2)>0$ for $q\geqq 3$.

Case 2.4.

If $M=E_6/(\mathrm{Spin}(10)\times T^1)$, $\kappa_2=12$ and $n=16$, thus

$$n^2+5n+6-2(\kappa_2^2+2\kappa_2)=4>0\,.$$

Case 2.5.

If $M=E_7/(E_6\times T^1)$, $\kappa_2=18$ and $n=27$, thus

$$n^2+5n+9-2(\kappa_2^2+2\kappa_2)=3^2+5\times 3^3+4>0\,.$$

Therefore the integral (6.30) is positive for the cases above.

Now we consider the cases $M=P^n(\boldsymbol{C})$ and $M=Q^n(\boldsymbol{C})$.

Case 2.6.

If $M=P^n(\boldsymbol{C})$, $\kappa_2=n+1$, and thus (6.30) is given by

$$\frac{1}{(n+3)(n+2)(n+1)}\{n^{n+1}9(n+1)(n+2)-(n+2)^{n+1}(n+1)(n+2)\}$$
$$=\frac{1}{n+3}(9n^{n+1}-(n+2)^{n+1})=\frac{n^{n+1}}{n+3}\left(9-\left(\frac{n+2}{n}\right)^{n+1}\right).$$

We define a function $p(y)$ $(y\geqq 2)$ by

$$p(y)=\left(\frac{y+1}{y-1}\right)^y. \tag{6.31}$$

Then it is not difficult to see that $p(y)$ is a monotone decreasing function. Therefore we see that the integral (6.30) is positive for $n\geqq 2$.

Case 2.7.

If $M=Q^n(\boldsymbol{C})$ $(n\geqq 3)$, $\kappa_2=n$ and thus (6.30) is given by

$$\frac{(n-1)^{n+1}(n^2-n-4)}{(n+3)(n+2)(n+1)}\left\{\frac{9n^2+19n+8}{n^2-n-4}-\left(\frac{n+1}{n-1}\right)^{n+1}\right\}.$$

We claim that $\dfrac{9n^2+19n+8}{n^2-n-4}-\left(\dfrac{n+1}{n-1}\right)^{n+1}>0$ for $n\geqq 3$. Since the function $p(y)$ defined by (6.31) is monotone decreasing, it is enough to show that

$$\frac{(9n^2+19n+8)(n-1)}{(n^2-n-4)(n+1)}>8 \qquad \text{for } n\geqq 3\,.$$

But this is obvious, since

$$(9n^2+19n+8)(n-1)-8(n+1)(n^2-n-4)=n^3+10n^2+29n+24>0\,.$$

Thus the integral (6.30) is positive for $n\geqq 3$. q.e.d.

Finally we give an example of Einstein Kähler manifold which is not of the type in Corollary 1 of Main Theorem.

EXAMPLE 6.4. Let M_1 be the complex Grassmann manifold $G_{6,2}(\boldsymbol{C})$ of 2-planes in $\boldsymbol{C}^6$ and M_2 the complex projective space $P^8(\boldsymbol{C})$. Note that in this case $\kappa_1=6$ and $\kappa_2=9$. Consider the $P^1(\boldsymbol{C})$-bundle $P(p_1^*L_1^2 \oplus p_2^*L_2^3)$ over $M_1 \times M_2$. Then the integral in Main Theorem is given by

$$\int_{-1}^{1}(6-2x)^8(9+3x)^8 x dx = 0 .$$

Thus $P(p_1^*L_1^2 \oplus p_2^*L_2^3)$ has an Einstein Kähler metric.

References

[1] T. Aubin: *Equations du type Monge-Ampère sur les variétés kählériennes compactes,* C.R. Acad. Sci. Paris **283** (1976), 119–121.

[2] L. Bérard Bergery: *Sur de nouvelles variétés riemanniennes d'Einstein,* Publ. de l'Inst. E. Cartan, No. 6, 1982.

[3] A. Besse: Manifolds all of whose geodesics are closed, Springer, Berlin-Heidelberg-New York, 1978.

[4] M.A. Blanchard: *Sur les variétés analytique complexes,* Ann. Sci. Ecole Norm. Sup. **73** (1956), 157–202.

[5] A. Borel and F. Hirzebruch: *Characteristic classes and homogeneous spaces* I, Amer. J. Math. **80** (1958), 458–538.

[6] A. Futaki: *An obstruction to the existence of Einstein Kähler metrics,* Invent. Math. **73** (1983), 437–443.

[7] M. Ise: *Some properties of complex analytic vector bundles over compact complex homogeneous spaces,* Osaka Math. J. **12** (1962), 217–252.

[8] K. Ishikawa and Y. Sakane: *On complex projective bundles over a Kähler C-space,* Osaka J. Math. **16** (1979), 121–132.

[9] G. Jensen: *Einstein metrics on principal fiber bundles,* J. Differential Geom. **8** (1973), 599–614.

[10] S. Kobayashi: Transformation groups in differential geometry, Springer, Berlin-Heidelberg-New York, 1972.

[11] S. Kobayashi and K. Nomizu: Foundation of differential geometry, vol. II, Interscience, New York, 1969.

[12] Y. Matsushima: *Sur la structure du groupe d'homéomorphisms analytiques d'une certaine variété kaehlérienne,* Nagoya Math. J. **11** (1957), 145–150.

[13] S. Murakami: Cohomology groups of vector-valued forms on symmetric spaces, Lecture notes, Chicago Univ., 1966.

[14] B. O'Neill: *The fundamental equations of a submersion,* Michigan Math. J. **13** (1966), 459–469.

[15] B. Shiffman and A.J. Sommese: Vanishing theorems on complex manifolds, Progress in Mathematics, 56, Birkhäuser Boston, 1985.

[16] M. Takeuchi: *Homogeneous Kähler submanifolds in complex projective spaces,* Japan. J. Math. **4** (1978), 171–219.

[17] N.R. Wallach: *Compact homogeneous Riemannian manifolds with strictly positive curvature,* Ann. of Math. **96** (1972), 277–295.

[18] H.C. Wang: *Closed manifolds with homogeneous complex structure,* Amer. J. Math. **76** (1954), 1–32.
[19] S.T. Yau: *On the curvature of compact Hermitian manifolds,* Invent. Math. **25** (1974), 213–239.
[20] ————: *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation* I, Comm. Pure Appl. Math. **31** (1978), 339–411.

Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560
Japan