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CUTTING AND PASTING OF PAIRS

Dedicated to Professor Itiro Tamura on his 60th birthday

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Introduction

For integers $m \geq n \geq 0$, an (m, n)-pair (M, N) is a pair of an m-dimensional closed smooth manifold M and an n-dimensional closed smooth submanifold N of M. In this paper we will consider for such pairs *cutting and pasting equivalence* (called briefly *SK-equivalence*) and *controllable cutting and pasting equivalence* (called briefly *SKK-equivalence*).

Karras-Kreck-Neumann-Ossa [1] considered the SK-equivalence and SKK-equivalence for closed smooth manifolds, and investigated the resulting SK-group SK_m and SKK-group SKK_m. As a natural extension of this notion we will define such equivalences for pairs and obtain the SK-group $SK_{m,n}$ and SKK-group $SKK_{m,n}$ of (m, n)-pairs. We will denote by $[M, N]^{SK}$ in $SK_{m,n}$ and $[M, N]^{SKK}$ in $SKK_{m,n}$ the class represented by an (m, n)-pair (M, N), respectively.

NOTE. We caution the reader that Karras-Kreck-Neumann-Ossa [1] uses the symbols SK^O and SKK^O to denote our SK and SKK. Their use of SK and SKK is for the oriented analogue. In the present paper we will only consider the unoriented case. So we drop the symbol "O" from the notation. Manifolds considered are all smooth (of class C^∞), and we also omit the term "smooth".

In section 1 we will consider the SK-equivalence, and obtain

Theorem 0.1. *There is a split short exact sequence*

$$0 \to SK_m \xrightarrow{i} SK_{m,n} \xrightarrow{j} SK_n \to 0$$

where the homomorphisms i and j are defined by $i([M]^{SK})=[M, \phi]^{SK}$ and $j([M, N]^{SK})=[N]^{SK}$, respectively.

Corollary 0.2. *The following* (i), (ii) *and* (iii) *are equivalent*:

(i) $[M, N]^{SK}=[M', N']^{SK}$ *in* $SK_{m,n}$,

(ii) $[M]^{SK}=[M']^{SK}$ *in* SK_m, *and* $[N]^{SK}=[N']^{SK}$ *in* SK_n,

(iii) $\chi(M)=\chi(M')$ *and* $\chi(N)=\chi(N')$, *where* $\chi(\ )$ *denotes the Euler characteristic.*

In section 2 we will consider the SKK-equivalence. In section 3 we will consider relations of the SKK-group $SKK_{m,n}$ and the unoriented cobordism group $\mathfrak{N}_{m,n}$ of (m, n)-pairs. Denote by $SKK_n(BO(m-n))$ the SKK-group of singular n-dimensional closed manifolds in the classifying space $BO(m-n)$ for $(m-n)$-dimensional vector bundles. We will then obtain

Theorem 0.3. *There is a split short exact sequence*

$$0 \to SKK_m \xrightarrow{i} SKK_{m,n} \xrightarrow{j} SKK_n(BO(m-n)) \to 0\,.$$

Here i is defined by $i([M]^{SKK})=[M, \phi]^{SKK}$. j is defined by $j([M, N]^{SKK})=[N, \nu_N]^{SKK}$, where $\nu_N\colon N \to BO(m-n)$ is a classifying map for the normal bundle of N in M.

Denote by $\mathfrak{N}_m$ the unoriented cobordism group of m-dimensional closed manifolds. Denote by $\mathfrak{N}_n(BO(m-n))$ the unoriented cobordism group of singular n-dimensional closed manifolds in $BO(m-n)$. Classes in these cobordism groups are denoted by $[\ ]^{\mathfrak{N}}$.

Corollary 0.4. *The following* (i)$\sim$(iv) *are equivalent*:

(i) $[M, N]^{SKK}=[M', N']^{SKK}$ *in* $SKK_{m,n}$,

(ii) $[M]^{SKK}=[M']^{SKK}$ *in* SKK_m, *and* $[N, \nu_N]^{SKK}=[N', \nu_{N'}]^{SKK}$ *in* $SKK_n(BO(m-n))$,

(iii) $[M]^{\mathfrak{N}}=[M']^{\mathfrak{N}}$ *in* $\mathfrak{N}_m$, $[N, \nu_N]^{\mathfrak{N}}=[N', \nu_{N'}]^{\mathfrak{N}}$ *in* $\mathfrak{N}_n(BO(m-n))$, $\chi(M)=\chi(M')$ *and* $\chi(N)=\chi(N')$,

(iv) $[M, N]^{\mathfrak{N}}=[M', N']^{\mathfrak{N}}$ *in* $\mathfrak{N}_{m,n}$, $\chi(M)=\chi(M')$ *and* $\chi(N)=\chi(N')$.

(S^m, S^n) denotes the standard pair of m-dimensional and n-dimensional spheres. Let $I_{m,n}$ be the subgroup of $SKK_{m,n}$ generated by $[S^m, S^n]^{SKK}$ and $[S^m, \phi]^{SKK}$. We will then obtain

Theorem 0.5. *There is a short exact sequence*

$$0 \to I_{m,n} \xrightarrow{i} SKK_{m,n} \xrightarrow{j} \mathfrak{N}_{m,n} \to 0$$

where i is the canonical inclusion, and j is defined by $j([M, N]^{SKK})=[M, N]^{\mathfrak{N}}$.

1. Cutting and pasting of pairs

Let X be a space. A *singular n-dimensional closed manifold* in X is an equivalence class (M, f), where M is an n-dimensional closed manifold, $f\colon M \to X$ is a map, and (M, f) is equivalent to (M', f') if there is a diffeomorphism

$\alpha\colon M\to M'$ such that $f=f'\circ\alpha$. Let $\mathcal{M}_n(X)$ be the set of singular n-dimensional closed manifolds in X. Let P and Q be n-dimensional compact manifolds, φ and $\psi\colon \partial P\to\partial Q$ be diffeomorphisms. Glueing P and Q along the boundary by φ and ψ, we then obtain n-dimensional closed manifolds $P\cup_\varphi Q$ and $P\cup_\psi Q$. Give $\mathcal{M}_n(X)$ the SK-equivalence relation $\sim$ generated by relations of the form

$$(P\cup_\varphi Q, f)\sim(P\cup_\psi Q, f'),$$

where $f\colon P\cup_\varphi Q\to X$ and $f'\colon P\cup_\psi Q\to X$ are maps for which there are homotopies $f|P\simeq f'|P$ and $f|Q\simeq f'|Q$. Then the quotient set $\mathcal{M}_n(X)/\sim$ becomes a semigroup with disjoint union as its group operation. $SK_n(X)$ is the Grothendieck group of the semigroup. If X is one point, we write SK_n for $SK_n(X)$.

Let $m\geq n\geq 0$ be integers. Let (P, Q) be a pair of an m-dimensional compact manifold P and an n-dimensional compact submanifold Q of P (with $\partial Q=Q\cap\partial P$). Let (P', Q') be another pair as above, and φ and $\psi\colon \partial P\to\partial P'$ be diffeomorphisms inducing diffeomorphisms $\varphi|\partial Q$ and $\psi|\partial Q\colon \partial Q\to\partial Q'$, respectively. We then obtain (m, n)-pairs

$$\begin{aligned}(P, Q)\cup_\varphi(P', Q') &= (P\cup_\varphi P', Q\cup_{\varphi|\partial Q}Q'), \quad\text{and}\\ (P, Q)\cup_\psi(P', Q') &= (P\cup_\psi P', Q\cup_{\psi|\partial Q}Q').\end{aligned}$$

Letting $\mathcal{M}_{m,n}$ be the set of diffeomorphism classes of (m, n)-pairs, we give $\mathcal{M}_{m,n}$ the SK-equivalence relation $\sim$ generated by relations of the form

$$(P, Q)\cup_\varphi(P', Q')\sim(P, Q)\cup_\psi(P', Q').$$

Then the quotient set $\mathcal{M}_{m,n}/\sim$ becomes a semigroup with respect to disjoint union. $SK_{m,n}$ is the Grothendieck group of the semigroup.

Theorem 1.1. *There is a split short exact sequence*

$$0\to SK_m\xrightarrow{i}SK_{m,n}\xrightarrow{j}SK_n\to 0$$

where the homomorphisms i and j are defined by $i([M]^{SK})=[M,\phi]^{SK}$ and $j([M,N]^{SK})=[N]^{SK}$, respectively.

Proof. It is clear that i is monic, j is epic and $j\circ i=0$. Letting $k\colon SK_{m,n}\to SK_m$ be the homomorphism defined by $k([M, N]^{SK})=[M]^{SK}$, we easily see that $k\circ i=$identity. Thus k gives the splitting of the sequence. It only remains to show that $\operatorname{Ker} j\subset\operatorname{Im} i$. This is proved as in Kosniowski [2; §2.6]. Every element of $SK_{m,n}$ is of the form $[M_1, N_1]^{SK}-[M_2, N_2]^{SK}$. If $[M_1, N_1]^{SK}-[M_2, N_2]^{SK}\in\operatorname{Ker} j$, then $[N_1]^{SK}=[N_2]^{SK}$ in SK_n. For $i=1, 2$ let $\nu(N_i)$ be the normal bundle of N_i in M_i, and $\nu_{N_i}\colon N_i\to BO(m-n)$ its classifying map. The augmentation homomorphism $\varepsilon\colon SK_n(BO(m-n))\to SK_n$ is an isomorphism (see Karras-Kreck-Neumann-Ossa [1; Theorem 2.11] or Kosniowski

[2; Theorem 3.5.1]). This implies that $[N_1, \nu_{N_1}]^{SK}=[N_2, \nu_{N_2}]^{SK}$ in $SK_n(BO(m-n))$, and further that

$$[RP(\nu(N_1)\oplus R), N_1]^{SK} = [RP(\nu(N_2)\oplus R), N_2]^{SK}$$

in $SK_{m,n}$. Here $RP(\ )$ denotes the associated real projective space bundle, R is the trivial line bundle over N_i, and the submanifold $RP(R)$ of $RP(\nu(N_i)\oplus R)$ is identified with N_i. Let T_i be a closed tubular neighborhood of N_i in M_i. Then we easily see that

$$(M_i, N_i) = (M_i-\mathring{T}_i, \phi)\cup(T_i, N_i)\,,$$

where $\mathring{T}_i$ denotes the interior of T_i. Letting T'_i be a closed tubular neighborhood of N_i in $RP(\nu(N_i)\oplus R)$, we see that T_i is diffeomorphic to T'_i. Let

$$K_i = (RP(\nu(N_i)\oplus R)-\mathring{T}'_i)\cup(M_i-\mathring{T}_i)\,, \quad \text{and}$$
$$K'_i = (M_i-\mathring{T}_i)\cup(M_i-\mathring{T}_i)\,.$$

It then follows that

$$[M_i, N_i]^{SK}+[K_i, \phi]^{SK} = [RP(\nu(N_i)\oplus R), N_i]^{SK}+[K'_i, \phi]^{SK}$$

in $SK_{m,n}$. Since

$$[RP(\nu(N_1)\oplus R), N_1]^{SK} = [RP(\nu(N_2)\oplus R), N_2]^{SK}\,,$$

then

$$[M_1, N_1]^{SK}-[M_2, N_2]^{SK} = [K'_1, \phi]^{SK}-[K_1, \phi]^{SK}-[K'_2, \phi]^{SK}+[K_2, \phi]^{SK}\,.$$

This shows that $[M_1, N_1]^{SK}-[M_2, N_2]^{SK}\in \mathrm{Im}\, i$. q.e.d.

From Theorem 1.1 and Karras-Kreck-Neumann-Ossa [1; Theorem 1.3a] or Kosniowski [2; Theorem 2.5.1] we obtain

Corollary 1.2. *The following* (i), (ii) *and* (iii) *are equivalent*:
(i) $[M, N]^{SK}=[M', N']^{SK}$ *in* $SK_{m,n}$,
(ii) $[M]^{SK}=[M']^{SK}$ *in* SK_m, *and* $[N]^{SK}=[N']^{SK}$ *in* SK_n,
(iii) $\chi(M)=\chi(M')$ *and* $\chi(N)=\chi(N')$.

2. Controllable cutting and pasting of pairs

We first define the SKK-group $SKK_n(X)$ of singular n-dimensional closed manifolds in a space X. Let P, P', Q and Q' be n-dimensional compact manifolds with $\partial P=\partial P'$ and $\partial Q=\partial Q'$. Let φ and ψ: $\partial P\to\partial Q$ be diffeomorphisms. Define $A_{(\varphi,\psi)}$ as the closed manifold obtained from the disjoint union of $\partial P\times[0, 1]$ and $\partial Q\times[0, 1]$ by identifying $\partial P\times\{0\}$ with $\partial Q\times\{0\}$ by φ and $\partial P\times\{1\}$ with $\partial Q\times\{1\}$ by ψ. Let f_1: $P\cup_\varphi Q\to X$, f_2: $P'\cup_\psi Q'\to X$, f_3: $P\cup_\psi Q$

$\to X$ and f_4: $P' \cup_\varphi Q' \to X$ be maps such that

(1) there are homotopies H_1: $f_1|P \simeq f_3|P$, H_2: $f_1|Q \simeq f_3|Q$, H_3: $f_2|P' \simeq f_4|P'$ and H_4: $f_2|Q' \simeq f_4|Q'$, and

(2) if F: $A_{(\varphi,\psi)} \to X$ is a map defined by $H_1|\partial P\times[0, 1]$ and $H_2|\partial Q\times[0, 1]$, and if F': $A_{(\varphi,\psi)} \to X$ is a map defined by $H_3|\partial P'\times[0, 1]$ and $H_4|\partial Q'\times[0, 1]$, then there is a homotopy $F \simeq F'$.

Then give $\mathcal{M}_n(X)$ the SKK-equivalence relation $\sim$ generated by the relations of the form

$$(*)\quad (P\cup_\varphi Q, f_1)+(P'\cup_\psi Q', f_2)\sim(P\cup_\psi Q, f_3)+(P'\cup_\varphi Q', f_4)\,,$$

where $+$ denotes the disjoint union. Define $SKK_n(X)$ as the Grothendieck group of the quotient semigroup $\mathcal{M}_n(X)/\sim$. If X is one point, we write SKK_n for $SKK_n(X)$.

Denote by $\mathfrak{N}_n(X)$ the unoriented cobordism group of singular n-dimensional closed manifolds in X.

Lemma 2.1. *There is a homomorphism $SKK_n(X)\to\mathfrak{N}_n(X)$ sending a class $[M, f]^{SKK}$ to the class $[M, f]^{\mathfrak{N}}$.*

Proof. It suffices to prove that the relation $(*)$ implies the equality

$$(**)\quad [P\cup_\varphi Q, f_1]^{\mathfrak{N}}+[P'\cup_\psi Q', f_2]^{\mathfrak{N}} = [P\cup_\psi Q, f_3]^{\mathfrak{N}}+[P'\cup_\varphi Q', f_4]^{\mathfrak{N}}$$

in $\mathfrak{N}_n(X)$. From Karras-Kreck-Neumann-Ossa [1; Lemma 1.9] we see that

$$[P\cup_\varphi Q, f_1]^{\mathfrak{N}} = [P\cup_\psi Q, f_3]^{\mathfrak{N}}+[A_{(\varphi,\psi)}, F]^{\mathfrak{N}}\,,\quad\text{and}$$
$$[P'\cup_\psi Q', f_2]^{\mathfrak{N}} = [P'\cup_\varphi Q', f_4]^{\mathfrak{N}}+[A_{(\varphi,\psi)}, F']^{\mathfrak{N}}$$

in $\mathfrak{N}_n(X)$. Since $F\simeq F'$, it follows that $[A_{(\varphi,\psi)}, F]^{\mathfrak{N}}=[A_{(\varphi,\psi)}, F']^{\mathfrak{N}}$. This shows the equality $(**)$ holds. q.e.d.

Now we define the SKK-group $SKK_{m,n}$ of (m, n)-pairs. Let (P_i, Q_i), $i=1, 2, 3, 4$, be pairs of m-dimensional compact manifolds P_i and n-dimensional compact submanifolds Q_i of P_i such that $(\partial P_1, \partial Q_1)=(\partial P_3, \partial Q_3)$ and $(\partial P_2, \partial Q_2)=(\partial P_4, \partial Q_4)$. Let φ and ψ: $\partial P_1\to\partial P_2$ be diffeomorphisms inducing diffeomorphisms $\varphi|\partial Q_1$ and $\psi|\partial Q_1$: $\partial Q_1\to\partial Q_2$, respectively. Give $\mathcal{M}_{m,n}$ the SKK-equivalence relation $\sim$ generated by the relations of the form

$$(***)\qquad \begin{aligned}&(P_1, Q_1)\cup_\varphi(P_2, Q_2)+(P_3, Q_3)\cup_\psi(P_4, Q_4)\\ &\sim(P_1, Q_1)\cup_\psi(P_2, Q_2)+(P_3, Q_3)\cup_\varphi(P_4, Q_4)\,.\end{aligned}$$

Define $SKK_{m,n}$ as the Grothendieck group of the quotient semigroup $\mathcal{M}_{m,n}/\sim$.

Let ν_1, ν_2, ν_3, and ν_4 be classifying maps of the normal bundles of $Q_1\cup_\varphi Q_2$ in $P_1\cup_\varphi P_2$, $Q_3\cup_\psi Q_4$ in $P_3\cup_\psi P_4$, $Q_1\cup_\psi Q_2$ in $P_1\cup_\psi P_2$, and $Q_3\cup_\varphi Q_4$ in $P_3\cup_\varphi P_4$, respectively. When the relation (***) holds, we see that in $\mathcal{M}_n(BO(m-n))$,

$$(Q_1 \cup_\varphi Q_2, \nu_1)+(Q_3 \cup_\psi Q_4, \nu_2) \sim (Q_1 \cup_\psi Q_2, \nu_3)+(Q_3 \cup_\varphi Q_4, \nu_4) .$$

From this we obtain a well-defined homomorphism $SKK_{m,n} \to SKK_n(BO(m-n))$ sending a class $[M, N]^{SKK}$ to the class $[N, \nu_N]^{SKK}$, where $\nu_N: N \to BO(m-n)$ is a classifying map of the normal bundle of N in M. We then obtain

Theorem 2.2. *There is a split short exact sequence*

$$0 \to SKK_m \xrightarrow{i} SKK_{m,n} \xrightarrow{j} SKK_n(BO(m-n)) \to 0$$

where i and j are the homomorphisms defined by $i([M]^{SKK})=[M, \phi]^{SKK}$ and $j([M, N]^{SKK})=[N, \nu_N]^{SKK}$.

Proof. It is easy to see that i is monic and $j \circ i=0$.

Given $(N, f) \in \mathcal{M}_n(BO(m-n))$, we take the pair $(RP(E \oplus R), N) \in \mathcal{M}_{m,n}$, where E is the pull-back by f of the universal $(m-n)$-dimensional vector bundle over $BO(m-n)$. This correspondence defines a homomorphism k: $SKK_n(BO(m-n)) \to SKK_{m,n}$, and k satisfies $j \circ k=$identity. This shows that j is epic and the sequence splits.

It now remains to show that $\operatorname{Ker} j \subset \operatorname{Im} i$. Suppose that $[M_1, N_1]^{SKK}-[M_2, N_2]^{SKK} \in \operatorname{Ker} j$. This implies that $[N_1, \nu_{N_1}]^{SKK}=[N_2, \nu_{N_2}]^{SKK}$ in $SKK_n(BO(m-n))$ and $[RP(\nu(N_1) \oplus R), N_1]^{SKK}=[RP(\nu(N_2) \oplus R), N_2]^{SKK}$ in $SKK_{m,n}$. Let T_i, T'_i, K_i and K'_i $(i=1, 2)$ be as in the proof of Theorem 1.1. We then see that in $\mathcal{M}_{m,n}$,

$$\begin{aligned}&((M_i, N_i)+(K_i, \phi))+((K_i, \phi)+(K'_i, \phi)) \\ &\quad \sim((RP(\nu(N_i) \oplus R), N_i)+(K'_i, \phi))+((K_i, \phi)+(K'_i, \phi)) .\end{aligned}$$

This shows that in $SKK_{m,n}$,

$$[M_i, N_i]^{SKK}+[K_i, \phi]^{SKK} = [RP(\nu(N_i) \oplus R), N_i]^{SKK}+[K'_i, \phi]^{SKK} .$$

From this we see that

$$\begin{aligned}&[M_1, N_1]^{SKK}-[M_2, N_2]^{SKK} \\ &= [K'_1, \phi]^{SKK}-[K_1, \phi]^{SKK}-[K'_2, \phi]^{SKK}+[K_2, \phi]^{SKK} \in \operatorname{Im} i .\end{aligned}$$

q.e.d.

Corollary 2.3. *$[M, N]^{SKK}=[M', N']^{SKK}$ holds in $SKK_{m,n}$ if and only if $[M]^{SKK}=[M']^{SKK}$ in SKK_m and $[N, \nu_N]^{SKK}=[N', \nu_{N'}]^{SKK}$ in $SKK_n(BO(m-n))$.*

Let X be a path connected space. Denote by I_n the subgroup of $SKK_n(X)$ generated by $[S^n, c]^{SKK}$, where c denotes a constant map (this map is unique up to homotopy). As in Karras-Kreck-Neumann-Ossa [1; Theorem 4.2] we obtain

Theorem 2.4. *Given a path connected space X, there is a short exact sequence*

$$0 \to I_n \xrightarrow{i} SKK_n(X) \xrightarrow{j} \mathfrak{N}_n(X) \to 0$$

where i is the canonical inclusion, and j is defined by $j([M, f]^{SKK})=[M, f]^{\mathfrak{N}}$ (see Lemma 2.1).

From Theorem 2.2, Theorem 2.4 and Karras-Kreck-Neumann-Ossa [1; Theorem 4.2] we obtain

Corollary 2.5. *$[M, N]^{SKK}=[M', N']^{SKK}$ holds in $SKK_{m,n}$ if and only if $[M]^{\mathfrak{N}}=[M']^{\mathfrak{N}}$ in $\mathfrak{N}_m$, $[N, \nu_N]^{\mathfrak{N}}=[N', \nu_{N'}]^{\mathfrak{N}}$ in $\mathfrak{N}_n(BO(m-n))$, $\chi(M)=\chi(M')$ and $\chi(N)=\chi(N')$.*

3. Cobordism of pairs

Two (m, n)-pairs (M, N) and $(M', N')\in\mathcal{M}_{m,n}$ are *cobordant*, if there exists a cobordism (K, L) between (M, N) and (M', N'), i.e., K is an $(m+1)$-dimensional compact manifold and L is an $(n+1)$-dimensional compact submanifold of K with $(\partial K, \partial L)=(M, N)+(M', N')$. The quotient set of $\mathcal{M}_{m,n}$ by this cobordism relation becomes a group with disjoint union as its group operation. We denote this group by $\mathfrak{N}_{m,n}$. Wall [3] showed that $[M, N]^{\mathfrak{N}}=[M', N']^{\mathfrak{N}}$ holds in $\mathfrak{N}_{m,n}$ if and only if $[M]^{\mathfrak{N}}=[M']^{\mathfrak{N}}$ in $\mathfrak{N}_m$ and $[N, \nu_N]^{\mathfrak{N}}=[N', \nu_{N'}]^{\mathfrak{N}}$ in $\mathfrak{N}_n(BO(m-n))$. From this fact and Corollary 2.5 we obtain

Proposition 3.1. *$[M, N]^{SKK}=[M', N']^{SKK}$ holds in $SKK_{m,n}$ if and only if $[M, N]^{\mathfrak{N}}=[M', N']^{\mathfrak{N}}$ in $\mathfrak{N}_{m,n}$, $\chi(M)=\chi(M')$, and $\chi(N)=\chi(N')$.*

From this proposition we obtain a well-defined homomorphism $j\colon SKK_{m,n} \to\mathfrak{N}_{m,n}$ sending a class $[M, N]^{SKK}$ to the class $[M, N]^{\mathfrak{N}}$, and obtain

Corollary 3.2. *If both m and n are odd, then the homomorphism $j\colon SKK_{m,n} \to\mathfrak{N}_{m,n}$ is an isomorphism.*

Denoting by $I_{m,n}$ the subgroup of $SKK_{m,n}$ generated by $[S^m, S^n]^{SKK}$ and $[S^m, \phi]^{SKK}$, we obtain

Theorem 3.3. *There is a short exact sequence*

$$0 \to I_{m,n} \xrightarrow{i} SKK_{m,n} \xrightarrow{j} \mathfrak{N}_{m,n} \to 0$$

where i is the canonical inclusion.

Proof. It is easy to see that i is monic, j is epic and $j\circ i=0$. To see that $\operatorname{Ker} j\subset\operatorname{Im} i$ let $[M, N]^{SKK}-[M', N']^{SKK}\in\operatorname{Ker} j$. Then (M, N) and (M', N')

are cobordant. Thus $\chi(M)-\chi(M')$ and $\chi(N)-\chi(N')$ are even. By Proposition 3.1 it follows that

$$[M, N]^{SKK}-[M', N']^{SKK}$$
$$=\frac{1}{2}(\chi(N)-\chi(N'))[S^m, S^n]^{SKK}+\frac{1}{2}(\chi(M)-\chi(M')-\chi(N)+\chi(N'))[S^m, \phi]^{SKK}.$$

This shows that $[M, N]^{SKK}-[M', N']^{SKK}\in \mathrm{Im}\, i$. q.e.d.

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